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AI

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO

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SUPPLEMENTO

AI

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO

DIRETTORE: **B. PETTINEO**
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PROCEEDINGS OF THE FIFTH INTERNATIONAL CONFERENCE ON FUNCTIONAL ANALYSIS AND APPROXIMATION THEORY

ACQUAFREDDA DI MARATEA (POTENZA - ITALY), JUNE 16 - 23, 2004

SERIE II - NUMERO 76 - ANNO 2005



PALERMO
SEDE DELLA SOCIETÀ
VIA ARCHIRAFI, 34

Center for Studies on
Functional Analysis and Approximation Theory
University of Basilicata
Potenza - Italy

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Preface

During the week of September 16 - 23, 2004, the Fifth International Conference on Functional Analysis and Approximation Theory was held at Acquafredda di Maratea (Potenza). Earlier meetings were held in 1989, 1992, 1996 and 2000 at the same place.

The Conference was organized by the Center for Studies on Functional Analysis and Approximation Theory of the University of Basilicata (Potenza), directed by Prof. Giuseppe Mastroianni, with the collaboration of the National Research Group "Operator Theory, Semigroups and Applications to Evolution Equations and Approximation Problems"(PRIN – COFIN 2003-2005), directed by Prof. Francesco Altomare.

Besides Francesco Altomare and Giuseppe Mastroianni, the organizing committee consisted also of Antonio Attalienti, Michele Campiti, Lorenzo D'Ambrosio, Sabrina Diomede, Donatella Occorsio and Maria Grazia Russo.

The scientific committee consisted of Francesco Altomare, Antonio Attalienti, Michele Campiti, Romulus Cristescu, Giuseppe Mastroianni, Ioan Raşa, József Szabados and Peter Vertesi,

The meeting was devoted to some significant aspects of contemporary mathematical research on Functional Analysis, Operator Theory and Approximation Theory including the applications of these fields in other areas such as partial differential equations, integral equations, numerical analysis and stochastic analysis.

A special session was dedicated to Professor Giuseppe Mastroianni on the occasion of his 65^{th} birthday, as a fair token of gratitude for his contribution to mathematics and to the organization of all the Maratea conferences (1989 – 2004).

The Conference was attended by over 180 mathematicians coming from 25 countries. The scientific program consisted of 15 plenary lectures, 4 joint section talks and about 130 communications. Several thematic sessions were arranged.

Among the main topics treated therein we mention:

- Banach spaces, approximation in normed spaces.
- Function spaces and approximation problems.
- Linear operators on Banach spaces. Positive operators. Spectral theory.
- Semigroups of operators, Feller semigroups and partial differential equations. Maximal regularity. Stochastic analysis.
- Real functions and inequalities. Complex approximation.
- Approximation by polynomials. Best approximation.
- Approximation by positive operators. Rate of convergence.
- Orthogonal polynomials.
- Integral equations.
- Interpolation and quadrature formulas.
- Integral operators, integral transforms and Fourier integrals.
- Wavelets and signal processing.
- Numerical analysis and stability

In this volume we collect some invited lectures together with a selection of the papers corresponding to the research talks, which have been carefully refereed. Our thanks go to the referees for their collaboration as well as for their accurate work.

We gratefully acknowledge the financial support of the Institutions listed below:

- Department of Mathematics of the University of Basilicata (Potenza),
- Department of Mathematics of the University of Bari,
- Department of Economics of the University of Bari,
- Department of Mathematics of the University of Lecce,
- Faculty of Sciences of the University of Basilicata,
- University of Basilicata,
- University of Bari,
- National Group for Mathematical Analysis, Probability and their Applications (G.N.A.M.P.A.),
- Basilicata Region Administration,
- Basilicata Tourism Board.

We are particularly indebted to Mirella Cappelletti Montano, Carmela De Bonis, Vita Leonessa, Sabina Milella, Graziana Musceo and Gabriella Sabino, who assisted and helped us in organizing the Conference secretariat services. To all of them we express our special thanks.

We are also grateful to Biancamaria Della Vecchia for her friendly collaboration.

Finally we would like to thank Professor P. Vetro for accepting to publish the Proceedings of the Conference in a supplement of "Rendiconti del Circolo Matematico di Palermo".

May 2005

Francesco Altomare
On behalf of the Organizing Committee

PROGRAM OF THE CONFERENCE

Thursday, June 17

☑ 9:00	OPENING CEREMONY

Plenary Lectures

	Chairman: **Vilmos Totik**
☑ 9:30-10:20	**Bernd Silbermann,** *Chemnitz, Germany* Approximation of spectra and numerical linear algebra
☑ 10:30-11:20	**Aldric Brown,** *London, United Kingdom* Metric projections in spaces of continuous functions- from Chebyshev and Haar to the present
☑ 11:30-12:00	*Break*
☑ 12:00-12:50	**Borislav Bojanov,** *Sofia, Bulgaria* Interpolation by bivariate polynomials

Section Talks

	SECTION **A** *Chairman:* **Zeev Ditzian**	SECTION **B** *Chairman:* **Romulus Cristescu**
☑ 15:00-15:30	**Peter Vertesi,** *Budapest, Hungary* Some Remarks on Paraorthogonal Polynomials	**Richard Becker,** *Paris VI, France* On normal cones and positive operators in Banach spaces
☑ 15:35-15:50	**Marie-Madeleine Derriennic,** *Rennes, France* Analogues in q-calculus of Bernstein-Durrmeyer polynomials and Jacobi polynomials	**Mariusz Szczepanik,** *Lublin, Poland* Nearly uniformly noncreasy spaces
☑ 15:55-16:10	**Ioan Rasa,** *Cluj-Napoca, Romania* Some C_0-Semigroups Related to Polynomial Operators	**Jan Fourie,** *Potchefstroom, South Africa* Absolutely (p,q)-summing multiplier sequences and functions with applications to tensor products
☑ 16:15-16:30	**Ulrich Abel,** *Friedberg, Germany* Asymptotic approximation by Dunmeyer variants of Jabinovshi-Leviatan operators	**Andrzej Kryczka,** *Madrid, Spain* Estimates of deviation from weak compactens for operators interpolated by polygon method
☑ 16:35-16:50	**Octavian Agratini,** *Cluj-Napoca, Romania* Rate of convergence of a class of Bézier-Durrmeyer type operators for functions of bounded variation	**Ilse Röntgen,** *Potchefstroom, South Africa* Strongly (p,q)-summing and cohen (p,q)-summing operators on Banach spaces
☑ 16:55-17:10	**Francoise Bastin,** *Liège, Belgium* A Riesz basis of wavelets with deficient splines and its dual	**Libor Vesely,** *Milano, Italy* One-Point Location Problem from a very General Point of View
☑ 17:10-17:30	*Break*	

	SECTION **A** *Chairman:* **Miguel Antonio Jimenez Pozo**	SECTION **B** *Chairman:* **Ehrhard Behrends**
☑ 17:30-17:45	**Dany Leviatan,** *Tel-Aviv, Israel* The Bramble-Hilbert Lemma for Convex Domains and Applications	**Jan Vybiral,** *Jena, Germany* Function Spaces with Dominating Mixed Smoothness, Decomposition Techniques, Entropy Numbers
☑ 17:50-18:05	**Marcel G. De Bruin,** *Delft, The Netherlands* Overconvergence of some next-to-interpolatory polynomials	**Carsten Michels,** *Oldenburg, Germany* Norms of tensor product identities
☑ 18:10-18:25	**Rupert Lasser,** *Munich, Germany* Homogeneous Banach spaces and Segal algebras with respect to orthogonal polynomials	**Maria J. Rivera,** *Valencia, Spain* On the natural norm in tensor products respect to complex Banach lattices
☑ 18:30-18:45	**Laura Angeloni,** *Perugia, Italy* Rate of approximation in modular spaces for nonlinear integral operators	**Christian Richter,** *Jena, Germany* Counterexamples to Knaster's conjecture
☑ 18:50-19:05	**Elena Berdysheva,** *Stuttgart, Germany* Bernstein-Durrmeyer type quasi-interpolants on intervals	**Trond Abrahamsen,** *Kristiansand, Norway* A test for bounded semivariation
☑ 19:10-19:25	**Sofiya Ostrovska,** *Ankara, Turkey* Properties of q-Bernstein polynomials	**Olav Nygaard,** *Kristiansand, Norway* Some Open Problems Concerning Massiveness

Friday, June 18

Plenary Lectures

	Chairman: **Lutz Weis**
☑ 9:00-9:50	**Nigel Kalton,** *Columbia, Missouri, U.S.A.* Greedy Algorithms and Bases
☑ 9:55-10:45	**Francisco Marcellan,** *Madrid, Spain* Asymptotics for Sobolev orthogonal polynomials: bounded and unbounded supports
☑ 10:50-11:20	*Break*
☑ 11:20-12:00	**Ehrhard Behrends,** *Berlin, Germany* The small ball property in Banach spaces (quantitative results)

Joint Section Talks

☑ 12:05-12:35	**Ursula Westphal,** *Hannover, Germany* K-Functionals related to semigroups of operators

Section Talks

	SECTION **A** *Chairman:* **Ioan Rasa**	SECTION **B** *Chairman:* **Witold Wnuk**	SECTION **C** *Chairman:* **Peter Vertesi**
☑ 15:00-15:30	**George A. Anastassiou,** *Memphis, U.S.A.* Fractional and other approximation of Csiszar's *f*-divergence	**Romulus Cristescu,** *Bucharest, Romania* Regular operators on spaces of vector valued functions	**Uwe Luther,** *Chemnitz, Germany* Weighted uniform convergence of collocation methods for Cauchy singular integral equations
☑ 15:35-15:50	**Miguel Antonio Jimenez Pozo,** *Puebla, Mexico* A class of splines with constrain length	**Oleksiy Karlovych,** *Lisbon, Portugal* Fredholmness of singular integral operators in generalized Lebesgue spaces with variable exponent	**Giuliana Criscuolo,** *Napoli, Italy* Numerical methods for nonlinear integral equations of Prandtl's type
☑ 15:55-16:10	**Ioan Gavrea,** *Cluj-Napoca, Romania* The rate of convergence by certain new Meyer-Konig and Zeller operators of finite type	**Thomas Kühn,** *Leipzig, Germany* Entropy Numbers of Embeddingd of Weighted Besov Spaces	**Alexandru Mihai Bica,** *Oradea, Romania* A new method to approximate the solutions of the first order delay neutral differential equations
☑ 16:15-16:30	**Alexander Stokolos,** *Chicago, Illinois, U.S.A.* On the rate of a.e. Approximations by Certain Integral Mean	**Miroslawa Zima,** *Rzeszow, Poland* On the Local Spectral Radius of Positive Operators	**Maria Rosaria Capobianco,** *Napoli, Italy* Some remarks on quadrature rules on unbounded intervals
☑ 16:35-16:50	**Yuliya Babenko,** *Vanderbilt, Tennessee, U.S.A.* On existence of functions with prescribed norms of its derivatives	**Vasile Postolica,** *Piatra Neamt, Romania* A survey on the best approximation in locally convex spaces	**Woula Themistoclakis,** *Napoli, Italy* Jackson Theorems via de la Vallée Poussin means
☑ 16:55-17:10	**Vasiliy Prokhorov,** *South Alabama, U.S.A.* On best approximation of analytic functions by rational functions	**Mirella Cappelletti Montano,** *Bari, Italy* Korovkin-type theorems in regular convex vector lattices of continuous functions	**Maria Italia Gualtieri,** *Cosenza, Italy* New automatic non interpolatory quadrature and cubature formulas

☑ 17:10-17:30	*Break*

	SECTION **A** *Chairman:* **József Szabados**	SECTION **B** *Chairman:* **Maria Girardi**
☑ 17:30-17:45	**Jacek Gilewicz,** *Marseille, France* Some new inequalities for N-point Padé approximation in the Stiltjes case	**Franciszek Hugon Szafraniec,** *Krakow, Poland* Operator of the q-oscillator and their relatives
☑ 17:50-18:05	**Maria Grazia Russo,** *Potenza, Italy* Some Remarks on a L^p estimate of the Lagrange interpolation error	**Frank Filbir,** *Neuherberg, Germany* Hypergroups with compact translation operators
☑ 18:10-18:25	**Donatella Occorsio,** *Potenza, Italy* An extension of Bernstein operator on the semiaxis	**Liliana Pavel,** *Bucharest, Romania* Invariant subsets of the L^p-spaces of a hypergroup
☑ 18:30-18:45	**Szilárd Révész,** *Budapest, Hungary* Turán-Markov inequalities for convex domains on the plane	**Carl Rohwer,** *Stellenbosch, South Africa* Approximation and Quasi-Inverses of Min/Max Operators
☑ 18:50-19:05	**Margit Lénárd,** *Safat, Kuwait* On weighted (0,2) interpolation	**Jasbir Singh Manhas,** *Alkhod, Oman* Composition Operators and Homomorphisms on Weighted Spaces of Analytic Functions
☑ 19:10-19:25	**Béla Nagy,** *Szeged, Hungary* Asymptotic real Bernstein inequality in L^α norm	**Francesco Tschinke,** *Palermo, Italy* C^*-Seminorms Generated by a Family of Biweights on Partial *-Algebras

Saturday, June 19

Plenary Lectures

	Chairman: **Irina Melnikova**
☑ 9:00-9:50	**Lutz Weis,** *Karlsruhe, Germany* The holomorphic H^∞-functional calculus
☑ 9:55-10:45	**Yuan Xu,** *Eugene, Oregon, U.S.A.* Polynomials of least deviation from zero in several variables
☑ 10:50-11:20	*Break*
☑ 11:20-12:00	**Niels Jacob,** *Swansea, Wales* Fractional derivatives and semigroup theory

Joint Section Talks

☑ 12:05-12:35	**Zeev Ditzian,** *Edmonton, Canada* Multivariate Averages Approximation

Section Talks

	SECTION **A** *Chairman:* **Kroó**	SECTION **B** *Chairman:* **G. Bucur**
☑ 15:00-15:15	**Francisco Javier Muñoz Delgado,** *Jaén, Spain* Hölder approximation by Berstein polynomials	**Witold Wnuk,** *Poznan, Poland* Topological Riesz spaces satisfying the Levi property
☑ 15:20-15:35	**Paolo Emilio Ricci,** *Roma, Italy* Operatorial Solutions of Boundary Value Problems in Half Space	**Nicolae Danet,** *Bucharest, Romania* Extensions of linear operators
☑ 15:40-15:55	**István Blahota,** *Nyiregyhaza, Hungary* On Vilenkin-like systems	**Florica Voicu,** *Bucharest, Romania* Fixed points for mappings in complete vector lattices
☑ 16:00-16:15	**Yamina Laskri,** *Annaba, Algeria* Asymptotics behavior for L^p extremal polynomials off a complex curve	**Angela Albanese,** *Lecce, Italy* Intersections of Fréchet spaces and (LB)-spaces
☑ 16:20-16:35	**György Gát,** *Nyiregyhaza, Hungary* On a.e. summability of Fourier series on Vilenkin groups	**José A. Conejero Casares,** *Valencia, Spain* On Mixing properties of semigroups of operators
☑ 16:40-16:55	**Rabah Khaldi,** *Annaba, Algeria* Strong Asymptotics of orthogonal polynomials on the segment [1,-1]: the case of a measure with infinite discrete part	**Alexander Brudnyi,** *Alberta, Canada* Grauert and Lax-Halmos-type theorems and extension of matrices with entries in H^∞
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **Yuan Xu**	SECTION **B** *Chairman:* **Ursula Westphal**
☑ 17:15-17:30	**Károly Nagy,** *Nyiregyhaza, Hungary* Some convergence properties of the Walsh-Kaczmarz system	**Irina Melnikova,** *Ekaterinburg, Russia* Semigroup and distribution methods as regularization for ill-posed abstract differential problems
☑ 17:35-17:50	**Miguel A. Piñar,** *Granada, Spain* Bivariate orthogonal polynomials satisfying second-order differencre equations	**Mihai Voicu,** *Bucharest, Romania* Pseudo-Resolvent on locally convex spaces
☑ 17:55-18:10	**Rodolfo Toledo,** *Nyiregyhaza, Hungary* Convergence in norm on the complete product of finite groups	**Assia Guezane-Lakoud,** *Annaba, Algeria* Boundary value problem for functional differential equations
☑ 18:15-18:30	**Joaquin F. Sanchez-Lara,** *Almeria, Spain* Computation of the Entropy of Orthogonal Polynomials	**Rachida Amiar,** *Annaba, Algeria* On the approximation of the solutions of elliptic parabolic equations by positive operators
☑ 18:35-18:50	**Lidia Fernandez,** *Granada, Spain* On multivariate classical orthogonal polynomials	**Roman Drnovšek,** *Ljublijana, Slovenia* Maximal dominated operator semigroups
☑ 18:55-19:10	**Marija Stanic,** *Kragujevac, Serbia* Multiple orthogonality and quadrature of gaussian type	**Maren Schmalmack,** *Hannover, Germany* Asymptotic behaviour of convolution semigroups
☑ 19:15-19:30	**Karol Dziedziul,** *Gdansk, Poland* Asymptotic formula in orthogonal projection	

Sunday, June 20

☑ 19:00	*Social Dinner*

Monday, June 21

Special Session dedicated to
Professor Giuseppe Mastroianni
on the occasion of his 65th birthday

Plenary Lectures

	Chairman: **Gradimir Milovanović**
☑ 9:00-9:50	**Francesco Altomare,** *Bari, Italy* On the scientific activity of Professor Giuseppe Mastroianni
☑ 9:55-10:45	**Vilmos Totik,** *Szeged, Hungary & South Florida, U.S.A.* Green's functions and approximation theory
☑ 10:50-11:20	*Break*
☑ 11:20-12:00	**Giovanni Monegato,** *Torino, Italy* Truncated interpolation processes on unbounded intervals and their applications

Joint Section Talks

☑ 12:05-12:35	**József Szabados,** *Budapest, Hungary* Weighted approximation by generalized Szàsz-Mirakjan operators

Section Talks

	SECTION **A** *Chairman:* **Marcel G. de Bruin**	SECTION **B** *Chairman:* **Gianluca Vinti**
☑ 15:00-15:15	**Sergiy Borodachov,** *Vanderbilt, Tennessee, U.S.A.* On optimization of approximate integration over a *d*-dimensional ball	**Gheorghe Bucur,** *Bucharest, Romania* Maps preserving potentials
☑ 15:20-15:35	**Michael-Ralf Skrzipek,** *Hagen, Germany* Shift strategies for Polynomial expansions	**Eggert Briem,** *Reykyavik, Iceland* Ultra non-vanish function spaces and operating functions
☑ 15:40-15:55	**Letizia Scuderi,** *Torino, Italy* Numerical Integration of Functions with Endpoints Singularities and/or Complex Poles in 3*D* Galerkin Bem	**Jacobus Grobler,** *Potchefstroom, Republic of South Africa* Embedded marginal spaces
☑ 16:00-16:15	**Włodzimierz Łenski,** *Zielona Gora, Poland* Pointwise strong approximation by orthogonal expansions	**Pawel Kolwicz,** *Poznan, Poland* Coefficient of orthogonal convexity of some Banach function spaces
☑ 16:20-16:35	**Maria Carmela De Bonis,** *Potenza, Italy* Numerical methods for CSIE in [-1,1]	**Ileana Bucur,** *Bucharest, Romania* Multiple sandwich theorem in linear ordered spaces
☑ 16:40-16:55	**Aleksandar Cvetkovic,** *Nis, Serbia* Numerical construction of generalized Hermite polynomials	**Luigi Muglia,** *Cosenza, Italy* About common solutions for differential systems
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **Ulrich Abel**	SECTION **B** *Chairman:* **Niels Jacob**
☑ 17:15-17:30	**Mircea Ivan,** *Cluj-Napoca, Romania* Mean value theorems and divided differences	**Faouzia Rebbani,** *Annaba, Algeria* Study of an abstract non-local boundary value problem
☑ 17:35-17:50	**Carmen Sanguesa,** *Zaragoza, Spain* A probabilistic approach to divided difference expansions	**Massimo Lanza de Cristoforis,** *Padova, Italy* Persistence of eigenvalues and multiplicity in the Neumann problem for the Laplace operator on nonsmooth domains
☑ 17:55-18:10	**Javier Cárcamo,** *Bilbao, Spain* Some optimal quadrature rules of probabilistic type	**David Jornet Casanova,** *Valencia, Spain* ω-hypoelliptic differential operators of constant strength
☑ 18:15-18:30	**Aicke Hinrichs,** *Jena, Germany* Covering numbers, Vapnik-Cervonenkis classes and bounds for the star-discrepancy	**Nikita Artamonov,** *Moscow, Russia* Estimates of solutions for some classes of second order differential equations in Hilbert space
☑ 18:35-18:50	**Mariusz Piotrowski,** *Jena, Germany* Fourier type 2 operators with respect to locally compact abelian groups	**Bjoern Böttcher,** *Swansea, Wales* A parametrix construction for the foundamental solution of the evolution equation associated with a pseudo-differential operator
☑ 18:55-19:10	**Iurie Caraus,** *Chisinau, Moldova* The numerical solution for the systems of singular integro- differential equations in Hölder spaces	**Alexander Potrykus,** *Swansea, Wales* Roth's method applied to some pseudo-differential operators with bounded symbols
☑ 19:15-19:30		**Anastasia Zakharova,** *Moscow, Russia* Frames and Their Generalisation

Tuesday, June 22

Plenary Lectures

	Chairman: **Dany Leviatan**
☑ 9:00-9:50	**Gradimir Milovanović,** *Niş, Serbia* Gaussian quadrature formulae of non-standard type
☑ 9:55-10:45	**Gianluca Vinti,** *Perugia, Italy* From Approximation theory to signal processing: recent developments
☑ 10:50-11:20	*Break*
☑ 11:20-12:00	**Andras Kroò,** *Budapest, Hungary* On recent Progress in Extremal Problems for Multivariate Polynomials

Joint Section Talks

☑ 12:05-12:35	**Maria Girardi,** *Karlsruhe, Germany* Integral operators with operator-valued kernels

Section Talks

	SECTION **A** *Chairman:* **Bernd Silbermann**	SECTION **B** *Chairman:* **Nigel Kalton**
☑ 15:00-15:15	**Yuri Brudnyi,** *Haifa, Israel* Interpolation and approximation of anisotropic Besov spaces	**Marian Nowak,** *Poznan, Poland* Topological properties of vector-valued function spaces
☑ 15:20-15:35	**Evgeniy Pustylnik,** *Haifa, Israel* A new class of approximation spaces	**Ahmed Zayed,** *Chicago, Illinois, U.S.A.* Interpolation of functions having values in a Banach space
☑ 15:40-15:55	**Ferenc Tookos,** *Tampa, South Florida, U.S.A.* Smoothness of Green's functions	**Camillo Trapani,** *Palermo, Italy* Recent results on *-representations of normed partial *-algebras
☑ 16:00-16:15	**Joan Cerdà,** *Barcelon, Spain* Approximation spaces, interpolation and commutators for multipliers on $\mathbf{R}^n$	**Salvatore Triolo,** *Palermo, Italy* Non-commutative L^p spaces as Banach C^*modules
☑ 16:20-16:35	**Sándor Szabó,** *Budapest, Hungary* An application of potential theory	**Eberhard Malkowsky,** *Niş, Serbia* Matrix transformations and measures of noncompactness
☑ 16:40-16:55	**Gunther Schmidt,** *Berlin, Germany* Numerical Solutions of the Lippman-Schwinger Equation	**Alessandro Trombetta,** *Cosenza, Italy* Rearrangements of functions and compactness in the space L^∞
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **Borislav Bojanov**	SECTION **B** *Chairman:* **Franciszek Hugon Szafraniec**
☑ 17:15-17:30	**Khosrow Maleknejad,** *Karaj, Iran* Numerical Solution of Non-Linear Hammerstein Equations by Using Piecewise Polynomials	**Giulio Trombetta,** *Cosenza, Italy* 1-ball contractive retractions onto the shepre
☑ 17:35-17:50	**Wolfgang zu Castell,** *Neuherberg, Germany* Positive definite functions on various domains	**Jurie Conradie,** *Cape Town, South Africa* Uniformly integrable sets and embeddings
☑ 17:55-18:10	**Agota Horváth,** *Budapest, Hungary* Weighted Fekete points as 1(W)-normal point system	**Jan Schneider,** *Jena, Germany* Function spaces of varying smoothness
☑ 18:15-18:30	**Joris van Deun,** *Heverlee, Belgium* Computing orthogonal rational functions on a halfline	

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The Small Ball Property in Banach Spaces - Quantitative Results

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Abstract. A metric space (M, d) is said to have *the small ball property (*sbp*)* if for every $\varepsilon_0 > 0$ there exists a sequence (r_n) of positive numbers with $r_n \leq \varepsilon_0$ for every n and $\lim r_n = 0$ such that M is the union of the closed balls $B(x_n, r_n), n = 1, 2, \ldots$ for suitable elements x_n of M.

In joint work with V. Kadets this property has been investigated systematically, some facts will be reviewed at the beginning of the present note.

The main results here concern two *quantitative versions* of this notion. We assign to M the infimum of the ε_0 with the above property. Surprisingly, for many natural subsets of Banach spaces this infimum is either zero (which corresponds to the *sbp*-case) or one. We also investigate the positive sequences (r_n) for which sequences (x_n) exist such that the union of the $B(x_n, r_n)$ covers M. The special case when this collection consists of *all* positive (r_n) is of particular interest since a long time. It is known that results into this direction can depend on the underlying set theory.

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1 Introduction

In many parts of mathematics one needs a notion to express the fact that some subset is "small" or "negligible". It depends, of course, on the particular structure of the sets under consideration which approach is appropriate. "Small" can mean that the measure is zero, that the set is of first category, that the Hausdorff dimension is zero or something else. Interesting connections between the classical notions of smallness are discussed in [8], for more recent results we refer the reader to chapter 6 of [2].

In a joint paper ([1]) of Vladimir Kadets and the author another notion of smallness has been studied. The definition can be found in the abstract, it applies to arbitrary metric spaces. The main results of [1] are the following:

1. The unit ball of an infinite dimensional Banach space does never have the *sbp*. However, there are incomplete spaces with an *sbp* unit ball.
2. If X is the bidual of an infinite dimensional Banach space then the collection of extreme points of the unit ball of X fails to have *sbp*.
3. σ-precompactness implies *sbp*, but the converse doesn't hold.
4. If K has *sbp* then this is not necessarily true for $K \times K$.

In this note we provide *quantitative* refinements of this approach. Here are the relevant definitions:

Definition 1.1 *Let (M, d) be a metric space and A a subset of M.*

(i) *Consider the collection of all $\varepsilon_0 > 0$ such that there exist a sequence (r_n) of positive numbers with $r_n \leq \varepsilon_0$ and $\lim r_n = 0$ and suitable $x_n \in M$ such that $A \subset \bigcup_n B(x_n, r_n)$; here $B(x, r)$ denotes the closed ball with center x and radius r.*
We denote by $sbp_M(A)$ the infimum of these ε_0.

(ii) *Let $\mathcal{S}_M(A)$ be the collection of decreasing sequences (r_n) of strictly positive numbers such that $A \subset \bigcup_n B(x_n, r_n)$ for suitable $x_n \in M$. If $\mathcal{S}_M(A)$ contains all positive decreasing sequences, then A is said to have the* Borel property*.*

Borel has asked in [3] whether only the countable subsets of $[\,0, 1\,]$ can satisfy the previous condition. This question has attracted the attention of many mathematicians, a final answer has been given in Laver's paper [6] where one also finds a sketch of the history of the subject: The assertion "Borel's conjecture is true" is consistent, i.e., the answer depends on the underlying set theory.

The *sbp*-function will be discussed in *section 2*, in *section 3* we investigate the sets $\mathcal{S}_M(A)$ and the Borel property.

2 Properties of the function $sbp_M(\cdot)$

General properties

The collection of the admissible ε_0 in the definition of $sbp_M(A)$ is always nonempty since we deal only with bounded sets A. (This boundedness condition will also be tacitly assumed in the sequel.) The function $sbp_M(\cdot)$ is obviously monotone, also it is clear that $sbp_M(A) \leq r$ if A is contained in a ball with radius r. Note that it might happen that $sbp_M(A) = r$ when A is a ball $B(x_0, r)$: see proposition 2.3 below.

Here are some further properties:

Lemma 2.1 *Let (M, d) be a metric space.*

(i) *For $A \subset M$ and $r > 0$ one has $sbp_M(A) \leq r$ iff one finds for any sequence $\varepsilon_1 \geq \varepsilon_2 \geq \cdots$ with $r \geq \varepsilon_1$ and $\varepsilon_n \to 0$ suitable finite sets $\Delta_n \subset M$ such that*

$$A \subset \bigcup_n \bigcup_{x \in \Delta_n} B(x, \varepsilon_n).$$

(ii) *If $A_1, \ldots, A_k$ are contained in M, then*

$$sbp_M\big(\bigcup_i A_i\big) = \sup_i sbp_M(A_i).$$

(iii) *The preceding formula also holds for countably many A_i's contained in M if there is a k_0 such that the A_i with $i > k_0$ have the small ball property. It is, however, not true in general.*

(iv) *Suppose that $A \subset M' \subset M$. Then*

$$sbp_M(A) \leq sbp_{M'}(A) \leq 2 sbp_M(A).$$

It is not true in general that $sbp_M(A) = sbp_{M'}(A)$.

Proof: (i) This is obvious.

(ii) By monotonicity we have $sbp_M(A_{i_0}) \leq sbp_M\big(\bigcup_i A_i\big)$ for every i_0 which proves that

$$sbp_M\big(\bigcup_i A_i\big) \geq \sup_i sbp_M(A_i).$$

Conversely, let $\varepsilon_0 > \sup_i sbp_M(A_i)$ be given. By assumption one finds positive sequences (r_n^i) tending to zero and bounded by ε_0 and centers x_n^i such that $A_i \subset \bigcup_n B(x_n^i, r_n^i)$ for every i. Any mixture of the r_n^i with

associated centers x_n^i gives rise to sequences (r_n) and (x_n) such that $r_n \to 0$, $r_n \leq \varepsilon_0$, $\bigcup_i A_i \subset \bigcup_n B(x_n, r_n)$. This proves the other inequality.

(iii) The inequality "$\geq$" is again obvious, for the reverse inequality one has to argue a little bit more subtly. Choose, for given $\varepsilon_0 > \sup_i sbp_M(A_i)$, sequences (r_n^i) and (x_n^i) for $i = 1, \ldots, k_0$ as in the preceding proof and also, for $i > k_0$, sequences (r_n^i) and centers (x_n^i) such that $\lim_n r_n^i = 0$, $r_n^i \leq \varepsilon_0/i$ and $A_i \subset \bigcup_n B(x_n^i, r_n^i)$. Then again any mixture of the r_n^i $(i, n = 1, 2, \ldots)$ will tend to zero, it will be bounded by ε_0 and $\bigcup_{n,i} B\big(x_n^i, r_n^i\big)$ will cover $\bigcup A_i$.

In order to prove that equality does not hold in general consider any infinite dimensional separable Banach space X. If B denotes the unit ball, then – as will be shown later in proposition 2.3 – one has $sbp_X(B) = 1$. Therefore $sbp_X(C) = 2$, if C stands for the ball with radius 2.

Since the space is separable C can be covered by a sequence (B_n) of translates of B. Each of these translates satisfies $sbp_X(B_n) = 1$, and this proves our claim.

(iv) The first inequality is obvious. For the proof of the second suppose that $A \subset \bigcup B(x_n, r_n)$, with $r_n \leq \varepsilon_0$, $r_n \to 0$ and $x_n \in M$. We may assume that all $B(x_n, r_n)$ meet M' (otherwise they are not necessary for the covering of A), and therefore we can choose $y_n \in M'$ with $d(x_n, y_n) \leq r_n$. Then it follows from the triangle inequality that $A \subset \bigcup_n B(y_n, 2r_n)$, and this proves the second part.

For an example where equality does not hold we consider a metric space where there exists an uncountable set A consistig of elements of mutual distance two such that A lies in a ball of radius one. (One could choose M as the Banach space $l^1(S)$ over an uncountable index set S and A as the collection of unit vectors.) Then $sbp_M(A) \leq 1$ but $sbp_A(A) = 2$. (For a more natural example see proposition 2.4.) □

Products

As it has been already noted, the product of two *sbp*-spaces needs not be *sbp*. The counterexample $K \subset X := l^\infty(l^\infty)$ in theorem 5.3 of [1] can be used to show that also the function $sbp_X(\cdot)$ behaves "badly" if products are considered.

Proposition 2.2 *The space K of [1], theorem 5.3, has the following properties:*

(i) *K lies in a ball with radius $1/2$.*

(ii) *$sbp_X(K) = 0$, i.e., K has the small ball property.*

(iii) *If $X \times X$ is provided with the maximum distance, then*

$$sbp_{X\times X}(K \times K) = 1/2,$$

i.e., this number is as large as possible.

Proof: (i) K lies in the ball $B(x, 1/2)$, where

$$x = \big((1/2, 1/2, 1/2, \ldots), (0, 0, 0, \ldots), (0, 0, 0, \ldots), \ldots\big).$$

(ii) This has been shown in [1], theorem 5.2.
(iii) It is clear that $sbp(K \times K) \leq 1/2$ since $K \times K$ lies in a ball with radius $1/2$. Let $\varepsilon_0 < 1/2$ be given. We have to prove that $K \times K$ cannot be covered by a sequence of balls in $X \times X$ for which the radii (r_n) tend to zero and are bounded by ε_0.

Let such a sequence (r_n) and centers $c_1, c_2, \ldots \in X \times X$ be given, we argue as in [1]. With the notation of this paper we have $K \times K = \bigcup_{i,j} K_i \times K_j$, where the $K_i \times K_j$ have mutual distance one. Thus every ball $B_n := B(c_n, r_n)$ will meet at most one $K_i \times K_j$. Choose m_1 such that $r_n \leq \varepsilon_0/2$ for $n \geq m_1$ and then an a_1 with the property $\big(K_{a_1} \times K_1\big) \cap \big(\bigcup_{n=1}^{m_1} B_n\big) = \emptyset$.

We write $K_{a_1} \times K_1$ as the disjoint union of the sets $K_{a_1} \times K_{1n}$. The mutual distance of these buildung blocks is $1/2$ so that every B_n with $n > m_1$ meets at most one of these. Select an $m_2 > m_1$ so that $r_n \leq \varepsilon_0/2^{a_1}$ for $n > m_2$ and then a $K_{a_1} \times K_{1a_2}$ which is disjoint to the $B_1, \ldots, B_{m_2}$.
If this construction is continued, one gets an

$$x = \big(\Phi(a_1 a_3 a_5 \ldots), \Phi(a_2 a_4 a_6 \ldots)\big)$$

such that $x \in K \times K$ but $x \notin \bigcup_n B_n$. □

Balls and spheres in Banach spaces

Unit balls in Banach spaces are always "big":

Proposition 2.3 *Let X be an infinite dimensional Banach space and B_X the closed unit ball of X. Then $sbp_X(B_X) = 1$, i.e. $sbp_X(B_X)$ is as large as possible.*

Remark: It has been kindly pointed out to us by the referee that this result is also an immediate consequence of theorem 1 in [4].

Proof: It is clear that all $\varepsilon_0 \geq 1$ are admissible so that $sbp_X(B) \leq 1$. Now let a number $\varepsilon_0 < 1$ be given. We have to show that, whenever (r_n)

is a sequences of positive numbers tending to zero and bounded by ε_0, it is impossible to find centers $x_1, x_2, \ldots$ in X such that $B_X \subset \bigcup_n B(x_n, r_n)$.

Let such a sequence (r_n) and any $x_1, x_2, \ldots \in X$ be given. We choose a positive δ such that $\varepsilon_0 + 2\delta < 1$ and then an index n_1 with the property that $r_n \leq \delta/10$ for $n \geq n_1$.

The Riesz lemma allows us to find a y_1 in B_X such that $\|y_1 - x_n\| \geq 1-\delta$ for every $n \leq n_1$; in particular we have

$$B(y_1, \delta) \cap B(x_n, r_n) = \emptyset$$

for these n.

Now we need the following consequence of the Riesz lemma: Whenever K is a ball with radius r there are $z_1, z_2, \ldots$ in K such that

- $B(z_n, r/10) \subset B$ for all n, and
- $\|z_n - z_m\| > 3r/10$ for $n \neq m$.

We apply this to the ball $B_1 := B(y_1, \delta)$. Since $r_n \leq \delta/10$ for $n \geq n_1$ we know that $B(x_n, r_n)$ will meet at most one ball $B(z_i, \delta/10)$, $i = i(n)$, for these n. Choose $n_2 > n_1$ such that $r_n \leq \delta/10^2$ for $n \geq n_2$. By the preceding observation there must be a ball[1] $B_2 := B(z_i, \delta/10)$ which has an empty intersection with the $B(x_n, r_n)$ for $n = n_1 + 1, \ldots, n_2$. One even has $B_2 \cap B(x_n, r_n) = \emptyset$ for all $n \leq n_2$ since $B(x_n, r_n) \cap B_1 = \emptyset$ and $B_2 \subset B_1$ hold.

It should be clear how to continue. We repeat the construction with B_2 instead of B_1 to get a B_3 of radius $\delta/10^2$ in B_2 such that

$$B_3 \cap \left(\bigcup_{n=1}^{n_3} B(x_n, r_n) \right) = \emptyset,$$

where n_3 is such that $r_n \leq \delta/10^3$ for $n \geq n_3$.

In this way we obtain a sequence $B_X \supset B_1 \supset B_2 \supset \cdots$ of closed balls the radii of which tend to zero such that

$$B_k \cap \left(\bigcup_{n=1}^{n_k} B(x_n, r_n) \right) = \emptyset,$$

where $n_1 < n_2 < \cdots$. By completeness there exists an x_0 in $\bigcap B_k \subset B_X$, and this x_0 cannot be contained in $\bigcup_n B(x_n, r_n)$. This completes the proof. □

With a similar proof one can provide a more natural example to the assertion made in lemma 2.1:

[1] There are even infinitely many such balls.

Proposition 2.4 *Let X be the Banach space l^1 and S the unit sphere of X. Then $sbp_X(S) = 1$ and $sbp_S(S) = 2$.*

Proof: Let $\varepsilon_0 < 2$ be given, (r_n) a sequence tending to zero with $r_n \leq \varepsilon_0$ and $x_1, x_2, \ldots$ any points in S. One has to find an $x \in S$ with $x \notin \bigcup_n B(x_n, r_n)$. Choose first a $\delta > 0$ with $\varepsilon_0 + 2\delta < 2$ and then an n_1 such that $r_n \leq \delta/10$ for $n \geq n_1$. The special structure of l^1 allows us to find y_1 in the unit sphere such that $\|y_1 - x_n\| \geq 2 - \delta$ for $n \leq n_1$; the vector y_1 could be chosen as a unit vector associated with a "large" index. Then continue as in the preceding proof, this time working in the sphere. □

Remark: More generally it can similarly be shown that $sbp_S(S) = 2^{1/p}$ for the unit sphere S in l^p, where $1 \leq p \leq \infty$.

Boundaries

We recall that a *boundary* of a Banach space X is a subset B of the dual unit ball such that for every $x \in X$ there exists $x' \in B$ such that $x'(x) = \|x\|$. It is an elementary exercise to show that the collection of extreme functionals is a boundary.

It depends on the geometry of X how large boundaries must be. Consider first the space $X = c_0$. Then $X' = l^1$, a space with countably many extreme points: this shows that boundaries can have the small ball property. On the other hand it is known ([7]) that the collection of extreme points in infinite dimensional reflexive Banach spaces is always uncountable. Theorem 4.2 of [1] explains this different behaviour: boundaries for infinite dimensional reflexive Banach spaces never have the small ball property, in particular they have to be uncountable.

Here we will show more: boundaries in infinite dimensional reflexive spaces are always "as big as possible":

Proposition 2.5 *Let X be an infinite dimensional reflexive Banach space and $B \subset X'$ a boundary. Then $sbp_{X'}(B) = 1$.*

The proof will depend on the following lemma which is a refinement of lemma 4.1 in [1]. (In [1] it was only necessary to deal with centers in $B_{X'}$, now they might be anywhere in X'.)

Lemma 2.6 *Let X be a Banach space and $Y \subset X$ an infinite dimensional closed affine subspace. Further let Δ be a finite subset of X' and a and ε numbers in $]\,0, 1\,[$ such that there exists $y_0 \in Y$ with $\|y_0\| < a$. Then there is an infinite dimensional closed affine subspace W of Y such that*

(i) *there exists* $w_0 \in W$ *with* $\|w_0\| < a + 2\varepsilon$*;*
(ii) *for every* $y' \in \bigcup_{x' \in \Delta} B(x', \varepsilon)$ *with* $\|y'\| \leq 1$ *and every* $y \in B_X \cap W$ *one has* $y'(y) < \|y\|$*, i.e. none of these* y' *recognizes* $\|y\|$*.*

Proof: Define Z as

$$Z := \{y \mid y \in Y,\ x'(y) = x'(y_0) \text{ for every } x' \in \Delta\}.$$

This is a closed infinite dimensional affine subspace of Y which contains y_0. The next steps are to select any $w_0 \in Z$ such that

$$\|y_0\| + 2\varepsilon < \|w_0\| < a + 2\varepsilon$$

and an $x_0' \in X'$ with $\|x_0'\| = 1$ and $x_0'(w_0) = \|w_0\|$. Then we define

$$W := \{y \mid y \in Z,\ x_0'(y) = x_0'(w_0)\};$$

we claim that W has the desired properties.

W is obviously a closed infinite dimensional affine subspace of Y, and $w_0 \in W$ has the claimed property. Now let $x' \in \Delta$ and z' with $\|z'\| \leq \varepsilon$ and $\|x' + z'\| \leq 1$ be given. We have to show that $x' + z'$ does not assume its norm at any $y \in W \cap B_X$.

Select any such y. Then, on the one hand, we have

$$\|y\| \geq |x_0'(y)| = |x_0'(w_0)| = \|w_0\| > \|y_0\| + 2\varepsilon.$$

On the other hand we can obtain the following inequality:

$$\begin{aligned}(x' + z')(y) &= x'(y) + z'(y) \\ &= x'(y_0) + z'(y) \\ &= (x' + z')(y_0) + z'(y) - z'(y_0) \\ &\leq \|y_0\| + 2\varepsilon.\end{aligned}$$

This completes the proof of the lemma. □

We now turn to the *proof of proposition 2.5*. Let $B \subset B_{X'}$ be a boundary for X. In order to show that $sbp_{X'}(B) = 1$ we will use lemma 2.1(i): for given $\varepsilon_0 < 1$ we have to provide a sequence (ε_n) with $\varepsilon_n \to 0$ and $\varepsilon_0 \geq \varepsilon_n$ such that for no choice of finite subsets $\Delta_n \subset X'$ one can have

$$B \subset \bigcup_n \bigcup_{x' \in \Delta_n} B(x, \varepsilon_n).$$

Choose for such an ε_0 a sequence (ε_n) with $\varepsilon_0 + 2\varepsilon_1 + 2\varepsilon_2 + \cdots < 1$. Suppose that $\Delta_1, \Delta_2, \ldots$ are any finite subsets of X'. First we apply the preceding lemma with $Y = Y_1 := X$, $a = \varepsilon_0$, $\varepsilon = \varepsilon_1$ and $\Delta := \Delta_1$. Let $Y_2 \subset Y_1$ be the space W of this lemma, we note that $Y_2 \cap B_X \neq \emptyset$ since $\|w_0\| < \varepsilon_0 + 2\varepsilon_1 < 1$.

We continue with another application of the lemma, this time with $Y := Y_2$, $a := \varepsilon_0 + 2\varepsilon_1$ and $\varepsilon := \varepsilon_2$. In this way we get a sequence $Y_1 \supset Y_2 \supset \cdots$ of closed infinite dimensional affine subspaces of X with the following properties:

- The sets $Y_n \cap B_X$ are nonempty and decreasing.
- No $y' \in B_{X'} \cap \big(\bigcup_{x' \in \Delta_n} B(x', \varepsilon_n)\big)$ assumes its norm on $Y_n \cap B_X$.

Now the reflexivity of X comes into play. Since the $Y_n \cap B_X$ are weakly compact one finds a y in the intersection of these sets. Since no y' in

$$B_{X'} \cap \big(\bigcup_n \bigcup_{x' \in \Delta_n} B(x', \varepsilon_n)\big)$$

assumes its norm at y, the set $\bigcup_n \bigcup_{x' \in \Delta_n} B(x', \varepsilon_n)$ cannot cover a boundary. This completes the proof of proposition 2.5. □

Remark: With a similar proof as in in the discussion of the small ball property in section 4 of [1] even more can be shown: boundaries B in infinite dimensional biduals satisfy $sbp(B) = 1$. The main technical difficulty is to guarantee that the space W of lemma 2.6 can be chosen to be weak*-closed if X is a dual space. This enables one to use the Alaoglu-Bourbaki theorem in order to show that the intersection of the $Y_n \cap B_X$ is nonempty.

3 The sets $\mathcal{S}_M(A)$ and the Borel property

General properties

To illustrate the definition we start with two simple *examples*.

1) First we consider $A = [\,0, 1\,]$ as a subset of the metric space $M = \mathbb{R}$. Let (r_n) be in $\mathcal{S}_M(A)$, i.e. $[\,0, 1\,] \subset \bigcup_n [\,x_n - r_n, x_n + r_n\,]$ for suitable $x_1, x_2, \ldots$ in $\mathbb{R}$. It follows that

$$1 = \lambda(A) \leq \sum_n \lambda\big([\,x_n - r_n, x_n + r_n\,]\big) = 2\sum_n r_n.$$

On the other hand, if positive r_n satisfy $1 = 2\sum_n r_n$, we may put

$$x_1 := 1 - r_1, x_k := r_2 + \cdots + r_k \text{ for } k \geq 2\ ;$$

then $[\,0,1\,] = \bigcup_n [\,x_n - r_n, x_n + r_n\,]$, and it follows that

$$\mathcal{S}_M(A) = \{(r_n) \mid \ (r_n) \text{ is decreasing, and } \textstyle\sum r_n \geq 1/2\}.$$

2) Let $C := \{0,1\}^{\mathbb{N}}$ be the *Cantor set*. As usual C is provided with the following product metric: $d\big((x_n),(y_n)\big)$ is zero if the sequences (x_n) and (y_n) are identical and 2^{1-k} otherwise, where k is the first index n with $x_n \neq y_n$.

On this metric space we consider the product measure μ associated with the uniform distribution on $\{0,1\}$. This μ is a probability measure on C for which a ball $B = B(x, 2^{-k})$ satisfies $\mu(B) = 2^{-k}$ $(k = 0, 1, \ldots)$. More generally $\mu\big(B(x,r)\big) \leq r$ is true, and this implies that $\sum_n r_n \geq 1$ whenever $C \subset \bigcup B(x_n, r_n)$ for suitable $x_1, x_2, \ldots$.

We denote, for $r > 0$, by $\alpha(r)$ the largest number 2^{-k} with $k \in \mathbb{N}_0$ and $2^{-k} \leq r$; this is a useful definition here since one has $B(x,r) = B\big(x, \alpha(r)\big)$ for all x and $r > 0$. With this notation we may rephrase the preceding observation as

$$\mathcal{S}_C(C) \subset \{(r_n) \mid \sum_n \alpha(r_n) \geq 1\}.$$

Suppose that $\sum \alpha(r_n) > 1$. Then there exists an m such that $\sum_{n=1}^m \alpha(r_n) \geq 1$ and it is easy to find centers $z_1, \ldots, z_m$ such that

$$C \subset \bigcup_{n=1}^m B\big(z_n, \alpha(r_n)\big) \subset \bigcup_{n=1}^m B(z_n, r_n).$$

If, e.g., $r_1 = 1/2$, $r_2 = r_3 = 1/4$ one could work with $z_1 = (0,0,0,\ldots)$, $z_2 = (1,1,1,\ldots)$ and $z_3 = (1,0,0,\ldots)$.

So far we have proved that $\mathcal{S}_C(C)$ contains the positive decreasing (r_n) with $\sum \alpha(r_n) > 1$ and is contained in the collection of the (r_n) with $\sum \alpha(r_n) \geq 1$.

Even more can be said. If $\sum_{n=1}^m \alpha(r_n) < 1$ for every m and $z_1, z_2, \ldots$ are abitrary, then $\big(\bigcup_{n=1}^m B(z_n, \alpha(r_n))\big)_{m=1,2,\ldots}$ is an increasing sequence of open proper subsets of C so that – by compactness – $\bigcup_n B(z_n, \alpha(r_n))$ cannot be all of C. In this way we arrive at a characterization of $\mathcal{S}_C(C)$: A decreasing sequence (r_n) of strictly positive numbers belongs to this set iff $\sum \alpha(r_n) > 1$.

In the definition of $\mathcal{S}_M(A)$ we have restricted our attention to *decreasing* strictly positive sequences. This is a reasonable restriction since positive sequences tending to zero have a decreasing rearrangemant. The following statements are obviously true:

Lemma 3.1 *Let (M,d) be a metric space and A, B bounded subsets of M such that $A \subset B$.*

(i) $\mathcal{S}_M(B) \subset \mathcal{S}_M(A)$.

(ii) *The collection $\mathcal{S}_M(A)$ has, as a subset of the space s of all sequences, the following properties:*

- *Let $(x_n) \in s$ be strictly positive and decreasing. If (x_n) contains a subsequence which is in $\mathcal{S}_M(A)$, then $(x_n) \in \mathcal{S}_M(A)$ also holds.*
- *$(x_n) \in \mathcal{S}_M(A)$ and $y_n \geq x_n$ (all n) imply that $(y_n) \in \mathcal{S}_M(A)$.*

Remark: It would be interesting to know which properties characterize the sets $\mathcal{S}_M(A)$. More precisely: Suppose $\Delta \subset s$ is a collection of positive decreasing sequences with the properties described in (ii). Is it true – and if not, which additional properties are necessary – that there exist (M,d) and $A \subset M$ such that $\mathcal{S}_M(A) = \Delta$?

The Borel property

In order to study the "size" of a set A by means of the collection $\mathcal{S}_M(A)$ it is necessary to know that it contains some nontrivial information. Clearly one can decide whether A has the small ball property and one can also calculate $sbp_M(A)$ if $\mathcal{S}_M(A)$ is known. In this subsection we deal with a very modest question: Does $\mathcal{S}_M(A)$ recognize whether A is countable? This is the Borel conjecture, the answer is rather complicated. Note that $\mathcal{S}_M(A)$ is the collection $\mathcal{S}$ of *all* decreasing strictly positive sequences if A is countable so that the problem can be rephrased by asking whether $\mathcal{S}_M(A)$ is a proper subset of $\mathcal{S}$ for uncountable A.

A canonical generalization of the ideas which have been used in the discussion of $[\,0,1\,]$ and the Cantor set $\{0,1\}^{\mathbb{N}}$ immediately leads to the following

Lemma 3.2 *Suppose that there exists a probability measure μ on M with the following properties:*

(i) $\mu(A) > 0$.

(ii) *There is a function $K :]\,0,+\infty\,[\to]\,0,+\infty\,[$ with $\lim_{r\to 0} K(r) = 0$ such that $\mu\big(B(x,r)\big) \leq K(r)$ for arbitrary x and r.*

Then $\mathcal{S}_M(A)$ is a proper subset of $\mathcal{S}$.

Proof: This is easy: $A \subset \bigcup B(x_n, r_n)$ implies $0 < \mu(A) \leq \sum_n K(r_n)$, and since $\lim K(r)$=0 this cannot be true for all positive sequences (r_n). □

There remain many cases where this idea cannot be applied directly. For example, what about the Cantor set when considered as a subset of

$[0,1]$ with the usual metric? Since the Lebesgue measure is zero one has to argue differently. In fact, the preceding idea can be used to treat the case of arbitrary uncountable compact sets:

Proposition 3.3 *Let K be an uncountable compact metric space. Then $\mathcal{S}_M(K)$ cannot be all of $\mathcal{S}$.*

Proof: First we will prove the following

Claim: If L is an uncountable compact metric space then there exist an $\varepsilon > 0$ and subsets L_1, L_2 of L with the following properties:

- Both L_1 and L_2 are uncountable and compact and the diameter is bounded by ε.
- $d(x,y) \geq \varepsilon$ for $x \in L_1$ and $y \in L_2$.

Proof of the claim: Suppose that the assertion is false. We will prove that L is countable.

Given $\varepsilon > 0$ we may cover L with compact subsets $L_1, \ldots, L_k$ of diameter at most ε. Suppose the first l of these sets are uncountable. Then they must lie in some ball with center in L_1 and radius 3ε since otherwise the claim were true. This shows that for every ε our space is countable up to a possible subset of diameter 3ε. Iterating this this procedure we know that L is a countable union of countable sets plus the intersection of possibly uncountable subsets the diameters of which tend to zero. Then L must be countable.

We now are ready for the proof of the proposition. By the claim one finds positive $\varepsilon_1 > \varepsilon_2 > \cdots$ and uncountable compact subsets $L_{i_1 i_2 \ldots i_k}$ of K for $k = 1, 2, \ldots$ and $i_\kappa \in \{0, 1\}$ with the following properties:

- The mutual distance of L_0 and L_1 is at least ε_1, and the mutual distance of $L_{i_1 i_2 \ldots i_{k-1} 0}$ and $L_{i_1 i_2 \ldots i_{k-1} 1}$ is at least ε_k for arbitrary $k \geq 2$ and $i_1, \ldots, i_{k-1}$.
- The diameter of $L_{i_1 i_2 \ldots i_k}$ is at most ε_k (all k and i_κ).
- $\lim \varepsilon_n = 0$.

The proof can now be completed easily. We note that for any $(i_n) \in C$ the intersection of the decreasing family $(L_{i_1 i_2 \ldots i_k})_{k=1,2,\ldots}$ contains precisely one point which we will call $x_{i_1 i_2 \ldots}$; here it is again essential that K is compact.

Define $\phi : C \to K$ by $(i_n) \mapsto x_{i_1 i_2 \ldots}$. It is then routine to show that ϕ is one-to-one and continuous. Denote by ν the image measure of the measure

μ on C which we have defined above. By construction, a ball of radius ε_n has measure 2^{-n}. Therefore the assertion follows from lemma 3.2. □

The Borel property obviously passes to subsets. Thus it follows from the preceding proposition that $\mathcal{S}_M(A)$ will be different from $\mathcal{S}$ whenever A contains an uncountable compact subset. This happens for "nearly all" uncountable sets. To make this precise it is necessary to repeat some notions from set theory.

Denote by $\mathcal{N}$ the space of sequences in $\mathbb{N}$, i.e. $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$; we provide $\mathcal{N}$ with the product topology. Call a subset A of a Polish space *analytic* if it is the continuous image of $\mathcal{N}$. This is a very general class of sets, it contains not only all Borel sets in arbitrary Polish spaces but even continuous images of such sets (Lemma 39.2 in [5]). It can be shown that every uncountable analytic set contains a homeomorphic image of the Cantor set C: cf. theorem 94 and lemma 4.2 in [5]. Therefore, by proposition 3.3, there are only countable sets which have the Borel property and are at the same time analytic.

To phrase it differently: For all analytic sets A one may hope to find essential properties of A encoded in $\mathcal{S}_M(A)$. Recall, however, that we have already noted that for sets which are not necessarily analytic this can not be guaranteed since the assertion

"Every set with the Borel property is countable" might be true or false depending on the model of set theory under consideration (see [6]).

References

[1] E. Behrends, V. Kadets, *Metric spaces with the small ball property*, Studia Math. **148 (3)** (2001), 275 – 287

[2] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis I*, AMS, Coll. Publ. 48, 2000.

[3] E. Borel, *Sur la classificaiton des ensembles de mesure nulle*, Bull. Soc. Math. France **47** (1919), 97-125.

[4] J. Connett, *On covering the unit ball in a Banach space*, J. London Math. Soc. (2) **7** (1973), 291–294.

[5] Th. Jech, *Set Theory*, Springer Verlag, Berlin-Heidelberg-New York, 1978.

[6] R. Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), 151–169.

[7] J. Lindenstrauss, R. R. Phelps, *Extreme point properties of convex bodies in reflexive Banach spaces*, Israel J. Math. **6** (1968), 39–48.

[8] J. C. Oxtoby, *Measure and Category*, Springer Verlag, Berlin-Heidelberg-New York, 1964.

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The Questions of Haar and Mairhuber for Five Possible Properties of a Metric Projection in a Space of Continuous Functions

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Abstract. The paper considers finite dimensional subspaces G of spaces $C_0(T)$ of real continuous functions vanishing at infinity on a locally compact Hausdorff space T and the associated set valued metric projections P_G of $C_0(T)$ onto G. The properties of the title are introduced and a summary of most of what is known concerning the properties and the two questions is presented. Thomas Fischer in 1987 defined, for compact T, a submapping of the metric projection, here called the *Fischer submapping*. The submapping and its properties are described; its application to the three properties of P_G being lower semi-continuous, having a unique continuous selection and having at least one continuous selection is discussed. The final section presents a structure theorem for T and G with the property that P_G is lower semi-continuous.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

1 Introduction

Throughout the paper T denotes a locally compact Hausdorff space, $C_0(T)$ the space of real continuous functions on T which vanish at infinity, G a finite dimensional subspace of $C_0(T)$ and $P_G : C_0(T) \to \mathcal{P}(G)$ the set valued metric projection of $C_0(T)$ onto G, that is $P_G(f)$ denotes the set of best uniform approximations from G to $f \in C_0(T)$.

The purpose of the paper is to tell a story, to add something to it and to demonstrate that it is unfinished. It begins with P. L. Chebyshev, from whose work the concept of a Chebyshev subspace G of $C_0(T)$ is derived, A. Haar, who characterised those G which are Chebyshev, and J. C. Mairhuber who determined for which spaces T there do exist non-trivial Chebyshev subspaces G of $C_0(T)$. Precise statements are given in Section 4.

The first of the five properties of the title is Chebyshevity, the others are the property of being almost Chebyshev, and the properties of having a metric projection which is lower semi-continuous, which has a unique continuous selection and which does have a continuous selection. Of each of these properties the Haar and Mairhuber questions can be asked: which G have the property ? and for which T do there exist non-trivial subspaces G of $C_0(T)$ with the property ? The five properties are introduced in Section 3. Section 4 presents most of what is in print concerning these properties and the two questions.

In 1987 Thomas Fischer, working in a more general optimisation context, defined a submapping of P_G which here is denoted P_G^F and is called the *Fischer submapping*. He established remarkable properties of P_G^F and related them to the existence of a continuous selection for P_G. In Section 5 a summary account of Fischer's work (retricted to the approximation theoretic context) is given and it is shown that it provides a new approach to the last three of the five properties. For the existence of a unique continuous selection for the metric projection and for the existence of a continuous selection the results will be described (the latter will appear elsewhere). Section 6 presents a new characterisation of those G for which the metric projection is lower semi-continuous and obtains from it a structure theorem for T and G which provides an answer to the Mairhuber question; the theorem is surprisingly and satisfyingly expressed in terms of the Haar condition.

2 Preliminaries

First we must establish our context and notation. Consider a real normed linear space $(X, \|\cdot\|)$ and a non-empty subset G of X. If $f \in X$ then

$$d(f, G) = \inf\{\|f - g\| : g \in G\}$$

is the distance of f from G and the set of best approximations to f from G is

$$\begin{aligned} P_G(f) &= \{g \in G : \|f - g\| = d(f, G)\} \\ &= B'(f, d(f, G)) \cap G \\ &\subseteq S(f, d(f, G)) \end{aligned}$$

(where $B'(f, d)$ and $S(f, d)$ denote the closed ball and the sphere in X of centre f and radius d). The set-valued mapping

$$P_G : X \to \mathcal{P}(G),$$

which in general may have empty values, is the metric projection of X onto G.

Our concern is with G which are finite dimensional linear subspaces of X. In this case for each $f \in X$ the set $P_G(f)$is a non-empty closed convex subset of G and the metric projection is upper semicontinuous.

Let T be a locally compact Hausdorff space and let $X = C_0(T)$, the space of those continuous real functions on T which vanish at infinity; let $C_0(T)$ be equipped with the uniform norm, that is $\|f\| = \max_{t \in T} |f(t)|$. Throughout the paper G will denote a finite dimensional subspace of a space $C_0(T)$ and we are concerned with properties of the metric projection P_G and their relations with properties of G and properties of T.

Later there will be a need for a natural mapping

$$e : T \to G^*$$

which will now be defined.

If X is a normed linear space let X^* denote its normed adjoint space and let $J : X \to X^{**}$ be the canonical embedding. If $\dim X < \infty$ then J is an isomorphism.

For any finite dimensional space G of real continuous functions on T, equipped with a norm topology, we introduce the mapping

$$e = e_G : T \to G^*$$

defined by $e(t)(g) = g(t)$ for each $g \in G$ and each $t \in T$; that is, $e(t)$ is the restriction to G of evaluation at t. The mapping is continuous and $\operatorname{sp} e(T) = G^*$, where $\operatorname{sp} S$ denotes the linear span of S. The space G can be recaptured from the mapping:

$$G = \{\phi \circ e : \phi \in G^{**}\}.$$

The space G is a subspace of $C_0(T)$ if and only if $e(t) \to 0$ as $t \to \infty$.

More generally if F is a finite dimensional real linear space, equipped with a norm topology, and $e : T \to F$ is a continuous mapping such that $\operatorname{sp} e(T) = F$, then G defined by

$$G = \{\phi \circ e : \phi \in F^*\}$$

and $e_G : T \to G^*$ are such that $e : T \to F$ is a copy of $e_G : T \to G^*$; if $J : F \to F^{**}$ is the canonical isomorphism and $\psi : G^* \to F^{**}$ is the adjoint of the isomorphism $F^* \overset{\phi \to \phi \circ e}{\longrightarrow} G$ then $J \circ e = \psi \circ e_G$.

Thus there is a correspondence between spaces G and mappings $e : T \to F$, and a property of G can be regarded as, or is equivalent to, a property of $e : T \to G^*$. In Section 6 the development will be largely in terms of $e : T \to G^*$.

Given a mapping $e : T \to F$ and $A \subseteq T$ we will write

$$E(A) = E_e(A) = \operatorname{sp} e(A),$$

and

$$E^u(A) = E_e^u(A) = \cap\{E_e(U) : A \subseteq \operatorname{int} U, U \subseteq T\}.$$

By the finite dimensionality of F, if $\dim E(U)$ is minimal subject to the condition $A \subseteq \operatorname{int} U$ then $E(U) = E^u(A)$.

3 Five possible properties of a metric projection

P. L. Chebyshev in the 1850s discovered the phenomenon of uniqueness of best uniform approximation from the space of polynomial functions of degree $\leq n$ to continuous funtions on an interval $[a, b]$. This was the origin of the first and strongest of the five properties with which we are concerned:

(P1) $\operatorname{card} P_G(f) = 1$ for all $f \in X$.

It is a classical result that for each non-negative integer n and each interval $[a, b]$, with $a < b$, the space of real polynomial functions of degree $\leq n$ has the property (P1) - it is now said to be a Chebyshev subspace of $C([a, b])$.

The set $P_G(f)$ is a convex set and so is a single point if and only if its dimension is zero. G. Sh. Rubinshtein ([27], see also [29]) generalised property (P1):

(P1$_k$) $\dim P_G(f) \leq k$ for all $f \in X$.

The next property was introduced and studied by A. L. Garkavi [16, 17, 15].

(P2) $X \setminus \{f : \operatorname{card} P_G(f) = 1\}$ is a meagre subset of X.

If G has the property (P2) it is said to be *almost Chebyshev*. If X is a Banach space, G is proximinal and P_G is upper semi-continuous then (P2) is equivalent to the formally weaker property:

(P2') $\{f : \operatorname{card} P_G(f) = 1\}$ is a dense subset of X.

A mapping $s : X \to G$ is a *selection* for P_G if $s(f) \in P_G(F)$ for each $f \in X$. The questions whether, and when, there exists a continuous selection for a metric projection arose naturally c. 1960 (v. [7, 8]). Sufficient conditions for the existence of a continuous selection for a set-valued mapping are provided by a celebrated theorem of Michael, of which one principal hypothesis is the *lower semi-continuity* of the set-valued mapping. Let X and Y be topological spaces and $P : X \to \mathcal{P}(Y)$ a set-valued mapping of X into Y.

Definition $P : X \to \mathcal{P}(Y)$ is *lower semi-continuous (lsc)* if $P(f) \neq \emptyset$ for all $f \in X$ and $\{f \in X : P(f) \cap V \neq \emptyset\}$ is an open subset of X for each open subset V of Y.

In our considerations Y is a metric (in fact Banach) space and in this case lower semi-continuity can be visualised as meaning that, for each $x \in X$,

$$d(y, P(x')) \to 0 \text{ as } x' \to x \text{ for each } y \in P(x).$$

Michael's Selection Theorem *If X is a paracompact Hausdorff (e.g. metric) space, Y is a Banach space and $P : X \to \mathcal{P}(Y)$ is a lower semi-continuous set-valued mapping such that $P(f)$ is a closed convex set for each $f \in X$ then there exists a continuous selection for P.*

There is a simple corollary to the theorem from which it follows that, under the hypotheses of the theorem, there is a unique continuous selection for P if and only if P is single point valued.

Corollary 3.1 *If the hypotheses of the theorem are satisfied, $f \in X$ and $g \in P(f)$ then there exists a continuous selection s for P such that $g = s(f)$.*

In the case of $G \subseteq C_0(T)$ all but one of the hypotheses of Michael's theorem are automatically satisfied by $X = C_0(T), Y = G$ and $P = P_G$ and the existence of a continuous selection for P_G is implied by the third of the properties we consider:

(P3) P_G is lower semi-continuous.

Relatively recently J. Blatter [3, 5, 4] investigated in detail a fourth property:

(P4) There exists a *unique* continuous selection for P_G.

Blatter, partly in colloboration with Fischer, has extended the classical theory of finite dimensional Chebyshev subspaces G of $C(T)$, where T is a compact space, to subspaces G of $C(T)$ which have property (P4). Bianchini [1] has further extended much of the theory to the case of locally compact T.

The last of the five properties to be formally listed is:

(P5) There exists a continuous selection for P_G.

If G is Chebyshev then the single point valued metric projection P_G is continuous. So (P1) implies each of (P2), (P3), (P4) and P(5). It is easily seen that P(2') and (P5) imply (P4). It has been noted that (P3) implies (P5).

4 The questions of Haar and Mairhuber

Haar's theorem [18] answered, for a compact subset T of a Euclidean space, the question 'What subspaces G of $C_0(T)$ have property (P1) ?' The result was extended to locally compact Hausdorff spaces by Phelps [26]. We state the result as:

(P1H) *A subspace G of $C_0(T)$ is Chebyshev if and only if*

$$\operatorname{card} g^{-1}(0) \leq \dim G - 1 \text{ for each } g \in G \setminus \{0\}$$

or, equivalently,

$$\operatorname{card} e^{-1}(M) \leq \dim G - 1 \text{ for each hyperplane } M \text{ of } G^*.$$

Either of these two conditions can be referred to as the 'Haar condition'.

The trivial cases of the theorem will be significant for us in Section 6. If $\dim G = 0$ then $G = \{0\}$ and G is by definition Chebyshev. If $\dim G = 1$ and $G = \mathrm{sp}\{g_0\}$ then G is Chebyshev if and only if $g_0^{-1}(0) = \emptyset$. Note that there exists such a $G \in C_0(T)$ if and only if T is σ-compact.

If, for example, $T = [0,1]^2$ and $\dim G \geq 2$ then there exists a $g \in G$ which changes sign and then g has a continuum of zeros, so that, by Haar's theorem, G cannot be Chebyshev. Mairhuber [25] answered, for compact T, the question 'For which spaces T does there exist a subspace G of $C_0(T)$, of finite dimension ≥ 2, with the property (P1) ?' His result was extended to locally compact T by Lutts [24]. We state the result as:

> **(P1M)** *There exists a Chebyshev subspace G of $C_0(T)$ of finite dimension ≥ 2 if and only if T is homeomorphic to a subspace of the circle S^1.*

One can in these two questions, originating with Haar and Mairhuber, substitute for (P1) any possible property of a metric projection P_G of $C_0(T)$ onto a subspace G. The rest of this section summarises most of what is known and in print about these two questions applied to ($\mathrm{P1}_k$), for $k \geq 0$, (P2), (P3), (P4) and (P5).

For T compact and for ($\mathrm{P1}_k$) the Haar question is answered [27, 29].

> **($\mathbf{P1}_k$H)** *If T is compact a subspace G has the property ($\mathrm{P1}_k$) if and only if*
>
> $$\dim G|_{\{t_1,\ldots,t_{n-k}\}} = n-k \text{ whenever } t_1, \ldots, t_{n-k} \text{ are distinct points of } T,$$
>
> *or, equivalently,*
>
> $$\dim E(A) = \mathrm{card}\, A \text{ whenever } \mathrm{card}\, A \leq n-k.$$

The Mairhuber question for ($\mathrm{P1}_k$) has been considered by Shashkin et al [6, 28]. The results are elegant but partial and the question might perhaps be reconsidered.

Garkavi obtained the following characterisation, in the case of compact metric T, of almost Chebyshev subspaces.

> **(P2H)** *If T is compact a subspace G of $C_0(T)$ is almost Chebyshev if and only if, for each open subset U of T*
>
> $$\dim\{g \in G : g|_U = 0\} \leq \max\{n - \mathrm{card}\, U, 0\}, \qquad (4.1)$$

or, equivalently, if, for each hyperplane M of G^*,

$$\operatorname{card} \operatorname{int} e^{-1}(M) = \dim E(\operatorname{int} e^{-1}(M)).$$

A proof of the characterisation for compact T, and further equivalent conditions, are given by Blatter in [4]. The author does not know whether the characterisation remains valid for locally compact T.

It is convenient to recall here the definition of a Z-space: G is a Z-space if for each $g \in G \setminus \{0\}$, $\operatorname{int} g^{-1}(0) = \emptyset$. A Z-space G satisfies Garkavi's condition.

The Haar question for lower semi-continuity has been considered by Wu Li [21] in greater generality than here. His results contain the following characterisation.

(P3H) *The metric projection P_G is lower semi-continuous if and only if, for every $g \in G \setminus \{0\}$,*

$$\operatorname{card} \operatorname{bdy} g^{-1}(0) \leq \dim\{p \in G : \operatorname{int} g^{-1}(0) \subseteq p^{-1}(0)\} - 1, \quad (4.2)$$

or, equivalently, for each hyperplane M of G^*

$$\operatorname{card} \operatorname{bdy} e^{-1}(M) + \dim E(\operatorname{int} e^{-1}(M)) \leq \dim G - 1. \quad (4.3)$$

This and a related condition are considered in detail in Section 6.

Li gives, as a corollary to the preceding result, the following partial answer to the Mairhuber question.

(pP3M) *If T is connected and P_G is lower semi-continuous then G satisfies the Haar condition. Consequently, if $\dim G \geq 2$, then by Mairhuber's theorem itself, T is either an interval or a circle.*

In Section 6 a more comprehensive result is obtained.

The following result appears to have been obtained independently by Li [22], for compact metric T, and Blatter [3], for compact T.

(P4H) *If T is compact and G satisfies (P4) then it satisfies (P2).*

There follows immediately the following equivalence.

If T is compact then a subspace G satisfies (P4) if and only if it satisfies both (P2) and (P5).

The Mairhuber question applied to (P4) is answered by a theorem of Blatter [3, p. 205]. We state here an abbreviation of Blatter's theorem; his theorem contains more detail - for example in the case that T is non-metrisable necessarily $\dim G = 2$.

> **(P4M)** *If T is compact and there exists a subspace G of $C_0(T)$, of dimension at least two, with property (P4) then either T is homeomorphic to a subspace of a circle or T is non-metrisable and is homeomorphic to a subspace of an 'interval with split points'.* (If $A \subseteq [0,1]$ then $[0,1] \times \{0\} \cup A \times \{1\}$, equipped with the lexicographic order toplogy, is said to be an interval with split points).

The conclusion of the theorem, as stated here, was obtained in [10] under the stronger assumptions that G is a Z-space with property (P5). Bianchini [2] has considered the extension of this result to locally compact T but his results are not complete.

Finally in this section the situation concerning property (P5) - there exists a continuous selection for P_G - will be described.

In 1971 an example was given in [9] of a subspace G of $C([-1,1])$, of dimension 5, which is not Chebyshev but has a metric projection for which there exists a unique continuous selection. The example was considered by G. Nürnberger and M. Sommer who, in the nineteen seventies, together and separately, investigated subspaces G of $C([a,b])$ with property (P5); they paid particular attention to spline spaces. Their work culminated in a paper by Sommer [30] which completed a characterisation of such subspaces. The result is complicated and the full account extends over several papers. For some, the work cried out for extension to spaces of functions on more general topolgical spaces. Wu Li, starting in part from the work and ideas of Nürnberger and Sommer, identified a condition on G which is necessary and sufficient for the metric projection to have a continuous selection.

> **(P5H)** *There exists a contiuous selection for P_G if and only if G is a* regularly weakly interpolating (RWI) *space.*

Li's account in his monograph [23] is long, complicated and fascinating, but difficult to read. It is also rather technical and not amenable to brief summary. A condition equivalent to 'not regularly weakly interpolating', equivalent to Li's definition and derivable in a straightforward way from it, though somewhat different in spirit, is stated in the next section.

The preceding results yield the following characterisation of those G with property (P4).

(P4H) *A subspace G has property (P4) if and only if it satisfies Garkavi's condition (4.1) and is regularly weakly interpolating.*

Blatter [3, 4] gives an alternative, equivalent, set of conditions characterising those G with (P5) which he names the *weak Haar conditions*. The conditions, obtained by Nürnberger and Sommer, which characterise those $G \subseteq C([a,b])$ with the property (P5) are not visible in the (RWI) condition. Li [23] discusses the relation between his (RWI) condition and those of Nürberger and Sommer for $C([a,b])$. Li proves, in particular, the following theorem which has an important position in his work.

If G is regularly weakly interpolating then for each $g \in G \setminus \{0\}$,

$$\operatorname{card} \operatorname{bdy} g^{-1}(0) \leq \dim\{p \in G : \operatorname{int} g^{-1}(0) \subseteq p^{-1}(0)\} \qquad (4.4)$$

or, equivalently, for each hyperplane M of G^*,

$$\operatorname{card} \operatorname{bdy} e^{-1}(M) + \dim E(\operatorname{int} e^{-1}(M)) \leq \dim G. \qquad (4.5)$$

The latter inequality should be compared with (4.3).

Li says that G is a *Quasi-Haar* space if it satisfies two conditions, both consequences of (RWI), of which (4.4) is the first. He gives an example which shows that in general a Quasi-Haar space need not be (RWI). However the second of Li's conditions is the first of a sequence of conditions, $\dim G$ in number, and all can be shown to be consequences of the (RWI) condition.

The theory so far has generated a small forest of conditions: the Haar condition, weak Chebyshevity (see, for example [30, 23], Blatter's weak Haar conditions, Li's Quasi-Haar conditions and their extension, the weakly interpolating condition [13] the *disjoint leaves condition* of Section 6 below (v. [12]) and others. It does not yet appear possible to give an efficient tour of all these conditions and the relations between them.

5 The Fischer submapping of P_G

Definition If $f \in C_0(T)$ and $g \in P_G(f)$ then, for $\eta = 1$ and $\eta = -1$,

$$\operatorname{crit}_\eta(f, \{g\}) = \{t \in T : f(t) - g(t) = \eta \|f - g\|\}.$$

If $f \in C_0(T)$ and $\emptyset \neq H \subseteq P_G(f)$ then, for $\eta = 1$ and $\eta = -1$,

$$\operatorname{crit}_\eta(f, H) = \cap_{g \in H} \operatorname{crit}_\eta(f, \{g\}).$$

If $f \notin G$ then $\operatorname{crit}_\eta(f, H)$ is compact, for $\eta = 1$ and $\eta = -1$. Write

$$\operatorname{crit}(f, H) = \operatorname{crit}_1(f, H) \cup \operatorname{crit}_{-1}(f, H).$$

If $g \in P_G(f)$ then $\operatorname{crit}(f, \{g\})$ is non-empty for each $f \in C_0(T)$.

If H is convex and $g \in \operatorname{relint} H$ then, for $\eta = 1$ and $\eta = -1$,

$$\operatorname{crit}_\eta(f, \{g\}) = \operatorname{crit}_\eta(f, H)$$

so that $\operatorname{crit}(f, H) \neq \emptyset$. It is easily shown that the restriction of H to $\operatorname{crit}(f, H)$ is a single function.

It is convenient to add here a new definition which is used in Section 6.

Definition If (C_1, C_{-1}) is a pair of compact subsets of T it will be said to be a *critical pair* if there exists $f \in C_0(T)$ such that $C_\eta = \operatorname{crit}_\eta(f, P_G(f))$ for $\eta = 1$ and -1.

The critical sets $\operatorname{crit}(f, H)$ play a fundamental role in the development which follows.

The rest of this section is a summary account of work of Thomas Fischer [14]. It differs from that in [14] in that Fischer's notation and terminology has been modified, it is restricted to the approximation problem whereas Fischer considers a more general optimisation problem, and it considers locally compact spaces T whereas Fischer's development is, formally, restricted to compact T. In the context of the approximation problem Fischer's development and the results of his which are quoted extend straightforwardly to the case of locally compact T. A self-contained exposition of all but Thm 5.6 below is given in [12].

Fischer's results, some results of Li [20] and the relations between them were considered in [11].

The foundation of Fischer's work is the sequence of definitions which now follows.

Let $f \in C_0(T)$ and let H be either $P_G(f)$ or a non-empty face, that is an extremal convex subset, necessarily closed, of $P_G(f)$. Then $\mathbb{R}(H - H)$ is a linear subspace of G. Define

$$F_H : T \to \mathbb{R}(H - H)^*$$

by

$$F_H(t)(h) = h(t), \text{ for } t \in T \text{ and } h \in \mathbb{R}(H - H).$$

If $t \in \operatorname{crit}(f, H)$ then $F_H(t) = 0$.

For $\eta = 1$ and $\eta = -1$ let $\mathcal{A}_\eta(f, H)$ be the set of $\xi \in \mathbb{R}(H - H)^* \setminus \{0\}$ such that there exist a net $((t_\alpha, \theta_\alpha) : \alpha \in \Lambda)$ in $(T \setminus \operatorname{crit}_\eta(f, H)) \times \mathbb{R}^+$ and a point $t \in \operatorname{crit}_\eta(f, H)$ such that

$$\xi = \lim_\Lambda \theta_\alpha \eta F_H(t_\alpha), \qquad t = \lim_\Lambda t_\alpha.$$

The set $\mathcal{A}_\eta(f, H)$ is a closed subset of $\mathbb{R}(H - H)^* \setminus \{0\}$. Let $\mathcal{A}(f, H) = \mathcal{A}_1(f, H) \cup \mathcal{A}_{-1}(f, H)$.

If $\mathcal{A}(f, H) \neq \emptyset$ and $\xi \in \mathcal{A}(f, H)$ choose any $h_0 \in H$ and let

$$\lambda_{f,\xi}(H) = \{h \in H : \xi(h - h_0) = \sup \xi(H - h_0)\}.$$

Then $\lambda_{f,\xi}(H)$ is a non-empty proper face of H; it is independent of the choice of h_0.

If $H' = \lambda_{f,\xi}(H)$ for some $\xi \in \mathcal{A}(f, H)$ we will say that H' is an *immediate successor* of H and will write $H' \prec H$. Define H' to be a *successor* of H if $H' = H_k \prec H_{k-1} \prec \ldots \prec H_0 = H$ for some $H_1, \ldots, H_{k-1}$. Let $\mathcal{H}^*(f, H)$ be the set of sequences $H = H_0 \succ H_1 \succ \ldots \succ H_k$. Then $\mathcal{H}^*(f, H)$ is a directed tree with root H.

If $(H_0, H_1, \ldots, H_k) \in \mathcal{H}^*(f, H)$ then

$$\dim H = \dim H_0 > \dim H_1 > \ldots > \dim H_k \geq 0$$

and so $k \leq \dim H \leq \dim G$. Therefore every sequence in $\mathcal{H}^*(f, H)$ can be extended to a maximal sequence, a *leaf* of the tree $\mathcal{H}^*(f, H)$. Clearly, a sequence $(H_0, H_1, \ldots, H_k) \in \mathcal{H}^*(f)$ is maximal if and only if $\mathcal{A}(f, H_k) = \emptyset$. In this case it will be said, abusing the language, that H_k is a *leaf* for (f, H). Fischer defines, but in our notation,

$$P^F(f, H) = \cap\{H : H \text{ is a leaf for } (f, H)\};$$

that is, $P^F(f, H)$ is the intersection of all successors of H. Further, we write

$$P^F_G(f) = P^F(f, P_G(f)).$$

It is appropriate to refer to the set valued mapping $P^F_G : C_0(T) \to \mathcal{P}(G)$ as the *Fischer submapping* of P_G, and if $P^F_G(f)$ is non-empty to refer to it as a *Fischer face* of $P_G(f)$.

The following lemma gives an important property of leaves.

Lemma 5.1 *If $f \in C_0(T)$ and H is either $P_G(F)$ or a face of $P_G(F)$ then the following statements are equivlent.*

(1) $P^F(f, H) = H$.

(2) $\mathcal{A}(f, H) = \emptyset$.

(3) *There exists a neighbourhood W of* $\operatorname{crit}(f, H)$ *such that $H|_W$ is a single function.*

A successor of H is a proper face of H and so the equivalence of (1) and (2) is immediate. It follows from the definition of $\mathcal{A}(f, H)$ that (3) implies (2). The proof that (2) implies (3) follows straightforwardly from the definition of the sets $\mathcal{A}(f, H)$.

To (3) of Lemma 5.1 we add the following lemma and definition from [12].

Lemma 5.2 *Suppose that $f \in C_0(T)$ and that H is a leaf for $(f, P_G(f))$. If $g \in G$ and*

$$\operatorname{crit}_\eta(f, H) \subseteq \operatorname{int}\{t : \eta g(t) \geq 0\} \text{ for } \eta = 1 \text{ and } \eta = -1 \tag{5.1}$$

then $\operatorname{crit}(f, H) \subseteq \operatorname{int} g^{-1}(0)$.

Definition A pair (C_1, C_{-1}) of subsets of T will be said to satisfy the *leaf condition* if C_1, C_{-1} are disjoint compact subsets of T, not both empty, such that if $g \in G$ and

$$C_\eta \subseteq \operatorname{int}\{t : \eta g(t) \geq 0\} \text{ for } \eta = 1 \text{ and } \eta = -1 \tag{5.2}$$

then $C_1 \cup C_{-1} \subseteq \operatorname{int} g^{-1}(0)$.

If one negates Li's definition of regularly weakly interpolating then a pair of finite sets satisfying the leaf condition appears. The following theorem is given by [12, Theorems 4.9 and 4.11].

Theorem 5.3 *Let G be a finite dimensional subspace of $C_0(T)$. The following three conditions are equivalent.*

(1) *G is not regularly weakly interplating.*

(2) *There exists a disjoint pair (A_1, A_{-1}) of finite subsets of T such that*

(nRWI)(1) *(A_1, A_{-1}) satisfies the leaf condition, and*

(nRWI)(2) *$E^u(A_1 \cup A_{-1}) \neq E(A_1 \cup A_{-1})$.*

(3) *There exists a disjoint pair (A_1, A_{-1}) of compact subsets of T such that the conditions (nRWI)(1) and (nRWI)(2) are satisfied.*

Two of Fischer's results, properties of the submapping P_G^F, are now stated.

Theorem 5.4 *Either all faces of $P_G(f)$ which are leaves for $(f, P_G(f))$ coincide and $P_G^F(f) \neq \emptyset$ or there exist leaves H and H' for $(f, P_G(f))$ which are disjoint.*

Theorem 5.5 *If $P_G^F(f) \neq \emptyset$ for all $f \in C_0(T)$ then the submapping $P_G^F : C_0(T) \to \mathcal{P}(G)$ of P_G is lower semi-continuous, and so, by Michael's Selection Theorem, there exists a continuous selection for P_G^F.*

Perturbation theorems and arguments have been a recurrent feature of this theory. The following theorem was proved in [11] under the assumption that T is compact. The proof extends to the case of locally compact T. The development of the theorem can be traced through the papers [19, 9, 14, 20].

Theorem 5.6 *If $f \in C_0(T)$, H is a successor of $P_G(f)$ for $(f, P_G(f))$ and $\epsilon > 0$ then there exists $f' \in C_0(T)$ such that*

(i) $\|f - f'\| \leq \epsilon$,

(ii) $P_G(f') = H$.

The following theorem is also due to Fischer [14].

Theorem 5.7 *If there exists a continuous selection for P_G then $P_G^F(f) \neq \emptyset$ for every $f \in C_0(T)$ and P_G^F is the largest lower semi-continuous submapping of P_G.*

Proof Suppose that $s : C_0(T) \to G$ is a continuous selection for P_G, that $f \in C_0(T)$ and that H is any successor of $P_G(f)$ for $(f, P_G(f))$. It follows from the perturbation theorem 5.6 that $s(f) \in H$. Consequently $s(f) \in P_G^F(f)$. Thus s is a submapping of P_G^F. It follows from the corollary to Michael's Selection Theorem that if P is a lower semi-continuous submapping of P_G then it is a submapping of P_G^F.

It is now time to return to properties (P3), (P4) and (P5) and to consider them in the light of the preceding Fischer theory.

Property (P3) The theorem which follows is an obvious consequence of Michael's Selection Theorem and Thm 5.7. The consequences of the theorem are the subject of Section 6.

Theorem 5.8 *The following conditions are equivalent.*

(1) *The subspace G has property (P3).*

(2) $P_G^F = P_G$.

(3) *$P_G(f)$ is a leaf for $(f, P_G(f))$ for each $f \in C_0(T)$.*

Property (P4) If G has property (P4) then $P_G^F(f) \neq \emptyset$ for all $f \in C_0(T)$ (Thm 5.7) and, by the corollary to Michael's Selection Theorem, $\operatorname{card} P_G^F(f) = 1$ for all $f \in C_0(T)$. Then by the perturbation theorem it follows that G has property (P2') and is almost Chebyshev, so that we have a further proof, but for locally compact T, of the fact that (P4) implies (P2).

Property (P5) The next theorem is contained in the results of Fischer [14]. It follows immediately from Theorems 5.4, 5.5 and 5.6.

Theorem 5.9 *There does not exist a continuous selection for P_G if and only if there exists $f \in C_0(T)$ such that there are disjoint leaves H_1 and H_2 for $(f, P_G(f))$.*

The theorem provides an alternative approach to the problem of characterising those subspaces G for which there is a continuous selection for P_G and an alternative solution which is more efficient than that of Li [23]. The first steps are straightforward.

Theorem 5.10 ([12]) *Let G be a finite dimensional subspace of $C_0(T)$. Suppose that $f \in C_0(T)$ and that H_1 and H_2 are disjoint leaves for $(f, P_G(f))$. If $g_j \in \operatorname{relint} H_j$, for $j = 1$ and 2, $g = g_1 - g_2$ and $C_{\epsilon,\eta} = \operatorname{crit}_\eta(f, H_\epsilon)$ for $\epsilon = 1$ and 2, and $\eta = 1$ and -1, then the pairs $(C_{1,1}, C_{1,-1})$ and $(C_{2,1}, C_{2,-1})$, together with the function g, satisfy the following conditions.*

(DLC)(1) *The pairs $(C_{1,1}, C_{1,-1})$ and $(C_{2,1}, C_{2,-1})$ satisfy the leaf condition.*

(DLC)(2) $\cup_{\eta=1,-1} C_{1,\eta} \cap C_{2,\eta} \not\subseteq \operatorname{int} g^{-1}(0)$.

(DLC)(3) **(a)** $\cup_{\eta=1,-1} C_{1,\eta} \cap C_{2,\eta} \subseteq g^{-1}(0)$,

(b) *For $\eta = 1$ and $\eta = -1$,*

$$\eta g(t) < 0 \text{ for } t \in C_{1,\eta} \setminus C_{2,\eta},$$
$$\eta g(t) > 0 \text{ for } t \in C_{2,\eta} \setminus C_{1,\eta},$$

(c) *For $\eta = 1$ and $\eta = -1$, if $t \in C_{1,-\eta} \cap C_{2,\eta}$ then $g(t) = \eta \|g\|$.*

The proof of the theorem is straightforward. The condition (DLC)(1) is a consequence of Lemma 5.2. Condition (DLC)(2) depends upon the definition of leaves and expresses the fact that $g_2 \in H_2$ but that $g_2 + g \notin H_2$. Conditions (DLC)(3) are simple consequences of the definitions of the critical sets $\mathrm{crit}_\eta(f, H_\epsilon) = \mathrm{crit}_\eta(f, \{g_\epsilon\})$. The definitions which follow are central to [12].

Definition If subsets $C_{\epsilon,\eta}$ of T and function $g \in G$ satisfy the three conditions of the theorem it will be said that they satisfy the *disjoint leaves conditions* (DLC)(1), (2) and (3). If such subsets and function exist then G will be said to satisfy the *disjoint leaves condition* (DLC).

It follows from Thm 5.9 and Thm 5.10 that if P_G has a continuous selection then G does not satisfy the disjoint leaves condition. The converse is true.

Theorem 5.11 (Alternative (P5H)) *There exists a continuous selection for the metric projection P_G if and only if the subspace G does not satisfy the disjoint leaves condition.*

The proof is completed by showing that if G satisfies the disjoint leaves condition then there exists a function $f \in C_0(T)$ such that no selection for P_G is continuous at f; the proof in [12] is a refinement of Li's proof in [23] that if G is not RWI then there is such a function f.

There now remains the question how one reconciles the two characterisations of (P5). It is easily shown that G is not RWI if and only if there are pairs $(C_{\epsilon,1}, C_{\epsilon,-1})$, $\epsilon = 1, 2$, and function $g \in C_0(T)$ which satisfy the disjoint leaves conditions together with the further conditions that the sets are finite and $C_{1,\eta} = C_{2,\eta}$ for $\eta = 1, -1$. Thus if G is not RWI then G satisfies the disjoint leaves condition. A substantial part of [12] is devoted to giving a direct proof of the converse and so of the following theorem, which, together with Thm 5.10, provides an alternative and significantly shorter proof of Li's characterisation (P5H).

Theorem 5.12 *A finite dimensional subspace G of $C_0(T)$ is not regularly weakly interpolating if and only if it satisfies the disjoint leaves condition.*

6 The function s and property (P3)

The aim of this section is to use the characterisation of those G with property (P3) obtained from Fischer's theory (Thm 5.8) to obtain an alternative

answer to Haar's question applied to (P3) (Thm 6.3) and a structure theorem (Thm 6.6) for the mapping $e : T \to G^*$ which provides an answer to the Mairhuber question.

First, working in the context of the mapping $e : T \to G^*$, the conditions (4.3) and (4.4)) are investigated.

Let $e : T \to F$, where F is a finite dimensional linear space, be a continuous mapping such that $E(T) = F$. We define a function s on the family of linear subspaces of F.

Definition If N is a linear subspace of F let

$$s(N) = \operatorname{card} \operatorname{bdy} e^{-1}(N) + \dim E(\operatorname{int} e^{-1}(N)) - \dim N. \tag{6.1}$$

Note that, by the condition $E(T) = F$, $s(F) = 0$.

If N is a subspace of F such that $N = E(e^{-1}(N))$ then there is a natural and useful interpretation of $s(N)$. In this case

$$N = E(e^{-1}(N)) = E(\operatorname{bdy} e^{-1}(N)) + E(\operatorname{int} e^{-1}(N))$$

and

$$s(N) \geq \dim E(\operatorname{bdy} e^{-1}(N)) + \dim E(\operatorname{int} e^{-1}(N)) - \dim N \geq 0. \tag{6.2}$$

Consider a basis of N chosen in the following way. Choose $t_1, \ldots, t_r$ in $\operatorname{int} e^{-1}(N)$ so that $e(t_1), \ldots, e(t_r)$ is a basis of $E(\operatorname{int} e^{-1}(N))$. Then choose $t_{r+1}, \ldots t_m$ so that $e(t_1), \ldots, e(t_r)$ is a basis of N. Then

$$N = E(\operatorname{int} e^{-1}(N)) \oplus E(\{t_{r+1}, \ldots t_m\}) \tag{6.3}$$

and

$$s(N) = \operatorname{card}(\operatorname{bdy} e^{-1}(N) \setminus \{t_{r+1}, \ldots t_m\}), \tag{6.4}$$

(s is for *surplus*). If $t \in \operatorname{bdy} \operatorname{int} e^{-1}(N) \subseteq \operatorname{bdy} e^{-1}(N)$ then $t \in (\operatorname{int} e^{-1}(N))^-$ so that $e(t) \in E(\operatorname{int} e^{-1}(N))^- = E(\operatorname{int} e^{-1}(N))$. Thus $\operatorname{bdy} \operatorname{int} e^{-1}(N)$ consists of surplus points and, if $N = E(e^{-1}(N))$ then

$$s(N) \geq \operatorname{card} \operatorname{bdy} \operatorname{int} e^{-1}(N). \tag{6.5}$$

Here we digress to give a now simple proof of a beautiful theorem of Li concerning the case $T = S^1$: in this case (P5) implies (P4).

Theorem 6.1 (Wu Li [23]) *If G is a finite dimensional subspace of $C(S^1)$ and there exists a continuous selection for the metric projection P_G then it is unique.*

Proof First note that G is a Z-space if and only if $\operatorname{int} e^{-1}(M) = \emptyset$ for each hyperplane M of G^*, and it is easily seen that this is so if and only if $\operatorname{int} e^{-1}(M) = \emptyset$ for each hyperplane M of G^* such that $M = E(e^{-1}(M))$.

Suppose that P_G has a continuous selection. Then Li's Quasi-Haar condition (4.5) is satisfied, that is $s(M) \leq 1$ for every hyperplane M of G^*. If $M = E(e^{-1}(M))$ then, by (6.5),

$$\operatorname{card} \operatorname{bdy} \operatorname{int} e^{-1}(M) \leq 1$$

while $e^{-1}(M)$ is a proper closed subset of S^1. However the circle has the property that if U is a non-empty, not dense open subset of S^1 then $\operatorname{card} \operatorname{bdy} U \geq 2$. Therefore $\operatorname{int} e^{-1}(M)$ is empty. It follows that G is a Z-space, is almost Chebyshev and the continuous selection for P_G is unique.

It will be shown (Thm 6.3) that, in the case $F = G^*$, the metric projection P_G is lower semi-continuous if and only if $e : T \to F$ stisfies the condition

(s_0) $s(N) = 0$ whenever $N \subseteq F$ and $N = E(e^{-1}(N))$.

If N is a subspace of F and $N' = E(e^{-1}(N))$ then $e^{-1}(N') = e^{-1}(N)$ and

$$s(N) = s(N') - (\dim N - \dim N') \leq s(N'). \tag{6.6}$$

It follows, appealing to (6.2), that (s_0) is equivalent to

(s_0') $s(N) \leq 0$ for each linear subspace N of F.

It will follow from Thm 6.3 and (P3H) that (s_0') and (4.3) are equivalent. It is not obvious that they are so. It can be shown that if $s(M) \leq k \in \mathbb{N}$ for each hyperplane M of F then, for each proper linear subspace N of F there exists a hyperplane M of F such that

$$s(N) \leq s(M) + (\dim M - \dim N) \tag{6.7}$$

If $N \subseteq M \subseteq F$ and $e^{-1}(M) = e^{-1}(N)$ then equality holds in (6.7). In general one can do no better than (6.7).

The following simple lemma is important for the development.

Lemma 6.2 *If N is a subspace of F and $N = E(e^{-1}(N))$ then the condition*

(1) $s(N) = 0$,

is equivalent to

(2) *(a)* $N = E(\operatorname{bdy} e^{-1}(N)) \oplus E(\operatorname{int} e^{-1}(N))$, *and*
(b) $\operatorname{card} \operatorname{bdy} e^{-1}(N) = \dim E(\operatorname{bdy} e^{-1}(N))$.

Furthermore, (1) implies

(3) $\operatorname{int} e^{-1}(N)$ *is an open and closed subset of* T.

Proof If $s(N) = 0$ then for any choice of $t_1, \ldots, t_m$, as above, $\operatorname{bdy} e^{-1}(N)$ has no surplus points so that $\{t_{r+1}, \ldots, t_m\} = \operatorname{bdy} e^{-1}(N)$. So (2)(a) follows by (6.3) and (2)(b) follows because $e(t_{r+1}), \ldots, e(t_m)$ are linearly independent.

Now suppose that (2) is satisfied and consider a choice of $t_1, \ldots, t_m$. By (2)(a), (6.3) and the inclusion $\{t_{r+1}, \ldots, t_m\} \subseteq \operatorname{bdy} e^{-1}(N)$ it follows that

$$E(\{t_{r+1}, \ldots, t_m\}) = E(\operatorname{bdy} e^{-1}(N))$$

and then by (2)(b) that $\{t_{r+1}, \ldots, t_m\} = \operatorname{bdy} e^{-1}(N)$ so that $s(N) = 0$.

(1) implies (3) by (6.5).

The equivalence of the lower semi-continuity of P_G and the condition (s_0) is the target of the next theorem whose statement, as the equivalence of six conditions, reflects the structure of the proof.

Theorem 6.3 *The condition*

(s_0) $s(N) = 0$ *for every subspace* N *of* G^* *such that* $N = E(e^{-1}(N))$,

is equivalent to each of the five conditions:

(1) P_G *is lower semicontinuous;*

(2) $P_G^F = P_G$*;*

(3) $P_G(f)$ *is a leaf for* (f, P_G) *for each* $f \in C_0(T)$*;*

(4) *If* (C_1, C_{-1}) *is a critical pair then* $E^u(C_1 \cup C_{-1}) = E(C_1 \cup C_{-1})$*;*

(5) *If* (C_1, C_{-1}) *is a critical pair then* $e^{-1}(E(C_1 \cup C_{-1}))$ *is an open and closed subset of* T.

Proof The theorem will follow from the implications

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (s_0) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3).$$

The equivalence of (1), (2) and (3) is Thm 5.8.

(3) *implies* (4). Suppose that condition (3) is satisfied. Then by the equivalence of (3) and (2), and Thm 5.5, there exists a continuous selection for P_G and, by Li's characterisation (P3H), G is (RWI). If (C_1, C_{-1}) is a critical pair then, by Lemma 5.2, (C_1, C_{-1}) satisfies the leaf condition. So

by Thm 5.3 ((1)⇒(3)) (C_1, C_{-1}) does not satisfy the condition (nRWI)(2') of that theorem, which means that (4) is satisfied.

The proof that (4) implies (s_0) requires a proposition.

Proposition 6.4 *If (A_1, A_{-1}) is a disjoint pair of finite sets, not both empty, and*

$$\sum_{\eta=1,-1} \sum_{t\in A_\eta} \alpha(t)\eta e(t) = 0 \tag{6.8}$$

for some coefficients $\alpha(t)$, positive for each $t \in A_1 \cup A_{-1}$, then there exists a critical pair (C_1, C_{-1}) such that $A_\eta \subseteq C_\eta \subseteq e^{-1}(E(A_1 \cup A_{-1}))$ for $\eta = 1$ and -1.

Proof The set $K = \{g \in G : g(A_1 \cup A_{-1}) = \{0\} \text{ and } \|g\| \leq 1\}$ is a compact subset of G. (The set K contains the zero function 0 and possibly $K = \{0\}$.) Let $g^*(t) = \sup\{g(t) : g \in K\}$. Then $g^* \in C_0(T)$ and $0 \leq g \leq 1$. If $t \in T$ then $g^*(t) = 0$ if and only if $g(t) = 0$ for all $g \in K$, from which it follows that $(g^*)^{-1}(0) = e^{-1}(E(A_1 \cup A_{-1}))$.

Let V_1 and V_{-1} be disjoint compact neighbourhoods of A_1 and A_{-1} respectively, such that $g^*(t) \leq 1/2$ for all $t \in V_1 \cup V_{-1}$. By Tietze's extension theorem there exists $\phi \in C_0(T)$ such that $\|\phi\| = 1$ and $\phi(t) = 1$ for $t \in V_1 \cup V_{-1}$, and there exists a continuous $\theta : T \to [0, 1]$ such that, for $\eta = 1$ and -1, $\theta(t) = (1 + \eta)/2$ for $t \in V_\eta$. Let

$$f = \phi(\theta(1 - g^*) + (1 - \theta)(-1 + g^*)) = -(1 - 2\theta)\phi(1 - g^*).$$

Then $f \in C_0(T)$, $\|f\| \leq 1$ and $\eta f(t) = 1$ for $t \in A_\eta$ and $\eta = 1$ and -1.

Suppose that $g \in G$ and $\|f - g\| \leq 1$. Then

$$f(t) - 1 \leq g(t) \leq 1 + f(t)$$

for all $t \in T$ and, for $\eta = 1$ and -1,

$$\eta g(t) \geq 0 \text{ for } t \in A_\eta.$$

Then, by (6.8),

$$\sum_{\eta=1,-1} \sum_{t\in A_\eta} \alpha(t)\eta g(t) = 0.$$

All the summands are non-negative so that all are zero and $g(t) = 0$ for all $t \in A_1 \cup A_{-1}$. It follows that $\|f - g\| = 1$. However $\|f - 0\| = 1$ so $d(f, G) = 1$ and $0 \in P_G(f)$.

Now, if $g \in P_G(f)$ then $\|f - g\| = 1$ and, for $t \in A_\eta$, where $\eta = 1$ or -1, $(f-g)(t) = f(t) = \eta$. Thus $A_\eta \subseteq \mathrm{crit}_\eta(f, P_G(f)) \subseteq \mathrm{crit}_\eta(f, \{0\}) = \{t : f(t) = \eta\} \subseteq (g^*)^{-1}(0) = e^{-1}(E(A_1 \cup A_{-1}))$. The proof of the proposition is complete.

Now suppose that (4) is satisfied.

If (C_1, C_{-1}) is a critical pair then, by (4), there is a neighbourhood V of $C_1 \cup C_{-1}$ such that $E(V) = E(C_1 \cup C_{-1})$ and so $C_1 \cup C_{-1} \subseteq V \subseteq e^{-1}(E(A_1 \cup A_{-1}))$ and $C_1 \cup C_{-1} \subseteq \operatorname{int} e^{-1}(E(A_1 \cup A_{-1}))$.

Now suppose that N is a subspace of G^* such that $N = E(e^{-1}(N))$. It will be shown that conditions (2)(a) and (2)(b) of Lemma 6.2 are satisfied. If either of them are not satisfied then there exist points $\tau_1, \ldots, \tau_r$ of $e^{-1}(N)$ and a relation

$$\sum_{i=1}^{r} \alpha_i e(\tau_i) = 0$$

in which the coefficients α_i are non-zero and $\{\tau_1, \ldots, \tau_r\} \cap \operatorname{bdy} e^{-1}(N) \neq \emptyset$. This relation can be rewritten in the form (6.8) with positive coefficients and

$$A_1 \cup A_{-1} = \{\tau_1, \ldots, \tau_r\} \not\subseteq \operatorname{int} e^{-1}(N).$$

By the proposition there exists a critical pair (C_1, C_{-1}) where $A_\eta \subseteq C_\eta$ for $\eta = 1$ and -1, and

$$C_1 \cup C_{-1} \subseteq e^{-1}(E(A_1 \cup A_{-1})) \subseteq e^{-1}(N)$$

and so, by the first paragraph of this proof,

$$A_1 \cup A_{-1} \subseteq C_1 \cup C_{-1} \subseteq \operatorname{int} e^{-1}(E(A_1 \cup A_{-1})) \subseteq \operatorname{int} e^{-1}(N)$$

which is a contradiction. This proves that (s_0) is satisfied.

(s_0) *implies* (5). Suppose that (s_0) is satisfied and that (C_1, C_{-1}) is a critical pair for $f \in C_0(T) \setminus G$. It may be supposed that $0 \in \operatorname{relint} P_G(f)$ so that $\mathrm{crit}_\eta(f, P_G(f)) = \mathrm{crit}_\eta(f, \{0\})$, for $\eta = 1$ and -1. Let $N = E(C_1 \cup C_{-1})$, so $C_1 \cup C_{-1} \subseteq e^{-1}(N)$ and $N = E(e^{-1}(N))$. Then, by (s_0) and Lemma 6.2 conditions (2)(a) and (b) of the lemma are satisfied. It will be shown that $N = E(e^{-1}(N))$ from which it will follow that all points of $\operatorname{bdy} e^{-1}(N)$ are surplus, so that, by (s_0), $\operatorname{bdy} e^{-1}(N) = \emptyset$ and (5) will be proved.

Suppose not, that

$$N = E(\operatorname{bdy} e^{-1}(N)) \oplus E(\operatorname{int} e^{-1}(N))$$

and that $E(\operatorname{bdy} e^{-1}(N)) \neq \{0\}$. The latter condition implies that

$$(C_1 \cup C_{-1}) \cap \operatorname{bdy} e^{-1}(N) \neq \emptyset$$

for, if not then $C_1 \cup C_{-1} \subseteq \operatorname{int} e^{-1}(N)$ and $N = E(C_1 \cup C_{-1}) = E(\operatorname{int} e^{-1}(N))$. Then

$$N^* = E(\operatorname{bdy} e^{-1}(N))^* \oplus E(\operatorname{int} e^{-1}(N))^*)$$

and there is a surjection

$$G \to \hat{G} = G^{**} \to N^*.$$

It follows that there exists a $g \in G$ such that

$$\begin{aligned} g(t) &= 0 \text{ for all } t \in \operatorname{int} e^{-1}(N)) \\ g(t) &= f(t) \text{ for all } t \in \operatorname{bdy} e^{-1}(N)) \end{aligned}$$

(by (2)(b) the images of the points of $\operatorname{bdy} e^{-1}(N)$ form a basis of $E(\operatorname{bdy} e^{-1}(N)))$. Now $\operatorname{int} e^{-1}(N))$ is an open and closed subset of T and $T = (\operatorname{int} e^{-1}(N)) \cup (T \setminus \operatorname{int} e^{-1}(N))$. It will be shown that if $\theta \geq 0$ is small then $\theta g \in P_G(f)$. But, if $t \in (C_1 \cup C_{-1}) \cap \operatorname{bdy} e^{-1}(N) \neq \emptyset$ then $|(f - \theta g)(t)| < |f(t)| = \|f\|$ which is a contradiction.

If $t \in \operatorname{int} e^{-1}(N)$ then $|(f - \theta g)(t)| = |f(t)| = \|f\| = d(f, G)$. If $t \notin \operatorname{int} e^{-1}(N)$ then either $t \in (C_1 \cup C_{-1}) \cap \operatorname{bdy} e^{-1}(N)$ and $|f(t)| = \|f\| \neq 0$, or $|f(t)| \leq \|f\|$. Now $g(t) = f(t)$ for all $t \in \operatorname{bdy} e^{-1}(N)$ and so $g(t)f(t) > 0$ for all t in some neighbourhood W of $(C_1 \cup C_{-1}) \cap \operatorname{bdy} e^{-1}(N)$. Now $\sup\{|f(t)| : t \in (T \setminus \operatorname{int} e^{-1}(N)) \setminus W\} < \|f\|$. It follows that if $\theta > 0$ is small then $|(f - \theta g)(t)| < \|f\|$ for all $t \in T \setminus \operatorname{int} e^{-1}(N)$. This proves that $\|f - \theta g\| \leq \|f\|$ and $\theta g \in P_G(f)$. This completes the proof that (s_0) implies (5).

(5) $\Rightarrow$ (4) $\Rightarrow$ (3). Suppose that (5) is satisfied and consider $f \in C_0(T)$. Let $C_\eta = \operatorname{crit}_\eta(f, P_G(f))$ for $\eta = 1$ and -1. Let W be the open and closed set $e^{-1}(E(C_1 \cup C_{-1}))$. However $C_1 \cup C_{-1} \subseteq W$ and so $E^u(C_1 \cup C_{-1}) \subseteq E(W) \subseteq E(C_1 \cup C_{-1})$ and the inclusions are equalities. This proves (4). If (4) is satisfied then

$$E(W) = E^u(C_1 \cup C_{-1}) = E(C_1 \cup C_{-1})$$

for some open neighbourhood W of $C_1 \cup C_{-1}$. If $g \in P_G(f) - P_G(f)$ then $g|_{C_1 \cup C_{-1}} = 0$ and it follows that $g|_W = 0$. Then $\mathcal{A}(f, P_G(f)) = \emptyset$ and $P_G(f)$ is a leaf for $(f, P_G(f))$ by Lemma 5.1. This proves (3) and the proof of the theorem is complete.

The final result is a structure theorem for a mapping $e : T \to F$ (as before, continuous and such that $E(T) = F$) which satisfies condition (s_0) and so also for T and subspaces G of $C_0(T)$ which have a lower semi-continuous metric projection. The theorem is a straightforward consequence of the following lemma. In the statement of the lemma $e : T \to F$ is replaced by $e^{(0)} : T^{(0)} \to F_0$. A quotient mapping which arises will be denoted $\pi_1 : F_0 \to F_0/F_1$.

Lemma 6.5 *Suppose that $e^{(0)} : T^{(0)} \to F_0$ satisfies the condition (s_0) and that $E(W) \neq F_0$ for some non-empty open subset W of T. Then there exists a subspace $F_1 \neq F_0$ of F_0 such that $T^{(1)} = e^{-1}(F_1)$ and $T_1 = T^{(0)} \setminus T^{(1)}$ are open and closed subsets of $T^{(0)}$, $E(T^{(1)}) = F_1$ and the mappings*

$$e_1 = \pi_1 e|_{T_1} : T_1 \to F_0 \to F_0/F_1 \tag{6.9}$$

and

$$e^{(1)} = e|_{T^{(1)}} : T^{(1)} \to F_1 \tag{6.10}$$

have the properties

(a) $\operatorname{int} e_1^{-1}(N) = \emptyset$ *whenever N is a proper subspace of F_0/F_1,*

(b) $e_1 : T_1 \to F_0/F_1$ *satisfies the Haar condition, and*

(c) $e^{(1)} : T^{(1)} \to F_1$ *satisfies the condition (s_0).*

Comversely, if F_1, T_1, $T^{(1)}$, e_1 and $E^{(1)}$ are related as in the preceding statement, have the properties (b) and (c) and, further,

$$F_0 = E(T_1) \oplus E(T^{(1)}), \tag{6.11}$$

then $e^{(0)} : T^{(0)} \to F_0$ satisfies the condition (s_0).

Proof For the duration of the proof the subscript and superscript in $e^{(0)} : T^{(0)} \to F_0$ will be suppressed.

Suppose that $e : T \to F$ satisfies (s_0) and that $\operatorname{int} e^{-1}(N) \neq \emptyset$ for some $N \subseteq F$ such that $N = E(e^{-1}(N)) \neq F$, and so $\operatorname{int} e^{-1}(N) \neq T$ and $E(\operatorname{int} e^{-1}(N) \neq F$. Choose a non-empty open subset W of T such that $\dim E(W)$ is maximal subject to the condition $E(W) \neq F$ and let $F_1 = E(W)$.

Now $W \subseteq \operatorname{int} e^{-1}(F_1)$ and

$$F_1 = E(W) \subseteq E(\operatorname{int} e^{-1}(F_1) \subseteq F_1 \neq F$$

so that $F_1 = E(\operatorname{int} e^{-1}(F_1)) = E(e^{-1}(F_1))$. Now, by (s_0), $s(F_1) = 0$ and so $\operatorname{card} \operatorname{bdy} e^{-1}(F_1) = 0$ and the subset $e^{-1}(F_1)$ of T is both closed and open.

Next (a) will be established; it is a consequence of the maximality property of $\dim F_1$. Suppose that $N \subseteq F/F_1$ and $N = E_{e_1}(e_1^{-1}(N)) \neq F/F_1$. Let $N' = \pi_1^{-1}(N) \supseteq F_1$. Then $N' \neq F$ and

$$e^{-1}(N') = e_1^{-1}(N) \cup T^{(1)}. \tag{6.12}$$

It follows that

$$\begin{aligned} E(e^{-1}(N')) &= E(e_1^{-1}(N)) + F_1 \\ &= \pi_1^{-1}(\pi_1 E(e_1^{-1}(N)) = \pi_1^{-1}(E_{e_1}(e_1^{-1}(N))) = N'. \end{aligned} \tag{6.13}$$

Now

$$\operatorname{int} e^{-1}(N') = T^{(1)} \cup \operatorname{int} e_1^{-1}(N)$$

and

$$E(\operatorname{int} e^{-1}(N')) = F_1 + E(\operatorname{int} e_1^{-1}(N)). \tag{6.14}$$

By the maximal property of $\dim F_1$ it follows that $E(\operatorname{int} e_1^{-1}(N)) \subseteq F_1$ and $\operatorname{int} e_1^{-1}(N) \subseteq T^{(1)}$, so that $\operatorname{int} e_1^{-1}(N) = \emptyset$. This proves (a)

Now by (a), (6.12), condition (s_0) for $e : T \to F$ applied to N', and (6.14),

$$\begin{aligned} \operatorname{card} e_1^{-1}(N) &= \operatorname{card} \operatorname{bdy} e_1^{-1}(N) = \operatorname{card} \operatorname{bdy} e^{-1}(N') \\ &= \dim N' - \dim E(\operatorname{int} e^{-1}(N')) = \dim N' - \dim F_1 = \dim N. \end{aligned}$$

This proves (b).

The condition (c), that is (s_0) for $e^{(1)} : T^{(1)} \to F_1$, is inherited from the condition (s_0) for $e : T \to F$.

Given (6.11), (b) and (c), the mappings $e_1; T_1 \to E(T_1) \cong F/F_1$ and $e^{(1)} : T^{(1)} \to E(T^{(1)}) = F_1$ both satisfy the condition (s_0) and it follows that $e : T \to F$ does so also.

The following structure theorem is now a straightforward consequence of the lemma. In the statement π_j denotes the quotient mapping $F_{j-1} \to F_{j-1}/F_j$.

Theorem 6.6 *If P_G is lower semi-continuous then there exist $r \geq 1$, disjoint non-empty open and closed subspaces $T_1, \ldots, T_r$ of T and subspaces $F = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{r-1} \supseteq F_r = \{0\}$ of $F = G^*$ such that*

(α) $T = T_1 \cup \ldots \cup T_r$,

(β) $F_j = E(T_{j+1} \cup \ldots \cup T_r) \neq F_{j-1}$ *for* $j = 1, \ldots r$, *and*

(γ) *the mappings*

$$e_j = \pi_j e|_{T_j} : T_j \to F_{j-1} \to F_{j-1}/F_j, \quad j = 1, \ldots, r,$$

all satisfy the Haar condition and $\operatorname{int} e_j^{-1}(N) = \emptyset$ *whenever* N *is a proper subspace of* F_{j-1}/F_j.

Conversely, if there exist such $r \geq 1$, $T_1, \ldots, T_r$ *and* $F_1, \ldots, F_{r-1}$ *and, further,*

(δ) $F = \oplus_{j=1}^{r} E(T_j)$,

then P_G *is lower semi-continuous.*

Proof By Theorem 6.3 the lower semi-continuity of P_G means that $e : T \to G^*$ satisfies the condition (s_0).

Note that if $e : T \to F$ satisfies (s_0) and $\operatorname{int} e^{-1}(N) = \emptyset$ whenever $N = E(e^{-1}(N)) \neq F$ then for each such N

$$\operatorname{card} e^{-1}(N) = \operatorname{card} \operatorname{bdy} e^{-1}(N) = \dim N,$$

so that the Haar condition is satisfied. Thus either there exists a non-empty open subset W of T such that $E(W) \neq F$ or there does not and the Haar condition is satisfied.

Now repeated application of the lemma to $e^{(0)} : T^{(0)} \to F_0 = G^*$, $e^{(1)} : T^{(1)} \to F_1$, ..., in turn, must terminate with an $e^{(r-1)} : T^{(r-1)} \to F_{r-1}$ satisfying the condition (γ). Let $T_r = T^{(r-1)}$. This proves the existence of elements satisfying (α), (β) and (γ). The final part of the theorem now follows directly from the lemma.

The theorem has some obvious consequences. The condition $s(\{0\}) = 0$ implies that $e^{-1}(0)$ is an open and closed subset of T. If $e^{-1}(0) \neq \emptyset$ then necessarily $F_{r-1} = \{0\}$ and $T_r = e^{-1}(0)$. In this case T_r may be any locally compact Hausdorff space. If $1 \leq j \leq r-1$ or $j = r$ and $\dim F_{r-1} \geq 1$ then T_j is a σ-compact locally compact Hausdorff space. If $1 \leq j \leq r$ and T_j contains an isolated point then, by (γ), $\dim F_{j-1}/F_j \leq 1$. If $\dim F_{j-1}/F_j > 1$ then, by Mairhuber's theorem T_j is homeomorphic to a subset of the circle S^1. The theorem provides an answer to the Mairhuber question applied to the property (P3). It falls short (perhaps by not much) of being a characterisation of those G for which P_G is lower semi-continuous.

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References

[1] W. Bianchini. Uma Extensão da Teoria de Chebyshev para Espaços Localmente Compactos. Ph.D. thesis, Instituto de Matemática da Universidade Federal do Rio de Janeiro (1994).

[2] W. Bianchini. *An extension of Mairhuber's theorem for subspaces of $C_0(X)$ with a unique continuous selection*. Rend. Circ. Mat. Palermo Serie II, Suppl.**51**, Vol. I (1998), 265–276.

[3] J. Blatter. *Unique continuous selections for metric projections of $C(X)$ onto finite-dimensional subspaces*. J. Approx. Th. **61** (1990), 194–221.

[4] J. Blatter. *An extension of the Chebyshev theory*. In J. Guddat et al., editors, Parametric Optimization and Related Topics III. Verlag Peter Lang, Frankfurt am Main, pages 35–77.

[5] J. Blatter and T. Fischer. *Unique continuous selections for metric projections of $C(X)$ onto finite-dimensional vector subspaces, II*. J. Approx. Th. **66** (1991), 229–265.

[6] V. G. Boltyanskii, S. S. Ryshkov and Y. A. Shashkin. *k-regular mappings and their application to the approximation of functions*. Uspekhi Mat. Nauk **15, no. 6(96)** (1960), 125–132.

[7] A. Brown. Some problems of linear analysis. Ph.D. thesis, University of Cambridge (1961).

[8] A. Brown. *Best n-dimensional approximation to sets of functions*. Proc. London Math. Soc. **14** (1964), 577–594.

[9] A. Brown. *On continuous selections for metric projections in spaces of continuous functions*. J. Funct. Analysis **8** (1971), 431–449.

[10] A. Brown. *An extension to Mairhuber's theorem. On metric projections and discontinuity of multivariate best uniform approximation*. J. Approx. Th. **36** (1982), 156–172.

[11] A. Brown. *Continuous selections for metric projections in spaces of continuous functions: on results of Thomas Fischer, results of Li*

Wu and the relation between them. In P. Papini, editor, Functional Analysis and Approximation Theory,Proceedings of an International Conference, Bagni di Lucca, 1988. Pitagora Editrice, Bologna, pages 125–163.

[12] A. L. Brown. *Continuous selections for metric projections in spaces of continuous functions and a disjoint leaves condition*. To appear.

[13] F. Deutsch and G. Nürnberger. *Weakly interpolating subspaces*. Numer.Funct. Anal. and Optimiz. **5** (1982-83), 267–288.

[14] T. Fischer. *Continuous selections for semi-infinite optimisation*. In Parametric Optimisation and Related Topics. Akademie-Verlag, Berlin, pages 95–112.

[15] A. Garkavi. *Almost Chebyshev systems of continuous functions*. Izv. Vyssh. Uchebn. Zaved. Mat. **45** (1965), 36–44. (= Amer. Math. Soc. Transl. 96(1970), 177-187).

[16] A. L. Garkavi. *On Chebyshev and almost Chebyshev subspaces*. Dok. Akad. Nauk SSSR **149** (1963), 1250–1252.

[17] A. L. Garkavi. *On Chebyshev and almost Chebyshev subspaces*. Izv. Akad. Nauk SSSR **28** (1964), 799–818.

[18] A. Haar. *Die Minkowskische Geometrie und die Annaherung an stetige Funktionen*. Math. Ann. **78** (1918), 294–311.

[19] A. Lazar, P. Morris and D. Wulbert. *Continuous selections for metric projections*. J. Functional Anal. **3** (1969), 193–216.

[20] W. Li. *The characterization of continuous selections for metric projections in $C(X)$*. Scientia Sinica (English) **XXXI,A:9** (1988), 1039–1052.

[21] W. Li. *An intrinsic characterization of lower semicontinuity of the metric projection in $C_0(T, X)$*. J. Approx. Th. **57** (1989), 136–149.

[22] W. Li. *Various continuities of metric projections in $C_0(T, X)$*. J. Approx. Th. **57** (1989), 150–168.

[23] W. Li. Continuous Selections for Metric Projections and Interpolating subspaces. Volume 1, Approximation and Optimization. Verlag Peter Lang, Frankfurt (1991).

[24] J. A. Lutts. *Topological spaces which admit unisolvent systems*. Trans. Amer. Math. Soc. **111** (1964), 440–448.

[25] J. C. Mairhuber. *On Haar's theorem concerning Chebychev approximation problems having unique solution*. Proc. Amer. Math. Soc. **7** (1956), 609–615.

[26] R. R. Phelps. *Uniqueness of Hahn-Banach extensions and unique best approximation*. Trans. Amer. Math. Soc. **95** (1960), 238–255.

[27] G. S. Rubinshtein. *On an extremal problem in normed linear spaces*. Dokl. Akad. Nauk SSSR **103** (1955), 451–454.

[28] Y. A. Shashkin. *Topological properties of sets connected with the theory of approximation of functions*. Izv. Akad. Nauk SSSR, ser. mat. **29** (1965), 1085–1094.

[29] I. Singer. Best approximation in normed linear spaces by elements of linear subspaces. Springer-Verlag, Berlin (1970).

[30] M. Sommer. *Continuous selections for metric projections*. In R. Devore and K. Scherer, editors, Quantitative Approximation. Academic Press, New York, pages 301–317.

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Roth's Method Applied to Some Pseudo-Differential Operators with Bounded Symbols. A Case Study.

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Abstract. We show that Roth's method can be used to construct Feller semigroups by starting with a pseudo-differential operator having a bounded continuous negative definite symbol.

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1 Introduction

By now it is a well-known fact that (nice) Feller semigroups on $C_\infty(\mathbb{R}^n)$ are generated by pseudo-differential operators with negative definite symbols, compare Ph. Courrège [3] to whom this result is due, see also [8] and the references given therein. A natural question is of course the converse: Given

$$-q(x,D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi, \tag{1.1}$$

where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous function such that $\xi \mapsto q(x,\xi)$ is negative definite, i.e. $\xi \mapsto e^{-tq(x,\xi)}$ is for all $t > 0$ positive definite. When does $-q(x,D)$ extend to a generator of a Feller semigroup, hence a Feller process?

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In [8] several results in this direction have been surveyed, for details we refer to [7], but not without mentioning the important contribution of W. Hoh [4] and [5]. For diffusions J.P. Roth in [11] suggested a further method to construct for a symbol with variable coefficients the semigroup by carefully handling the family of semigroups (and their generators) obtained when freezing the coefficients. This procedure we will call *Roth's method*. For pseudo-differential operators E. Popescu [10] has taken up this idea, but in the end he failed to obtain the necessary uniform bounds to get the limiting semigroup.

In this note we will use Roth's method to construct for bounded symbols, which are C^2-functions with respect to the co-variable, the corresponding Feller semigroups. Some of our work relies on ideas of E. Popescu [10], especially the use of a theorem due to P.R. Chernoff [2], see Theorem 2.5.

A word is in order on the main result, Theorem 2.6 and its corollary. Under our conditions it is possible to deduce the existence of the Feller semigroups by simpler means: $-q(x,D)$ is a bounded operator! Hence $e^{-tq(x,D)}$ is well defined and from Courrège's result it follows that $\left(e^{-tq(x,D)}\right)_{t\geq 0}$ must be a Feller semigroup. Of course the Chernoff approximation is new and might be of interest when studying path properties of the corresponding process by using the symbol, compare for this the central contribution of R. Schilling [12] and [13].

But our main purpose is to make Roth's method work with the aim to get insight how to extend or how to use the result for unbounded symbols. In this respect it is noteworthy that the Yosida approximation of any symbol is a bounded one.

2 Constructing the Feller semigroup

Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bounded continuous function such that $\xi \mapsto q(x,\xi)$ is negative definite for every $x \in \mathbb{R}^n$. In addition we assume further that $\xi \mapsto q(x,\xi) \in C^2(\mathbb{R}^n)$ and it holds

$$|\partial_\xi^\alpha q(x,\xi)| \leq C_\alpha \qquad \text{for } \alpha \in \mathbb{N}_0^n\,,\ |\alpha| \leq 2. \tag{2.1}$$

Note that for a negative definite function $\psi \in C^2(\mathbb{R}^n)$ we have always $|\partial^\alpha \psi(\xi)| \leq C_\alpha(1+\psi^{\frac{1}{2}}(\xi))$ for $|\alpha| = 1$, and $|\partial^\alpha \psi(\xi)| \leq C_\alpha$ for $|\alpha| = 2$. Thus in (2.1) the main condition is that $\xi \mapsto q(x,\xi)$ is C^2 and C_α is independent of, hence uniformly in, x. From Chr. Berg and G. Forst [1],

Proposition 7.13 we know that for $x_0 \in \mathbb{R}^n$ there exists $m(x_0) > 0$ such that $\xi \mapsto m(x_0) - q(x_0, \xi)$ is positive definite. Hence by Bochner's theorem there exists a bounded measure $\nu^{x_0} \in \mathcal{M}_+^b(\mathbb{R}^n)$ such that

$$\widehat{\nu^{x_0}}(\xi) = m(x_0) - q(x_0, \xi). \tag{2.2}$$

We assume that $m(x_0)$ and therefore also the total mass of ν^{x_0} is uniformly bounded with respect to x_0, i.e.

$$|m(x_0)| \leq K \quad \text{and} \quad \|\nu^{x_0}\| \leq (2\pi)^{\frac{n}{2}} K. \tag{2.3}$$

For $x_0 \in \mathbb{R}^n$ fixed we may associate with $\xi \mapsto q(x_0, \xi)$ a convolution semigroup $(\mu_t^{x_0})_{t\geq 0}$, $\widehat{\mu}_t^{x_0}(\xi) = (2\pi)^{-\frac{n}{2}} e^{-tq(x_0,\xi)}$, and hence a Feller semigroup $(V_t^{x_0})_{t\geq 0}$, where

$$V_t^{x_0} u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t^{x_0}(\mathrm{d}y)\,,\ u \in C_\infty(\mathbb{R}^n). \tag{2.4}$$

On $\mathcal{S}(\mathbb{R}^n)$ we find for the generator A of $(V_t^{x_0})_{t\geq 0}$, compare [6], Example 4.1.12,

$$Au(x) = -q(x_0, D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x_0, \xi)\widehat{u}(\xi)\,\mathrm{d}\xi. \tag{2.5}$$

Using (2.2) we find further for $u \in \mathcal{S}(\mathbb{R}^n)$ that

$$\begin{aligned} &-q(x_0, D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x_0, \xi)\widehat{u}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\big(m(x_0) - q(x_0, \xi)\big)\widehat{u}(\xi)\mathrm{d}\xi - (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(x_0)\widehat{u}(\xi)\mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x-y)\nu^{x_0}(\mathrm{d}y) - m(x_0)u(x). \end{aligned}$$

Hence we have on $C_\infty(\mathbb{R}^n)$ the representation

$$-q(x_0, D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x-y)\nu^{x_0}(\mathrm{d}y) - m(x_0)u(x), \tag{2.6}$$

and A is a bounded operator (as expected). We will write in the following $-q(x_0, D)$ for the righthand side in (2.6), but note that the representation (2.5) holds only (in a classical sense) for functions u such that $\widehat{u} \in L^1(\mathbb{R}^n)$, for example we may take the Wiener algebra, compare [6], p.87. From (2.6) we can easily derive a bound for $q(x_0, D)$:

$$|q(x_0, D)u(x)| \leq (2\pi)^{-\frac{n}{2}} \nu^{x_0}(\mathbb{R}^n)\|u\|_\infty + m(x_0)\|u\|_\infty \leq 2K\|u\|_\infty,$$

which yields

$$\|q(x_0, D)u\|_\infty \leq C_0\|u\|_\infty\,,\; C_0 = 2K. \tag{2.7}$$

Next we define on $\mathcal{S}(\mathbb{R}^n)$ the pseudo-differential operators

$$W_t u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-tq(x,\xi)} \widehat{u}(\xi)\,\mathrm{d}\xi. \tag{2.8}$$

Lemma 2.1 *For $u \in \mathcal{S}(\mathbb{R}^n)$ the function $W_t u$ belongs to $C_\infty(\mathbb{R}^n)$.*

Proof. Since $\widehat{u} \in \mathcal{S}(\mathbb{R}^n)$ the properties of $x \mapsto q(x,\xi)$ imply that $x \mapsto W_t u(x)$ is continuous. Further, in order to prove that $W_t u(x) \to 0$ as $x \to \infty$ observe that

$$\begin{aligned} |x|^2 W_t u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |x|^2 e^{ix\cdot\xi} e^{-tq(x,\xi)} \widehat{u}(\xi)\,\mathrm{d}\xi \\ &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \Delta_\xi(e^{ix\cdot\xi}) e^{-tq(x,\xi)} \widehat{u}(\xi)\,\mathrm{d}\xi \\ &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Delta_\xi\big(e^{-tq(x,\xi)} \widehat{u}(\xi)\big)\,\mathrm{d}\xi. \end{aligned}$$

But from (2.1) and $\widehat{u} \in \mathcal{S}(\mathbb{R}^n)$ we deduce that $\Delta_\xi(e^{-tq(x,\xi)}\widehat{u}(\xi)$, is bounded by an L^1-function which is independent of x. Hence $\big||x|^2 W_t u(x)\big| \leq$ constant which implies the lemma. □

Note that

$$W_t u(x) = V_t^x u(x). \tag{2.9}$$

For $x_0 \in \mathbb{R}^n$ it follows that

$$|V_t^{x_0} u(x)| = \left| \int_{\mathbb{R}} u(x-y)\mu_t^{x_0}(\mathrm{d}y) \right| \leq \|u\|_\infty, \tag{2.10}$$

or

$$\sup_{x\in\mathbb{R}^n} |V_t^x u(x)| \leq \|u\|_\infty,$$

implying for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\|W_t u\|_\infty = \sup_{x\in\mathbb{R}^n} |W_t u(x)| = \sup_{x\in\mathbb{R}^n} |V_t^x u(x)| \leq \|u\|_\infty. \tag{2.11}$$

From (2.11) it is clear that we may extend W_t as a contraction from $C_\infty(\mathbb{R}^n)$ into itself and we will denote this extension once again by W_t. We need

Lemma 2.2 *Let $\big(A_1, D(A_1)\big)$ and $\big(A_2, D(A_2)\big)$ be linear bounded generators of strongly continuous contraction semigroups $(e^{tA_1})_{t\geq 0}$ and $(e^{tA_2})_{t\geq 0}$ on the Banach space $(X, \|\cdot\|)$ such that $D(A_1) = D(A_2) = X$. Then*

$$\|e^{tA_1}x - e^{tA_2}x\| \leq t \sup_{\|y\|\leq 1} \|A_1 y - A_2 y\| \|x\|$$

holds for all $t \geq 0$ and $x \in X$.

Lemma 2.3 *Let A be a linear and bounded operator on the Banach space $(X, \|\cdot\|)$ such that $\|A\| \leq 1$. Then*

$$\|e^{n(A-id)}x - A^n x\| \leq \sqrt{n}\|Ax - x\|$$

for all $n \geq 1$ and for all $x \in X$.

Lemma 2.2 is taken from E. Popescu [10] and Lemma 2.3 is taken from A. Pazy [9]. In the proof of the following proposition we follow closely E. Popescu [10]:

Proposition 2.4 *Let $q(x, \xi)$ be as above, especially assume (2.1), and consider the operators $\big(W_t, C_\infty(\mathbb{R}^n)\big)$ and $\big(V_t^{x_0}, C_\infty(\mathbb{R}^n)\big)$ as introduced above. Then it holds*

$$\lim_{\substack{m\to\infty \\ l\to\infty}} \left\|(W_{\frac{t}{m}})^m u - (W_{\frac{t}{l}})^l u\right\|_\infty = 0 \tag{2.12}$$

for $u \in C_\infty(\mathbb{R}^n)$ uniformly for t in compact intervals.

Proof. Let $u \in C_\infty(\mathbb{R}^n)$ and $t \geq 0$. We observe that

$$\begin{aligned}\left\|(W_{\frac{t}{m}})^m u - (W_{\frac{t}{l}})^l u\right\|_\infty &\leq \left\|(W_{\frac{t}{m}})^m u - e^{m(W_{\frac{t}{m}} - \mathrm{id})}u\right\|_\infty \\ &+ \left\|e^{m(W_{\frac{t}{m}} - \mathrm{id})}u - e^{l(W_{\frac{t}{l}} - \mathrm{id})}u\right\|_\infty + \left\|(W_{\frac{t}{l}})^l u - e^{l(W_{\frac{t}{l}} - \mathrm{id})}u\right\|_\infty .\end{aligned}$$

For the linear operator W_t we have already shown that $\|W_t u\|_\infty \leq \|u\|_\infty$. Applying Lemma 2.3 to $A = W_{\frac{t}{m}}, n = m$, and to $A = W_{\frac{t}{l}}, n = l$, we arrive at

$$\left\|(W_{\frac{t}{m}})^m u - e^{m(W_{\frac{t}{m}} - \mathrm{id})}u\right\|_\infty \leq \sqrt{m}\left\|W_{\frac{t}{m}}u - u\right\|_\infty \tag{2.13}$$

and

$$\left\|(W_{\frac{t}{l}})^l u - e^{l(W_{\frac{t}{l}} - \mathrm{id})}u\right\|_\infty \leq \sqrt{l}\left\|W_{\frac{t}{l}}u - u\right\|_\infty , \tag{2.14}$$

respectively. Now, for $x_0 \in \mathbb{R}^n$ fixed we get

$$\begin{aligned}\left|V^{x_0}_{\frac{t}{m}}u(x) - u(x)\right| &= \left|\int_0^{\frac{t}{m}} V^{x_0}_s q(x_0, D)u(x)\,\mathrm{d}s\right| \\ &\leq \frac{t}{m}\|V^{x_0}_s\|\,\|q(x_0, D)u\|_\infty \\ &\leq \frac{t}{m}C_0\|u\|_\infty,\end{aligned} \tag{2.15}$$

and

$$\left|V^{x_0}_{\frac{t}{l}}u(x) - u(x)\right| \leq \frac{t}{l}C_0\|u\|_\infty, \tag{2.16}$$

respectively. For this we used that $(V^{x_0}_t)_{t\geq 0}$ is generated by $-q(x_0, D)$ and we applied Lemma 4.1.14 in [6]. It follows that

$$\left|W_{\frac{t}{m}}u(x) - u(x)\right| = \left|V^x_{\frac{t}{m}}u(x) - u(x)\right| \leq \frac{t}{m}C\|u\|_\infty$$

and

$$\left|W_{\frac{t}{l}}u(x) - u(x)\right| \leq \frac{t}{l}C\|u\|_\infty$$

with some $C > 0$. Taking the supremum over $x \in \mathbb{R}^n$ we find

$$\left\|W_{\frac{t}{m}}u - u\right\|_\infty \leq \frac{t}{m}C\|u\|_\infty \qquad \text{and} \qquad \left\|W_{\frac{t}{l}}u - u\right\|_\infty \leq \frac{t}{l}C\|u\|_\infty. \tag{2.17}$$

Using (2.13) and (2.14) we conclude

$$\lim_{m\to\infty}\left\|(W_{\frac{t}{m}})^m u - e^{m(W_{\frac{t}{m}}-\mathrm{id})}u\right\|_\infty \leq \lim_{m\to\infty}\frac{t}{\sqrt{m}}C\|u\|_\infty = 0$$

as well as

$$\lim_{l\to\infty}\left\|(W_{\frac{t}{l}})^l u - e^{l(W_{\frac{t}{l}}-\mathrm{id})}u\right\|_\infty \leq \lim_{l\to\infty}\frac{t}{\sqrt{l}}C\|u\|_\infty = 0,$$

and both limits are uniform for t in compact intervals. It remains to show that

$$\lim_{\substack{m\to\infty\\ l\to\infty}}\left\|e^{m(W_{\frac{t}{m}}-\mathrm{id})}u - e^{l(W_{\frac{t}{l}}-\mathrm{id})}u\right\|_\infty = 0$$

holds uniformly for t in compact intervals. Applying Lemma 2.2 we find (with $r = \frac{t}{m}$, $s = \frac{t}{l}$)

$$\begin{aligned}
&\left\| e^{t\frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}}u - e^{t\frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}}u \right\|_\infty = \left\| e^{t\frac{W_r-\mathrm{id}}{r}}u - e^{t\frac{W_s-\mathrm{id}}{s}}u \right\|_\infty \\
\leq &\sup_{\|v\|_\infty\leq 1} \left\| e^{t\frac{W_r-\mathrm{id}}{r}}v - e^{t\frac{W_s-\mathrm{id}}{s}}v \right\|_\infty \|u\|_\infty \qquad (2.18)\\
\leq &t \sup_{\|v\|_\infty\leq 1} \left\| \frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}v - \frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}v \right\|_\infty \|u\|_\infty.
\end{aligned}$$

In order to handle

$$\left\| \frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}v - \frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}v \right\|_\infty$$

we want to use the fact that $(V_t^{x_0})_{t\geq 0}$, $x_0 \in \mathbb{R}^n$, is a uniformly continuous semigroup on $C_\infty(\mathbb{R}^n)$ which follows from the fact that its generator $\big(-q(x_0,D), C_\infty(\mathbb{R}^n)\big)$ is a bounded operator, compare A. Pazy [9]. Writing for a moment $V_t^{x_0} = e^{-tq(x_0,D)}$ we find, compare again A. Pazy,

$$\|V_t^{x_0} - \mathrm{id}\| = \|e^{-tq(x_0,D)} - \mathrm{id}\| \leq t\|q(x_0,D)\|e^{t\|q(x_0,D)\|}, \qquad (2.19)$$

and from this we derive

$$\begin{aligned}
&\left| \frac{V_{\frac{t}{m}}^{x_0}-\mathrm{id}}{\frac{t}{m}}v(x) - \frac{V_{\frac{t}{l}}^{x_0}-\mathrm{id}}{\frac{t}{l}}v(x) \right| \\
= &\left| \frac{m}{t}\int_0^{\frac{t}{m}} V_s^{x_0}q(x_0,D)v(x)\,\mathrm{d}s - \frac{l}{t}\int_0^{\frac{t}{l}} V_s^{x_0}q(x_0,D)v(x)\,\mathrm{d}s \right| \\
\leq &\left| \frac{m}{t}\int_0^{\frac{t}{m}} \big(V_s^{x_0}q(x_0,D)v(x) - q(x_0,D)v(x)\big)\,\mathrm{d}s \right| \\
&+ \left| \frac{l}{t}\int_0^{\frac{t}{l}} \big(V_s^{x_0}q(x_0,D)v(x) - q(x_0,D)v(x)\big)\,\mathrm{d}s \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{m}{t} \int_0^{\frac{t}{m}} \big(V_s^{x_0} - \mathrm{id}\big) q(x_0, D) v(x) \, \mathrm{d}s \right| + \left| \frac{l}{t} \int_0^{\frac{t}{l}} \big(V_s^{x_0} - \mathrm{id}\big) q(x_0, D) v(x) \, \mathrm{d}s \right| \\
&\leq \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leq s \leq \frac{t}{m}}} \big(\|V_s^{x_0} - \mathrm{id}\| \, \|q(x_0, D) v\|_\infty \big) + \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leq s \leq \frac{t}{l}}} \big(\|V_s^{x_0} - \mathrm{id}\| \, \|q(x_0, D) v\|_\infty \big) \\
&\leq \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leq s \leq \frac{t}{m}}} \big(s \|q(x_0, D)\|^2 e^{s \|q(x_0, D)\|} \big) \|v\|_\infty + \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leq s \leq \frac{t}{l}}} \big(s \|q(x_0, D)\|^2 e^{s \|q(x_0, D)\|} \big) \|v\|_\infty \\
&\leq \frac{t}{m} C_0^2 e^{\frac{t}{m} C_0} \|v\|_\infty + \frac{t}{l} C_0^2 e^{\frac{t}{l} C_0} \|v\|_\infty .
\end{aligned}
$$

It follows that

$$
\lim_{\substack{m \to \infty \\ l \to \infty}} \Big(\frac{t}{m} C_0^2 e^{\frac{t}{m} C_0} \|v\|_\infty + \frac{t}{l} C_0^2 e^{\frac{t}{l} C_0} \|v\|_\infty \Big) = 0. \tag{2.20}
$$

Now,

$$
\begin{aligned}
\left\| \frac{\frac{V_{\frac{t}{m}}^{x_0} - \mathrm{id}}{\frac{t}{m}} v - \frac{\frac{V_{\frac{t}{l}}^{x_0} - \mathrm{id}}{\frac{t}{l}} v}{\frac{t}{l}} \right\|_\infty &= \sup_{x \in \mathbb{R}^n} \left| \frac{\frac{V_{\frac{t}{m}}^{x_0} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{\frac{V_{\frac{t}{l}}^{x_0} - \mathrm{id}}{\frac{t}{l}} v(x)}{\frac{t}{l}} \right| \\
&\leq \frac{t}{m} C_0^2 e^{\frac{t}{m} C_0} \|v\|_\infty + \frac{t}{l} C_0^2 e^{\frac{t}{l} C_0} \|v\|_\infty
\end{aligned}
$$

and thus

$$
\lim_{\substack{m \to \infty \\ l \to \infty}} \sup_{x_0 \in \mathbb{R}^n} \left\| \frac{\frac{V_{\frac{t}{m}}^{x_0} - \mathrm{id}}{\frac{t}{m}} v - \frac{\frac{V_{\frac{t}{l}}^{x_0} - \mathrm{id}}{\frac{t}{l}} v}{\frac{t}{l}} \right\|_\infty = 0
$$

uniformly for t in compact intervals. Since

$$
\left| \frac{\frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{\frac{W_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v(x)}{\frac{t}{l}} \right| = \left| \frac{\frac{V_{\frac{t}{m}}^{x} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{\frac{V_{\frac{t}{l}}^{x} - \mathrm{id}}{\frac{t}{l}} v(x)}{\frac{t}{l}} \right|
$$

we find

$$
\begin{aligned}
&\lim_{\substack{m \to \infty \\ l \to \infty}} \left\| \frac{\frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{\frac{W_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v}{\frac{t}{l}} \right\|_\infty \\
&\leq \lim_{\substack{m \to \infty \\ l \to \infty}} \sup_{x_0 \in \mathbb{R}^n} \left\| \frac{\frac{V_{\frac{t}{m}}^{x_0} - \mathrm{id}}{\frac{t}{m}} v - \frac{\frac{V_{\frac{t}{l}}^{x_0} - \mathrm{id}}{\frac{t}{l}} v}{\frac{t}{l}} \right\|_\infty = 0.
\end{aligned}
$$

Remembering (2.18) we finally get

$$\begin{aligned}
&\lim_{\substack{m\to\infty\\ l\to\infty}} \|e^{m(W_{\frac{t}{m}}-\mathrm{id})}u - e^{l(W_{\frac{t}{l}}-\mathrm{id})}u\|_\infty \\
=&\lim_{\substack{m\to\infty\\ l\to\infty}} \left\| e^{t\frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}}u - e^{t\frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}}u \right\|_\infty \\
\leq& t \lim_{\substack{m\to\infty\\ l\to\infty}} \sup_{\|v\|_\infty\leq 1} \left\| \frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}v - \frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}v \right\|_\infty \|u\|_\infty \\
\leq& t \sup_{\|v\|_\infty\leq 1} \lim_{\substack{m\to\infty\\ l\to\infty}} \left\| \frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}v - \frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}v \right\|_\infty \|u\|_\infty \\
=&0
\end{aligned}$$

uniformly for t in compact intervals. □

We want to combine Proposition 2.4 with a result due to P.R. Chernoff [2], Theorem 2.5 below, in order to prove that $-q(x,D)$ generates a Feller semigroup.

Theorem 2.5 (P.R. Chernoff) *Let $(S_t)_{t\geq 0}$ be a family of strongly continuous linear contractions on a Banach space $(X, \|\cdot\|)$ with $S_0 = id$. Assume that the strong derivative S_0' is densely defined and suppose that*

$$\lim_{m\to\infty} \left\| (S_{\frac{t}{m}})^m u - T_t u \right\| = 0 \tag{2.21}$$

holds for all $u \in X$. Then $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on X and its generator A extends S_0'. Moreover the convergence of $(S_{\frac{t}{m}})^m$ to T_t is uniform for t in compact intervals.

Theorem 2.6 *Let $q(x,\xi)$ satisfy the assumptions of Proposition 2.4 and define on $\mathcal{S}(\mathbb{R}^n)$ the operator $-q(x,D)$ by*

$$-q(x,D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi. \tag{2.22}$$

Then $-q(x,D)$ extends to the generator $\left(A, C_\infty(\mathbb{R}^n)\right)$ of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $C_\infty(\mathbb{R}^n)$.

Proof. Let $\left(W_t, C_\infty(\mathbb{R}^n)\right)$ and $\left(V_t^{x_0}, C_\infty(\mathbb{R}^n)\right)$ be defined as in (2.8) and (2.4), respectively. In order to apply Theorem 2.5 to the family $(W_t)_{t\geq 0}$

we need to prove first that $(W_t)_{t\geq 0}$ is a family of strongly continuous linear contractions on $C_\infty(\mathbb{R}^n)$ with $W_0 = \mathrm{id}$ and that $\big((W_{\frac{t}{m}})^m u\big)_{m\in\mathbb{N}}$ converges strongly to some limit $T_t u$, i.e. we have as strong limit $T_t u = \lim_{m\to\infty}(W_{\frac{t}{m}})^m u$. Since

$$W_0 u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-0q(x,\xi)} \widehat{u}(\xi)\, \mathrm{d}\xi = F^{-1}(\widehat{u})(x) = u(x),$$

it follows that $W_0 = \mathrm{id}$. The contraction property of W_t was proved in (2.11) and Proposition 2.4, especially (2.12), says that $\big((W_{\frac{t}{m}})^m\big)_{m\in\mathbb{N}}$ is a strong Cauchy sequence in $C_\infty(\mathbb{R}^n)$, i.e. for each $u \in C_\infty(\mathbb{R}^n)$ a Cauchy sequence is given by $\big((W_{\frac{t}{m}})^m u\big)_{m\in\mathbb{N}}$. Hence for $t > 0$ we may define on $C_\infty(\mathbb{R}^n)$ the operator

$$T_t u := \lim_{m\to\infty} (W_{\frac{t}{m}})^m u. \tag{2.23}$$

Next we show that the family $(W_t)_{t\geq 0}$ is strongly continuous. For this we need to check

$$\lim_{s\uparrow t} \|W_t u - W_s u\|_\infty = 0 \tag{2.24}$$

and

$$\lim_{s\downarrow t} \|W_t u - W_s u\|_\infty = 0. \tag{2.25}$$

Now, for $s < t$ and fixed $x_0 \in \mathbb{R}^n$ we have

$$\|V_t^{x_0} u - V_s^{x_0} u\|_\infty \leq \|V_s^{x_0}\|\, \|V_{t-s}^{x_0} u - u\|_\infty \leq \|V_{t-s}^{x_0} u - u\|_\infty$$

and by (2.7) it follows that

$$\sup_{x_0\in\mathbb{R}^n} \|V_r^{x_0} u - u\|_\infty \leq r C_0 \|u\|_\infty,$$

implying

$$\lim_{s\uparrow t} \sup_{x_0\in\mathbb{R}^n} \|V_t^{x_0} u - V_s^{x_0} u\|_\infty = 0.$$

Observing that

$$|W_t u(x) - W_s u(x)| = |V_t^x u(x) - V_s^x u(x)|$$

we may deduce

$$\lim_{s\uparrow t} \|W_t u - W_s u\|_\infty = 0$$

proving (2.24). In order to prove (2.25) just observe that for $s > t$ we have

$$V_t^{x_0} u - V_s^{x_0} u = V_t^{x_0}(V_{s-t}^{x_0} u - u)$$

and we may argue as above. Finally, in order to apply Theorem 2.5 we have to show that the strong derivate W_0' is densely defined. For this we are going to prove for $u \in \mathcal{S}(\mathbb{R}^n)$ (in fact $u \in C_\infty(\mathbb{R}^n)$ would be possible) that

$$\lim_{t\to 0} \left\| \frac{W_t u - u}{t} + q(x,D)u \right\|_\infty = 0 \tag{2.26}$$

holds which will give $A = W_0' = -q(x,D)$. Since $(V_t^{x_0})_{t\geq 0}$ is a uniformly continuous semigroup, using (2.19), we get

$$\begin{aligned}
&\left| \frac{V_t^{x_0} u(x) - u(x)}{t} + q(x_0,D)u(x) \right| \\
=&\left| \frac{1}{t}\int_0^t V_s^{x_0} q(x_0,D)u(x)\,\mathrm{d}s - q(x_0,D)u(x) \right| \\
=&\left| \frac{1}{t}\int_0^t (V_s^{x_0} - \mathrm{id}) q(x_0,D)u(x)\,\mathrm{d}s \right| \\
\leq&\frac{1}{t} t \Big(\sup_{\substack{x_0\in\mathbb{R}^n \\ 0\leq s\leq t}} \|V_s^{x_0} - \mathrm{id}\| \Big) \|q(x_0,D)u\|_\infty \\
=&\Big(\sup_{\substack{x_0\in\mathbb{R}^n \\ 0\leq s\leq t}} s\|q(x_0,D)\| e^{s\|q(x_0,D)\|} \Big) \|q(x_0,D)u\|_\infty \\
\leq& tC_0^2 e^{tC_0} \|u\|_\infty,
\end{aligned}$$

where we used in the last step (2.7). Thus we find

$$\lim_{t\to 0} \sup_{x_0\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} \left| \frac{V_t^{x_0} u(x) - u(x)}{t} + q(x_0,D)u(x) \right| = 0. \tag{2.27}$$

Since

$$\begin{aligned}
&\left| \frac{W_t u(x) - u(x)}{t} + q(x,D)u(x) \right| \\
=&\left| \frac{V_t^x - u(x)}{u(x)} + q(x,D)u(x) \right| \\
\leq& tC_0^2 e^{tC_0} \|u\|_\infty
\end{aligned}$$

we arrive at

$$\lim_{t\to 0}\left\|\frac{W_t u - u}{t} + q(x,D)u\right\|_\infty = 0$$

proving the theorem. □

Corollary 2.7 *In the situation of Theorem 2.6 the semigroup $(T_t)_{t\geq 0}$ is a Feller semigroup.*

Proof. For this we have only to note that $A = -q(x,D)$ satisfies on $C_\infty(\mathbb{R}^n)$ the positive maximum principle, compare Ph. Courrège [3], or [6]. □

Remark 2.8 *Note that Theorem 2.6 gives more than an existence result. We have also an approximation procedure: The symbol of $(T_t)_{t\geq 0}$,*

$$\sigma(T_t)(x,\xi) = e^{-ix\xi}\Big(T_t e^{i(\cdot,\xi)}\Big)(x)$$

is in general not explicitly known to us, and clearly we are interested in

$$T_t u(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(T_t)(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi,$$

compare the introduction. However the symbol of W_t is known explicitly.

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References

[1] Berg, Chr. and G. Forst, *Potential Theory on Locally Compact Abelian Groups,* Ergebnisse der Mathematik und ihre Grenzgebiete (Ser. II), **87,** Springer Verlag, Berlin 1975.

[2] Chernoff, P. R.*Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators,* Memoirs of the American Mathematical Society Number, **140,** American Mathematical Society, Providence R. I. 1974.

[3] Courrège, Ph. Sur la forme intégro–différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum, Sém. Théorie du Potentiel Exposé, **2,**(1965/66), 38pp.

[4] Hoh, W., Pseudo differential operators with negative definite symbols and the martingale problem, *Stoch. and Stoch. Rep.* **55,** (1995), 225–252.

[5] Hoh, W., A symbolic calculus for pseudo differential operators generating Feller semigroups, *Osaka J. Math.* **35,** (1998), 798–820.

[6] Jacob, N., *Pseudo–Differential Operators and Markov Processes Vol. I. Fourier Analysis and Semigroups,* Imperial College Press, London 2001.

[7] Jacob, N., *Pseudo–Differential Operators and Markov Processes Vol. II. Generators and Their Potential Theory,* Imperial College Press, London 2002.

[8] Jacob, N. and R. L. Schilling, Lévy-type processe and pseudo-differential operators, *O. E. Barndorff-Nielsen et.al. (eds.): Lévy Processes—Theory and Applications,* Birkhäuser Verlag, Basel 2001, 139–168.

[9] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Applied Mathematical Sciences **44,** Springer Verlag, Berlin 1983.

[10] Popescu, E., Sur les sémi-groupes de Feller et les opérateurs pseudo–différentiels, *Rev. Roumaine Math. Pures Appl.* **36,** (1991), 181–191.

[11] Roth, J. P., Les opérateurs elliptiques comme générateurs infinitésimaux de semi–groupes de Feller, *Sém. Théorie du Potentiel, Paris, No.3 (eds. M. Brelot, G. Choquet et J. Deny), Lecture Notes in Mathematics* **681,** Springer Verlag, Berlin 1978, 234–251.

[12] Schilling, R. L. Comparable processes and the Hausdorff dimension of their sample paths, Stoch. and Stoch. Rep. **57,** (1996), 89–110.

[13] Schilling, R. L. Growth and Hölder conditions for the sample paths of Feller processes, Probab. Theory Relat. Fields **112,** (1998), 565–611.

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Newman-type Inequalities for Multivariate Polynomials

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Abstract. By the famous Newman Inequality for any $p \in \text{span}\{x^{\lambda_j} : 0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n\},\ x \in \mathbb{R}$,

$$x|p'(x)| \le c\sum_{j=0}^{n} \lambda_j \|p\|_{[0,1]}, \quad 0 < x \le 1 \ .$$

In this paper we study various extensions of this inequality to multivariate setting.

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1 Introduction.

Let P_n^d denote the set of real polynomials of d variables and total degree at most n. Furthermore, denote by $\|.\|_K$ the usual supremum norm on the set $K \subset \mathbb{R}^d$. The classical inequalities of Markov and Bernstein give sharp estimates for the derivatives of univariate polynomials via their supremum norms:

$$\|p'\|_{[a,b]} \le \frac{2n^2}{b-a}\ \|p\|_{[a,b]}, \quad p \in P_n^1 \ , \tag{1.1}$$

$$|p'(x)| \le \frac{n}{\sqrt{(b-x)(x-a)}}\ \|p\|_{[a,b]}, \quad p \in P_n^1 \ , \quad x \in (a,b) \ . \tag{1.2}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Newman [6] gave an elegant extension of Markov inequality (1.1) for Müntz polynomials

$$M_1(\Lambda_n) := \operatorname{span}\{x^{\lambda_j},\ 0 \le j \le n\}\ , \tag{1.3}$$

where $\Lambda_n := \{0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$. According to Newman Inequality for every $p \in M_1(\Lambda_n)$ and $x \in [0,1]$

$$x|p'(x)| \le 11 \sum_{j=0}^{n} \lambda_j\ \|p\|_{[0,1]}\ . \tag{1.4}$$

Note that in case of integer λ_j's the Markov Inequality (1.1) yields a bound of order λ_n^2 while $\sum_{j=0}^{n} \lambda_j \le n\,\lambda_n$ in (1.4) is of smaller order if $\lambda_n/n \to \infty$.

Another advantage of Newman Inequality (1.4) consists in the fact that it applies for noninteger λ_j's, as well. However, it should be also noted that (1.4) gives an estimate of the derivative $p'(x)$ with a weight x. It is known that this weight becomes superfluous when $\lambda_j \ge j$, $0 \le j \le n$, (see [2, p. 287]).

Moreover, a version of Newman Inequality holds on $[a,1]$ with $0 < a < 1$. That is for every $p \in M_1(\Lambda_n)$

$$\|p'\|_{[a,1]} \le c \left(\sum_{j=0}^{n} \lambda_j + n^2 \right) \|p\|_{[a,1]} \tag{1.5}$$

with a constant $c > 0$ depending only on and a (see [2] for details).

2 Main Results.

There are various extensions of Markov and Bernstein inequalities (1.1), (1.2) to multivariate setting, see for instance the survey [3]. In this paper we shall study possible generalizations of Newman Inequality (1.4) for multivariate Müntz polynomials. As usual, in case of several variables the results will depend on the geometry of the underlying set.

For $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \subset \mathbb{Z}_+^1$ consider the corresponding space of multivariate Müntz polynomials

$$M_d(\Lambda_n) :=$$
$$= \operatorname{span}\{\mathbf{x}^{\mathbf{k}} : \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d, k_1 + \cdots + k_d = \lambda_j, 0 \le j \le n\},\ \mathbf{x} \in \mathbb{R}^d\ .$$

Given $p \in M_d(\Lambda_n)$ and $\boldsymbol{\omega} \in S^{d-1}$, the unit sphere in $\mathbb{R}^d$, let $D_{\boldsymbol{\omega}}p$ denote the derivative of p in direction $\boldsymbol{\omega}$, i.e., $D_{\boldsymbol{\omega}}(p) = \langle \boldsymbol{\omega}, \partial p \rangle$, where ∂p is the gradient of p. Furthermore, let for any $\mathbf{x} \in K$

$$Dp := \sup_{\boldsymbol{\omega} \in S^{d-1}} |D_{\boldsymbol{\omega}} p| = |\partial p|$$

be the Eucledian norm of the gradient.

Let now $K \subset \mathbb{R}^d$ be a compact set with nonempty interior, Int $K \neq \emptyset$, and $\mathbf{0} \in$ Int K. We shall call K *star-like* if for every $\mathbf{x} \in K$ and $0 \leq t < 1$ we have $t\mathbf{x} \in$ Int K. Furthermore, denote by

$$r_K^*(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}$$

the Minkowski functional of K centered at $\mathbf{0}$. Clearly $r_K^*(\mathbf{x}) < 1$ if and only if $\mathbf{x} \in$ Int K, and $r_K^*(\mathbf{x}) = 1$ for $\mathbf{x} \in$ Bd K, the boundary of K. Moreover, by the above definition $r_K^*(t\mathbf{x}) = t r_K^*(\mathbf{x}),\ t > 0$.

Furthermore, if $r_K^* \in C^1$, denote by

$$\omega(\partial r_K^*, t) = \sup\{|\partial r_K^*(\mathbf{x}_1) - \partial r_K^*(\mathbf{x}_2) : \mathbf{x}_1, \mathbf{x}_2 \in K, |\mathbf{x}_1 - \mathbf{x}_2| \leq t\}$$

and $\omega_K(t) := t\omega(\partial r_K^*, t)$ the moduli of continuity and smoothness of ∂r_K^*, respectively. In what follows $\mathbf{c}$ will stand for (possibly different) positive constant depending only on K. Moreover, ω_K^{-1} will denote the inverse function of ω_K.

THEOREM 1. *Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a star-like domain with $r_K^* \in C^1$. Then for every $p \in M_d(\Lambda_n)$*

$$\|Dp\|_K \leq c \left\{ \frac{1}{\omega_K^{-1}(\lambda_n^{-2})} + \sum_{j=0}^{n} \lambda_j \right\} \|p\|_K \ . \tag{2.1}$$

Note that $\omega_K(t) = o(t)$ when $r_K^* \in C^1$, i.e., $1/\omega_K^{-1}(\lambda_n^{-2}) = o(\lambda_n^2)$. In addition whenever $\lambda_n/n \to \infty$ $(n \to \infty)$ we also have $\sum\limits_{j=0}^{n} \lambda_j = o(\lambda_n^2)$ and then the estimate in (2.1) is of order $o(\lambda_n^2)$, which is better that $O(\lambda_n^2)$ provided in this case by a Markov-type inequality.

Furthermore, if $r_K^* \in C^2$ then $\omega_K(t) = O(t^2)$, i.e.,

$$\frac{1}{\omega_K^{-1}(\lambda_n^{-2})} \leq c\lambda_n \leq \sum_{j=0}^{n} \lambda_j \ .$$

Thus we obtain from (2.1) the next

Corollary. *Let $K \subset \mathbb{R}^d$ be star-like and $r_K^* \in C^2$. Then for every $p \in M_d(\Lambda_n)$*

$$\|Dp\|_K \le c \sum_{j=0}^{n} \lambda_j \ \|p\|_K \ . \tag{2.2}$$

This corollary gives an analogue of Newman Inequality (1.4) under the condition of C^2-smoothness of the domain. The estimate (2.1) which holds for C^1-domains contains an extra term $1/\omega_K^{-1}(\lambda_n^{-2})$. Our next example shows that the above corollary is not true, in general, for C^1-domains.

Example 2. Let $\Lambda_0 = \{m\}$, $m \in \mathbb{N}$. Then $M_d(\Lambda_0) = H_m^d$, the set of homogeneous polynomials of degree m. It is shown in [5] that there exists a convex body $K \subset \mathbb{R}^d$ with $r_K^* \in C^1$ and $p_m \in H_m^d$, $m \in \mathbb{N}$ such that

$$\|Dp_m\|_K \ge m\sqrt{\log m}\ \|p_m\|_K, \quad m \in \mathbb{N} \ .$$

Since $\sum\limits_{j=0}^{n} \lambda_j = m$, (2.2) evidently fails.

In order to verify Theorem 1 we shall need a result from [4]. It is shown in [4] (see [4, Theorem 1, p. 237]) that whenever $r_K^* \in C^1$ then for any $p \in P_n^d$, $\mathbf{x}_0 \in \mathrm{Bd}\ K$ and $\boldsymbol{\omega} \in S^{d-1}$ orthogonal to $\partial r_K^*(\mathbf{x}_0)$ we have

$$D_{\boldsymbol{\omega}} p(\mathbf{x}_0) \le c_0 \frac{1}{\omega_K^{-1}(n^{-2})}\ \|p\|_K \ , \tag{2.3}$$

where $c_0 > 0$ depends only on K. Estimate (2.3) improves the usual Markov-type inequality in case when only *tangential* derivatives are considered. (Note that $\partial r_K^*(\mathbf{x}_0)$ is the outer normal to K and therefore orthogonality of $\boldsymbol{\omega}$ to $\partial r_K^*(\mathbf{x}_0)$ means that $\boldsymbol{\omega}$ is a tangential direction.)

Proof of Theorem 1. Set $\widetilde{K} := \{\mathbf{x} \in K : r_K^*(\mathbf{x}) \le 1/2\}$. The continuity of r_K^* implies that $\widetilde{K} \subset \mathrm{Int}\ K$ is compact. Therefore there exist a $\delta > 0$ such that for any $\mathbf{x}_0 \in \widetilde{K}$ the ball $B(\mathbf{x}_0, \delta)$ centered at $\mathbf{x}_0$ and radius δ is contained in K. By the Bernstein Inequality (1.2) for any univariate polynomial q of degree at most n

$$|q'(0)| \le \frac{n}{\delta}\ \|q\|_{[-\delta,\delta]} \ .$$

This immediately implies that for any $\boldsymbol{\omega} \in S^{d-1}$ and $p \in M_d(\Lambda_n)$ we have

$$|D_{\boldsymbol{\omega}} p(\mathbf{x}_0)| \le \frac{\lambda_n}{\delta}\ \|p\|_{B(\mathbf{x}_0,\delta)} \le \frac{\lambda_n}{\delta}\|p\|_K \ .$$

Hence (2.1) holds in this case.

Assume now that $\mathbf{x}_0 \in K \setminus \widetilde{K}$, i.e., $r_K^*(\mathbf{x}_0) > \dfrac{1}{2}$. For any $p \in M_d(\Lambda_n)$ set $q(\mathbf{x}) := p(r_K^*(x_0)\mathbf{x})$. Clearly, $q \in M_d(\Lambda_n)$, and

$$\|q\|_K \le \|p\|_K, \quad Dq\left(\frac{\mathbf{x}_0}{r_K^*(\mathbf{x}_0)}\right) = r_K^*(\mathbf{x}_0)Dp(\mathbf{x}_0) \ .$$

Since $r_K^*(\mathbf{x}_0) > 1/2$ and $\mathbf{x}_0/r_K^*(\mathbf{x}_0) \in \mathrm{Bd}\ K$ we have

$$\sup_{p\in M_d(\Lambda_n)} \frac{Dp(x_0)}{\|p\|_K} \ \le \ 2 \sup_{q\in M_d(\Lambda_n)} \frac{\|Dq\|_{\mathrm{Bd}\,K}}{\|q\|_K} \ .$$

This shows that without loss of generality we may assume that $\mathbf{x}_0 \in \mathrm{Bd}\ K$. Clearly any $p \in M_d(\Lambda_n)$ has total degree at most λ_n, i.e., whenever $\boldsymbol{\omega} \perp \partial r_K(\mathbf{x}_0)$ ($\boldsymbol{\omega} \in S^{d-1}$) we have by (2.3)

$$D_{\boldsymbol{\omega}} p(\mathbf{x}_0) \le \frac{c_0}{\omega_K^{-1}(\lambda_n^{-2})} \ \|p\|_K \ . \tag{2.4}$$

Now set $g(t) := p(t\mathbf{x}_0), \quad t \in \mathbb{R}$. Clearly, $g \in \mathrm{span}\{t^{\lambda_j} : 0 \le j \le n\}$ and

$$\|g\|_{[0,1]} \le \|p\|_K \ . \tag{2.5}$$

Hence setting $\boldsymbol{\omega}_d := \mathbf{x}_0/|\mathbf{x}_0| \in S^{d-1}$ we obtain by (1.4) and (2.5)

$$\begin{aligned} |D_{\boldsymbol{\omega}_d} p(\mathbf{x}_0)| &= \frac{1}{|\mathbf{x}_0|}|\langle \partial p(\mathbf{x}_0), \mathbf{x}_0\rangle| = \\ &= \frac{1}{|\mathbf{x}_0|}|g'(1)| \le \frac{11}{|\mathbf{x}_0|}\sum_{j=0}^{n} \lambda_j \|g\|_{[0,1]} \le c_1 \sum_{j=0}^{n} \lambda_j \|p\|_K \ , \end{aligned} \tag{2.6}$$

where $c_1 := 11/\delta$. Since $r_K^*(t\mathbf{x}) = t r_K^*(\mathbf{x})$ for every $t > 0$ it follows that

$$r_K^*(\mathbf{x}) = \frac{d}{dt} r_K^*(t\mathbf{x}) = \langle \partial r_K^*(t\mathbf{x}), \mathbf{x}\rangle \ .$$

Setting $t = 1$ and recalling that $r_K^*(\mathbf{x}_0) = 1$ for $\mathbf{x}_0 \in \mathrm{Bd}\ K$ we have

$$\langle \partial r_K^*(\mathbf{x}_0), \mathbf{x}_0\rangle = r_K^*(\mathbf{x}_0) = 1 \ . \tag{2.7}$$

Consider now the space

$$L_{d-1} := \{\mathbf{u} \in \mathbb{R}^d : \mathbf{u} \perp \partial r_K^*(\mathbf{x}_0)\} \ .$$

Let $\boldsymbol{\omega}_j$, $1 \le j \le d-1$, be an orthonormal basis in L_{d-1}. By (2.7), $\mathbf{x}_0 \notin L_{d-1}$ and hence an arbitrary $\boldsymbol{\omega} \in S^{d-1}$ can be written as

$$\boldsymbol{\omega} = \sum_{j=1}^{d} a_j \, \boldsymbol{\omega}_j \ , \qquad a_j \in \mathbb{R} \ .$$

Since $\boldsymbol{\omega}_d = \mathbf{x}_0/|\mathbf{x}_0|$ we have by (2.7)

$$\begin{aligned} \langle \boldsymbol{\omega}, \partial r_K^*(\mathbf{x}_0)\rangle = a_d \langle \boldsymbol{\omega}_d, \ \partial r_K^*(\mathbf{x}_0)\rangle = \\ = \frac{a_d}{|\mathbf{x}_0|}\langle \mathbf{x}_0, \partial r_K^*(\mathbf{x}_0)\rangle = \frac{a_d}{|\mathbf{x}_0|} \ . \end{aligned} \tag{2.8}$$

In addition, for any $1 \le j \le d-1$

$$\langle \boldsymbol{\omega}, \boldsymbol{\omega}_j\rangle = a_j + a_d \langle \boldsymbol{\omega}_d, \boldsymbol{\omega}_j\rangle \ . \tag{2.9}$$

Thus by (2.8) and (2.9)

$$|a_d| \le |\mathbf{x}_0| \ |\partial r_K^*(\mathbf{x}_0)| \le M := \max_{\mathbf{x}\in K} |\mathbf{x}| \ |\partial r_K^*(\mathbf{x})| \ ,$$

$$|a_j| \le | + |a_d| \le 1 + M, \qquad 1 \le j \le d-1 \ .$$

Combining these estimates with (2.4) and (2.6) we arrive at

$$\begin{aligned} |D_{\boldsymbol{\omega}} p(\mathbf{x}_0)| = |\langle \partial p(\mathbf{x}_0), \boldsymbol{\omega}\rangle| \le \sum_{j=1}^{d} |a_j| \ |D_{\boldsymbol{\omega}_j} p(\mathbf{x}_0)| \le \\ \le c \left\{ \frac{1}{\omega_K^{-1}(\lambda_n^{-2})} + \sum_{j=0}^{n} \lambda_j \right\} \|p\|_K, \end{aligned}$$

with some $c > 0$ depending only on K. This completes the proof of Theorem 1.

Theorem 1 above gives a multivariate analogue of the Newman Inequality in case when the origin is *inside* the domain. Now we shall consider the more difficult case when the origin is on the *boundary* of the domain.

In order to obtain an analogue of (1.4) in this case we shall restrict ourselves to convex domains. Let $K \subset \mathbb{R}^d$ be a convex body with $\mathbf{0} \in \mathrm{Bd}\ K$. For a given $\mathbf{x}_0 \in \mathrm{Int}\ K$ consider the Minkowski functional of K centered at $\mathbf{x}_0$

$$r_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}_0 + \frac{\mathbf{x} - \mathbf{x}_0}{\alpha} \in K\}, \qquad \mathbf{x} \in \mathbb{R}^d.$$

Again $r_K(\mathbf{x}) \le 1$ if and only if $\mathbf{x} \in K$.

We shall say that $K \in C^2$ if r_K has continuous second partial derivatives in $\mathbb{R}^d$. Moreover, K is called *nonflat* at $\mathbf{0}$ if there exists a ball $B_0 \supset K$ with $\mathbf{0} \in \text{Bd } B_0$. If H_0 is the tangent plane to B_0 at $\mathbf{0}$ denote by $|\mathbf{x}|_K$ the distance from $\mathbf{x} \in \mathbb{R}^d$ to H_0.

THEOREM 2. *Let $K \in C^2$ be a strictly convex body in $\mathbb{R}^d$ which is nonflat at $\mathbf{0} \in Bd\ K$. Then for any $p \in M_d(\Lambda_n)$*

$$|\mathbf{x}|_K\ Dp(\mathbf{x}) \le c \sum_{j=0}^{n} \lambda_j \|p\|_K, \qquad \mathbf{x} \in K\ . \tag{2.10}$$

For K as in Theorem 2 and $0 < \xi < 1$ denote $K_\xi := \{\xi \mathbf{x} : \mathbf{x} \in K\}$.

Lemma 1. *Let K be as in Theorem 2. Then for every $\mathbf{x} \in K_{1/2}$*

$$dist(\mathbf{x}, Bd\ K) \ge c|\mathbf{x}|_K\ . \tag{2.11}$$

Proof. Since K is nonflat at $\mathbf{0} \in \text{Bd } K$ there exists a ball $B_0 \supset K$ with $\mathbf{0} \in \text{Bd } B_0$. Let H_0 be its tangent plane at $\mathbf{0}$. Without loss of generality we may assume that $H_0 = \{(0, x_2, \dots, x_d) : x_j \in \mathbb{R},\ 2 \le j \le d\}$. Then $|\mathbf{x}|_K = \text{dist}(\mathbf{x}, H_0) = x_1$ for every $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$. We can also assume that $x_1 \ge 0$ whenever $(x_1, \dots, x_d) \in K$. Furthermore differentiating with respect to $t > 0$ the relation

$$r_K(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)) = t\ r_K(\mathbf{x})\ ,$$

and setting $\mathbf{x} = (\mathbf{0})$ and $t = 1$ yields $1 = r_K(\mathbf{0}) = \langle \partial r_K(\mathbf{0}), -\mathbf{x}_0 \rangle$, i.e., $\partial r_K(\mathbf{0}) \ne 0$. Moreover, using that $\partial r_K(\mathbf{0})$ is the outer normal to K, and hence to H_0, as well, we obtain that $\partial r_K(\mathbf{0}) = (-\xi, 0, \dots, 0)$ where $\xi > 0$ is a constant depending on K. Recalling that $K \subset B_0$, where B_0 is a ball of radius R and with tangent plane H_0 at the origin, it follows that B is centered at $(R, 0, \dots, 0)$, i.e., $B = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|^2 \le 2Rx_1\}$. In particular, since $K \subset B_0$

$$|\mathbf{x}|^2 \le 2R\, x_1, \quad \mathbf{x} \in K\ . \tag{2.12}$$

Let $\mathbf{x} \in K_{1/2}$, i.e., $r_K(2\mathbf{x}) \le 1$. Then we have by the Taylor Theorem

$$1 \ge r_K(2\mathbf{x}) = 1 + \langle \partial r_K(\mathbf{0}), 2\mathbf{x} \rangle + \frac{1}{2} \langle \partial^2 r_K(\mathbf{0}) 2\mathbf{x}, 2\mathbf{x} \rangle + o(|\mathbf{x}|^2) \tag{2.13}$$

$$r_K(\mathbf{x}) = 1 + \langle \partial r_K(\mathbf{0}), \mathbf{x} \rangle + \frac{1}{2} \langle \partial^2 r_K(\mathbf{0}) \mathbf{x}, \mathbf{x} \rangle + o(|\mathbf{x}|^2)\ . \tag{2.14}$$

It follows from (2.13) that

$$\langle \partial^2 r_K(\mathbf{0})\mathbf{x}, \mathbf{x}\rangle \le -\langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle + o(|\mathbf{x}|^2) \ . \tag{2.15}$$

Substituting (2.15) into (2.14) yields

$$r_K(\mathbf{x}) \le 1 + \frac{1}{2}\langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle + o(|\mathbf{x}|^2) \ .$$

Finally, recalling that $\partial r_K(\mathbf{0}) = (-\xi, 0, \dots, 0)$ and using (2.12) in the above relation we obtain that

$$r_K(\mathbf{x}) \le 1 - \frac{1}{2}\ \xi x_1 + o(x_1) \le 1 - \frac{1}{4}\ \xi x_1 \tag{2.16}$$

whenever $x_1 \le \delta$, and δ depending on K is chosen sufficiently small. Thus by (2.16) for every $\mathbf{y} \in \text{Bd } K$

$$\frac{1}{4}\ \xi|\mathbf{x}|_K = \frac{1}{4}\ \xi x_1 \le 1 - r_K(\mathbf{x}) = r_K(\mathbf{y}) - r_K(\mathbf{x}) \le M|\mathbf{x} - \mathbf{y}|$$

where $M := \sup\{|\partial r_K(\mathbf{x})| : \mathbf{x} \in K\}$. Clearly, the last inequality yields (2.11) for $\mathbf{x} \in K_{1/2}$, $x_1 \le \delta$. The smoothness and nonflatness of K yields that Bd $K \cap K_{1/2} = \{\mathbf{0}\}$, i.e., (2.11) is trivial whenever $\mathbf{x} \in K_{1/2}$, $x_1 \ge \delta$. This completes the proof of Lemma.

Proof of Theorem 2. First we consider the case when $\mathbf{x} \in K_{1/2}$. Then by (2.11) the ball $B(\mathbf{x}, c|\mathbf{x}|_K)$ centered at $\mathbf{x}$ and of radius $c|\mathbf{x}|_K$ is contained in K. Using that any $p \in M_d(\Lambda_n)$ restricted to lines is a univariate polynomial of degree $\le \lambda_n$ we obtain by the Bernstein Inequality (1.2)

$$|D_{\boldsymbol{\omega}} p(\mathbf{x})| \le \frac{\lambda_n}{c|\mathbf{x}|_K}\|p\|_K, \qquad \boldsymbol{\omega} \in S^{d-1} \ .$$

Hence (2.10) holds whenever $\mathbf{x} \in K_{1/2}$.

Let now $\mathbf{x} \in K \setminus K_{1/2}$. Then $\mathbf{x} \in \text{Bd } K_{\xi_0}$ with some $1/2 < \xi_0 \le 1$. Setting $g(\mathbf{y}) := p(\xi_0 \mathbf{y}),\ y \in K$ we have

$$\|g\|_K \le \|p\|_K,\ |\partial p(\mathbf{x})| \le \frac{1}{\xi_0}\left|\partial g\left(\frac{\mathbf{x}}{\xi_0}\right)\right| \le 2\left|\partial g\left(\frac{\mathbf{x}}{\xi_0}\right)\right| \ .$$

This shows that it suffices to estimate $|\partial g|$ at $\mathbf{x}/\xi_0 \in \text{Bd } K$, i.e., without loss of generality one can assume in this case that $\mathbf{x} \in \text{Bd } K$. Again, as in the proof of Lemma 1 we shall assume that B_0 is a ball of radius R containing K, $\mathbf{0} \in \text{Bd } B_0$, $H_0 = \{(0, x_2, \dots, x_d) : x_j \in \mathbb{R},\ 2 \le j \le d\}$ is the tangent plane to B_0 (and hence to K, as well) at $\mathbf{0}$, and $x_1 \ge 0$

whenever $\mathbf{x} \in K$. Then again $\partial r_K(\mathbf{0}) = (-\xi, 0, \dots, 0)$ with some $\xi > 0$, and $|\mathbf{x}|_K = x_1$.

We shall verify that with some $c_1 > 0$ depending only on K

$$|\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle| \geq c_1 x_1, \qquad \mathbf{x} \in \text{Bd } K \ . \tag{2.17}$$

By the Taylor Formula

$$r_K(\mathbf{x}) = r_K(\mathbf{0}) + \langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle + \frac{1}{2}\langle \partial^2 r_K(\mathbf{0})\mathbf{x}, \mathbf{x}\rangle + o(|\mathbf{x}|^2) \ .$$

Since $r_K(\mathbf{0}) = r_K(\mathbf{x}) = 1$ for $\mathbf{0}, \mathbf{x} \in \text{Bd } K$ we obtain

$$\langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle = -\frac{1}{2}\langle \partial^2 r_K(\mathbf{0})\mathbf{x}, \mathbf{x}\rangle + o(|\mathbf{x}|^2) \ . \tag{2.18}$$

Furthermore, applying Taylor Formula for ∂r_K

$$\partial r_K(\mathbf{x}) = \partial r_K(\mathbf{0}) + \partial^2 r_K(\mathbf{0})\mathbf{x} + o(|\mathbf{x}|) \ .$$

Combining this with (2.18) yields

$$\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle = \langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle + \langle \partial^2 r_K(\mathbf{0})\mathbf{x}, \mathbf{x}\rangle o(|\mathbf{x}|^2) = \frac{1}{2}\langle \partial^2 r_K(\mathbf{0})\mathbf{x}, \mathbf{x}\rangle + o(|\mathbf{x}|^2) \ . \tag{2.19}$$

On the other hand by (2.18)

$$\langle \partial^2 r_K(\mathbf{0})\mathbf{x}, \mathbf{x}\rangle = -2\langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle + o(|\mathbf{x}|^2) \ .$$

Hence applying the last realation in (2.19) yields

$$\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle = -\langle \partial r_K(\mathbf{0}), \mathbf{x}\rangle + o(|\mathbf{x}|^2) = \xi x_1 + o(|\mathbf{x}|^2) \ .$$

Using this and (2.12)

$$\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle = \xi x_1 + o(x_1) \geq \xi x_1/2 \tag{2.20}$$

whenever $\mathbf{x} \in \text{Bd } K$, $|\mathbf{x}| \leq \delta$ with some $\delta > 0$ depending only on K.

Furthermore, by nonflatness of K we have $\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle \neq 0$ whenever $\mathbf{x} \in \text{Bd } K \setminus \{\mathbf{0}\}$. Hence

$$\inf\{|\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle| : \mathbf{x} \in \text{Bd } K, \ |\mathbf{x}| \geq \delta\} > 0 \ .$$

This together with (2.20) yields (2.17). Now we shall proceed similarly to the proof of Theorem 1. Since any $p \in M_d(\Lambda_n)$ has total degree $\leq \lambda_n$, whenever $\widetilde{\boldsymbol{\omega}} \perp \partial r_K(\mathbf{x})$ we have by (2.3) and $r_K \in C^2$

$$|D_{\widetilde{\boldsymbol{\omega}}} p(\mathbf{x})| \leq c_0 \, \lambda_n \, \|p\|_K \ . \tag{2.21}$$

Clearly, with $\boldsymbol{\omega}_d := \mathbf{x}/|\mathbf{x}|$ we have as in (2.6)

$$|D_{\boldsymbol{\omega}_d} p(\mathbf{x})| \le c_1 \sum_{j=0}^{n} \lambda_j \, \|p\|_K \ . \tag{2.22}$$

Since $\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle \neq 0$ any $\boldsymbol{\omega} \in S^{d-1}$ can be written as

$$\boldsymbol{\omega} = \sum_{j=1}^{d} a_j \, \boldsymbol{\omega}_j, \qquad a_j \in \mathbb{R} \ ,$$

where $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{d-1}$ is an orthonormal basis in $\partial r_K(\mathbf{x})^{\perp}$. Thus by (2.17)

$$|\langle \partial r_K(\mathbf{x}), \boldsymbol{\omega}\rangle| = |a_d \langle \partial r_K(\mathbf{x}), \boldsymbol{\omega}_d\rangle| = \frac{|a_d|}{|\mathbf{x}|} |\langle \partial r_K(\mathbf{x}), \mathbf{x}\rangle| \ge \frac{c_1 |a_d| x_1}{|\mathbf{x}|}.$$

Hence

$$|a_d| \le \frac{|\mathbf{x}| \ |\partial r_K(\mathbf{x})|}{c_1 \ x_1} \le \frac{c_2}{x_1} \ , \tag{2.23}$$

with some $c_2 > 0$ depending only on K. Since $\langle \boldsymbol{\omega}, \boldsymbol{\omega}_j\rangle = a_j + a_d \langle \boldsymbol{\omega}_d, \boldsymbol{\omega}_j\rangle$, $1 \le j \le d-1$, we have by (2.23)

$$|a_j| \le 1 + |a_d| \le 1 + c_2/x_1 \le \frac{c_3}{x_1} \ . \tag{2.24}$$

Using (2.21)-(2.24) we arrive at

$$|D_{\boldsymbol{\omega}} p(\mathbf{x})| = |\langle \partial p(\mathbf{x}), \boldsymbol{\omega}\rangle| \le \sum_{j=1}^{d} |a_j| \cdot |\langle \partial p(\mathbf{x}), \boldsymbol{\omega}_j\rangle| \le$$

$$\le \frac{c}{x_1} \ \sum_{j=0}^{n} \ \lambda_j \ \|p\|_K \ ,$$

with some $c > 0$ depending only on K. Since $|\mathbf{x}|_K = x_1$ this yields (2.10). □

Now we shall consider the case when $\mathbf{0} \in \mathrm{Bd}\ K$ and K is not smooth. We shall present a result of this type for star-like domains in $\mathbb{R}^2_+$. Based on this result one can obtain corresponding estimates for domains in $\mathbb{R}^d_+$, $d > 2$.

Let $(\lambda_j, \mu_j) \in \mathbb{R}^2_+$, $1 \le j \le n$, and set

$$\widetilde{P}^2_n := \mathrm{span}\{x^{\lambda_j} y^{\mu_j} : 0 \le j \le n\}, \quad (x, y) \in \mathbb{R}^2 \ . \tag{2.25}$$

This is a general Müntz-type polynomial space. Note that the exponents no longer are assumed to be integers.

Further, let K be a star-like domain in $\mathbb{R}^2$ with Minkowski functional r_K^* centered at $\mathbf{0}$. For given $0 < \alpha < \beta$ denote

$$\begin{aligned} D_{\alpha\beta} &:= \{(x,y) \in \mathbb{R}^2_+ : \ 0 \le \alpha x \le y \le \beta x\}, \\ K_{\alpha\beta} &:= K \cap D_{\alpha\beta} \ . \end{aligned} \tag{2.26}$$

Moreover, we shall say that $r_K^* \in \ \mathrm{Lip}_M \ 1$ on $K_{\alpha\beta}$ with some $M > 0$ if whenever $(x_1, y_1), \ (x_2, y_2) \in K_{\alpha\beta}$

$$|r_K^*(x_1,y_1) - r_K^*(x_2,y_2)| \le M(|x_1 - x_2| + |y_1 - y_2|) \ .$$

THEOREM 3. *Let $\widetilde{P}_n^2$ be given by* (2.25) *and assume that $r_K^* \in Lip_M \ 1$. Then for every $p \in \widetilde{P}_n^2$*

$$\| \sqrt{x^2 + y^2} \ Dp \ \|_{K_{\alpha\beta}} \le c \left(\sum_{j=0}^{n} (\lambda_j + \mu_j) + n^2 \right) \|p\|_{K_{\alpha\beta}} \tag{2.27}$$

where $c > 0$ depends only on $K_{\alpha\beta}$.

Note that estimate (2.27) is weaker than (2.10) in the sense that in the sum on the right hand side of (2.27) *all* exponents have to be accounted for. For instance, if $(\lambda_j, \mu_j) = (j, n-j), \ 0 \le j \le n$, i.e., $\widetilde{P}_n^2 = H_n^2$ then (2.27) yields a constant of order n^2. On the other hand if $K \in C^2$ then from (2.10) we obtain an estimate of order n. ($H_n^2 = M_d(\Lambda_0)$ with $\lambda_0 = n$.) Our next example shows that without proper smoothness only weaker Newman-type inequalities similar to (2.27) can hold, in general.

Example. Let K be a triangle in $\mathbb{R}^2$ given by $K = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y \le 1, \ 0 \le x \le 1\}$, and consider the polynomial $p(x,y) := y^n \ T_n(x/y) \in H_2^2$, where $T_n(t) = \mathrm{cosnarccos}\ (2t-1)$ is the Chebyshev polynomial on $[0,1]$. Evidently,

$$\|p\|_K \le \|T_n\|_{[0,1]} \le 1, \quad \frac{\partial p}{\partial x}(1,1) = T_n'(1) = 4n^2 \ .$$

Hence in this case inequality (2.27) can be reversed (with another constant).

Proof of Theorem 3. Let $(\tilde{x}, \tilde{y}) \in K_{\alpha\beta}$, $p \in \widetilde{P}_n^2$. Set $g(t) := p(t\tilde{x}, t\tilde{y})$, $0 \le t \le 1$. Then $g \in \mathrm{span}\{t^{\lambda_j + \mu_j} : 0 \le j \le n\}$, and

$$\|g\|_{[0,1]} \le \|p\|_{K_{\alpha\beta}}. \tag{2.28}$$

Then we obtain by (1.4) and (2.28)

$$\left|\tilde{x}\frac{\partial p}{\partial x}(\tilde{x},\tilde{y})+\tilde{y}\frac{\partial p}{\partial y}(\tilde{x},\tilde{y})\right| = |g'(1)| \le 11\sum_{j=0}^{n}(\lambda_j+\mu_j)\|p\|_{K_{\alpha\beta}} \ . \tag{2.29}$$

Furthermore, there exists a $\eta_K > 0$ depending only on $K_{\alpha\beta}$ such that

$$\Delta^* := \{(x,y)\in\mathbb{R}^2 : 0 \le \alpha\, x \le y \le \beta\, x,\ 0 \le x \le \eta_K\} \subset K_{\alpha\beta}.$$

Now we shall distinguish between 2 cases.

Case 1. $\tilde{x} \le \dfrac{\alpha}{\beta}\,\eta_K$. Consider the curve

$$\tilde{\gamma} := \left\{\left(\frac{\tilde{x}^2}{\tilde{y}}t,\ \frac{\tilde{x}^2}{\tilde{y}}t^2\right) : \alpha \le t \le \beta\right\} \ .$$

Clearly, $\tilde{\gamma}$ passes through $(\tilde{x},\tilde{y})$ (when $t = \tilde{y}/\tilde{x}$) and $\tilde{\gamma} \subset D_{\alpha\beta}$. Moreover, since $\tilde{x} \le \alpha\eta_K/\beta$ it follows that

$$\frac{\tilde{x}^2}{\tilde{y}}t \le \frac{\alpha\,\eta_K}{\beta}\cdot\frac{\tilde{x}t}{\tilde{y}} \le \frac{\alpha\,\eta_K}{\beta}\cdot\frac{\beta}{\alpha} = \eta_K \ ,$$

i.e., $\tilde{\gamma} \subset \Delta^*$. Set now

$$\tilde{g}(t) := p\left(\frac{\tilde{x}^2}{\tilde{y}}t,\ \frac{\tilde{x}^2}{\tilde{y}}t^2\right) \in \operatorname{span}\{t^{\lambda_j+2\mu_j} : 0 \le j \le n\} \ .$$

Since $\tilde{\gamma} \subset \Delta^* \subset K_{\alpha\beta}$ we have

$$\|\tilde{g}\|_{[\alpha,\beta]} \le \|p\|_{K_{\alpha\beta}}. \tag{2.30}$$

Hence we obtain by (1.5) and (2.30) for any $t \in [\alpha,\beta]$

$$\begin{aligned}|\tilde{g}'(t)| &= \left|\frac{\partial p}{\partial x}\,\frac{\tilde{x}^2}{\tilde{y}} + \frac{\partial p}{\partial y}\,2t\,\frac{\tilde{x}^2}{\tilde{y}}\right| \le \\ &\le c\sum_{j=0}^{n}(\lambda_j+2\mu_j)\|g\|_{[\alpha,\beta]} \le c_1\left(\sum_{j=0}^{n}(\lambda_j+\mu_j)+n^2\right)\|p\|_{K_{\alpha\beta}} \ ,\end{aligned}$$

where $c_1 > 0$ depends only on $K_{\alpha\beta}$. Setting $t = \tilde{y}/\tilde{x}$ in the above estimate yields

$$\begin{aligned}\left|\tilde{x}\frac{\partial p}{\partial x}\,(\tilde{x},\tilde{y}) + \frac{\partial p}{\partial y}\,(\tilde{x},\tilde{y})2\tilde{y}\right| &= \frac{\tilde{y}}{\tilde{x}}\left|\tilde{g}'\left(\frac{\tilde{y}}{\tilde{x}}\right)\right| \le \\ &\le \beta c_1\left(\sum_{j=0}^{n}(\lambda_j+\mu_j)+n^2\right)\|p\|_{K_{\alpha\beta}} \ .\end{aligned} \tag{2.31}$$

Using (2.29) and (2.31) we arrive at

$$\left|\tilde{y}\frac{\partial p}{\partial y}\,(\tilde{x},\tilde{y})\right| \le (11+\beta c_1)\left(\sum_{j=0}^{n}(\lambda_j+\mu_j)+n^2\right)\|p\|_{K_{\alpha\beta}}\ ,$$

$$\left|\tilde{x}\frac{\partial p}{\partial y}\,(\tilde{x},\tilde{y})\right| \le (22+\beta c_1)\left(\sum_{j=0}^{n}(\lambda_j+\mu_j)+n^2\right)\|p\|_{K_{\alpha\beta}}\ .$$

Since $|x| \ge \sqrt{x^2+y^2}\ /\ \sqrt{1+\beta^2}$, $|y| \ge \alpha\sqrt{x^2+y^2}\ /\ \sqrt{1+\alpha^2}$ whenever $(x,y) \in K_{\alpha\beta}$ the above estimates yield (2.27).

Case 2. $\tilde{x} > \alpha\,\eta_K/\beta$. We may assume without loss of generality, that $\tilde{y}/\tilde{x} \ge (\alpha+\beta)/2$. (The case $\tilde{y}/\tilde{x} \le (\alpha+\beta)/2$ can be treated similarly.)

For some $1 < \gamma \le 2$ to be specified below consider the plane curve

$$x = \tilde{x}t,\ y = \tilde{y}\,t^{\gamma}, \quad 0 \le t \le 1\ , \tag{2.32}$$

which connects $(\tilde{x},\tilde{y})$ to the origin. We need to decide for what values of t this curve is contained in $K_{\alpha\beta}$. Clearly, $y/x \le \beta$ on this curve. In order to show that this curve belongs to the needed angle we also need to check the condition $y/x \ge \alpha$. Set $c_0 := \left(\frac{2\alpha}{\alpha+\beta}\right)^{1/(\gamma-1)} < 1$, and assume that $t \ge c_0$. Then using that $\tilde{y}/\tilde{x} \ge (\alpha+\beta)/2$ we have by (2.32)

$$\frac{y}{x} = \frac{\tilde{y}}{\tilde{x}}\,t^{\gamma-1} \ge \frac{\alpha+\beta}{2}c_0^{\gamma-1} \ge \alpha\ .$$

Hence whenever $c_0 \le t \le 1$ we have $\alpha \le y/x \le \beta$. In order to see if $(x,y) \in K_{\alpha\beta}$ we also need to check the condition $r_K^*(x,y) \le 1$ which ensures that $(x,y) \in K$. Using that $r_K^*(\tilde{x},\tilde{y}) \le 1$ and $r_K^* \in \mathrm{Lip}_M 1$ we have for any (x,y) satisfying (2.32)

$$\begin{aligned} &r_K^*(x,y) = r_K^*(\tilde{x}\,t,\tilde{y}\,t^{\gamma}) = t\,r_K^*(\tilde{x},\tilde{y}\,t^{\gamma-1}) \le \\ &\le r_K^*(\tilde{x},\tilde{y}\,t^{\gamma-1}) - r_K^*(\tilde{x},\tilde{y}) + 1 - (1-t)r_K^*(\tilde{x},\tilde{y}\,t^{\gamma-1}) \le \\ &\le M\tilde{y}|1-t^{\gamma-1}| + 1 - (1-t)r_K^*(\tilde{x},\tilde{y}\,t^{\gamma-1})\ . \end{aligned} \tag{2.33}$$

Let $c_0 \le t \le 1$. Then with some constant $c_1 > 0$ depending only on K

$$\tilde{y}(1-t^{\gamma-1}) \le c_1(\gamma-1)(1-t), \qquad t \in [c_0,1]\ . \tag{2.34}$$

Moreover since $\tilde{x} > \alpha\,\eta_K/\beta$ it follows that

$$r_K^*(\tilde{x},\tilde{y}\,t^{\gamma-1}) \ge c_2 \tag{2.35}$$

with some $c_2 > 0$ depending only on $K_{\alpha\beta}$. Combining (2.33)-(2.35) yields

$$r_K^*(x,y) \le 1 + M\, c_1(\gamma-1)(1-t) - c_2(1-t) = 1 + (1-t)(M\, c_1(\gamma-1) - c_2).$$

Thus setting $\gamma := \min\{2, 1 + c_2/2Mc_1\} > 1$ we obtain that $r_K^*(x,y) \le 1$ on the curve (2.32) for $t \in [c_0, 1]$. Hence with this choice of γ curve (2.32) is contained in $K_{\alpha\beta}$ for $t \in [c_0, 1]$. Set now $g(t) := p(\tilde{x}t, \tilde{y}\, t^\gamma)$, where $p \in \widetilde{P}_n^2$. Then by the above observation

$$\|g\|_{[c_0,1]} \le \|p\|_{K_{\alpha,\beta}} \ . \tag{2.36}$$

Moreover, $g \in \operatorname{span}\{t^{\lambda_j + \gamma\,\mu_j} : 0 \le j \le n\}$. Thus using again the Newman-type Inequality (1.5) we obtain by (2.36)

$$|g'(1)| \le c \left(\sum_{j=0}^{n} (\lambda_j + \gamma\,\mu_j) + n^2 \right) \|g\|_{[c_0,1]} \le 2c \left(\sum_{j=0}^{n} (\lambda_j + \mu_j) + n^2 \right) \|p\|_{[K_{\alpha\beta}]} \tag{2.37}$$

Obviously,

$$g'(1) = \tilde{x} \frac{\partial p}{\partial x}(\tilde{x}, \tilde{y}) + \gamma \tilde{y} \frac{\partial p}{\partial y}(\tilde{x}, \tilde{y}) \ .$$

This and (2.37) yields

$$\left| \tilde{x} \frac{\partial p}{\partial x}(\tilde{x}, \tilde{y}) + \gamma \tilde{y} \frac{\partial p}{\partial y}(\tilde{x}, \tilde{y}) \right| \le 2c \left(\sum_{j=0}^{n} (\lambda_j + \mu_j) + n^2 \right) \|p\|_{K_{\alpha\beta}} \ . \tag{2.38}$$

Thus with $A := \tilde{x} \frac{\partial p}{\partial x}(\tilde{x}, \tilde{y})$ and $B := \tilde{y} \frac{\partial p}{\partial y}(\tilde{x}, \tilde{y})$ we have by (2.38) and (2.29)

$$A + B = C_1, \quad A + \gamma B = C_2 \tag{2.39}$$

with some $C_1, C_2 > 0$ satisfying

$$|C_j| \le c^* \left(\sum_{j=0}^{n} (\lambda_j + \mu_j) + n^2 \right) \|p\|_{K_{\alpha\beta}}, \qquad j = 1, 2 \tag{2.40}$$

and $c^* > 0$ depending only on K. Thus by (2.39) and (2.40)

$$\left| \tilde{y} \frac{\partial p}{\partial y}(\tilde{x}, \tilde{y}) \right| = |B| = \frac{1}{\gamma - 1} |C_2 - C_1| \le \frac{2c^*}{\gamma - 1} \left(\sum_{j=0}^{n} (\lambda_j + \mu_j) + n^2 \right) \|p\|_{K_{\alpha\beta}} \ . \tag{2.41}$$

Using above estimate and (2.40)

$$\left|\tilde{x}\frac{\partial p}{\partial x}(\tilde{x},\tilde{y})\right| = |A| \le |B| + |C_1| \le \frac{\gamma+1}{\gamma-1}c^*\left(\sum_{j=0}^{n}(\lambda_j+\mu_j)+n^2\right)\|p\|_{K_{\alpha\beta}}\ . \tag{2.42}$$

Using again that

$$|x| \ge \frac{\sqrt{x^2+y^2}}{\sqrt{1+\beta^2}}, \quad |y| \ge \frac{\alpha\sqrt{x^2+y^2}}{\sqrt{1+\alpha^2}}$$

whenever $(x,y) \in K_{\alpha,\beta}$ estimates (2.41)-(2.42) yield the desired inequality (2.27). This completes the proof of Theorem 3.

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References

[1] D. Benko, T. Erdélyi, J. Szabados, The full Markov-Newman Inequality for Müntz polynomials on positive intervals, *Proc. Amer. Math. Soc.* **131** (2003), 2385-2391.

[2] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, New York, Springer Verlag, 1995.

[3] A. Kroó, Classical Polynomial Inequalities in Several Variables, *In Constructive Theory of Functions* (B. D. Bojanov, ed.) Darba, Sofia, (2003), 19-32.

[4] A. Kroó, Markov-type Inequalities for surface gradients of multivariate polynomials, *J. Approx. Th.* **118** (2002), 235-245.

[5] A. Kroó, Markov-type inequalities for homogeneous polynomials on smooth convex bodies,*East J. Approx.* **9** (2003), 487-500.

[6] D. J. Newman, Derivative bounds for Müntz polynomials, *J. Approx. Th.* **18** (1976), 360-362.

[7] D. R. Wilhelmsen, A Markov inequality in several dimensions, *J. Approx. Th.* **11** (1974), 216-220.

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Multiple Orthogonality and Quadratures of Gaussian Type

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Abstract. In this paper we consider multiple orthogonal polynomials on the real line, defined using orthogonality conditions spread out over r different measures. Such polynomials are generalization of the ordinary orthogonal polynomials (case $r = 1$).

Using the discretized Stieltjes–Gautschi procedure, we compute recursive coefficients of corresponding recurrence relation for such polynomials, and also, zeros of multiple orthogonal polynomials. We construct and consider the corresponding quadrature formulas of Gaussian type. Specially, we consider quadratures of Gauss–Lobatto and Gauss–Radau type. Some numerical examples are also included.

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1 Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy r orthogonality conditions.

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Starting with a problem that arise in the evaluation of computer graphics illumination models, Borges [6] has examined the problem of numerically evaluating a set of r definite integrals taken with respect to distinct weight functions but related by a common integrand and interval of integration. The nodes of an optimal set of such quadratures (quadratures of Gaussian type) are the zeros of type II multiple orthogonal polynomials. However, Borges has not used multiple orthogonality. In [12] Milovanović and Stanić have presented effective numerical method for constructing type II multiple orthogonal polynomials, and the corresponding Gaussian quadratures, using *discretized Stieltjes–Gautschi procedure* [7].

In [13] Milovanović and Stanić have investigated multiple orthogonal polynomials on the semicircle, as a generalization of orthogonal polynomials on the semicircle, introduced by Gautschi and Milovanović [10] – complex polynomials orthogonal with respect to the complex-valued inner products $[f,g]_k = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w_k(e^{i\theta})\,d\theta$, for $k = 1, 2, \ldots, r$. Also, a numerical method for constructing these polynomials and corresponding Gaussian quadratures has presented.

In this paper we repeat some basic results on multiple orthogonal polynomials, their numerical construction, and consider quadratures of Gauss–Lobatto and Gauss–Radau type, including some numerical examples. The paper is organized as follows. First, some basic facts about the type II multiple orthogonal polynomials are given in Section 2. In Section 3 we give some properties of type II multiple orthogonal polynomials with nearly diagonal multi-indices and numerical procedure for their construction, based on the discretized Stieltjes–Gautschi procedure. In the same section an optimal set of quadrature formulas and the corresponding method for calculating the nodes and weight coefficients of such quadratures are considered. In Section 4 we consider quadrature formulas of Gaussian type with preassigned nodes. Specially, we consider quadratures of Gauss–Lobatto and Gauss–Radau type. Also, some numerical examples are included.

2 Type II Multiple Orthogonal Polynomials

Let $r \geq 1$ be an integer and let $w_1, w_2, \ldots, w_r$ be r weight functions on the real line so that the support of each w_i is a subset of an interval E_i. Let $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a vector of r nonnegative integers, which is called a *multi-index* with the length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$.

There are two types of multiple orthogonal polynomials:

1° *Type I multiple orthogonal polynomials.* Here we want to find a vector

of polynomials $(A_{\vec{n},1}, A_{\vec{n},2}, \ldots, A_{\vec{n},r})$ such that each $A_{\vec{n},i}$ is a polynomial of degree $n_i - 1$ and the following orthogonality conditions hold:

$$\sum_{j=1}^{r} \int_{E_j} A_{\vec{n},j}\, x^k w_j(x) dx = 0, \quad k = 0, 1, \ldots, |\vec{n}| - 2. \tag{2.1}$$

Each $A_{\vec{n},i}$ has n_i coefficients and the type I vector is completely determined if we can find all the $|\vec{n}|$ unknown coefficients. The orthogonality relations (2.1) give $|\vec{n}| - 1$ homogenous linear equations for these $|\vec{n}|$ coefficients. If the matrix of coefficients has full rank, then we can determine the type I vector uniquely up to a multiplicative factor.

For $r = 1$ we have the case of ordinary orthogonal polynomials.

2° *Type II multiple orthogonal polynomials.* The type II multiple orthogonal polynomial is a monic polynomial $\pi_{\vec{n}}$ of degree $|\vec{n}|$ such that the following orthogonality conditions:

$$\int_{E_1} \pi_{\vec{n}}(x)\, x^k w_1(x) dx = 0, \qquad k = 0, 1, \ldots, n_1 - 1, \tag{2.2}$$

$$\int_{E_2} \pi_{\vec{n}}(x)\, x^k w_2(x) dx = 0, \qquad k = 0, 1, \ldots, n_2 - 1, \tag{2.3}$$

$$\vdots$$

$$\int_{E_r} \pi_{\vec{n}}(x)\, x^k w_r(x) dx = 0, \qquad k = 0, 1, \ldots, n_r - 1, \tag{2.4}$$

are satisfied.

Again, for $r = 1$ we have the ordinary orthogonal polynomials.

The conditions (2.2)–(2.4) give $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients $a_{k,\vec{n}}$ of the polynomial $\pi_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k,\vec{n}}\, x^k$, where $a_{|\vec{n}|,\vec{n}} = 1$. But the matrix of coefficients of this system can be singular and we need some additional conditions on the r weight functions to provide the uniqueness of the multiple orthogonal polynomial. If the polynomial $\pi_{\vec{n}}(x)$ is unique, then we say that $\vec{n}$ is a *normal multi-index* and if all multi-indices are normal then we have a *complete system.*

In this paper we consider only the type II multiple orthogonal polynomials. For each of the weight functions w_k, $k = 1, 2, \ldots, r$,

$$(f, g)_k = \int_{E_k} f(x) g(x) w_k(x) dx \tag{2.5}$$

denotes the corresponding inner product of the functions f and g.

Our interest is in systems of r weight functions for which all multi-indices are normal. There are two distinct cases for which the type II multiple orthogonal polynomials are given.

1. *Angelesco systems* for which the intervals E_i, on which the weight functions are supported, are disjoint, i.e., $E_i \cap E_j = \emptyset$ for $1 \le i, j \le r$, $i \ne j$.

2. *AT system* is such that all weight functions are supported on the same interval E and we also require that the $|\vec{n}|$ functions

$$w_1(x),\, xw_1(x),\, \ldots,\, x^{n_1-1}w_1(x),\;\; w_2(x),\, xw_2(x),\, \ldots, x^{n_2-1}w_2(x),\; \ldots,\, w_r(x),\, xw_r(x),\, \ldots,\, x^{n_r-1}w_r(x)$$

form a Chebyshev system on E for each multi-index $\vec{n}$. This means that every linear combination

$$\sum_{j=1}^{r} Q_{n_j-1}(x)w_j(x),$$

where Q_{n_j-1} is a polynomial of degree at most $n_j - 1$, has at most $|\vec{n}| - 1$ zeros on E.

The following two theorems hold (for proof see [16]):

Theorem 2.1 *In an Angelesco system the type II multiple orthogonal polynomial $\pi_{\vec{n}}(x)$ factors into r polynomials $\prod_{j=1}^{r} q_{n_j}(x)$, where each q_{n_j} has exactly n_j zeros on E_j.*

Theorem 2.2 *In an AT system the type II multiple orthogonal polynomial $\pi_{\vec{n}}(x)$ has exactly $|\vec{n}|$ zeros on E.*

3 Type II Multiple Orthogonal Polynomials with Nearly Diagonal Multi-index

Let $n \in \mathbb{N}$ and write it as $n = \ell r + \nu$, with $\ell = [n/r]$ and $0 \le \nu < r$. The nearly diagonal multi-index $\vec{s}(n)$ corresponding to n is given by

$$\vec{s}(n) = (\underbrace{\ell+1, \ell+1, \ldots, \ell+1}_{\nu \text{ times}}, \underbrace{\ell, \ell, \ldots, \ell}_{r-\nu \text{ times}}).$$

Denote the corresponding type II multiple (monic) orthogonal polynomials by

$$\pi_n(x) = \pi_{\vec{s}(n)}(x).$$

Type II multiple orthogonal polynomials with nearly diagonal multi-index satisfy the following recurrence relation of order $r+1$[1]

$$x\pi_m(x) = \pi_{m+1}(x) + \sum_{i=0}^{r} \alpha_{m,r-i}\pi_{m-i}(x)\,, \quad m \geq 0, \tag{3.1}$$

with initial conditions $\pi_0(x) = 1$ and $\pi_i(x) = 0$ for $i = -1, -2, \ldots, -r$ (see [15]).

Setting $m = 0, 1, \ldots, n-1$ in (3.1), we get

$$x \begin{bmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1}(x) \end{bmatrix} = H_n \begin{bmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1}(x) \end{bmatrix} + \pi_n(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

i.e.,

$$H_n \mathbf{p}_n(x) = x\,\mathbf{p}_n(x) - \pi_n(x)\mathbf{e}_n, \tag{3.2}$$

where

$$\mathbf{p}_n(x) = \begin{bmatrix}\pi_0(x) & \pi_1(x) & \ldots & \pi_{n-1}(x)\end{bmatrix}^T, \qquad \mathbf{e}_n = [0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1]^T,$$

and H_n is the following lower (banded) Hessenberg matrix[2] of order n

$$H_n = \begin{bmatrix}
\alpha_{0,r} & 1 & & & & & & \\
\alpha_{1,r-1} & \alpha_{1,r} & 1 & & & & & \\
\vdots & \ddots & \ddots & \ddots & & & & \\
\alpha_{r,0} & \cdots & \alpha_{r,r-1} & \alpha_{r,r} & 1 & & & \\
 & \alpha_{r+1,0} & \cdots & \alpha_{r+1,r-1} & \alpha_{r+1,r} & 1 & & \\
 & & \ddots & & \ddots & \ddots & \ddots & \\
 & & & \alpha_{n-2,0} & \cdots & \alpha_{n-2,r-1} & \alpha_{n-2,r} & 1 \\
 & & & & \alpha_{n-1,0} & \cdots & \alpha_{n-1,r-1} & \alpha_{n-1,r}
\end{bmatrix}.$$

[1]It is known that ordinary orthogonal polynomials on the real line always satisfy a three-term recurrence relation.

[2]This kind of matrix was obtained also in construction of orthogonal polynomials on the radial rays in the complex plane (see [11]).

Let $\xi_\nu \equiv \xi_\nu^{(n)}$ $(\nu = 1, \ldots, n)$ be zeros of $\pi_n(x)$. Then (3.2) reduces to the eigenvalue problem

$$\xi_\nu \mathbf{p}_n(\xi_\nu) = H_n \mathbf{p}_n(\xi_\nu).$$

Thus, ξ_ν are eigenvalues of the matrix H_n and $\mathbf{p}_n(\xi_\nu)$ are the corresponding eigenvectors.

For computing zeros of $\pi_n(x)$ as the eigenvalues of the matrix H_n we use the EISPACK routine COMQR [3, pp. 277–284]. Notice that this routine needs an upper Hessenberg matrix, i.e., H_n^T. Also, the MATLAB or MATHEMATICA can be used.

Our aim here is to compute the recurrence coefficients in (3.1), i.e., the elements of the Hessenberg matrix H_n. Only for the simplest case of multiple orthogonality, i.e., when $r = 2$, for some classical weight functions (Jacobi, Laguerre, Hermite) one can find explicit formulas for the recurrence coefficients (see [14], [16]). In [12] the elements of H_n for arbitrary r have been calculated numerically, using the discretized Stieltjes–Gautschi procedure [7].

At first, we express the elements of H_n in terms of the inner products (2.5), and then we use the corresponding Gaussian formulas to discretize these inner products. Of course, we suppose that the type II multiple orthogonal polynomials exist with respect to the inner products $(\cdot\,,\cdot)_k$, $k = 1, 2, \ldots, r$, given by (2.5).

Taking that for inner products $(\cdot\,,\cdot)_{j+mr} = (\cdot\,,\cdot)_j$ $(m \in \mathbb{Z})$, the following result holds:

Theorem 3.1 *The type II multiple monic orthogonal polynomials* $\{\pi_n\}$*, with nearly diagonal multi-index, satisfy the recurrence relation*

$$\pi_{n+1}(x) = (x - \alpha_{n,r})\pi_n(x) - \sum_{k=0}^{r-1} \alpha_{n,k}\pi_{n-r+k}(x), \quad n \geq 0, \tag{3.3}$$

where

$$\alpha_{n,0} = \frac{\left(x\pi_n, \pi_{[(n-r)/r]}\right)_{\nu+1}}{\left(\pi_{n-r}, \pi_{[(n-r)/r]}\right)_{\nu+1}}$$

and

$$\alpha_{n,k} = \frac{\left(x\pi_n - \sum_{i=0}^{k-1} \alpha_{n,i}\pi_{n-r+i}, \pi_{[(n-r+k)/r]}\right)_{\nu+k+1}}{\left(\pi_{n-r+k}, \pi_{[(n-r+k)/r]}\right)_{\nu+k+1}}, \quad k = 1, 2, \ldots, r$$

Here, we put $n = \ell r + \nu$, *where* $\ell = [n/r]$ *and* $\nu \in \{0, 1, \ldots, r-1\}$ ($[t]$ *is integer part of* t).

Proof of the previous theorem one can find in [12].

We use alternatively recurrence relation and given formulas for coefficients. Knowing π_0 we compute $\alpha_{0,r}$, then knowing $\alpha_{0,r}$ we compute π_1, and then again $\alpha_{1,r}$ and $\alpha_{1,r-1}$, etc. Continuing in this manner, we can generate as many polynomials, and therefore as many of the recurrence coefficients as are desired.

All of the necessary inner products can be computed exactly, except for rounding errors, by using the Gauss–Christoffel quadrature formulas with respect to the corresponding weight function w_i, $i = 1, 2, \ldots r$.

3.1 Optimal Set of Quadratures

Denote with $W = \{w_1, w_2, \ldots, w_r\}$ an AT system.

Following [6, Definition 3] we introduced the following definition (see [12]):

Definition 3.2 *Let* W *be an AT system (the weight functions* w_i, $i = 1, 2, \ldots, r$ *are supported on the interval* E), $\vec{n} = (n_1, n_2, \ldots, n_r)$ *be a multi-index, and* $n = |\vec{n}|$. *A set of quadrature formulas of the form:*

$$\int_E f(x)\, w_m(x)dx \approx \sum_{i=1}^{n} A_{m,i} f(x_i), \quad m = 1, 2, \ldots, r \tag{3.4}$$

will be called an optimal set (quadratures of Gaussian type) with respect to $(W, \vec{n})$ *if and only if the weight coefficients,* $A_{m,i}$, *and the nodes,* x_i, *satisfy the following equations:*

$$\begin{aligned} \sum_{i=1}^{n} A_{m,i} &= \int_E w_m(x)dx \\ \sum_{i=1}^{n} A_{m,i}\, x_i &= \int_E x\, w_m(x)dx \\ &\vdots \\ \sum_{i=1}^{n} A_{m,i}\, x_i^{n+n_m-1} &= \int_E x^{n+n_m-1}\, w_m(x)dx \end{aligned} \tag{3.5}$$

for $m = 1, 2, \ldots, r$.

For these optimal set of quadratures the next generalization of fundamental theorem of Gauss–Christoffel quadrature formulas holds (see [12]):

Theorem 3.3 *Let W be an AT system, $\vec{n} = (n_1, n_2, \ldots, n_r)$, $n = |\vec{n}|$. Consider the quadrature formulas:*

$$\int_E f(x)\, w_m(x)dx \approx \sum_{i=1}^{n} A_{m,i} f(x_i) \tag{3.6}$$

where $m = 1, 2, \ldots, r$.

These formulas form an optimal set with respect to $(W, \vec{n})$ if and only if:

1° *They are exact for all polynomials of degree $\leq n-1$.*

2° *The polynomial $q(x) = \prod_{i=1}^{n}(x - x_i)$ is the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to W.*

For the case of the nearly diagonal multi-indices $\vec{s}(n)$ the nodes x_i, $i = 1, 2, \ldots, n$, of the Gaussian type quadrature formulas can be computed as eigenvalues of the corresponding banded Hessenberg matrix H_n.

The weight coefficients $A_{m,i}$ can be computed by requiring that each rule correctly generate the first n modified moments. Denote by $V_n = \left[\mathbf{p}_n(x_1)\ \mathbf{p}_n(x_2)\ \ldots\ \mathbf{p}_n(x_n)\right]$ the matrix of the eigenvectors of matrix H_n, each normalized so that the first component is equal to 1. Then, the weight coefficients $A_{m,i}$ can be found by solving

$$V_n \cdot \begin{bmatrix} A_{m,1} \\ A_{m,2} \\ \vdots \\ A_{m,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{*(m)} \\ \mu_1^{*(m)} \\ \vdots \\ \mu_{n-1}^{*(m)} \end{bmatrix}, \quad m = 1, 2, \ldots, r\,, \tag{3.7}$$

where

$$\mu_i^{*(m)} = \int_E \pi_i(x)\, w_m(x)dx\,, \quad m = 1, 2, \ldots, r, \quad i = 0, 1, \ldots, n-1\,,$$

are modified moments and $\pi_i = \pi_{\vec{s}(i)}$.

All of the modified moments can be computed exactly, except for rounding errors, by using the Gauss–Christoffel quadrature formulas with respect to the corresponding weight function w_m, $m = 1, 2, \ldots r$.

4 Quadrature Formulae of Gaussian Type with Preassigned Nodes

Let $W = \{w_1, w_2, \ldots, w_r\}$ be an AT system.

Following Definition 3.2 and ordinary quadrature formulas of Gaussian type with preassigned abscissas (see for example [1, Subsection 2.2.1]) we introduce the following definition:

Definition 4.1 *Let W be an AT system (the weight functions w_i, $i = 1, 2, \ldots, r$ are supported on the interval E), $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index, $n = |\vec{n}|$. A set of quadrature formulas of the form:*

$$\int_E f(x) w_m(x) \approx \sum_{i=1}^{k} a_{m,i} f(y_i) + \sum_{i=1}^{n} A_{m,i} f(x_i), \quad m = 1, 2, \ldots, r, \tag{4.1}$$

where the nodes $y_i \in E$, $i = 1, 2, \ldots, k$ are fixed and prescribed in advance, will be called an optimal set with preassigned nodes $\{y_i\}_{i=1}^{k}$ with respect to $(W, \vec{n})$ if and only if the weight coefficients, $a_{m,i}, A_{m,i}$, and the nodes, x_i, satisfy the following equations:

$$\begin{aligned}
\sum_{i=1}^{k} a_{m,i} + \sum_{i=1}^{n} A_{m,i} &= \int_E w_m(x)dx \\
\sum_{i=1}^{k} a_{m,i}\, y_i + \sum_{i=1}^{n} A_{m,i}\, x_i &= \int_E x\, w_m(x)dx \\
&\vdots \\
\sum_{i=1}^{k} a_{m,i}\, y_i^{n+n_m+k-1} + \sum_{i=1}^{n} A_{m,i}\, x_i^{n+n_m+k-1} &= \int_E x^{n+n_m+k-1}\, w_m(x)dx
\end{aligned} \tag{4.2}$$

for $m = 1, 2, \ldots, r$.

For a set of preassigned nodes $\{y_i\}_{i=1}^{k}$ we introduce $s(x)$ as a polynomial of degree k, with zeros at y_i, $i = 1, 2, \ldots, k$.

Denote

$$\widetilde{W} = \{\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_r\}, \quad \widetilde{w}_m(x) = s(x) w_m(x), \ \ m = 1, 2, \ldots, r. \tag{4.3}$$

Theorem 4.2 *Let W be an AT system, $\vec{n} = (n_1, n_2, \ldots, n_r)$, $n = |\vec{n}|$. Suppose that for preassigned nodes, $\{y_i\}_{i=1}^{k}$, $\widetilde{W}$ is also AT system. The set of quadrature formulas* (4.1) *form the optimal set with preassigned nodes $\{y_i\}_{i=1}^{k}$ with respect to $(W, \vec{n})$ if and only if:*

1° *They are exact for all polynomials of degree* $\leq n+k-1$.

2° *The polynomial* $q(x)=\prod_{i=1}^{n}(x-x_i)$ *is the type II multiple orthogonal polynomial* $\pi_{\vec{n}}$ *with respect to* $\widetilde{W}$.

Proof. Suppose first that the quadrature formulas (4.1) form the optimal set with preassigned nodes $\{y_i\}_{i=1}^{k}$ with respect to $(W,\vec{n})$.

In order to prove 1° we note that for each $m=1,2,\ldots,r$, the corresponding quadrature formula (4.1) is exact for all polynomials of degree $\leq n+n_m+k-1$ and then it is exact for those of degree $\leq n+k-1$.

To prove 2°, for $m=1,2,\ldots r$, assume that $p_m(x)$ is a polynomial of degree $\leq n_m-1$. Then the polynomial $q(x)p_m(x)s(x)$ has degree $\leq n+n_m+k-1$. Since the corresponding quadrature formula is exact for all such polynomials, it follows that

$$\begin{aligned}\int_E q(x)p_m(x)\,s(x)w_m(x)dx \;&=\sum_{i=1}^{k}a_{m,i}q(y_i)p_m(y_i)s(y_i)\\ &\quad+\sum_{i=1}^{n}A_{m,i}q(x_i)p_m(x_i)s(x_i).\end{aligned}\tag{4.4}$$

Since $s(y_i)=0$ for $i=1,2,\ldots,k$ and $q(x_i)=0$ for $i=1,2,\ldots,n$, the both sums on the right hand side in (4.4) are identically zero. Thus, we have

$$\int_E q(x)p_m(x)\,s(x)w_m(x)dx=0,\quad m=0,1,\ldots,r$$

and 2° follows.

Suppose now that for quadrature formulas (4.1) 1° and 2° hold.

For $m=1,2,\ldots,r$, let $t_m(x)$ be a polynomial of degree $\leq n+n_m+k-1$. We can write $t_m(x)=u_m(x)\cdot q(x)s(x)+v(x)$, where $u_m(x)$ is a polynomial of degree $\leq n_m-1$ and $v(x)$ is a polynomial of degree $\leq n+k-1$. It is easy to see that

$$\begin{aligned}t_m(y_i)&=v(y_i),\qquad i=1,2,\ldots,k,\\ t_m(x_i)&=v(x_i),\qquad i=1,2,\ldots,n.\end{aligned}\tag{4.5}$$

Then, we obtain

$$\begin{aligned}\int_E t_m(x)\,w_m(x)dx \;&=\int_E[u_m(x)q(x)s(x)+v(x)]\,w_m(x)dx\\ &=\int_E q(x)u_m(x)\,s(x)w_m(x)dx+\int_E v(x)\,w_m(x)dx\,.\end{aligned}$$

According to 2° we have $\int_E q(x)u_m(x)\, s(x)w_m(x)dx = 0$ and therefore we obtain

$$\int_E t_m(x)\, w_m(x)dx = \int_E v(x)\, w_m(x)dx\,.$$

Since $v(x)$ is a polynomial of degree $\le n+k-1$, it follows from 1° that

$$\int_E v(x)\, w_m(x)dx = \sum_{i=1}^{k} a_{m,i}v(y_i) + \sum_{i=1}^{n} A_{m,i}v(x_i)$$

and hence using (4.5) we obtain

$$\begin{aligned}\int_E t_m(x)\, w_m(x)dx &= \sum_{i=1}^{k} a_{m,i}v(y_i) + \sum_{i=1}^{n} A_{m,i}v(x_i)\\ &= \sum_{i=1}^{k} a_{m,i}t_m(y_i) + \sum_{i=1}^{n} A_{m,i}t_m(x_i).\end{aligned}$$

This proves that for each $m = 1, 2, \ldots, r$, the corresponding quadrature formula is exact for all polynomials of degree $\le n + n_m + k - 1$. □

According to Theorem 4.2, the nodes x_i, $i = 1, 2, \ldots, n$, of the optimal set of quadrature formulas (4.1) are the zeros of the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to the AT system $\widetilde{W}$. For computing these zeros in the case of nearly diagonal multi-index we use the discretized Stieltjes–Gautschi procedure (see [12, 13]). When we find the nodes, then for $m = 1, 2, \ldots, r$ we can choose the weight coefficients $a_{m,i}$, $i = 1, 2, \ldots, k$ and $A_{m,i}$, $i = 1, 2, \ldots, n$, so that they satisfy the following Vandermonde system of equations

$$V(y_1, \ldots, y_k, x_1, \ldots, x_n) \begin{bmatrix} a_{m,1} \\ \vdots \\ a_{m,k} \\ A_{m,1} \\ \vdots \\ A_{m,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{(m)} \\ \mu_1^{(m)} \\ \vdots \\ \mu_{n+k-1}^{(m)} \end{bmatrix}, \quad m = 1, 2, \ldots, r, \tag{4.6}$$

where

$$\mu_i^{(m)} = \int_E x^i\, w_m(x)dx, \quad m = 1, 2, \ldots, r, \quad i = 0, 1, \ldots, n+k-1,$$

are moments which can be computed exactly, except for rounding errors, by using the Gauss–Christoffel quadrature formulas with respect to the corresponding weight function w_m, $m = 1, 2, \dots r$.

Each of Vandermonde systems (4.6) has unique solution if all of preassigned nodes are distinct from the zeros of type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to $\widetilde{W}$. This is always satisfied in cases when the preassigned nodes are at the end points of the interval E.

4.1 Quadrature Formulae of Gauss–Radau and Gauss–Lobatto Type

a) Jacobi weight functions

In the case of Gauss–Radau type quadratures we have only one preassigned node, $y_1 = -1$, so that $s(x) = x + 1$. For each AT system W consisting of Jacobi weight functions [12]

$$w_m(x) = (1-x)^{\alpha}(1+x)^{\beta_m}, \;\; m = 1, 2, \dots, r,$$

$\alpha, \beta_m > -1$, $m = 1, 2, \dots, r$; $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, we have

$$\widetilde{w}_m(x) = (1-x)^{\alpha}(1+x)^{\beta_m+1}, \;\; m = 1, 2, \dots, r,$$

and corresponding set $\widetilde{W}$ is also AT system. The Vandermonde system (4.6) (for $n+1$ unknown weights) has a unique solution for each $m = 1, 2, \dots, r$.

In Table 1 the nodes and weights for quadrature formulas of Gauss–Radau type with respect to an AT system of Jacobi weights and nearly diagonal multi-index are given. Numbers in parentheses denote decimal exponents.

In the case of Gauss–Lobatto type quadratures we have two preassigned nodes, $y_1 = -1$ and $y_2 = 1$; $s(x) = 1 - x^2$, and for AT system W of Jacobi weight functions set $\widetilde{W}$ with elements

$$\widetilde{w}_m(x) = (1-x)^{\alpha+1}(1+x)^{\beta_m+1}, \;\; m = 1, 2, \dots, r,$$

is again AT system and Vandermonde system (4.6) (for $n + 2$ unknown weights) also has unique solution for each $m = 1, 2, \dots, r$.

In Table 2 the nodes and weights for quadrature formulas of Gauss–Lobatto type with respect to an AT system of Jacobi weights and nearly diagonal multi-index are given.

b) Laguerre weight functions

Table 1: Quadrature formulas of Gauss–Radau type, $r = 3$, $\alpha = -1/4$, $\beta_1 = -1/2$, $\beta_2 = 1/4$, $\beta_3 = 1$, when $n = 10$ and $n = 16$

	y_1, x_i	$a_{1,1}$, $A_{1,i}$	$a_{2,1}$, $A_{2,i}$	$a_{3,1}$, $A_{3,i}$
1	$-1.$	$1.007145484059894(-1)$	$2.062876155609072(-4)$	$-9.562972005938697(-7)$
1	$-9.833138304264925(-1)$	$2.435730645050421(-1)$	$1.129361273403368(-2)$	$5.241035783221104(-4)$
2	$-9.164273880324528(-1)$	$2.936452002423003(-1)$	$4.564271303762693(-2)$	$7.094582228332064(-3)$
3	$-7.794038086868831(-1)$	$3.223986760309847(-1)$	$1.037748068284181(-1)$	$3.340336785072431(-2)$
4	$-5.678600313784354(-1)$	$3.348502881280588(-1)$	$1.784715467514467(-1)$	$9.512338968694678(-2)$
5	$-2.928754974215839(-1)$	$3.340611167810139(-1)$	$2.576011782190145(-1)$	$1.986413967159547(-1)$
6	$2.129858755086223(-2)$	$3.214765372195719(-1)$	$3.265982323756700(-1)$	$3.318015253350430(-1)$
7	$3.412119085055776(-1)$	$2.974582873741706(-1)$	$3.707226525059520(-1)$	$4.620321265761330(-1)$
8	$6.294703272423117(-1)$	$2.612115619838483(-1)$	$3.767274261216885(-1)$	$5.433279924982002(-1)$
9	$8.505734897833458(-1)$	$2.096384020761912(-1)$	$3.326217108650106(-1)$	$5.277525560393169(-1)$
10	$9.765092384227134(-1)$	$1.306461010903117(-1)$	$2.177813020250095(-1)$	$3.630318479902930(-1)$
1	$-1.$	$4.955072915498704(-2)$	$3.270184865300507(-5)$	$-4.351164761828841(-8)$
1	$-9.957694114109361(-1)$	$1.262846817355126(-1)$	$2.093519371442199(-3)$	$3.471023381962368(-5)$
2	$-9.773102983214776(-1)$	$1.601763480957459(-1)$	$9.364154511359857(-3)$	$5.474482568814665(-4)$
3	$-9.360947441653007(-1)$	$1.838016263570097(-1)$	$2.336155953750918(-2)$	$2.969301411091863(-3)$
4	$-8.662104158604537(-1)$	$1.998820166290791(-1)$	$4.421711974927511(-2)$	$9.781538692113400(-3)$
5	$-7.646982637559852(-1)$	$2.102958318380690(-1)$	$7.104761664033177(-2)$	$2.400315683002052(-2)$
6	$-6.315299779035927(-1)$	$2.161684249080054(-1)$	$1.022336528091070(-1)$	$4.834989093124742(-2)$
7	$-4.694526660368208(-1)$	$2.181821415446974(-1)$	$1.356320427214107(-1)$	$8.431510884534577(-2)$
8	$-2.837170944030709(-1)$	$2.167503150578853(-1)$	$1.687614224963532(-1)$	$1.313973532900836(-1)$
9	$-8.168730183681652(-2)$	$2.121093566498412(-1)$	$1.989769497308692(-1)$	$1.866576144939541(-1)$
10	$1.276618526350826(-1)$	$2.043629625254307(-1)$	$2.236332403317656(-1)$	$2.447205969383093(-1)$
11	$3.343397209281050(-1)$	$1.934927571945595(-1)$	$2.402229655222859(-1)$	$2.982389315292906(-1)$
12	$5.279967697041811(-1)$	$1.793346219930818(-1)$	$2.464654052473528(-1)$	$3.387254246200834(-1)$
13	$6.986047395245323(-1)$	$1.614985143761679(-1)$	$2.402911524197785(-1)$	$3.575255051372864(-1)$
14	$8.371173741032776(-1)$	$1.391503514761979(-1)$	$2.195770701687902(-1)$	$3.464891696815435(-1)$
15	$9.360696233793310(-1)$	$1.103159913709379(-1)$	$1.810627867925640(-1)$	$2.971802397248579(-1)$
16	$9.900928342242115(-1)$	$6.831711355306033(-2)$	$1.144681096278244(-1)$	$1.917959855196322(-1)$

Table 2: Quadrature formulas of Gauss–Lobato type, $r = 2$, $\alpha = -1/4$, $\beta_1 = 1$, $\beta_2 = -1/2$, $n = 18$

	y_i, x_i	$a_{1,i}$, $A_{1,i}$	$a_{2,i}$, $A_{2,i}$
1	−1.	−5.427654059531051(−7)	6.278044907035962(−2)
2	1.	5.253978709241791(−2)	1.857561986737788(−2)
1	−9.940683226494066(−1)	6.258461212382826(−5)	1.366066177674554(−1)
2	−9.973295322435293(−1)	6.724998794425119(−4)	1.541068169804305(−1)
3	−9.330999859656411(−1)	2.894341625676589(−3)	1.672669872578609(−1)
4	−8.706256860528897(−1)	8.198883072026320(−3)	1.761905550567668(−1)
5	−7.847743707278070(−1)	1.815730482233846(−2)	1.818487840801168(−1)
6	−6.759923125940263(−1)	3.409994333062372(−2)	1.848930926177310(−1)
7	−5.460968834029435(−1)	5.679958534303609(−2)	1.857378515147516(−1)
8	−3.981042546631459(−1)	8.622406731661059(−2)	1.846488449675086(−1)
9	−2.360418319851534(−1)	1.213912667316848(−1)	1.817954776087856(−1)
10	−6.474035626481439(−2)	1.603454558780371(−1)	1.772794371553346(−1)
11	1.103930065314497(−1)	2.002575860755584(−1)	1.711488163863528(−1)
12	2.836154392458143(−1)	2.376344935881942(−1)	1.634018925424208(−1)
13	4.490957313876524(−1)	2.686044097870690(−1)	1.539811371776487(−1)
14	6.011595130523827(−1)	2.892279000156788(−1)	1.427539994158158(−1)
15	7.345236460917255(−1)	2.957584275197416(−1)	1.294692974446239(−1)
16	8.445112388464046(−1)	2.847195061026688(−1)	1.136567577648302(−1)
17	9.272392748240476(−1)	2.524575707052816(−1)	9.435935045249755(−2)
18	9.797689999534666(−1)	1.926868609416974(−1)	6.917199832716925(−2)

For Laguerre weights we can construct Gauss–Radau type quadratures. The preassigned node is $y_1 = 0$, and for each AT system of generalized Laguerre weight functions [12]

$$w_m(x) = x^{s_m} e^{-x}, \quad m = 1, 2, \ldots, r$$

$s_m > -1$, $m = 1, 2, \ldots, r$; $s_i - s_j \notin \mathbb{Z}$ whenever $i \neq j$, set $\widetilde{W}$ with weights

$$\widetilde{w}_m(x) = x^{s_m+1} e^{-x}, \quad m = 1, 2, \ldots, r$$

is also AT system.

The Vandermonde system (4.6) (for $n+1$ unknown weights) has a unique solution for each $m = 1, 2, \ldots, r$.

References

[1] P.J. Davis, P. Rabinowitz, Methods of Numerical Integration, *Academic Press, New York, San Francisco, London, 1975.*

[2] W. Gautschi, Orthogonal Polynomials, Computation and Approximation, *Oxford University Press, 2004.*

[3] B.T. Smith, *et al.* Matrix Eigensystem Routines – EISPACK Guide, *Lect. Notes Comp. Science Vol.* **6** *Springer Verlag, Berlin – Heidelberg – New York, 1976.*

[4] A.I. Aptekarev, *Multiple orthogonal polynomials,* J. Comput. Appl. Math. **99** (1998), 423–447.

[5] A.I. Aptekarev, A. Branquinho, W. Van Assche, *Multiple orthogonal polynomials for classical weights,* Trans. Amer. Math. Soc. **355** (2003), 3887–3914.

[6] C.F. Borges, *On a class of Gauss-like quadrature rules,* Numer. Math. **67** (1994), 271–288.

[7] W. Gautschi, *Orthogonal polynomials: applications and computation,* Acta Numerica (1996), 45–119.

[8] W. Gautschi, *High–order Gauss–Lobatto formulae,* Numerical Algorithms **25** (2000), 213–222.

[9] W. Gautschi, *Gauss–Radau formulae for Jacobi and Laguerre weight functions* 1999 International Symposium on Computational Sciences, to honor John R. Rice (West Lafayette, IN), Math. Comput. Simulation **54** (4–5) (2000), 403–412.

[10] W. Gautschi, G.V. Milovanović, *Polynomials orthogonal on the semicircle,* J. Approx. Theory **46** (1986), 230–250.

[11] G.V. Milovanović, *Orthogonal polynomials on the radial rays in the complex plane and applications,* Rend. Circ. Mat. Palermo, Serie II, Suppl. **68** (2002), 65–94.

[12] G.V. Milovanović, M. Stanić, *Construction of multiple orthogonal polynomials by discretized Stieltjes-Gautschi procedure and corresponding Gaussian quadratures,* Facta Univ. Ser. Math. Inform. **18** (2003), 9–29.

[13] G.V. Milovanović, M. Stanić, *Multiple orthogonal polynomials on the semicircle and corresponding quadratures of Gaussian type,* Math. Balkanica (N.S.) **18** (2004), 373-387.

[14] W. Van Assche, *Multiple orthogonal polynomials, irratinality and transcendence,* In: Continued Fractions: From Analytic Number Theory to Constructive Approximation (B.C. Berndt and F. Gesztesy, eds.), Contemporary Mathematics **236** (1999), 325–342.

[15] W. Van Assche, *Non-symetric linear difference equations for multiple orthogonal polynomials,* CRM Proceedings and Lecture Notes **25** Amer. Math. Soc., Providence, RI, 2000, pp. 391–405.

[16] W. Van Assche, E. Coussement, *Some classical multiple orthogonal polynomials,* J. Comput. Appl. Math. **127** (2001), 317–347.

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Markov Inequality and Green Functions

Ferenc Toókos and Vilmos Totik

Dedicated to Giuseppe Mastroianni on his 65th birthday

Abstract. A characterization is given for compact sets $E \subset \mathbf{C}$ satisfying the Markov inequality $||P_n'||_E \le Cn||P_n||_E$ for any polynomial P_n of degree $\le n$. It is shown that this Markov condition is equivalent to the Lipschitz continuity of the Green function $g_{\overline{\mathbf{C}}\setminus E}$, as well as to a Lipschitz type condition on the equilibrium measure. We also give an example for such a set with infinitely many connected components.

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1 Definitions and results

Let Π_n denote the set of algebraic polynomials of degree $\le n$. Markov's inequality is a basic result comparing the supremum norm of a polynomial $P_n \in \Pi_n$ to the supremum norm of its derivative:

$$||P_n'||_{[-1,1]} \le n^2||P_n||_{[-1,1]}.$$

If $C_1(0)$ is the unit circle, then the corresponding inequality

$$||P_n'||_{C_1(0)} \le n||P_n||_{C_1(0)}$$

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is due to Bernstein. Let us also remark that this is in some sense the optimal case, for if E is any compact set on the complex plane then there are polynomials $P_n \in \Pi_n$, $n = 1, 2, \ldots$ for which

$$||P_n'||_E \geq cn||P_n||_E$$

with some constant $c > 0$. Indeed, let D be a disk containing E with the smallest possible radius. Then $\partial D \cap E$ is not empty, say z_0 is a point in this set. If a is the center of D and r is its radius, then for $P_n(z) = (z-a)^n$ we have

$$|P_n'(z_0)| = \frac{n}{r}\|P_n\|_E.$$

Let $E \subset \mathbf{C}$ be compact with positive logarithmic capacity (see [2], [3] or [5] for the relevant concepts of potential theory). We say that E satisfies the Markov inequality with a polynomial factor if there exist $C,\ k > 0$ such that

$$||P_n'||_E \leq Cn^k||P_n||_E \tag{1.1}$$

holds for every n and $P_n \in \Pi_n$.

Inequality (1.1) is strongly related to the smoothness properties of the Green function belonging to E. Let Ω be the outer domain of E, i.e. the unbounded component of $\overline{\mathbf{C}} \setminus E$, and let $g_\Omega(z)$ denote Green's function of Ω with pole at infinity. Green's function g_Ω is said to be Hölder continuous if there exist $C_1,\ \alpha > 0$ such that

$$g_\Omega(z) \leq C_1\Big(\operatorname{dist}(z, E)\Big)^\alpha. \tag{1.2}$$

for all $z \in \mathbf{C}$. It is known that in certain cases the Markov inequality is equivalent to the Hölder continuity of the Green function. Totik (see [4]) proved that this is true for Cantor-type sets, i.e. (1.1) is equivalent to (1.2) if E is Cantor-type. It is an open problem if (1.1) and (1.2) are equivalent for any compact set E. In this paper our aim is to show that in the optimal cases $k = 1$ and $\alpha = 1$ they are, indeed, equivalent.

It is known that Hölder continuity of Green's function can be characterized by smooth behavior of the equilibrium measure μ_E of E, namely g_Ω is Hölder continuous (with some exponent α) if and only if

$$\mu_E\Big(D_\delta(z)\Big) \leq C_2\delta^\kappa \tag{1.3}$$

for every $z \in \mathbf{C}$ and $\delta > 0$ with some $\kappa > 0$, where $D_\delta(z)$ denotes the disk centered at z with radius δ. We shall also prove that aforementioned

equivalence of (1.1) and (1.2) in the $k = 1$, $\alpha = 1$ case is equivalent to the fact that in (1.3) we have $\kappa = 1$.

Recall that Ω is said to be regular (with respect to the Dirichlet problem) if g_Ω is continuous on ∂G.

Theorem 1 *Let E be a compact subset of the plane such that the unbounded component Ω of $\overline{\mathbf{C}} \setminus E$ is regular. Then the following are pairwise equivalent.*

i) *Optimal Markov inequality holds on E, i.e. there exists a $C > 0$ such that*

$$||P_n'||_E \le Cn||P_n||_E \tag{1.4}$$

for every polynomial $P_n \in \Pi_n$, $n = 1, 2, \ldots$.

ii) *Green's function g_Ω is Lipschitz continuous, i.e. there exists a $C_1 > 0$ such that*

$$g_\Omega(z) \le C_1 \mathrm{dist}(z,\ E) \tag{1.5}$$

for every $z \in \mathbf{C}$.

iii) *The equilibrium measure μ_E of E satisfies a Lipschitz type condition, i.e. there exists a $C_2 > 0$ such that*

$$\mu_E\Big(D_\delta(z)\Big) \le C_2\delta \tag{1.6}$$

for every $z \in E$ and $\delta > 0$.

If, in addition, Ω is simply connected, then **i)**–**iii)** *are also equivalent to*

iv) *The conformal mapping Φ from Ω onto the exterior of the unit disk is Lipschitz continuous, i.e.*

$$|\Phi(z_1) - \Phi(z_2)| \le C_3|z_1 - z_2|, \qquad z_1, z_2 \in \Omega.$$

We mention that each of **i), ii)** and **iv)** implies regularity, so in their equivalence the regularity assumption is not needed. However, **iii)** may be true without Ω being regular, in which case **iii)** is not equivalent to the other statements. Consider e.g. as E the unit disk together with the single point 2. In this case μ_E is the normalized arc measure on the unit circle and the one point set $\{2\}$ does not carry any mass. Thus, **iii)** holds, but the other statements in the theorem are not true.

There is a local version of our theorem which we formulate now. We say that E has the optimal local Markov property at the point $z_0 \in \partial\Omega$ if there is a constant C such that

$$|P_n^{(k)}(z_0)| \le C^k n^k \|P_n\|_E, \qquad P_n \in \Pi_n,\ n = 1, 2, \ldots$$

for all $k = 1, 2, \ldots$.

Theorem 2 *Let E be a compact subset of the plane, Ω the unbounded component of $\overline{\mathbf{C}} \setminus E$, and suppose that $z_0 \in \partial\Omega$ is a regular boundary point of Ω (i.e. $g_\Omega(z_0) = 0$). Then the following are equivalent.*

i) *E has the optimal Markov property at z_0.*

ii) *Green's function g_Ω is Lipschitz continuous at z_0, i.e.*

$$g_\Omega(z) \leq C_1|z - z_0|$$

with some constant C_1.

iii) *The equilibrium measure μ_E of E satisfies a Lipschitz type condition at z_0, i.e. there exists a $C_2 > 0$ such that*

$$\mu_E\Big(D_\delta(z_0)\Big) \leq C_2\delta$$

for every $\delta > 0$.

If, in addition, Ω is simply connected, then **i)**–**iii)** *are also equivalent to*

iv) *The conformal mapping Φ from Ω onto the exterior of the unit disk is Lipschitz continuous at z_0.*

In the last statement we think of Φ as being extended continuously onto the boundary of Ω.

It is worth noticing that much more is true than the equivalence of **ii)** and **iii)**, namely we can give a very precise two sided estimate for Green's function in terms of the equilibrium measure.

Theorem 3 *Let E be a compact subset of the plane, Ω the unbounded component of $\overline{\mathbf{C}} \setminus E$, and suppose that $z_0 \in \partial\Omega$ is a regular boundary point of Ω (i.e. $g_\Omega(z_0) = 0$). Then for every $0 < r < 1$ we have*

$$\int_0^r \frac{\mu_E(D_t(z_0))}{t} dt \leq \sup_{|z-z_0|=r} g_\Omega(z) \leq 3\int_0^{4r} \frac{\mu_E(D_t(z_0))}{t} dt. \tag{1.7}$$

Let F be a connected component of E which is of positive distance from the set $E \setminus F$. Then on F the Lipschitz continuity of Green's functions g_Ω and $g_{\overline{\mathbf{C}}\setminus F}$ are equivalent, and for the latter one can use the conformal mapping characterization given in Theorem 1, **iv)**. In particular, if g_Ω is Lipschitz, then so is every $g_{\overline{\mathbf{C}}\setminus F}$ for every component F which is of positive distance from F. In Section 5 (Example 2) we shall show that this need not

be true for components of F that are not of positive distance from $E \setminus F$, even if they are consisting of more than one point. This will be based on a construction in Example 1, which exhibits a set E with infinitely many connected components and Lipschitz-continuous Green function, which is an interesting fact in itself (note that the simplest example of a set E satisfying Theorem 1 is any finite union of disjoint smooth simple closed curves, and one is tempted to think that sets appearing in Theorem 1 can have only finitely many connected components).

2 Notations

For the following well known notations and results see either of the books [2], [3] or [5].

We shall use $c, c_0, c_1, c_2, \ldots, C, C_0, C_1, C_2, \ldots$ and $d_1, d_2, \ldots$ to denote positive constants. $\mathbf{T}$ denotes the unit circle, $\mathbf{D}$ the unit disk, $C_r(a)$ resp. $D_r(a)$ denote the circle resp. open disk centered at a with radius r. For the notions of logarithmic potential theory see e.g. [2] or [3]. In what follows E denotes a compact set of positive capacity, μ_E denote its equilibrium measure,

$$U^\nu(z) = \int \log \frac{1}{|z-t|} d\nu(t)$$

the logarithmic potential of the measure ν, $g_\Omega(z)$ the Green function of the domain Ω with pole at ∞, $\mathrm{cap}(E)$ the logarithmic capacity of E and $\omega(x, H, G)$ the harmonic measure in G corresponding to the set $H \subseteq \partial G$. These are related by the equation

$$g_\Omega(z) = \log \frac{1}{\mathrm{cap}(E)} - U^{\mu_E}(z), \tag{2.1}$$

which is used to define g_Ω for all $z \in \mathbf{C}$. We shall need the Bernstein-Walsh inequality

$$|P_n(z)| \le e^{n g_\Omega(z)} \|P_n\|_E \tag{2.2}$$

valid for all polynomials P_n of degree $n = 1, 2, \ldots$, as well as its sharpness:

$$e^{g_\Omega(z)} = \sup_{P_n \in \Pi_n,\ \|P_n\|_E \le 1} |P_n(z)|^{1/n} \tag{2.3}$$

valid for any $z \in \Omega$.

We shall also use Harnack's inequality: if u is a positive harmonic function in the unit disk and $|z| < 1$, then

$$\frac{1-|z|}{1+|z|} u(0) \le u(z) \le \frac{1+|z|}{1-|z|} u(0) \tag{2.4}$$

3 Proof of Theorems 1 and 2

We shall only prove Theorem 1, the proof of the local version (Theorem 2) is similar.

First we show that **(i)** is equivalent to **(ii)**. Suppose (1.5) holds true. Let $z \in \mathbf{C}$ and apply Cauchy's formula to $P_n \in \Pi_n$ on $C_r(z)$:

$$P_n'(z) = \frac{1}{2\pi i} \int_{C_r(z)} \frac{P_n(\xi)}{|\xi - z|^2} d\xi = \frac{1}{2\pi} \int_0^{2\pi} \frac{P_n(z + re^{it})}{re^{it}} dt.$$

Taking absolute value on both sides and setting $r = 1/n$ we get

$$|P_n'(z)| \le \frac{n}{2\pi} \int_0^{2\pi} |P_n(z + \frac{1}{n} e^{it})| dt. \tag{3.1}$$

For $z \in E$ the Bernstein-Walsh inequality (2.2) and (1.5) give

$$\begin{aligned} |P_n(z + \frac{1}{n} e^{it})| &\le ||P_n||_E e^{n g_\Omega(z + \frac{1}{n} e^{it})} \\ &\le ||P_n||_E e^{n C_1 \frac{1}{n}} = ||P_n||_E e^{C_1}. \end{aligned}$$

This and (3.1) prove **(i)**.

Conversely, suppose (1.4). Then it follows by induction that

$$||P_n^{(m)}||_E \le C^m n^m ||P_n||_E, \qquad m = 1, 2, \dots. \tag{3.2}$$

(1.5) is obviously true for $z \in E$, therefore we can assume $z \in \mathbf{C} \setminus E$. Choose $z_0 \in E$ such that $\operatorname{dist}(z,\ E) = |z - z_0|$. Suppose $P_n \in \Pi_n$ and $||P_n||_E \le 1$. We can use the (finite) Taylor-expansion of P_n around z_0 and (3.2) to obtain:

$$\begin{aligned} |P_n(z)| &= |P_n(z_0)| + \Big| \sum_{m=1}^{\infty} \frac{P_n^{(m)}(z_0)}{m!} (z - z_0)^m \Big| \\ &\le 1 + \sum_{m=1}^{\infty} \frac{C^m n^m}{m!} |z - z_0|^m = e^{Cn|z - z_0|}. \end{aligned}$$

Thus $|P_n(z)|^{1/n} \le e^{C|z - z_0|}$ and this, using the representation (2.3) and the choice of z_0, proves **(ii)**.

The equivalence of **ii)** and **iii)** follows from Theorem 3.

Finally, suppose that Ω is simply connected, and Φ is the conformal map of Ω onto the complement of the unit disk. Then $g_\Omega(z)$ is the real part of

$\log \Phi(z)$ (this follows e.g. from the defining properties of g_Ω), hence the Lipschitz property of Φ implies the Lipschitz property of g_Ω. Conversely, suppose that g_Ω is Lipschitz continuous. Since g_Ω is infinitely differentiable, this happens precisely if the partial derivatives $\partial g_\Omega/\partial x$ and $\partial g_\Omega/\partial y$ are bounded in Ω. But then using the Cauchy-Riemann equations it follows that the partial derivatives of $\Im \log \Phi(z)$ are also bounded, hence $\log \Phi$ is a Lipschitz function. But then so is Φ in any bounded set. On the other hand, around infinity the derivative of Φ tends to a constant (recall that $\lim_{z\to\infty} F(z)/z$ exists and apply Cauchy's formula for the derivative), and the proof is complete.

■

4 Proof of Theorem 3

Without loss of generality let $z_0 = 0$, and $r > 0$. It follows from the representation (2.1) and the assumed regularity ($g_\Omega(0) = 0$) that

$$g_\Omega(re^{i\varphi}) = g_\Omega(re^{i\varphi}) - g_\Omega(0) = \int \log \frac{|re^{i\varphi} - t|}{|t|} d\mu_E(t).$$

Since (see [2, p. 29])

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{|re^{i\varphi} - t|}{|t|} d\varphi = \log^+ \frac{r}{|t|},$$

we get with Fubini's Theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g_\Omega(re^{i\varphi}) d\varphi \;\; = \;\; \int \left(\log^+ \frac{r}{|t|}\right) d\mu_E(t).$$

With $S(t) = \mu_E(D_t(0))$ the last integral can be written as

$$\int_0^r \log \frac{r}{t} dS(t) = \int_0^r \frac{S(t)}{t} dt,$$

where the equality follows by integration by parts. Thus,

$$\int_0^r \frac{S(t)}{t} dt \le \sup_\varphi g_\Omega(re^{i\varphi}),$$

which is the left inequality in (1.7).

To prove the right inequality we write again for $|z| \leq 2r$

$$\begin{aligned} 0 \leq g_\Omega(z) &= g_\Omega(z) - g_\Omega(0) = \int \log \frac{|z-t|}{|t|} d\mu_E(t) \\ &= \int_{D_{2r}(0)} + \int_{E \setminus D_{2r}(0)} = h_1(z) + h_2(z) \end{aligned}$$

where both functions h_1 and h_2 vanish at 0. Clearly, for $|z| \leq 2r$ we have

$$\begin{aligned} h_1(z) = \int_{D_{2r}(0)} \log \frac{|z-t|}{|t|} d\mu_E(t) &\leq \int_{D_{2r}(0)} \log \frac{4r}{|t|} d\mu_E(t) \\ &\leq \int_{D_{4r}(0)} \log \frac{4r}{|t|} d\mu_E(t), \end{aligned}$$

and with the function $S(t)$ defined above this can be written again as $h_1(z) \leq m$ where

$$m = \int_0^{4r} \frac{S(t)}{t} dt.$$

Therefore, we have for all $|z| \leq 2r$

$$0 \leq h_1(z) + h_2(z) \leq m + h_2(z).$$

But the function on the right hand side is harmonic in $D_{2r}(0)$ and takes the value $m + h_2(0) = m$ at the origin, therefore we obtain from Harnack's inequality (2.4) (applied to the function $u(w) = m + h_2(w2r)$ and to the point $z/2r$) that for $|z| = r$ the inequality $m + h_2(z) \leq 3(m + h_2(0)) = 3m$ holds. Together with this we have for $|z| = r$

$$g(z) = h_1(z) + h_2(z) \leq m + 2m = 3m,$$

and this is the right inequality in (1.7).

■

5 Construction of Example 1

We are going to construct the first connected component E_0 of E in the following way. Let $0 < \theta < \pi$ be arbitrary. Set $J_\theta = \{e^{it} \ : \ \theta \leq |t| \leq \pi\}$ and

$$A_\theta = \bigcup_{a \in J_\theta} \overline{D}_{1/10}(a).$$

Thus, A_θ is a "thickened arc" of the unit circle $\mathbf{T}$. There is a largest θ^* such that the complement of A_{θ^*} is not connected (the two "arms" of A_{θ^*} touch each other).

Let $\omega(z, J, G)$ be the harmonic measure corresponding to the set $J \subseteq \partial G$ in the domain G. Clearly $\omega(0, \mathbf{T} \setminus A_\theta, \mathbf{D}) \to 0$ if $\theta \searrow \theta^*$. Fix $\theta_0 > \theta^*$ such that $\omega(0, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/9$.

Set $E_0 = A_{\theta_0}$, for every integer $n \geq 1$ define $E_n = (1/3)^n E_0$, and let $E = \bigcup_{n=1}^{\infty} E_n \cup \{0\}$. This is a compact set consisting of infinitely many components such that $\Omega = \overline{\mathbf{C}} \setminus E$ is connected. Below we show that g_Ω is Lipschitz.

Since $\omega(0, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/9$, it follows from Harnack's inequality ((2.4) with $|z| = 1/2$) that $\omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/3$ for all $z \in D_{1/2}(0)$, in particular this is true for $z \in C_{1/3}(0)$. Both $\omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D})$ and $g_{\overline{\mathbf{C}} \setminus E}(z)$ are positive harmonic functions in $\mathbf{D} \setminus E$,

$$g_{\overline{\mathbf{C}} \setminus E}(z) \leq g_{\overline{\mathbf{C}} \setminus E_0}(z) \leq \omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D})$$

on $\mathbf{T} \setminus A_{\theta_0}$ and $g_{\overline{\mathbf{C}} \setminus E}$ vanishes on E, while $\omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D})$ is nonnegative there. Thus, the same inequality holds true everywhere in $\mathbf{D} \setminus E$ by the maximum principle for harmonic functions. Hence

$$g_{\overline{\mathbf{C}} \setminus E}(z) \leq 1/3, \qquad z \in D_{1/3}(0) \setminus E. \tag{5.1}$$

Now consider the domain $D_{1/3}(0) \setminus E$ and the positive harmonic functions $\omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big)$ and $3g_{\overline{\mathbf{C}} \setminus E}(z)$. Since $3g_{\overline{\mathbf{C}} \setminus E}(z)$ vanishes on E and, in view of (5.1), it is at most 1 on $C_{1/3}(0) \setminus E$, where $\omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big)$ takes the (boundary) value 1, we have

$$3g_{\overline{\mathbf{C}} \setminus E}(z) \leq \omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big)$$

in $D_{1/3}(0) \setminus E$. Because of similarity,

$$\omega\Big(0, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big) = \omega(0, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/9,$$

which implies via Harnack's inequality as before that

$$\omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big) < 1/3, \qquad z \in D_{1/3}(0).$$

Thus, we can conclude $3g_{\overline{\mathbf{C}} \setminus E}(z) \leq 1/3$ if $z \in C_{1/9}(0)$, i.e.

$$g_{\overline{\mathbf{C}} \setminus E}(z) \leq 1/9, \qquad z \in D_{1/9}(0) \setminus E.$$

In a similar manner it follows by induction that

$$g_{\overline{\mathbf{C}}\setminus E}(z) \le \left(\frac{1}{3}\right)^n, \qquad z \in D_{(1/3)^n} \setminus E. \tag{5.2}$$

We are going to show that Green's function $g_{\overline{\mathbf{C}}\setminus E}$ is Lipschitz continuous. Let $z_0 \in \mathbf{C} \setminus E$ be arbitrarily chosen. Suppose the component closest to z_0 is E_{n_0}, i.e. $\operatorname{dist}(z_0,\ E) = \operatorname{dist}(z_0,\ E_{n_0})$. We will compare $g_{\overline{\mathbf{C}}\setminus E_{n_0}}$ and $g_{\overline{\mathbf{C}}\setminus E}$. It follows from the construction of E that E_{n_0} is included in the disk $D_{1/(2\cdot 3^{n_0-1})}(0)$. Thus the monotonicity of Green functions gives for $|z| = (1/3)^{n_0-1}$

$$g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z) \ge g_{\overline{\mathbf{C}}\setminus \overline{D}_{1/(2\cdot 3^{n_0-1})}(0)}(z) = \log \frac{(1/3)^{n_0-1}}{1/(2\cdot 3^{n_0-1})} = \log 2. \tag{5.3}$$

On the other hand, according to (5.2), $g_{\overline{\mathbf{C}}\setminus E}(z) \le (1/3)^{n_0-1}$ holds for $z \in C_{(1/3)^{n_0-1}}$. Hence

$$g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z) \ge \frac{3^{n_0-1}}{\log 2} g_{\overline{\mathbf{C}}\setminus E}(z), \qquad z \in C_{(1/3)^{n_0-1}}. \tag{5.4}$$

Since both sides are positive and harmonic in $D_{(1/3)^{n_0-1}} \setminus E$, inequality (5.4) remains true throughout this domain, and by the definition of n_0 the point z_0 is included in this set.

Let C denote the Lipschitz-constant of E_0, i.e.

$$g_{\overline{\mathbf{C}}\setminus E_0}(z) \le C\operatorname{dist}(z,\ E_0).$$

Using the fact $g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z) = g_{\overline{\mathbf{C}}\setminus E_0}(3^{n_0}z)$ we obtain

$$\begin{aligned} g_{\overline{\mathbf{C}}\setminus E}(z_0) &\le \frac{\log 2}{3^{n_0-1}} g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z_0) = \frac{\log 2}{3^{n_0-1}} g_{\overline{\mathbf{C}}\setminus E_0}(3^{n_0}z_0) \\ &\le \frac{\log 2}{3^{n_0-1}} C\operatorname{dist}(3^{n_0}z_0,\ E_0) \\ &= \frac{\log 2}{3^{n_0-1}} C 3^{n_0}\operatorname{dist}(z_0,\ E_{n_0}) \\ &= 3C\log 2 \cdot \operatorname{dist}(z_0,\ E). \end{aligned}$$

Therefore $g_{\overline{\mathbf{C}}\setminus E}(z)$ is Lipschitz continuous with Lipschitz constant $3C\log 2$.

■

6 Construction of Example 2

Let E be the set from the preceding section (Example 1), and let

$$E^* = E \cup [0,2] \cup \overline{D_1(3)},$$

i.e. we add to E the segment $[0,2]$ and to that attach the disk $\overline{D_1(2)}$. Then $F = [0,2] \cup \overline{D_1(3)}$ is a connected component of E^* for which $g_{\overline{\mathbf{C}}\setminus F}$ is not Lipschitz (in fact, around 0 this behaves like $g_{\mathbf{C}\setminus[0,2]}$, which is only Lipschitz $1/2$ smooth at 0). Thus, it is left to show that $g_{\overline{\mathbf{C}}\setminus E^*}$ is a Lipschitz function.

Since $g_{\overline{\mathbf{C}}\setminus E^*}$ is bounded by either of $g_{\overline{C}\setminus E}$, $g_{\overline{\mathbf{C}}\setminus[0,3]}$ and $g_{\overline{\mathbf{C}}\setminus\overline{D_1(3)}}$, the Lipschitz property is clear on E (Example 1), on $[1,2]$ and on $\overline{D_1(3)}$.

Thus, we have to worry about points close to the segment $[0,1]$. Let $z_0 \notin E^*$ be an arbitrary point, and let x_0 be a point in E that is closest to z_0. If $x_0 \in E \cup [1,2] \cup \overline{D_1(3)}$, then according to what we have just said, $g_{\overline{\mathbf{C}}\setminus E^*}(z) \le C|z - x_0|$ with some constant C. Thus, let us assume that $x_0 \in (0,1)$. Choose n_0 so that $3^{-n_0-1} \le x_0 < 3^{-n_0}$. Since $g_{\overline{\mathbf{C}}\setminus[-1,1]}(z) \ge 1$ for $|z| = 2$, we have

$$g_{\overline{\mathbf{C}}\setminus[0,2\cdot 3^{-n_0}]}(z) \ge 1, \qquad |z| = 3^{-n_0+1}.$$

(5.2) implies

$$3^{n_0-1} g_{\overline{\mathbf{C}}\setminus E^*}(z) \le 3^{n_0-1} g_{\overline{\mathbf{C}}\setminus E}(z) \le 1, \qquad |z| = 3^{-n_0+1},$$

hence by the comparison technique applied several times before we get that

$$3^{n_0-1} g_{\overline{\mathbf{C}}\setminus E^*}(z) \le g_{\overline{\mathbf{C}}\setminus[0,2\cdot 3^{-n_0}]}(z)$$

for all $|z| \le 3^{-n_0+1}$. In particular,

$$g_{\overline{\mathbf{C}}\setminus E^*}(z_0) \le 3^{-n_0+1} g_{[0,2\cdot 3^{-n_0}]}(z_0) = 3^{-n_0+1} g_{\overline{\mathbf{C}}\setminus[0,2]}(3^{n_0} z_0).$$

Here $3^{n_0} z_0$ lies in the annulus $1/3 \le |w| \le 1$, on which $g_{\overline{\mathbf{C}}\setminus[0,2]}$ is Lipschitz continuous, and the closest point in $[0,2]$ to $3^{n_0} z_0$ is $3^{n_0} x_0$. Hence we get with some constant C

$$g_{\overline{\mathbf{C}}\setminus E^*}(z_0) \le 3^{-n_0+1} C |3^{n_0} z_0 - 3^{n_0} x_0| = 3C|z_0 - x_0|$$

and the proof is over.

■

References

[1] L. Carleson and V. Totik, Hölder continuity of Green's functions, *Acta Sci. Math. (to appear).*

[2] T. Ransford, *Potential Theory in the Complex plane,* Cambridge: Cambridge University Press, 1995.

[3] E. B. Saff, V. Totik, *Logarithmic Potentials with External Fields,* New York/Berlin: Springer-Verlag, 1997.

[4] V. Totik, Markoff constants for Cantor sets, *Acta Sci. Math. (Szeged)* **60** (1995), 715–734.

[5] M. Tsuji, *Potential Theory in Modern Function Theory,* Maruzen, Tokyo, 1959.

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Approximation in Orlicz Spaces for Linear Integral Operators and Applications

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Dedicated to my daughter Margherita,
who was born during the final redaction of this paper

Abstract. We study the problem of the rate of approximation in Orlicz spaces for a family of linear integral operators of the form

$$(T_w f)(s) = \int_H K_w(s - h_w(t)) f(h_w(t)) d\mu_H(t), \quad w > 0, \ s \in G,$$

where G and H are locally compact Hausdorff topological groups, $f : G \to \mathbb{R}$ is a measurable function, $\{h_w\}_{w>0}$ is a family of homeomorphisms $h_w : H \to h_w(H) \subset G$ and $\{K_w\}_{w>0}$, with $K_w : G \to \mathbb{R}$, is a family of kernel functions. The general form of the operators allow us to give a unifying approach to the study of the rate of approximation for several classes of well-known integral operators, as the convolution ones, the Mellin convolution ones and a class of discrete operators, called "generalized sampling series", very important in the theory of signal processing. Moreover the setting of Orlicz spaces allows to deduce directly the results concerning the rate of approximation in L^p-spaces and this frame is also useful in the application to signal analysis since it gives the possibility to treat signals not necessarily bandlimited, nor with finite energy, as happens in the classical signal theory.

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1 Introduction

The problem of the convergence for linear integral operators of the form

$$(T_w f)(s) = \int_H K_w(s - h_w(t)) f(h_w(t)) d\mu_H(t), \quad w > 0, \ s \in G, \qquad \text{(I)}$$

where G and H are Hausdorff topological groups, $f : G \to \mathbb{R}$ is a measurable function, $\{h_w\}_{w>0}$ is a family of homeomorphisms $h_w : H \to h_w(H) \subset G$ and $\{K_w\}_{w>0}$ with $K_w : G \to \mathbb{R}$, is a family of kernel functions satisfying suitable assumptions, has been studied in [9] with respect to the uniform convergence and in the setting of Orlicz spaces (i.e. with respect to the modular convergence).

The above integral operators have been introduced in order to give a unified approach to the study of the convergence of several classical integral operators, as the classical convolution integral operators (see [19, 3, 7, 10, 43, 6]), the Mellin convolution ones (see [2, 8, 17, 41]) and a class of discrete operators which in particular contains the generalized sampling series of f, i.e.

$$(S_w^K f)(s) = \sum_{k \in \mathbb{Z}} K(sw - k) f(\frac{k}{w}), \quad s \in \mathbb{R}, \ w > 0,$$

where K is an integrable function on $\mathbb{R}$ (see a.e. [6, 1]), obtained as particular cases of the family (I) with particular choices of the groups G and H, of their respective measures and of the family of the homeomorphisms h_w.

In this paper we give the first unifying approach for the above family (I), in order to study the degree of approximation in the setting of Orlicz spaces. A partial result in this respect has been given in [1] in the particular case of the generalized sampling series. As concernes the general theory of Orlicz-type spaces, see [34, 37, 33, 35, 39, 6].

In order to attain our goal, we need to introduce a Lipschitz class which takes into account the setting of Orlicz spaces and some natural assumptions on the family of kernels involved in our operators, which are satisfied when the absolute moment of order α of the kernels are finite, as happens in the classical cases. The latter condition is satisfied by many well-known classes of kernels.

The importance of the study of the generalized sampling series relies on its application to the theory of "signal processing". Indeed, such discrete

operators have been introduced and studied by P.L. Butzer and his school (see e.g. [13, 20, 40, 14, 21, 22, 16, 15, 23, 24, 17, 18] and [11]) in order to solve the many disadvantages, from the point of view of the applications, of the formula given by the classical Witthaker-Kotelnikov-Shannon sampling theorem (see [42, 12]); for references to the classical sampling theory, see [32, 31, 25, 28, 12, 29, 30]. With the tools of approximation theory, it is possible to reconstruct signals, in an approximate sense, avoiding the band-limitation and the finiteness of the energy of the signal, as happens in the sampling theorem. The signals considered in the Butzer approach are supposed to be uniformly continuous and bounded and the theory enables also the prediction of the signals.

Our setting in Orlicz spaces allows us to generalize the theory to signals which belong to more general function spaces, so including in particular the L^p-spaces, the latter being a very interesting case from the point of view of the applications.

The paper is organized as follows. After the general notations and definitions of the Orlicz spaces (Section 2), we give a boundedness result for our family of integral operators in Orlicz spaces in Section 3, while in Section 4 we establish the result concerning the order of approximation. Here we introduce the Lipschitz class and the singularity assumptions in a general form ($\xi-$singularity assumption), from which we can deduce, as particular cases, the classical ones. In Section 5 we discuss the case of L^p-spaces, so important in the applications, and in Section 6 we furnish classes of operators to which our theory can be applied. Finally, in Section 7 we show some concrete example of kernels for which our assumptions are fully satisfied.

2 Notations and definitions

Let G and H be locally compact Hausdorff topological groups, where G is Abelian, and μ_G and μ_H are $\sigma-$finite Haar measures on the classes of Borel sets, denoted by $\mathcal{B}(G)$ and $\mathcal{B}(H)$, respectively. We recall that unimodularity for the group G is sufficient for our theory.

By $\mathcal{U}$ we shall denote the neighborhood base of the neutral element $\theta \in G$ and, due to the locally compactness of G, we can assume that $\mathcal{U}$ is a base of measurable symmetric compact neighborhoods of θ.

Let $\{h_w\}_{w>0}$ be a family of functions $h_w : H \to G$ such that h_w is a homeomorphism between H and $h_w(H) \subset G$ and let $X(G)$ and $X(H)$ be the spaces of all the Borel measurable real-valued functions defined on

G and H, respectively.

Since Orlicz spaces can be framed in the theory of *modular spaces*, we recall here some definitions.

A *modular* is a functional $\rho : X(G) \to \overline{\mathbb{R}_0^+}$, being $\overline{\mathbb{R}_0^+} = [0, \infty]$, satisfying the following properties:

(1) $\rho[f] = 0$ if and only if $f = 0$;

(2) $\rho[-f] = \rho[f]$, for every $f \in X(G)$;

(3) $\rho[\alpha f + \beta g] \leq \rho[f] + \rho[g]$, for every $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $f, g \in X(G)$.

If (3) is replaced by

(3′) $\rho[\alpha f + \beta g] \leq \alpha\rho[f] + \beta\rho[g]$, for every $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $f, g \in X(G)$,

then the modular is said to be *convex*. Moreover, if (3) is replaced by

(3″) there exists $D > 0$ such that $\rho[\alpha f + \beta g] \leq D(\alpha\rho[Df] + \beta\rho[Dg])$, for every $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $f, g \in X(G)$,

then ρ is said to be *quasi-convex*.

The *modular space* generated by ρ is the subspace

$$L^\rho(G) = \{f \in X(G) : \lim_{\lambda \to 0^+} \rho[\lambda f] = 0\}$$

(see [38, 37, 33, 6]). When the modular ρ is a convex modular (or quasi-convex), then the modular space can be defined as

$$L^\rho(G) = \{f \in X(G) : \rho[\lambda f] < +\infty, \text{ for some } \lambda > 0\}.$$

Let now $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a $\varphi-$function, i.e. φ satisfies the following properties:

i) φ is a non decreasing, continuous function;

ii) $\varphi(0) = 0$, $\varphi(u) > 0$ if $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$.

From now on, if φ is a $\varphi-$function, then we will write $\varphi \in \Phi$. If φ will be also a convex function, then we will write $\varphi \in \widetilde{\Phi}$.

We now consider the modular ρ of the form:

$$I_\varphi^G[f] := \int_G \varphi(|f(s)|) d\mu_G(s),$$

with $\varphi \in \Phi$.

The modular space generated by the above modular, is called the *Orlicz space* and it is defined as

$$L^\varphi(G) = \{f \in X(G) : \int_G \varphi(\lambda|f(s)|) d\mu_G(s) < +\infty, \text{ for some } \lambda > 0\}.$$

Hereafter, we will denote by I_φ^G and I_φ^H the modulars defined on $X(G)$ and $X(H)$ generating the Orlicz spaces $L^\varphi(G)$ and $L^\varphi(H)$, respectively.

A notion of convergence in modular spaces, called modular convergence, is introduced in [38]; in some sense it is more natural than the norm-convergence induced by the Luxemburg norm. Namely, in our setting of Orlicz spaces, it reads as follows:

we say that a sequence $\{f_n\}_n \subset L^\varphi(G)$ is *modular convergent* to $f \in L^\varphi(G)$, if there exists $\lambda > 0$ such that

$$I_\varphi^G[\lambda(f_n - f)] = \int_G \varphi(\lambda|f_n(s) - f(s)|)d\mu_G(s) \to 0, \quad n \to +\infty. \tag{1}$$

If (1) is satisfied for every $\lambda > 0$, this means that f_n converges to f as $n \to +\infty$ with respect to the Luxemburg norm, i.e. (see e.g. [37])

$$\|f_n - f\|_\rho := \inf\left\{u > 0 : \int_G \varphi\left(\left|\frac{f_n(s) - f(s)}{u}\right|\right) d\mu_G(s) \le u\right\} \to 0, \ \ n \to +\infty.$$

Below, follows some important examples of modular spaces (see e.g. [37, 6]).

1. L^p-**spaces.**
   In this case the generating modular is the functional
   $$I_p^G[f] := \int_G |f(s)|^p d\mu_G(s), \ p \ge 1.$$
   This is obviously a particular case of Orlicz spaces, obtained taking $\varphi(u) = u^p, p \ge 1$. Here the Luxemburg norm is given by :
   $$\begin{aligned} \|f\|_{I_p^G} &= \inf\{u > 0 : \int_G \left|\frac{f(s)}{u}\right|^p d\mu_G(s) \le 1\} = \Big(\int_G |f(s)|^p d\mu_G(s)\Big)^{\frac{1}{p}} \\ &= \|f\|_{L^p(G)}. \end{aligned}$$

2. **Musielak-Orlicz spaces.**
   These spaces are obtained taking the modular ρ of the form
   $$\widetilde{I}_\varphi^G[f] := \int_G \varphi(s, |f(s)|)d\mu_G(s),$$
   where $\varphi \in \Phi$, being now Φ the class of the $\varphi-$functions depending on a parameter, i.e. $\varphi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a measurable function of

$s \in G$, for every fixed $u \in \mathbb{R}_0^+$ and it is a $\varphi-$function of u for every $s \in G$ (see [37, 6]). Obviously, taking $\varphi(s,u) = \varphi(u), s \in G,\ u \in \mathbb{R}_0^+$, we obtain, as particular case, the classical Orlicz spaces.

We now introduce the class of the kernels involved in our operators. Let $\{K_w\}_{w>0}$ be a family of measurable functions $K_w : G \to \mathbb{R}$ such that $K_w \in L^1_{\mu_G}(G)$, for every $w > 0$ and

K$_\mathbf{w}$.1) for every $s \in G$ and $w > 0$, $K_w(s - h_w(\cdot)) \in L^1_{\mu_H}(H)$;

K$_\mathbf{w}$.2) there exists a constant $M > 0$ such that

$$\int_H |K_w(s - h_w(t))| d\mu_H(t) \leq M$$

for every $s \in G$ and $w > 0$.

In the following, we will write $\{K_w\}_{w>0} \subset \mathcal{K}_w$ if $\{K_w\}_{w>0}$ satisfies assumptions $K_w.i)$, for $i = 1, 2$.

3 Estimates in Orlicz spaces

From now on, for $\{K_w\}_{w>0} \subset \mathcal{K}_w$ and under the previous setting, we will consider the following family of linear integral operators

$$(T_w f)(s) = \int_H K_w(s - h_w(t)) f(h_w(t)) d\mu_H(t), \quad w > 0,\ s \in G,$$

where $f \in X(G)$ is such that the measurable function $(T_w f)(s) < +\infty$, a.e. $s \in G$.

Measurability of $(T_w f)(s)$ derives from standard arguments of measure theory taking into account the Borel measurability of the function $K_w(s-u)$ (see [27], pag. 396) and of f, and being h_w an homeomorphism.

We are going to study a kind of boundedness property for T_w in the setting of Orlicz spaces.

We may establish the following estimate for $T_w f$.

Theorem 1 *Let $\varphi \in \widetilde{\Phi}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$. Then we have the following estimate:*

$$I^G_\varphi[c(T_w f)] \leq M^{-1} \|K_w\|_{L^1_{\mu_G}(G)} I^H_\varphi[cM(f \circ h_w)],$$

for some $c > 0$, for every $w > 0$ and being M the constant of $K_w.2$). Hence, if $(f \circ h_w) \in L^\varphi(H)$ then $T_w f \in L^\varphi(G)$.

Proof. First we denote by

$$\mu_H^s(A) := \int_A |K_w(s - h_w(t))| d\mu_H(t),$$

and by $\widetilde{\mu}_H(A) := \dfrac{\mu_H^s(A)}{M}$, $A \in \mathcal{B}(H), s \in G$, so that $\widetilde{\mu}_H(H) = \mu_H^s(H)/M \leq 1$. Now, being $\varphi \in \widetilde{\Phi}$, applying monotonicity of φ, Jensen inequality and Fubini-Tonelli theorem (twice), we may write, for $c > 0$ and setting in the integration with respect to $d\mu_G(s)$, $z = s - h_w(t)$,

$$\begin{aligned}
& I_\varphi^G[c(T_w f)] = \int_G \varphi(c\ |(T_w f)(s)|) d\mu_G(s) \\
\leq\ & \int_G \varphi\left[c\ \left(\int_H |K_w(s - h_w(t))|\ |f(h_w(t)| d\mu_H(t)\right)\right] d\mu_G(s) \\
=\ & \int_G \varphi\left[c\ \left(M \int_H |f(h_w(t)| d\widetilde{\mu}_H(t)\right)\right] d\mu_G(s) \\
\leq\ & \frac{1}{M} \int_G \left\{\int_H |K_w(s - h_w(t))| \varphi(c\ M |f(h_w(t)|) d\mu_H(t)\right\} d\mu_G(s) \\
=\ & \frac{1}{M} \int_H \left\{\int_G |K_w(s - h_w(t))| \varphi(c\ M |f(h_w(t)|) d\mu_G(s)\right\} d\mu_H(t) \\
=\ & \frac{1}{M} \int_H \left\{\int_G |K_w(z)| \varphi(c\ M |f(h_w(t)|) d\mu_G(z)\right\} d\mu_H(t) \\
=\ & \frac{1}{M} \int_G |K_w(z)| \{\int_H \varphi(c\ M |f(h_w(t)|) d\mu_H(t)\} d\mu_G(z) \\
=\ & M^{-1} \|K_w\|_{L^1_{\mu_G}(G)} I_\varphi^H[cM(f \circ h_w)].
\end{aligned}$$

Therefore, the assertion follows □

4 Order of approximation in Orlicz spaces

We will now study the rate of approximation for the family of linear integral operators (I).

As it is natural, we introduce a Lipschitz class which takes into account the modular of the Orlicz space involved.
For $f \in X(G)$, $w > 0$, we denote by

$$I_{\varphi,w}^H[f] := I_\varphi^H[f \circ h_w]$$

the Orlicz modular on H applied to $f \circ h_w$.

Let $\mathcal{T}$ be the class of all measurable functions $\tau : G \to \mathbb{R}_0^+$, continuous at $t = \theta$ such that $\tau(\theta) = 0$ and $\tau(t) > 0$ for $t \neq \theta$. Now, for a given $\tau \in \mathcal{T}$, we define the Lipschitz class $Lip_\varphi^H(\tau)$ as

$$Lip_\varphi^H(\tau) = \{f \in X(G) : \exists\, \lambda > 0 \text{ s.t. } I_{\varphi,w}^H[\lambda(f(\cdot) - f(\cdot + t))] = O(\tau(t)),\ \text{as } t \to \theta\}$$

uniformly with respect to $w > 0$, where for any two functions $f, g \in X(G)$, $f(t) = O(g(t))$ as $t \to \theta$ means that there exists a constant $N > 0$ and a neighborhood U of θ such that $|f(t)| \leq N|g(t)|$, for $t \in U$.
We point out that in the particular case of $\tau(t) = |t|^\alpha$, $t \in \mathbb{R}$, $\alpha > 0$, we have the Zygmund-type classes.

Given the modular $I_\varphi^H : X(H) \to \overline{\mathbb{R}_0^+}$, we denote by $\Upsilon_\varphi(G)$ the subspace of $X(G)$ consisting of all measurable functions f such that the following condition holds:
there exist a constant $C \geq 1$ and a net of functions $l_w : G \to \mathbb{R}_0^+$ such that, for every $z \in G$ and $w > 0$,

$$I_\varphi^H[f(z + h_w(\cdot))] \leq I_\varphi^H[C(f \circ h_w)] + l_w(z). \qquad (\star)$$

Note that for any choice of h_w, $\Upsilon_\varphi(G)$ is always nonempty since it contains at least the null functions.
Moreover, we point out that in the particular case of $G = H$ and $h_w(t) = t$, $t \in H$, the above definition is always satisfied with $C = 1, l_w(z) = 0$ and $\Upsilon_\varphi(G) = X(G)$, being the modular of the Orlicz space translation invariant. From now on, we will work on a suitable subspace of $\Upsilon_\varphi(G)$.

For the next result, we will need of the following definition.

Definition 1 *We say that $(f \circ h_w) \in L^\varphi(H)$ uniformly with respect to $w > 0$, if there exists a constant $\nu > 0$ such that*

$$I_\varphi^H[\nu(f \circ h_w)] \leq R,$$

for every $w > 0$ and for an absolute constant $R > 0$.

In the particular case of $h_w(t) = t$, $t \in H$, the above definition is always fulfilled.

Let now Ξ be the class of all the functions $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ξ is continuous at $u = 0$, $\xi(0) = 0$ and $\xi(u) > 0$ for $u > 0$.
For a fixed $\xi \in \Xi$, we say that K_w is a **ξ-singular kernel** if, for every $V \in \mathcal{U}$,

$$\int_{G \setminus V} |K_w(s)|\ [l_w(s) + 1]\, d\mu_G(s) = O(\xi(w^{-1})),\ \text{as } w \to +\infty \qquad (2)$$

being l_w the net of functions of condition $(\star)$ and, setted $\Omega_w(s) := A_w(s)-1$, where
$A_w(s) := \int_H K_w(s-h_w(t))d\mu_H(t)$, there holds

$$\Omega_w(s) = \left[\int_H K_w(s-h_w(t))d\mu_H(t) - 1\right] = O(\xi(w^{-1})), \quad \text{as } w\to+\infty \qquad (2')$$

uniformly with respect to $s \in G$.
Note that taking $\xi(t) = t^\alpha$, $t \in \mathbb{R}_0^+$, $\alpha > 0$, $h_w(t) = t$ and with particular choices of the groups G and H and of their respective measures, we reproduce the classical situation of the theory for convolution integral operators (see Sections 6.1 and 6.2 where condition $(\star)$ is satisfied with $l_w(s) = 0$).

Now we may state the result on the rate of approximation for $(T_w f - f)$, with $f \in Lip_\varphi^H(\tau)$.

Theorem 2 *Let $\varphi \in \widetilde{\Phi}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$ be a ξ-singular kernel. Moreover suppose that $f \in L^\varphi(G) \cap Lip_\varphi^H(\tau) \cap \Upsilon_\varphi(G)$, $\tau \in \mathcal{T}$ and $(f \circ h_w) \in L^\varphi(H)$ uniformly with respect to $w > 0$. If there exists a neighborhood V of θ such that*

$$\int_V |K_w(s)|\tau(s)d\mu_G(s) = O(\xi(w^{-1})), \quad \text{as } w\to+\infty, \qquad (3)$$

then there is a constant $c > 0$ such that

$$I_\varphi^G[c(T_w f - f)] = O(\xi(w^{-1})),$$

for sufficiently large $w > 0$.

Proof. For every $s \in G$, we may write,

$$\begin{aligned}
|(T_w f)(s) - f(s)| &= \Big| \int_H K_w(s-h_w(t))f(h_w(t))d\mu_H(t) - f(s)\Big| \\
&= \Big| \int_H K_w(s-h_w(t))[f(h_w(t)) - f(s)]d\mu_H(t) + A_w(s)f(s) - f(s)\Big| \\
&\le \int_H |K_w(s-h_w(t))| \cdot |f(h_w(t)) - f(s)|d\mu_H(t) + |f(s)| \ \cdot |A_w(s) - 1|.
\end{aligned}$$

Now, for $c > 0$, by convexity of φ, we have

$$\begin{aligned}
I_\varphi^G[c(T_w f - f)] \le \\
&\le \frac{1}{2}\Big\{ \int_G \varphi\Big(2c \int_H |K_w(s-h_w(t))||f(h_w(t)) - f(s)|d\mu_H(t)\Big)d\mu_G(s) \\
&\qquad + \int_G \varphi\Big(2c|f(s)||A_w(s) - 1|\Big)d\mu_G(s)\Big\} =: \frac{1}{2}(J_1 + J_2).
\end{aligned}$$

We now estimate J_1 and J_2, separately.

By Jensen's inequality and Fubini-Tonelli theorem, arguing as in Theorem 1, we have

$$\begin{aligned} J_1 &= \int_G \varphi\Big(2c\int_H |K_w(s-h_w(t))||f(h_w(t))-f(s)|d\mu_H(t)\Big)d\mu_G(s) \\ &\leq \frac{1}{M}\int_G\int_H |K_w(s-h_w(t))|\varphi\Big(2cM|f(h_w(t))-f(s)|\Big)d\mu_H(t)d\mu_G(s) \\ &= \frac{1}{M}\int_H\int_G |K_w(s-h_w(t))|\varphi\Big(2cM|f(h_w(t))-f(s)|\Big)d\mu_G(s)d\mu_H(t). \end{aligned}$$

Now, putting $z = s - h_w(t)$ in the integration with respect to $d\mu_G(s)$, and using again the Fubini-Tonelli theorem, we have

$$\begin{aligned} J_1 &= \frac{1}{M}\int_H\int_G |K_w(z)|\varphi\Big(2cM|f(h_w(t))-f(z+h_w(t))|\Big)d\mu_G(z)d\mu_H(t) \\ &= \frac{1}{M}\int_G |K_w(z)|\Big[\int_H \varphi\Big(2cM|f(h_w(t))-f(z+h_w(t))|\Big)d\mu_H(t)\Big]d\mu_G(z). \end{aligned}$$

Taking now $c>0$ in such a way that $2cM \leq \lambda$, being λ the constant of the class $Lip_\varphi^H(\tau)$, (λ can be taken such that $\lambda \leq \dfrac{\nu}{2C}$, where C and ν are the constants of condition $(\star)$ and Definition 1, respectively), we obtain

$$J_1 \leq \frac{1}{M}\int_G |K_w(z)|\Big[\int_H \varphi\Big(\lambda|f(h_w(t))-f(z+h_w(t))|\Big)d\mu_H(t)\Big]d\mu_G(z).$$

We may write,

$$\begin{aligned} J_1 &\leq \frac{1}{M}\Big\{\int_V |K_w(z)|\int_H \varphi(\lambda|f(h_w(t))-f(z+h_w(t))|)d\mu_H(t)d\mu_G(z) \\ &+ \int_{G\setminus V} |K_w(z)|\int_H \varphi(\lambda|f(h_w(t))-f(z+h_w(t))|)d\mu_H(t)d\mu_G(z)\Big\} \\ &=: \frac{1}{M}(J_1^1+J_1^2), \end{aligned}$$

where V is the neighborhood of the Lipschitz class which, without any loss of generality, can be chosen in such a way that (3) holds; therefore, being $f \in Lip_\varphi^H(\tau)$, for some $A, N>0$ and for sufficiently large $w>0$,

$$J_1^1 \leq N\int_V |K_w(z)|\tau(z)d\mu_G(z) \leq NA\ \xi(w^{-1}).$$

As concernes J_1^2, by the monotonicity and being $f \in \Upsilon_\varphi(G)$, we have

$$J_1^2 \leq \int_{G\setminus V} |K_w(z)| \Big\{ \int_H \varphi\Big(2\lambda C|(f\circ h_w)(t)|\Big) d\mu_H(t) + l_w(z) \Big\} d\mu_G(z),$$

being $C \geq 1$. Hence, by (2) of the $\xi-$singularity assumption, there exists a constant $B > 0$ such that

$$\begin{aligned} J_1^2 &\leq I_\varphi^H[2\lambda C(f\circ h_w)] \int_{G\setminus V} |K_w(z)| d\mu_G(z) + \int_{G\setminus V} |K_w(z)| l_w(z) d\mu_G(z) \\ &\leq \big\{ I_\varphi^H[2\lambda C(f\circ h_w)] + 1 \big\} \ B\ \xi(w^{-1}) \end{aligned}$$

for sufficiently large $w > 0$.

We now estimate $J_2 := \displaystyle\int_G \varphi\Big(2c|A_w(s) - 1||f(s)|\Big) d\mu_G(s)$.

By $(2')$ of $\xi-$singularity, there exists $P > 0$ such that

$$J_2 \leq \int_G \varphi\Big(2cP|f(s)|\xi(w^{-1})\Big) d\mu_G(s),$$

for sufficiently large $w > 0$. Now, since $\xi \in \Xi$, and so it is continuous, by convexity of φ, we have

$$J_2 \leq I_\varphi^G[2cPf]\xi(w^{-1}),$$

for sufficiently large $w > 0$.

In conclusion, for sufficiently large $w > 0$, we obtain

$$\begin{aligned} I_\varphi^G[c(T_w f - f)] \ \leq\ & \frac{1}{2} \Big\{ \Big(NAM^{-1} + BM^{-1}\ I_\varphi^H[2\lambda C(f\circ h_w)] \\ & + BM^{-1} + I_\varphi^G[2cPf] \Big) \xi(w^{-1}) \Big\}. \end{aligned}$$

Therefore, since $f \in L^\varphi(G)$, $(f\circ h_w) \in L^\varphi(H)$ uniformly with respect to $w > 0$, which means that $I_\varphi^H[2\lambda C(f\circ h_w)] \leq I_\varphi^H[\nu(f\circ h_w)] \leq R$, being R the constant of Definition 1, we conclude that

$$I_\varphi^G[c(T_w f - f)] = O(\xi(w^{-1})), \qquad \text{as } \ w \to +\infty$$

□

Remark 1. Taking into account Definition $(3'')$ of Section 2 applied to the $\varphi-$function φ, it is easy to note that the previous result holds also in case when φ is a quasi-convex function.

5 Estimates and Degree of Approximation in L^p-spaces

We shall now apply our theory to the important case of L^p-spaces, with $1 \leq p < +\infty$, i.e. we are in the case of $\varphi(u) = u^p, p \geq 1$. We may state the following:

Proposition 1 *Let $\varphi \in \widetilde{\Phi}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$. Then we have the following estimate:*

$$\|T_w f\|_{L^p(G)} \leq M^{1-\frac{1}{p}} \|K_w\|^{1/p}_{L^1(G)} \|(f \circ h_w)\|_{L^p(H)}.$$

As a consequence, if $(f \circ h_w) \in L^p(H)$, then $T_w f \in L^p(G)$.

Proof. We give a direct proof of the above inequality. Being $\varphi \in \widetilde{\Phi}$, we have, by Jensen's inequality, Fubini-Tonelli theorem and setting $z = s - h_w(t)$ in the integration with respect to $d\mu_G(s)$,

$$\begin{aligned}
& \|(T_w f)\|^p_{L^p(G)} = \int_G |(T_w f)(s)|^p d\mu_G(s) \\
\leq\ & \frac{1}{M} \int_G \left[\int_H |K_w(s - h_w(t))| M^p |f(h_w(t)|^p d\mu_H(t) \right] d\mu_G(s) \\
=\ & \frac{1}{M} \int_H \left[\int_G |K_w(s - h_w(t))| M^p |f(h_w(t)|^p d\mu_G(s) \right] d\mu_H(t) \\
=\ & \frac{1}{M} \int_H \left[\int_G |K_w(z)| M^p |f(h_w(t)|^p d\mu_G(z) \right] d\mu_H(t) \\
=\ & \frac{1}{M} \int_G |K_w(z)| \{ \int_H M^p |f(h_w(t)|^p d\mu_H(t) \} d\mu_G(z) \\
=\ & M^{p-1} \|K_w\|_{L^1(G)} \|(f \circ h_w)\|^p_{L^p(H)}.
\end{aligned}$$

Therefore the assertion follows □

As concernes the result about the order of approximation, we may deduce, as a direct application of Theorem 2, the following corollary in which we will denote by $\Upsilon_p(G)$ the corresponding subspace of $X(G)$ introduced in Section 4.

Corollary 1 *Let $\varphi \in \widetilde{\Phi}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$ be a ξ-singular kernel. Moreover suppose that $f \in L^p(G) \cap Lip^H_p(\tau) \cap \Upsilon_p(G)$, $\tau \in \mathcal{T}$ and $(f \circ h_w) \in$*

$L^p(H)$ uniformly with respect to $w > 0$, where

$$Lip_p^H(\tau) := \Big\{ f \in X(G) : \int_H |f(h_w(t)) - f(h_w(t)+z)|^p d\mu_H(t) = O(\tau(z)), \ \text{as } z \to \theta \Big\}$$

uniformly with respect to $w > 0$. If there exists a neighborhood V of θ such that

$$\int_V |K_w(s)|\tau(s)d\mu_G(s) = O(\xi(w^{-1})), \quad \text{as } w \to +\infty, \qquad (3')$$

then

$$\|(T_w f - f)\|_{L^p(G)} = O(\xi(w^{-1})),$$

for sufficiently large $w > 0$.

L^p-spaces are important examples of Orlicz spaces and they are also of great importance from the point of view of the applications to the sampling theory (in this case $G = \mathbb{R}$ and $H = \mathbb{Z}$, as showed in Section 6.3); indeed, in the L^2-case, the result above stated (Corollary 1) gives information about the degree of approximation for signals with finite energy, while in the L^p-case, with $p \neq 2$, we cover different classes of signals, as for example the power signals and others.

6 Applications

Here we discuss some important classes of operators which can be deduced by the family (I), suitably chosing the groups G, H and the homeomorphisms $\{h_w\}_{w>0}$.

1. Take $G = H = (\mathbb{R}^N, +)$, provided with the Lebesgue measure dt and $h_w(t) = t$, $t \in \mathbb{R}^N$, $w > 0$; the family of integral operators (I) becomes now
$$(T_w f)(s) = \int_{\mathbb{R}^N} K_w(s-t)f(t)dt, \quad s \in \mathbb{R}^N,$$
for $f : \mathbb{R}^N \to \mathbb{R}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$; namely, we obtain the classical linear convolution integral operators. For references see [19, 37].
The assumptions of the class $\mathcal{K}_w$ together with (2) and (2') give the natural setting for the study of the rate of approximation for the above operators (see [19, 37]), taking also into account that condition $(\star)$ is satisfied as an equality with $C = 1$, $l_w(s) = 0$ and $\Upsilon_\varphi(G) = X(G)$,

for every $\varphi \in \Phi$. Moreover Definition 1 is obviously satisfied. Therefore, applying Theorems 1 and 2, we deduce a boundedness property and a result about the rate of approximation for the family of linear convolution integral operators in Orlicz spaces.

2. Take now $G = H = (\mathbb{R}^+, \cdot)$ provided with the log-measure $\mu_G = \mu_H = \int t^{-1}dt$ and $h_w(t) = t$, $t \in \mathbb{R}^+$, for every $w > 0$: we obtain the Mellin convolution operators and, in this case, the family of operators (I) takes the form

$$(T_w f)(s) = \int_{\mathbb{R}^+} K_w(st^{-1}) f(t) \frac{dt}{t}, \quad s \in \mathbb{R}^+$$

with $f : \mathbb{R}^+ \to \mathbb{R}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$. Similar reasonings of the previous class can be repeated here, including the above remarks about condition $(\star)$ and Definition 1, taking into account the multiplicative operation of the group G.
Furthermore our theory covers also the case of multidimensional Mellin convolution operators, taking $G = H = (\mathbb{R}_N^+, \cdot)$, where $\mathbb{R}_N^+ = (]0, +\infty[)^N$ and the inner operation "$\cdot$" is defined by $s \cdot t = (s_1 t_1, \ldots, s_N t_N)$, for $s = (s_1, \ldots, s_N) \in \mathbb{R}_N^+$ and $t = (t_1, \ldots, t_N) \in \mathbb{R}_N^+$. The neutral element θ of G is $\theta = 1 = (1, \ldots, 1)$, the inverse of $t \in \mathbb{R}_N^+$ is given by $t^{-1} = (t_1^{-1}, \ldots, t_N^{-1})$ and the Haar measure $\mu_G = \mu_H$ is defined as

$$\mu_G = \mu_H = \int \frac{dt}{< t >},$$

where $< t >= \prod_{k=1}^N t_k$, and dt is the Lebesgue measure.
So, from Theorems 1 and 2 we obtain results about the continuity and the rate of approximation for the above linear Mellin convolution operators in Orlicz spaces. For references, see [3, 8, 9, 17, 18, 43].

More in general, note that the above considerations of Sections 6.1 and 6.2 are still valid taking only $G = H$ and $h_w(t) = t, t \in G$.

3. Take now $G = (\mathbb{R}, +)$ provided with the Lebesgue measure dt, $H = (\mathbb{Z}, +)$ provided with the counting measure $\mu_{\mathbb{Z}}$ and $h_w : \mathbb{Z} \to \mathbb{R}$.
The family of operators (I) takes now the discrete form

$$(T_w f)(s) = \sum_{k=-\infty}^{+\infty} K_w(s - h_w(k)) f(h_w(k)), \quad s \in \mathbb{R}, \ w > 0,$$

where $f : \mathbb{R}\to\mathbb{R}$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$ with $K_w : \mathbb{R}\to\mathbb{R}$.
The assumptions $K_w.1)$ and $K_w.2)$ on the class $\mathcal{K}_w$ become now that $K_w(s - h_w(\cdot)) \in L^1_{\mu_\mathbb{Z}}(\mathbb{Z})$ $(K_w.1))$ and there exists a constant $M > 0$ such that

$$\sum_{k=-\infty}^{+\infty} |K_w(s - h_w(k))| \leq M,$$

for every $s \in \mathbb{R}$ and for every $w > 0$ $(K_w.2))$.

Among such classes of discrete operators, there is a fundamental example given by the "generalized sampling operators" which plays an important rule in the theory of signal processing.
This is obtained by taking $h_w(k) = \dfrac{k}{w}$, for every $w > 0,\ k \in \mathbb{Z},$ and $K_w : \mathbb{R}\to\mathbb{R}$ of the form $K_w(s) = K(ws),$ for every $w > 0, s \in \mathbb{R}$ where $K : \mathbb{R}\to\mathbb{R}$. The above discrete family of operators becomes now

$$(S^K_w f)(s) = \sum_{k=-\infty}^{+\infty} K(ws - k) f(\frac{k}{w})), \quad s \in \mathbb{R},\ w > 0,$$

which represents the "generalized sampling series" of f; for the linear case, see [13, 40, 15, 23, 24, 9], while for the nonlinear one's, see [44, 5, 36, 1].
Assumptions $K_w.1)$ and $K_w.2)$ become now that for every $s \in \mathbb{R}$ and $w > 0$,
$K(ws - (\cdot)) \in L^1_{\mu_\mathbb{Z}}(\mathbb{Z})$ $(K_w.1))$ and there exists $M > 0$ such that

$$\sum_{k=-\infty}^{+\infty} |K(ws - k)| \leq M,$$

for every $s \in \mathbb{R}$, $w > 0$ $(K_w.2))$, i.e.

$$\sum_{k=-\infty}^{+\infty} |K(t - k)| \leq M$$

for every $t \in \mathbb{R}$.
Therefore, our assumptions on the class $\mathcal{K}_w$ reproduce, for the generalized sampling series, those ones used in classical signal theory (see e.g. [24]). As to the $\xi-$ singularity assumptions, (2) means that for

every $\delta > 0$,

$$\int_{|s|\geq\delta} |K_w(s)|[l_w(s)+1]ds = \int_{|s|\geq\delta} |K(ws)|[l(ws)+1]ds$$
$$= \frac{1}{w}\int_{|z|\geq w\delta} |K(z)|[l(z)+1]dz = O(\xi(w^{-1})), \quad \text{as } w\to+\infty,$$

where in this case we have setted $l_w(s) = l(ws)$, for every $w > 0$ and $s \in \mathbb{R}$.

Similarly, assumptions (2') and (3) of Theorem 2 can be deduced.

In this case the subspace $\Upsilon_\varphi(G)$ of $X(G)$ is a nonempty subspace; as example we may take the function $f : \mathbb{R} \to \mathbb{R}$, with $f(t) = 1$, for $t \in [-M, M]$, for some $M > 0$ and zero outside and it is possible to see that for this function condition $(\star)$ is satisfied as an equality with $C = 1$ and $l_w(s) = 0$, for every $\varphi \in \Phi$. More in general we can deduce that $\Upsilon_\varphi(\mathbb{R})$ contains the space of bounded functions with compact support on $\mathbb{R}$.

Moreover, we point out that in this case, since Definition 1 is not meaningful in general, we can use the weaker condition that

$$f\left(\frac{(\cdot)}{w}\right) \in L^\varphi(\mathbb{Z})$$

in a weak sense, i.e. there exist constants $\lambda, \beta > 0$ with $\beta \geq 1$ such that

$$I_\varphi^{\mathbb{Z}}\left[\lambda\left(f\left(\frac{(\cdot)}{w}\right)\right)\right] \leq R\ w^\beta,$$

for sufficiently large $w > 0$ and for an absolute constant $R > 0$.

Indeed in the final estimation of Theorem 2, we can dominate from above the term $BM^{-1}I_\varphi^{\mathbb{Z}}[2\lambda C(f(\frac{k}{w}))]\xi(w^{-1})$ by $BM^{-1}Rw^\beta\xi(w^{-1})$; therefore, for a suitable choiche of the function ξ, we have that $\lim_{w\to+\infty} w^\beta\xi(w^{-1}) = 0$. As example, we may take $\xi(w) = w^\alpha$, with $\alpha > \beta$.

Due to the unifying approach of our theory, based on function spaces acting on groups, we can also cover the case of linear *multivariate* sampling series. Namely, we may take $G = (\mathbb{R}^N, +)$, $H = (\mathbb{Z}^N, +)$, $N \geq 1$ with the Lebesgue measure and the counting measure, respectively. The generalized sampling operators become now

$$(S_w^K f)(\mathtt{s}) = \sum_{\mathtt{k}\in\mathbb{Z}^N} K\Big(\mathtt{w}\cdot\mathtt{s}-\mathtt{k}\Big) f\Big(\frac{\mathtt{k}}{\mathtt{w}}\Big),$$

for $\mathtt{s} \in \mathbb{R}^N$, $\mathtt{w} \in \mathbb{R}_N^+$, $K : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \to \mathbb{R}$.
Here $\mathtt{w}$ is a vector, i.e. $\mathtt{w} = (w_1, \dots, w_N) \in \mathbb{R}_N^+$, and $w^1 \leq w^2$ means that $w_i^1 \leq w_i^2$ for $i = 1, 2, \dots N$. Moreover if $\mathtt{w} = (w_1, \dots, w_N)$, $\mathtt{s} = (s_1, \dots, s_N)$, $\mathtt{k} = (k_1, \dots, k_N)$, we set $\mathtt{w} \cdot \mathtt{s} = (w_1 s_1, \dots, w_N s_N)$, $\frac{\mathtt{k}}{\mathtt{w}} = \left(\frac{k_1}{w_1}, \dots, \frac{k_N}{w_N}\right)$ and $\mathtt{w} \to +\infty$ means that $w_i \to +\infty$ for each $i = 1, \dots, N$.
So, our theory can be applied also to this case, obtaining a result on the order of approximation in multivariate setting.
We remark that the multidimensional case is of fundamental interest for image enhancement; indeed, in two-dimensional setting (static image) the function $f(x, y)$ represents the luminance of the image, i.e. the value of the pixels-image in terms of their grey-levels, while in three-dimensional setting the function $f(x, y, t)$, where t is the time, represents a dinamic image, i.e. a movie. So it is clear that in this setting the approximation results mean the degree of approximation in order to reconstruct an image (static or dimanic), starting by the samples values of the image itself.

7 Final remarks

Here we show that in case of $\xi(w) = w^\alpha, \alpha \in \mathbb{R}^+$ and with particular choices of the groups G and H, of their measures and of the function $\tau \in \mathcal{T}$, under the usual assumption that the absolute moment of order α is finite, assumptions (2) of $\xi-$singularity (with $l_w(s) = 0$) and (3) of Theorem 2 are fulfilled (see [4]).

As example, let us take $G = H = (\mathbb{R}^N, +)$ provided with the Lebesgue measure, $\tau(\mathtt{s}) = |\mathtt{s}|^\alpha$, $\alpha \in \mathbb{R}^+$, $\mathtt{s} = (s_1, \dots, s_N) \in \mathbb{R}^N$, $N \geq 1$ and $h_w(\mathtt{t}) = \mathtt{t}$, $\mathtt{t} \in \mathbb{R}^N$, i.e. we deal with the linear convolution integral operators in $\mathbb{R}^N$, $N \geq 1$, as in Section 6.1.

Let us now take into consideration a family of kernels of Fejér-type, i.e. $\{K_w\}_{w>0}$ is of the form $K_w(\mathtt{t}) = w^N K(w\mathtt{t})$, $\mathtt{t} \in \mathbb{R}^N$, $K \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} K(\mathtt{t}) d\mathtt{t} = 1$ (that is $(2')$ of $\xi-$singularity is satisfied) and the absolute moment of order $\alpha > 0$ of the function K is defined by

$$m(K, \alpha) := \int_{\mathbb{R}^N} |\mathtt{t}|^\alpha |K(\mathtt{t})| d\mathtt{t}.$$

There holds the following proposition which we restate here for completeness (see also [4]).

Proposition 2 *Let $K \in L^1(\mathbb{R}^N)$ with $m(K,\alpha) < +\infty$ for some $\alpha > 0$. If $\{K_w\}_{w>0}$ is the associated family of kernels of Fejér-type, then:*

(a) for every $\delta > 0$,

$$\int_{|\mathtt{t}|\geq\delta} |K_w(\mathtt{t})| d\mathtt{t} = O(w^{-\alpha}), \text{ as } w \to +\infty,$$

and hence (2) of $\xi-$singularity is satisfied with $\xi(w) = w^\alpha$, $w, \alpha \in \mathbb{R}^+$;

(b) there exists $\delta > 0$ such that

$$\int_{|\mathtt{t}|\leq\delta} |K_w(\mathtt{t})||\mathtt{t}|^\alpha d\mathtt{t} = O(w^{-\alpha}), \text{ as } w \to +\infty,$$

and therefore (3) of Theorem 2 is satisfied with $\xi(w) = w^\alpha$, $w, \alpha \in \mathbb{R}^+$.

Proof. (a) For every $\delta > 0$ we have, putting $w\mathtt{t} = \mathtt{u}$,

$$\begin{aligned} int_{|\mathtt{t}|\geq\delta} |K_w(\mathtt{t})| d\mathtt{t} &= w^{N-1} \int_{|\mathtt{t}|\geq\delta} \frac{|w\mathtt{t}|^\alpha}{|\mathtt{t}|^\alpha w^{\alpha-1}} |K(w\mathtt{t})| d\mathtt{t} \\ &\leq \frac{w^{N-1}}{\delta^\alpha w^{\alpha-1}} \int_{|\mathtt{t}|\geq\delta} |w\mathtt{t}|^\alpha |K(w\mathtt{t})| d\mathtt{t} = \frac{1}{\delta^\alpha w^\alpha} \int_{|\mathtt{u}|\geq\delta w} |\mathtt{u}|^\alpha |K(\mathtt{u})| d\mathtt{u} \\ &\leq \delta^{-\alpha} w^{-\alpha} m(K,\alpha); \end{aligned}$$

hence the assertion (a) follows.

(b) For some $\delta > 0$ we have, putting $w\mathtt{t} = \mathtt{u}$,

$$\begin{aligned} \int_{|\mathtt{t}|\leq\delta} |K_w(\mathtt{t})||\mathtt{t}|^\alpha d\mathtt{t} &= w^N \int_{|\mathtt{t}|\leq\delta} |K(w\mathtt{t})||\mathtt{t}|^\alpha d\mathtt{t} \\ &= \frac{1}{w^\alpha} \int_{|\mathtt{u}|\leq\delta w} |\mathtt{u}|^\alpha |K(\mathtt{u})| d\mathtt{u} \leq w^{-\alpha} m(K,\alpha), \end{aligned}$$

and therefore assertion (b) follows and the Proposition is proved □

In conclusion, it is now clear that under the classical condition that $m(K,\alpha) < +\infty$, all the assumptions of our theory are satisfied, for example by the family of kernels $\{K_w\}_{w>0}$ of Fejér-type.

Similar reasonings hold in case of $G = H = (\mathbb{R}^+_N, \cdot)$, provided with the log-measure and $h_w(\mathtt{t}) = \mathtt{t}$, $w > 0$, i.e. in case of Mellin-type convolution integral operators with $\xi(w) = w^{\alpha-N}$, $N \geq 1$ and $\alpha > N$.

As concernes the case of generalized sampling operators with $G = (\mathbb{R}^N, +)$, we have kernels which are not properly of Fejér-type since, $K_w(\mathtt{t}) = K(w\mathtt{t})$, $\mathtt{t} \in \mathbb{R}^N$, $w > 0$. Anyway it can be showed that if $m(K, \alpha) < +\infty$ for some $\alpha > 0$, then (2) of $\xi-$singularity and (3) of Theorem 2 are satisfied taking $\xi(w) = w^{\alpha+N}$, $w, \alpha \in \mathbb{R}^+$ and $N \geq 1$.

Among such kernels for which $m(K, \alpha) < +\infty$, there are the Gauss-Weierstrass kernels, the box splines kernels and others, (see e.g. [4]). For example the Gauss-Weierstrass kernels are defined as follows.
We define (see Fig. 1 for $w = 10$)

$$G_w(\mathtt{t}) := w^N \frac{1}{\sqrt{\pi^N}}\ e^{-w^2|\mathtt{t}|^2},\ \mathtt{t} \in \mathbb{R}^N.$$

Setting $K(\mathtt{t}) := \pi^{-\frac{N}{2}}\ e^{-|\mathtt{t}|^2}$, we obtain a Fejér-type kernel $\{K_w\}_{w>0}$.

It is possible to show that the absolute moment of order $\alpha > 0$, $m(K, \alpha)$, is finite for every $\alpha > 0$ (see [19, 4]).

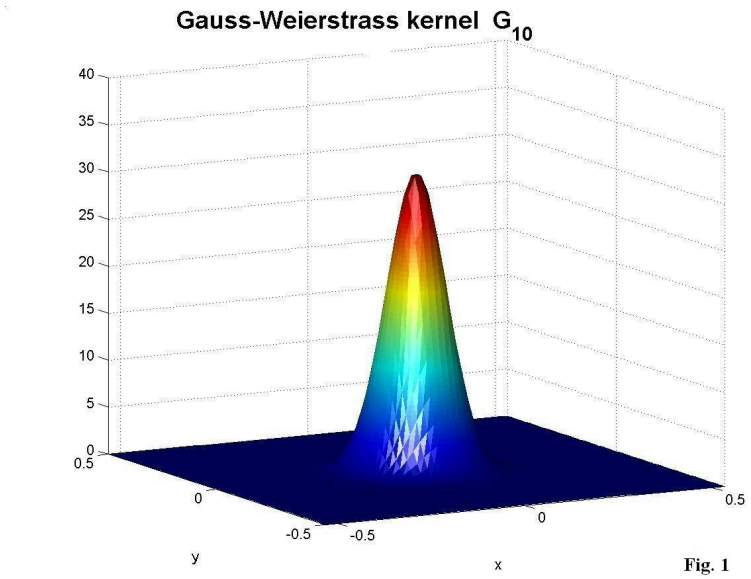


Fig. 1

Moreover it is also possible to take into consideration kernels not necessarily of Fejer type. In order to do this, we take into consideration, as example, a family of kernels $\{K_w\}_{w>0} \subset L^1_{2\pi}$ (L^1-space of $2\pi-$periodic functions) with $\int_{-\pi}^{\pi} K_w(t)dt = 1$ (so that $(2')$ is satisfied) and $G = H = (S^1, \cdot)$ where $S^1 = \{z \in \mathbb{C} :\ |z| = 1\}$, $h_w(t) = t$, $t \in [-\pi, \pi]$ with the Lebesgue

measure dt, i.e. we deal with periodic linear convolution integral operators

$$(T_w f)(s) = \int_{-\pi}^{\pi} K_w(s-t) f(t) dt, \ s \in \mathbb{R},$$

being f extended to the whole real axis with $2\pi-$periodicity. In this case $\tau(t) = |t|^\alpha, t \in [-\pi, \pi]$ and the absolute moment of order $\alpha \in \mathbb{R}^+$ of the kernels $\{K_w\}_{w>0}$ is defined by

$$m_{2\pi}(K_w, \alpha) := \int_{-\pi}^{\pi} |t|^\alpha |K_w(t)| dt.$$

Therefore (2) of $\xi-$singularity becomes, in the particular case of $\xi(w) = w^\alpha$, $\alpha \in \mathbb{R}^+$

$$\int_{\delta \le |t| \le \pi} |K_w(t)| dt = O(w^{-\alpha}), \text{ as } w \to +\infty,$$

for every $\delta > 0$ and (3) of Theorem 2 means that there exists $\delta > 0$ such that

$$\int_{-\delta}^{\delta} |K_w(t)| \ |t|^\alpha dt = O(w^{-\alpha}), \text{ as } w \to +\infty.$$

As before, for completeness we restate the following proposition (see also [4]).

Proposition 3 *For every $\delta > 0$ there holds*

$$\int_{\delta \le |t| \le \pi} |K_w(t)| dt \le \delta^{-\alpha} m_{2\pi}(K_w, \alpha).$$

Proof. For every $\delta > 0$ and for every $w > 0$, there holds

$$\begin{aligned} \int_{\delta \le |t| \le \pi} |K_w(t)| dt &= \frac{1}{\delta^\alpha} \int_{\delta \le |t| \le \pi} \delta^\alpha |K_w(t)| dt \\ &\le \frac{1}{\delta^\alpha} \int_{\delta \le |t| \le \pi} |t|^\alpha |K_w(t)| dt \le \delta^{-\alpha} m_{2\pi}(K_w, \alpha) \end{aligned}$$

□

Therefore, by the above Proposition 3, we deduce that, under the classical assumption that $m_{2\pi}(K_w, \alpha) = O(w^{-\alpha})$, $\alpha > 0$, as $w \to +\infty$, (2) of $w^\alpha-$singularity and (3) of Theorem 2 hold.

Among such kernels for which $m_{2\pi}(K_w, \alpha) = O(w^{-\alpha})$, $\alpha > 0$, as $w \to +\infty$, there are the Abel-Poisson kernels, the Fejer-Korovkin kernels, the integral means and others.

For example, the Abel-Poisson kernels are defined as follows (see Fig. 2):

$$p_r(t) := \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos t + r^2}, \ t \in \mathbb{R}, \ r \in [0, 1],$$

and it is proved that (see [19])

$$\int_{1-r}^{1} t^{\alpha} p_r(t) dt = O((1 - r)^{\alpha})$$

as $r \to 1^-$, $0 < \alpha < 1$; therefore $m_{2\pi}(p_r, \alpha) = O((1 - r)^{\alpha})$, as $r \to 1^-$, $0 < \alpha < 1$, $(\xi(r) = (1 - r)^{-\alpha})$.

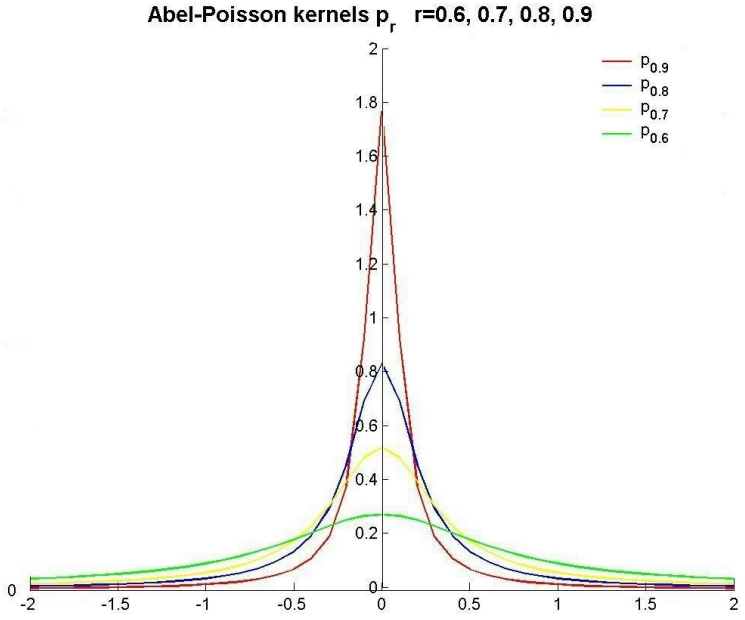


Fig. 2

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References

[1] L. ANGELONI - G. VINTI, *A unified approach to approximation results with applications to nonlinear sampling theory, Int. Journal of Math. Sciences,* **3** (2004), 1, 93-128.

[2] F. BARBIERI, *Approssimazione mediante nuclei momento, Atti Sem. Mat. Fis. Univ. Modena,* **32** (1983), 308-328.

[3] C. BARDARO, *On Approximation Properties for Some Classes of Linear Operators of Convolution Type, Atti Sem. Mat. Fis. Univ. Modena,* **33** (1984), 329-356.

[4] C. BARDARO - P.L. BUTZER - R.L. STENS - G. VINTI, *Convergence in variation and rates of approximation for Bernstein-type polynomials and singular convolution integrals, Analysis,* **23** (2003), 299-340.

[5] C. BARDARO - I. MANTELLINI, *On some Estimates for General Sampling Operators and Approximation Properties, Int. Journal of Math. Sciences,* **2** (2003),2, 289-326.

[6] C. BARDARO - J. MUSIELAK - G. VINTI, Nonlinear Integral Operators and Applications, *De Gruyter Series in Nonlinear Analysis and Applications,* **9***, De Gruyter, New York, Berlin, 2003.*

[7] C. BARDARO - J. MUSIELAK - G. VINTI, *Approximation by nonlinear integral operators in some modular function spaces, Annales Polonici Math.,* **53** (1996), 173-182.

[8] C. BARDARO - G. VINTI, *Modular convergence in generalized Orlicz spaces for moment type operators, Applicable Analysis,* **32** (1989), 265-276.

[9] C. BARDARO - G. VINTI, *A general approach to the convergence theorems of generalized sampling series, Applicable Analysis,* **64** (1997), 203-217.

[10] C. BARDARO - G. VINTI, *On the order of modular approximation for nets of integral operators in modular Lipschitz classes, Funct. Approx. Comment. Math.,* **26** (1998), 139-154.

[11] C. BARDARO - G. VINTI, *An Abstract Approach to Sampling Type Operators Inspired by the Work of P.L. Butzer - Part I - Linear Operators, Sampling Theory in Signal and Image Processing,* **2** (2003), 3, 271-296.

[12] L. BEZUGLAYA - V. KATSNELSON, *The sampling theorem for functions with limited multi-band spectrum I, Zeitschrift für Analysis und ihre Anwendungen,* **12** (1993), 511-534.

[13] P.L. BUTZER, *A survey of the Whittaker-Shannon sampling theorem and some of its extensions, J. Math. Res. Exposition,* **3** (1983), 185-212.

[14] P.L. BUTZER - W. ENGELS - S. RIES - R.L. STENS, *The Shannon sampling series and the reconstruction of signals in terms of linear, quadratic and cubic splines, SIAM J. Appl. Math.,* **46** (1986), 299-323.

[15] P.L. BUTZER - A. FISHER - R.L. STENS, *Generalized sampling approximation of multivariate signals: theory and applications, Note di Matematica,* **10**, (1990), Suppl. No. 1, 173-191.

[16] P.L. BUTZER - G. HINSEN, *Reconstruction of bounded signal from pseudo-periodic, irregularly spaced samples, Signal Processing,* **17** (1989), 1-17.

[17] P.L. BUTZER - S. JANSCHE, *A direct approach to the Mellin Transform, J. Fourier Anal. Appl.,* **3** (1997), 325-376.

[18] P.L. BUTZER - S. JANSCHE, *The exponential sampling theorem of signal analysis, Atti Sem. Mat. Fis. Univ. Modena,* Suppl. Vol. **46** (1998), a special issue of the International Conference in Honour of Prof. Calogero Vinti, 99-122.

[19] P.L. BUTZER - R.J. NESSEL, Fourier Analysis and Approximation, I, *Academic Press, New York-London, 1971.*

[20] P.L. BUTZER - S. RIES - R.L. STENS, *Shannon's sampling theorem, Cauchy's integral formula, and related results,* In: Anniversary Volume on Approximation Theory and Functional Analysis, (Proc. Conf., Math. Res. Inst. Oberwolfach, Black Forest, July 30-August 6, 1983), P.L. Butzer, R.L. Stens and B.Sz.-Nagy (Eds.),*Internat. Schriftenreihe Numer. Math.* **65***, Birkhäuser, Basel, 1984, 363-377.*

[21] P.L. BUTZER - S. RIES - R.L. STENS, *Approximation of continuous and discountinuous functions by generalized sampling series, J. Approx. Theory,* **50** (1987), 25-39.

[22] P.L. BUTZER - W. SPLETTSTÖßER - R.L. STENS, *The sampling theorem and linear prediction in signal analysis, Jahresber. Deutsch. Math.-Verein,* **90** (1988), 1-70.

[23] P.L. BUTZER - R.L. STENS, *Sampling theory for not necessarily band-limited functions: a historical overview, SIAM Review,* **34** (1992),1, 40-53.

[24] P.L. BUTZER - R.L. STENS, *Linear prediction by samples from the past,* Advanced Topics in Shannon Sampling and Interpolation Theory, *(editor R.J. Marks II), Springer-Verlag, New York, 1993.*

[25] M.M. DODSON - A.M. SILVA, *Fourier Analysis and the Sampling Theorem, Proc. Ir. Acad.,* **86** (1985), A, 81-108.

[26] G.B. FOLLAND, Real Analysis: Modern techniques and their applications, *Wiley and Sons, 1984.*

[27] E. HEWITT - K. STROMBERG, Real and Abstract Analysis, *Springer-Verlag, New York and Berlin, 1975.*

[28] J.R. HIGGINS, *Five short stories about the cardinal series, Bull. Amer. Math. Soc.,* **12** (1985), 45-89.

[29] J.R. HIGGINS, Sampling Theory in Fourier and Signal Analysis: Foundations, *Oxford Univ. Press, Oxford, 1996.*

[30] J.R. HIGGINS - R.L. STENS (Eds.), Sampling Theory in Fourier and Signal Analysis: advanced topics, *Oxford Science Publications, Oxford Univ. Press, Oxford, 1999.*

[31] A.J. JERRY, *The Shannon sampling-its various extensions and applications: a tutorial review, Proc. IEEE,* **65** (1977), 1565-1596.

[32] I. KLUVÁNEK, *Sampling theory in abstract harmonic analysis, Mat.-Fyz.Časopis Sloven. Akad. Vied,* **15** (1965), 43-48.

[33] W.M. KOZLOWSKI, Modular Function Spaces, *(Pure Appl. Math.) Marcel Dekker, New York and Basel, 1988.*

[34] M.A. KRASNOSEL'SKIĬ - YA.B. RUTICKIĬ, Convex Functions and Orlicz Spaces,*P. Noordhoff Ltd. - Groningen - The Netherlands, 1961.*

[35] L. MALIGRANDA, Orlicz Spaces and Interpolation, *Seminarios de Matematica, IMECC, Campinas, 1989.*

[36] I. MANTELLINI - G. VINTI, *Approximation results for nonlinear integral operators in modular spaces and applications, Ann. Polon. Math.,* **81** (1) (2003), 55-71.

[37] J. MUSIELAK, Orlicz Spaces and Modular Spaces, *Lecture Notes in Math.* **1034**,*Springer-Verlag, 1983.*

[38] J. MUSIELAK - W. ORLICZ, *On modular spaces, Studia Math.,* **28** (1959), 49-65.

[39] M.M. RAO - Z.D. REN, Theory of Orlicz Spaces, *Pure and Appl. Math., Marcel Dekker Inc., New York-Basel-Hong Kong, 1991.*

[40] S. RIES - R.L. STENS, *Approximation by generalized sampling series, Constructive Theory of Functions'84,* Sofia, (1984), 746-756.

[41] S. SCIAMANNINI - G. VINTI, *Convergence and rate of approximation in BV_φ for a class of integral operators, Approx. Theory Appl.,* **17** (2001), 17-35.

[42] C.E. SHANNON, *Communication in the presence of noise, Proc. I.R.E.,* **37** (1949), 10-21.

[43] C. VINTI, *A Survey on Recent Results of the Mathematical Seminar in Perugia, inspired by the Work of Professor P.L. Butzer, Result. Math.,* **34** (1998), 32-55.

[44] G. VINTI, *A general approximation result for nonlinear integral operators and applications to signal processing, Appl. Anal.,* **79** (2001), 217-238.

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Best Approximation of Monomials in Several Variables

Yuan Xu

Abstract. For a monomial x^α in d variables, the problem of best approximation to x^α by polynomials of lower degrees is studied on the unit sphere, the unit ball and the standard simplex. For the uniform norm we discuss what is known in $d = 2$, for which complete solutions are known, and in $d \geq 3$ for which only a few cases have been successfully solved. For the L^2 norm, we present the complete solution, including explicit formulas for the error of best approximation.

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1 Introduction

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we use the standard multi-index notation for the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The degree of x^α is $|\alpha| = \alpha_1 + \ldots + \alpha_d$. The base $\{x^\alpha : |\alpha| \leq n\}$ spans the space Π_n^d of polynomials of degree at most n in d variables.

Let Ω be a region in $\mathbb{R}^d$. For $f \in C(\Omega)$, the best approximation of f from Π_n^d in the uniform norm is the quantity

$$E_n(f; \Omega) = \inf_{p \in \Pi_{n-1}^d} \|f - p\|_{C(\Omega)}, \tag{1.1}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\|f\|_{C(\Omega)} = \max_{x\in\Omega} |f(x)|$. If $E_n(f;\Omega) = \|f - p^*\|_{C(\Omega)}$ then we call p^* an extremal polynomial for f. We are interested in the case that f is a monomial or some other fixed polynomials. In this case $f - p^*$ is often called a polynomial of least deviation from zero, which we shall call a *least polynomial* for f.

For $d = 1$ and $\Omega = [-1, 1]$ it is well-known that the best approximation to x^n is given in terms of the Chebyshev polynomial of the first kind

$$T_n(x) = \cos n(\arccos x) = 2^{n-1}x^n + q(x), \qquad q \in \Pi_{n-1};$$

in fact, $p^*(x) = x^n - 2^{1-n}T_n(x)$ and $E_n(x^n; [-1, 1]) = 2^{1-n}$. In the case of Ω being a cube in $\mathbb{R}^d$, the best approximation to x^α is given by the product of Chebyshev polynomials. We are interested in the cases that Ω is either the unit sphere $S^{d-1} = \{x : \|x\| = 1\}$ of $\mathbb{R}^d$, where $\|x\|$ denotes the usual Euclidean norm of $x \in \mathbb{R}^d$, the unit ball $B^d = \{x : \|x\| \le 1\}$ or the standard simplex $T^d = \{x : x_1 \ge 0, \ldots, x_d \ge 0, 1 - x_1 - \ldots - x_d \ge 0\}$ of $\mathbb{R}^d$.

To solve the problem of best approximation to the monomials, we need to complete two steps. The first one is to find the value of $E_n(f;\Omega)$ and the second is to find an explicit formula for the extremal polynomials. For $d = 2$ the best approximation to monomials are known explicitly for these regions (see [16, 8, 17]). We explain the result in Section 2 and indicate how the first step can follow from an elementary simpleminded way, which essentially reduce the problem to that of one variable. In the background of the elementary approach, however, is the theory of extremal signature of Rivlin and Shapiro [18], which will play an essential role in the case of $d \ge 3$. The elementary approach is no longer enough for $d \ge 3$ and this is due to an essential difficulty. In some simple cases we will use symmetry of the regions and of the polynomials to simplify the task of finding the extremal polynomials. This and the the extremal signature will be explained in Section 3. For $d > 3$ the extremal polynomials are known only in a few cases (see [1, 2, 19, 25]), these will be given in Section 4 and the method used to derive them will be discussed. Finally, in Section 5, we state the extremal polynomials and the error of the best approximation in L^2 norm, which are known explicitly in all three regions.

2 Best approximation in two variables

For $d = 2$ the regions are the unit circle S^1, the unit disk B^2 and the triangle $T^2 = \{(x, y) : x, y \ge 0, x + y \le 1\}$. We will denote the monomials by $x^k y^{n-k}$ for $0 \le k \le n$.

2.1 The unit circle S^1

The best approximation on the circle can be deduced from that of trigonometric polynomials on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Denote by $\mathcal{T}_n$ the space of trigonometric polynomials of degree at most n. Using $(x, y) = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi]$, every polynomial $\Pi_n^2(S^1)$ becomes a trigonometric polynomial in $\mathcal{T}_n$. In particular, using $e^{i\theta} = \cos\theta + i\sin\theta$, it is easy to see that

$$x^k y^{n-k} = t^*_{k,n}(\theta) + s_{k,n}(\theta),$$

where $s_{k,n} \in \mathcal{T}_{n-1}$ and

$$t^*_{k,n}(\theta) = \frac{1}{2^{n-1}} \begin{cases} \cos n\theta, & \text{if } n-k \text{ is even,} \\ \sin n\theta, & \text{if } n-k \text{ is odd.} \end{cases} \tag{2.1}$$

Since it is well known that the best approximation to $\cos n\theta$ from $\mathcal{T}_{n-1}$ is zero and $\sin\theta = \cos(\pi - \theta)$, it follows that

$$\inf_{p\in\Pi^2_{n-1}} \|x^k y^{n-k} - p\|_{C(S^1)} = \inf_{t\in\mathcal{T}_{n-1}} \frac{1}{2^{n-1}} \|\cos n\theta - t(\theta)\|_{C(\mathbb{T})} = \frac{1}{2^{n-1}} \tag{2.2}$$

and the infimum is attained by $t^*_{k,n}(\theta)$ in the trigonometric case. The extremal polynomial for $x^k y^{n-k}$ is obtained from rewriting $t_{k,n}(\theta)$ in (2.1) as a polynomial in x, y using $x = \cos\theta$ and $y = \sin\theta$.

2.2 The unit disk B^2

For the monomial $x^k y^{n-k}$ it is easy to get a lower bound for the value of $E_n(x^k y^{n-k}; B^2)$. In fact, since S^1 is a subset of B^2, we have

$$\begin{aligned} E_n(x^k y^{n-k}; B^2) &= \inf_{p\in\Pi^2_{n-1}} \|x^k y^{n-k} - p\|_{C(B^2)} \\ &\ge \inf_{p\in\Pi^2_{n-1}} \|x^k y^{n-k} - p\|_{C(S^1)} = \frac{1}{2^{n-1}} \end{aligned} \tag{2.3}$$

Using (2.2). In order to show that the equality holds, we need to find one polynomial $G_{k,n}(x, y) = x^k y^{n-k} - p(x, y)$ with $p \in \Pi^2_{n-1}$ such that $\|G_{k,n}\|_{C(B^2)} = 2^{1-n}$. The formula in (2.3) also shows that such a polynomial should satisfy $\|G_{k,n}\|_{C(B^2)} = \|G_n\|_{C(S^1)}$, which means that $G_{k,n}$ should satisfy

$$G_{k,n}(\cos\theta, \sin\theta) = 2^{1-n}\varepsilon_n t^*_{k,n}(\theta), \qquad 0 \le \theta \le 2\pi, \tag{2.4}$$

where $\varepsilon = \pm 1$ and $t^*_{k,n}$ is defined in (2.1). Of course the restriction of G on S^1 will not determine G completely. Such a polynomial was found in [8] in terms of U_n, the Chebyshev polynomial of the second kind,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta.$$

We summarize the result in the following:

Theorem 2.1 *For $n \ge k \ge 0$, $E_n(x^k y^{n-k}; B^2) = 2^{1-n}$ and*

$$G_{k,n}(x,y) = \frac{1}{2^n}\left(U_n(x)U_{n-k}(y) + U_{n-2}(x)U_{n-k-2}(x)\right).$$

is a least polynomial for $x^k y^{n-k}$ on B^2.

The elementary trigonometric identity shows that $G_{k,n}$ given in the theorem satisfies the equation (2.4). The most difficult part of the proof comes down to show that

$$|G_{k,n}(x,y)| \le 2^{1-n}, \qquad (x,y) \in B^2.$$

In contrast to the case of one variable, polynomials of best approximation in several variables are not unique in general. For the monomial $x^k y^{n-k}$ the uniqueness holds only if $k = 0$ or $k = n$, or $n = 1$ ([8, Theorem 2.2]). Furthermore, it is known that the difference between two least polynomials are necessarily of the form $(1 - x^2 - y^2)q(x,y)$ with $q \in \Pi^2_{n-3}$.

Using the generating functions, another construction of the least polynomial for $x^k y^{n-k}$ was given in [17]. Define a family of polynomials $R_\alpha \in \Pi^d_n$, $\alpha \in \mathbb{N}^d_0$, by

$$\|t\|^n T_n\left(\frac{\langle t, x\rangle}{\|t\|}\right) = \sum_{|\alpha|=n} R_\alpha(x) t^\alpha, \qquad t, x \in \mathbb{R}^d,$$

where $\langle t, x\rangle$ is the usual inner product in $\mathbb{R}^d$. Then $R_\alpha(x) = 2^{|\alpha|-1}\binom{|\alpha|}{\alpha} x^\alpha + p_\alpha$, $p_\alpha \in \Pi^d_{n-1}$. For $d = 2$ we write $\alpha = (k, n-k)$. It turns out that

$$\frac{1}{2^{n-1}}\binom{n}{k} R_{k,n-k}(x,y) = x^k y^{n-k} + p_k(x,y), \qquad p_k \in \Pi^2_{n-1},$$

is a least polynomial on B^2. This family is essentially different from that of $G_{k,n}$ in Theorem 2.1.

One can also generate further family of least polynomials from the existing ones. It is observed in [4] that if $p_{k,n}$ is a least polynomial for $x^k y^{n-k}$ then the polynomial

$$\frac{1}{2^{(m-1)n}} p_{k,n}\left(T_m(x), y U_{m-1}(x)\right) \in \Pi^2_{mn}$$

is a least polynomial for $x^{mn-n+k}y^{n-k}$, which can be different from either $G_{k,n}$ or the multiple of $R_{k,n-k}$ given above.

Another direction is to generate least polynomials for other homogeneous polynomials from the ones for monomials. As an example, one result [8, Theorem 3.2] states that if $p_{k,n-k}$ is the least polynomial for $x^k y^{n-k}$ on B^2 and k is even, then

$$q_{k,n}(x,y) = (-1)^{k/2} p_{k,n}\left((1-x^2-y^2)^{1/2}, y\right)$$

is a least polynomial for $(x^2+y^2)^k y^{n-k}$ on B^2. Finally let us mention that the extremal polynomial for $x^k y^{n-k}$ on B^2 is also the Kergin interpolant of $x^k y^{n-k}$ ([5, 10, 13]).

The standard triangle T^2. Just as in the case of B^2, it is easy to get a lower bound for the value of $E_n(x^k y^{n-k}; T^2)$. We use the fact that

$$\inf_{p\in\Pi_{n-1}} \|x^n - p\|_{C([0,1])} = \frac{1}{2^{2n-1}} \|T_n(2x-1)\|_{C([0,1])} = \frac{1}{2^{2n-1}}$$

instead of best approximation by trigonometric polynomials. We have

$$\begin{aligned} E_n(x^k y^{n-k}; T^2) &= \inf_{p\in\Pi^2_{n-1}} \|x^k y^{n-k} - p\|_{C(T^2)} \\ &\ge \inf_{p\in\Pi_{n-1}} \|x^k (1-x)^{n-k} - p(x, 1-x)\|_{C([0,1])} \\ &= \inf_{p\in\Pi_{n-1}} \|x^n - p\|_{C([0,1])} = \frac{1}{2^{2n-1}}. \end{aligned} \tag{2.5}$$

In order to show that the equality holds, we need to find one least polynomial $T_{k,n}$ for $x^k y^{n-k}$. Since the least polynomial for x^n on $[0,1]$ is given by $2^{1-2n}T_n(2x-1)$, the polynomial $T_{k,n}$ should agree with the Chebyshev polynomial $T_n(2x-1)$ on $\{(x,1-x) : 0 \le x \le 1\}$ which is part of the boundary of T^2. Such a polynomial was found in [16]. We summarize the result as the following:

Theorem 2.2 *For $m \ge k \ge 0$, define*

$$T_{k,m}(x,y) = \begin{cases} T_{k-m}(2y-1)T_m(8xy-1) \\ \quad +8xy(2y-1)U_{k-m-1}(2x-1)U_{m-1}(8xy-1), & \text{if } k > m \\ T_k(8xy-1), & \text{if } k = m \\ T_{m-k}(2x-1)T_k(8xy-1) \\ \quad +8xy(2x-1)U_{m-k-1}(2y-1)U_{k-1}(8xy-1), & \text{if } k < m. \end{cases}$$

Then $2^{1-2n}T_{k,n-k}$ is a polynomial of least deviation from zero for $x^k y^{n-k}$ on T^2 and $E_n(x^k y^{n-k}; T^2) = 2^{1-2n}$.

The difficult part is to recognize that $T_{k,n-k}$ as given above is the correct function. Once the formula is identified, the only difficult part of the proof is to show that $|T_{k,m}(x,y)| \le 1$ on T^2.

It was pointed out in [4] that there is a close relation between best approximation on T^2 and that on B^2. In fact, if $p(x,y)$ is a least polynomial for $x^k y^{n-k}$ on the triangle T^2 then $p(x^2, y^2)$ is a least polynomial for $x^{2k}y^{2n-2k}$ on B^2 and the relation holds conversely. A more precise statement will be given in the following section. This implies, in particular, that we can deduce the least polynomials for $x^k y^{n-k}$ on T^2 from those for $x^{2k}y^{2n-2k}$ on B^2. The exact relation between the polynomials $T_{k,n-k}$ and the polynomials $G_{k,n}$ is established in [4].

3 Extremal signature and invariant polynomials

In this section we discuss some of the tools that can be used to study the best approximation by polynomials.

3.1 Extremal signature

What is hidden in the elementary way of getting to $E_n(f; B^2)$ is the characterization of the extremal polynomials in terms of the extremal signature, which is the natural extension of the Chebyshev alternating theorem in one variable. In this section we review the extremal signatures.

The original study in [18] is in the general setting of approximation from a finite dimensional subspace of $C(\Omega)$ on a compact Hausdorff space Ω. We will state the version only in the case that we are interested in. Let Ω be an infinite compact set in $\mathbb{R}^d$. For our purpose it is either S^{d-1}, B^d or T^d. A *signature* σ on the set Ω is a function with finite support

$\{x_1, \ldots, x_N\}$ where $x_i \in \Omega$, such that $\sigma(x_i) = \pm 1$ for $1 \le i \le N$ and $\sigma(x) = 0$ if $x \in \Omega \setminus \{x_1, \ldots, x_N\}$. A signature σ is *extremal* with respect to Π_n^d if there exists a subset $\mathcal{S}$ in the support of σ and positive numbers λ_v, $v \in \mathcal{S}$, such that

$$\sum_{v \in \mathcal{S}} \lambda_v \sigma(v) p(v) = 0, \qquad \text{for all } p \text{ in } \Pi_n^d.$$

Let $r > 0$ be a fixed number. For each $p \in \Pi_{n-1}^d$, we denote by $\mathcal{S}_r(p; f)$ the set

$$\mathcal{S}_r(p; f) = \{x \in \Omega : |f(x) - p(x)| = r\}.$$

In the case of $r = \|f - p\|_{C(\Omega)}$, $\mathcal{S}_r(p; f)$ is the set of extremal points of $f - p$ and we denote it by $\mathcal{S}(p; f)$. The characterization of the best approximation of f from Π_n^d is given by the following theorem.

Theorem 3.1 *1. A polynomial p^* in Π_n^d satisfies $\|f - p^*\|_{C(\Omega)} = E_n(f; \Omega)$ if and only if there exists an extremal signature σ with support in $\mathcal{S}(p^*; f)$ such that $\sigma(v) = \operatorname{sign}(f - p^*)(v)$ for all $v \in \mathcal{S}(p^*; f)$.*

2. Suppose there exist a polynomial $p^* \in \Pi_n^d$ and an extremal signature σ supported on $\mathcal{S}_r(p^*; f)$. Then $E_n(f; \Omega) \ge r$.

The theorem states that if p is a polynomial of best approximation on Ω then it is so on a finite subset of Ω. For the proof of this theorem, see [18]. The part two of the statement is useful for computing a lower bound for $E_n(f; \Omega)$, especially since we often do not know the polynomial of best approximation. For various examples of extremal signatures, see [8, 15, 18, 21, 22].

In the case of best approximation to $x^k y^{n-k}$ on the disk B^2, one extremal signature agrees with that of extremal signature on the circle S^1. This signature is described as follows:

Let $\theta_1 < \theta_2 < \ldots < \theta_{2n}$ be the $2n$ extremal points of the trigonometric polynomial $t_{k,n}^*$ defined in (2.1). Let $\mathcal{S} = \{(\cos\theta_j, \sin\theta_j) : 1 \le j \le 2n\}$ and define the signature σ by $\sigma(\cos\theta_j, \sin\theta_j) = (-1)^j$; then σ is an extremal signature for $x^k y^{n-k}$ on B^2.

The proof follows from simple trigonometric identities, such as

$$\frac{1}{n} \sum_{k=0}^{n-1} e^{2ijk\pi/n} = \delta_{0,j} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} e^{ij(2k+1)\pi/n} = \delta_{0,j}, \quad 0 \le j \le n-1,$$

where $\delta_{0,j} = 1$ if $j = 0$ and $\delta_{0,j} = 0$ otherwise. Furthermore we notice that the extremal signature is connected to the quadrature formulas. In fact, in

the case that $\theta_1, \ldots, \theta_{2n}$ are zeros of $\sin n\theta$, $\theta_k = 2k\pi/n$, it follows from the above two sums that the following quadratures hold,

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{2k\pi}{n}\right) := \mathcal{L}_1 f, \qquad f \in \mathcal{T}_{n-1},$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{(2k+1)\pi}{n}\right) := \mathcal{L}_2 f, \qquad f \in \mathcal{T}_{n-1}.$$

Thus, σ is an extremal signature since the sum $\mathcal{L}f := \mathcal{L}_1 f - \mathcal{L}_2 f$ satisfies $\mathcal{L}f = 0$ for $f \in \Pi^2_{n-1}$. The case that θ_k are zeros of $\cos n\theta$ works similarly.

3.2 Invariant polynomials

For $d \geq 3$ one can often use the symmetry of the functions and the set Ω to gain some knowledge of the least polynomials. In essence, one can state that if Ω is invariant under a finite group G and f is also invariant under the same group, then the polynomial of best approximation has to be a polynomial invariant under G. This ideal is not new (see, for example, [7]) and the idea of making using of symmetry has been widely used in dealing with the cubature formulas (see for example, [11, 14, 20]). In [1] it is used to obtain least polynomials on the unit sphere and on the unit ball. We shall state the result for the general domain.

Let Ω be a region in $\mathbb{R}^d$ that is invariant under a finite group G; that is, $\Omega g := \{xg : g \in G, x \in \Omega\}$ coincides with Ω. For a function f defined on Ω we defined the action of $g \in G$ on f by $R(g)f(x) = f(xg)$. A function f on Ω is said to be invariant under G if $R(g)f = f$ for all $g \in G$. Let $G\Pi^d_n$ denote the space of polynomials in Π^d_n that are invariant under G.

The domains S^{d-1} and B^d are rotation invariant, we can take G as any finite subgroup of the rotation group $O(d)$. For the simplex the largest group under which T^d is invariant is the symmetric group $\mathcal{S}(T^d)$ that contains all permutations of the vertices of the simplex T^d. If f is invariant under $\mathcal{S}(T^d)$ then it is a symmetric function of $x_1, \ldots, x_d, 1 - x_1 - \ldots - x_d$. The symmetric group $\mathcal{S}_d$ of d coordinates is a subgroup of $\mathcal{S}(T^d)$.

Since the proof is essentially the same we shall state the result for the L^p norm for $1 \leq p \leq \infty$. For $1 \leq p < \infty$ let $\|f\|_{L^p(\Omega)}$ denote the L^p norm of f on Ω with respect to an absolutely continuous finite Borel measure $d\mu = \mu' dx$

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p d\mu(x)\right)^{1/p}$$

and we assume that the measure μ is invariant under G. We take the convention that the norm for $p = \infty$ is the $C(\Omega)$ norm.

Theorem 3.2 *Let Ω be a region in $\mathbb{R}^d$ invariant under a finite group G. If f is invariant under G, then*

$$\inf_{p\in\Pi^d_{n-1}} \|f(x)-p(x)\|_{L^p(\Omega)} = \inf_{p\in G\Pi^d_{n-1}} \|f(x)-p(x)\|_{L^p(\Omega)}.$$

Proof. Since $G\Pi^d_{n-1} \subset \Pi^d_{n-1}$, we evidently have

$$\inf_{p\in\Pi^d_{n-1}} \|f(x)-p(x)\|_{L^p(\Omega)} \le \inf_{p\in G\Pi^d_{n-1}} \|f(x)-p(x)\|_{L^p(\Omega)}.$$

For a given function F we denote by F_G the symmetric function

$$F_G(x) = \frac{1}{|G|}\sum_{g\in G} R(g)F(x)$$

where $|G|$ denote the number of elements in G. Clearly if F is invariant under G, then $F_G = F$. Using the triangle inequality and the fact that f is invariant under G we have

$$\begin{aligned}\|f-p\|_{L^p(\Omega)} &= \frac{1}{|G|}\sum_{g\in G}\|R(g)f - R(g)p\|_{L^p(\Omega)}\\ &\ge \|f_G - p_G\|_{L^p(\Omega)} = \|f-p_G\|_{L^p(\Omega)}.\end{aligned}$$

Taking infimum over p completes the proof.

In [1] this theorem is used for the sphere S^{d-1} and the ball B^d with G being a finite subgroup of the rotation group $O(d)$ to obtain the polynomial of best approximation for a number of special polynomials. For example, it is proved there that

$$\inf_{p\in\Pi^d_{d-1}} \|x_1\cdots x_d - p\|_{L^p(S^{d-1})} = \|x_1\cdots x_d\|_{L^p(S^{d-1})}, \tag{3.1}$$

where G is the symmetric group $\mathcal{S}_d$, and

$$\inf_{p\in\Pi^d_3} \|x_1^4+\cdots+x_d^4 - p\|_{L^p(S^{d-1})} = \inf_{c\in\mathbb{R}} \|x_1^4+\cdots+x_d^4 - c\|_{L^p(S^{d-1})}, \tag{3.2}$$

where G is the hyperoctahedral group $\mathcal{B}_d$, which is the group of symmetry of the unit cube $\{\pm1,\ldots,\pm1\}$ in $\mathbb{R}^d$ and it contains permutations of coordinates and the sign changes (the semi-product of the symmetric group $\mathcal{S}_d$

and the abelian group $\mathbb{Z}_2^d$). There are also several other least polynomials in [1], obtained by taking the above or the other finite subgroups of $O(d)$.

Since $\mathbb{Z}_2^d$ is a subgroup of $O(d)$ which contains all possible sign changes of the coordinates, one consequence of the theorem states that if f is even in each of its variables, so is a polynomial p of best approximation to f. Let $\mathbb{R}_+ := \{x \in \mathbb{R}^d : x_1 \ge 0, \dots, x_d \ge 0\}$ and let ϕ denote the mapping

$$\phi : \mathbb{R}^d \mapsto \mathbb{R}_+^d : x \mapsto \phi(x) = (x_1^2, \dots, x_d^2).$$

We note that ϕ is one-to-one from T^d to $B_+^d = B^d \cap \mathbb{R}_+^d$. It also induces a one-to-one mapping from Π_n^d to $G\Pi_{2n}^d$ with $G = \mathbb{Z}_2^d$, since every polynomial $p \in G\Pi_{2n}^d$ can be written uniquely as $p_e \circ \phi$ for some $p_e \in \Pi_n^d$ and, conversely, $q \circ \phi \in G\Pi_{2n}^d$ for every $q \in \Pi_n^d$. For each f that is even in each of its variables, we also write $f = f_e \circ \phi$.

Let us write in the following $L^p(\Omega, d\mu)$ to emphasis the dependence on the measure $d\mu$. For an invariant measure $d\mu = \mu' dx$ defined on B^d we define another measure $d\nu$ by

$$d\nu(x) = \mu'_e(x)/\sqrt{x_1 \cdots x_d}\, dx, \qquad x \in T^d.$$

The factor $1/\sqrt{x_1 \cdots x_d}$ is the Jacobian of changing variables $y \mapsto x = \phi(y)$.

Theorem 3.3 *Let $f \in L^p(B^d, d\mu)$ be invariant under $\mathbb{Z}_2^d$. If $p^*(x)$ is an extremal polynomial for f in $L^p(B^d, d\mu)$, then $p_e^*(x)$ is an extremal polynomial for f_e in $L^p(T^d, d\nu)$. On the other hand, if $g \in L^p(T^d, d\nu)$ and q^* is an extremal polynomial for f in $L^p(T^d, d\nu)$, then $q^* \circ \phi$ is an extremal polynomial for $f \circ \phi$ in $L^p(B^d, d\mu)$.*

Proof. If $f \in C(B^d)$ is invariant under $\mathbb{Z}_2^d$ then Theorem 3.2 implies that a polynomial p^* of best approximation to f on B^d is even in each of its variables. Hence, it can be written as $p_e^* \circ \phi$. Using the elementary formula

$$\int_{B^d} f(x_1^2, \dots, x_d^2) d\mu(x) = \int_{T^d} f(x_1, \dots, x_d) d\nu(x) \tag{3.3}$$

it follows that p_e^* is the polynomial of best approximation to f on T^d. On the other hand, if q^* is an extremal polynomial for g on T^d, then $q^* \circ \phi$ is a polynomial in $G\mathcal{P}_{2n}^d$ for $G = \mathbb{Z}_2^d$. Using (3.3) and then Theorem 3.2, we see that $q^* \circ \phi$ is an extremal polynomial for $f \circ \phi$ on B^d.

For $d = 2$ this theorem is proved in [4]. The idea of using the correspondence between polynomials on these domains has been used in dealing with orthogonal polynomials and cubature formulas on B^d and T^d ([24]) and it has also been used in [26] to relate the direct and the inverse type theorems for the best approximation on T^d and on B^d.

4 Best approximation in more than two variables

Comparing to two variables, there is little known in several variables. We discuss what is known and what is involved.

4.1 Some basic results

It is easy to see that the following proposition holds (see, for example, [2]).

Proposition 4.1 *If $F(x_1, \ldots, x_d) = f(x_1, \ldots, x_m, 0, \ldots, 0)$ for some $m < d$, then $E_n(F; \Omega^d) = E_n(f; \Omega^m)$ for Ω^d being the unit ball B^d, the simplex T^d or the sphere S^{d-1}.*

Proof. Since B^m is a subset of B^d, it follows readily that

$$\|F - p\|_{C(B^d)} \ge \|f - p(x_1, \ldots, x_m, 0, \ldots, 0)\|_{C(B^d)} \ge E_n(f; B^m)$$

for any $p \in \Pi_n^d$. Hence, $E_n(F; B^d) \ge E_n(f; B^m)$. On the other hand, if $p^* \in \Pi_n^m$ is an extremal polynomial for f on B^m, then $P^*(x_1, \ldots, x_d) = p^*(x_1, \ldots, x_m)$ is a polynomial in Π_n^d and $E_n(F; B^d) \le \|F - P^*\|_{C(B^d)} = \|f - p^*\|_{C(B^d)} = E_n(f; B^m)$. The proof for T^d and S^{d-1} is similar.

In particular, using the result in the previous section, this shows that if $\alpha = (\alpha_1, \alpha_2, 0, \ldots, 0)$ then

$$E_n(x^\alpha; B^d) = E_n(x^\alpha; S^{d-1}) = 2^{1-n}, \qquad n = \alpha_1 + \alpha_2 \tag{4.1}$$

and

$$E_n(x^\alpha; T^d) = 2^{1-2n}, \qquad n = \alpha_1 + \alpha_2. \tag{4.2}$$

Note that both these quantities do not depend on the dimension.

The first nontrivial example appears early in [19], which states that

$$\inf_{p \in \Pi_{n-1}^d} \|x_1 \ldots x_d - p\|_{C(S^{d-1})} = \|x_1 \ldots x_d\|_{C(S^{d-1})} = d^{-d/2}.$$

Only in the fairly recent work [1], polynomials of best approximation to other monomials on B^d and on S^{d-1} are found using the invariance of Theorem 3.2. It particular it is shown there that

$$\inf_{p \in \Pi_{n-1}^d} \|x_1 \ldots x_d - p\|_{C(B^d)} = \inf_{p \in \Pi_{n-1}^d} \|x_1 \ldots x_d - p\|_{C(S^{d-1})} = d^{-d/2}. \tag{4.3}$$

Apart from (3.1) and (3.2) there are several other polynomials whose extremal polynomials are found, including $(x_1^4 + \ldots x_d^4)$ among others. However, the only monomial in the list is $x_1 \cdots x_d$.

In [2] the best approximation is found for another monomial $x_1^2 x_2 \ldots x_d$ on the ball and on the sphere,

$$\begin{aligned} E_{d+1}(x_1^2 x_2 \cdots x_d, B^d) &= E_{d+1}(x_1^2 x_2 \cdots x_d, S^{d-1}) \\ &= \|(x_1^2 - a_d) x_2 \cdots x_d\|_{C(S^{d-1})} = \frac{a_d}{(d-1)^{\frac{d-1}{2}}} \end{aligned} \tag{4.4}$$

where $a = a_d$ is the solution of the equation

$$\frac{2}{d+1} \left(\frac{d-1}{d+1} \right)^{\frac{d-1}{2}} (1-a)^{\frac{d+1}{2}} = a.$$

In particular, $a_2 = 1/4$, $a_3 = 3 - \sqrt{8}$, $a_4 = (38 + 5 \times 10^{1/3} - 10^{5/3})/18$.

Recently in [25], we found a family of symmetric polynomials as extremal polynomials for $x_1^2 \cdots x_d^2$ in B^d and in S^{d-1}. This work depends heavily on the theorems in the previous section. It starts with several reductions. First, by Theorem 3.3, we only need to consider the best approximation to $x_1 \cdots x_d$ on the simplex T^d. Second, by Theorem 3.2, an extremal polynomial can be chosen to be symmetric on T^d.

As we mentioned before that there are two steps in dealing with the problem of least polynomials; one is to find a lower bound for the quantity $E_n(x_1 \cdots x_3; T^3)$ and the other is to identify an extremal polynomial that will attain the lower bound. In the case of two variables, we found the lower bound in (2.5) by taking the maximal on a subset that is part of the boundary of T^2. This way we reduce the problem to a problem in one variable that is already solved. We can attempt to apply the same method here and take the maximal on the face

$$T_0^d = \{(x_1, \ldots, x_d) \in T^d : x_1 + \ldots + x_d = 1\}$$

of T^d. Evidently $x \in T_0^d$ is the same as $(x_1, \ldots, x_{d-1}) \in T^{d-1}$. Let us define $X := (x_1, \ldots, x_{d-1}, 1 - x_1 - \ldots - x_{d-1})$. For $\alpha \in \mathbb{N}_0^d$ define

$$X^\alpha = x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} (1 - x_1 - \cdots - x_{d-1})^{\alpha_d}, \qquad x \in T^d.$$

Then the restriction leads to

$$\begin{aligned} E_n(x^\alpha; T^d) &= \inf_{p \in \Pi_n^d} \|x^\alpha - p\|_{C(T^d)} \\ &\ge \inf_{p \in \Pi_n^{d-1}} \|X^\alpha - p\|_{C(T^{d-1})} = E_n(X^\alpha; T^{d-1}). \end{aligned} \tag{4.5}$$

However, this does not reduce the problem to a lower dimension problem of least polynomial for the monomials, unless α_d (or one of the other component of α) is zero. If none of the components of α is zero, then X^α vanishes on the entire boundary of T^d. We cannot further reduce the problem by restricting to the boundary of the simplex.

In the simple case of $x_1x_2\cdots x_d$ we discuss what to do in the following two subsections. We consider the case $d = 3$ first to illustrate the idea.

4.2 Extremal polynomial for $x_1x_2x_3$ on T^3

In the case of $x_1x_2x_3$ on T^3 the right hand side of the equation (4.5) is a two dimensional problem on T^2. Recall that in the previous section we deduced the lower bound of $E_n(x^k y^{n-k}; T^2)$ by further restricting the maximal on the boundary of T^2 so that it becomes a problem in one variable. For the function $X_1(x) := x_1x_2(1-x_1-x_2)$, however, such a method will not work as the function is zero on the entire boundary of the triangle T^2.

Since X_1 is invariant under the symmetric group $\mathcal{S}(T^2)$, Theorem 3.1 shows that the optimal polynomial has to be a symmetric polynomial in $x_1, x_2, 1-x_1-x_2$. Such a polynomial, called $U_3(x_1, x_2)$, is discovered in [25]. It agrees with the Chebyshev polynomial $T_2(2x-1)$ of degree 2 on the boundary of the triangle T^2; that is

$$U_3(x,0) = U_3(0,x) = U_3(x,1-x) = T_2(2x-1).$$

Proposition 4.2 *Let $X_1(x) = x_1x_2(1-x_1-x_2)$. Define the polynomial*

$$U_3(x_1,x_2) = 72x_1x_2(1-x_1-x_2) - 3 + 4(x_1^2+x_2^2+(1-x_1-x_2)^2).$$

Then

$$E_2(X_1,T^2) = \inf_{p\in T_2^2} \|X_1 - p\|_{C(T^2)} = 72^{-1}\|U_3\|_{C(T^2)} = 72^{-1}.$$

Proof. Once the polynomial is identified, we can find the extremal signature. Indeed, define

$$\mathcal{S}_+ = \{x : |U_3(x_1,x_2)| = 1\} = \left\{(0,0),(1,0),(0,1),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$
$$\mathcal{S}_- = \{x : |U_3(x_1,x_2)| = -1\} = \left\{\left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

and let $\sigma(v) = 1$ on $\mathcal{S}_+$ and $\sigma(v) = -1$ on $\mathcal{S}_-$. Then σ is an extremal signature since the sum $Lf = L_1 f - L_2 f$ defined by

$$L_1 f = \frac{3}{4} f\left(\frac{1}{3}, \frac{1}{3}\right) + \frac{1}{12}(f(0,0) + f(1,0) + f(0,1))$$

and

$$L_2 f = \frac{1}{3}\left(f\left(\frac{1}{2}, 0\right) + f\left(0, \frac{1}{2}\right) + f\left(\frac{1}{2}, \frac{1}{2}\right)\right)$$

satisfies $Lf = 0$ for $f \in \Pi_2^2$. Consequently, since $|U_3(x_1, x_2)| = 72^{-1}$ on T^2, it follows from the part 2 of Theorem 3.1 that $E_2(X_1, T^2) \geq 72^{-1}$. To complete the proof we only need to show that $\|U_2\|_{C(T^2)} = 1$, which is an easy exercise.

According to the above proposition and Theorem 3.3, to find the extremal polynomial for $x_1 x_2 x_3$ on T^3 amounts to find a symmetric polynomial $R_3(x_1, x_2, x_3)$ such that R_3 agrees with $U(x_1, x_2)$ on the face T_0^3 of T_3. Although this requirement does not determine R_3 uniquely, it helps to guide our search. The outcome is given below.

Theorem 4.3 *Define the polynomial $R_3(x)$ by*

$$\begin{aligned} R_3(x) =& 72x_1x_2x_3 - 4(x_1 + x_2 + x_3) + 4(x_1 + x_2 + x_3)^2 \\ & - 8(x_1x_2 + x_2x_3 + x_1x_3) + 1. \end{aligned}$$

Then $72^{-1}R_3(x)$ is the least polynomial for $x_1x_2x_3$ on T^3 and

$$E_2(x_1x_2x_3; T^3) = E_2(x_1x_2(1 - x_1 - x_2); T^2) = 72^{-1}\|R_3\|_{C(T^3)} = 72^{-1}$$

Furthermore, $72^{-1}R_3(x_1^2, x_2^2, x_3^2)$ is a least polynomial for $x_1^2x_2^2x_3^2$ on B^3 and S^2, and

$$E_5(x_1^2x_2^2x_3^2; B^3) = E_5(x_1^2x_2^2x_3^2; S^2) = 72^{-1}\left\|R_3(x_1^2, x_2^2, x_3^2)\right\|_{C(B^3)} = 72^{-1}.$$

By Theorem 3.3 all we need to prove is that $R_3(x_1, x_2, x_3)$ is bounded in T^3. The polynomial R_3 is bounded on the boundary of T^3, since $R_3(x, y, 0) = R_3(x, 0, y) = R_3(0, x, y) = (1 - 2x)^2 + (1 - 2y)^2 - 1$ and $R_3(x, y, 1 - x - y) = U_3(x, y)$. Furthermore, solving critical points shows that R_3 attains its maximum and minimum on the boundary.

4.3 Extremal polynomial for $x_1 \cdots x_d$ on T^d

We now consider the case $d > 3$. By Theorem 3.2 an extremal polynomial can be taken as a symmetric polynomial in $x_1, \ldots, x_d$. It is well known that every symmetric polynomial can be written in terms of elementary symmetric polynomials $e_1, e_2, \ldots$.

Let $N \in \mathbb{N}$. The elementary symmetric polynomials of degree k in variables $x_1, x_2, \ldots, x_N$ are defined by

$$e_k(x) = \sum_{1 \le i_1 < \ldots < i_k \le N} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad 1 \le k \le N.$$

We assume that N is sufficiently large and do not write the dependence of e_k on the number of variables. Using the elementary symmetric polynomials, R_3 in Theorem 4.3 takes the form

$$R_3(x) = 72e_3(x) - 4e_1(x) + 4e_1^2(x) - 8e_2(x) + 1, \qquad x \in \mathbb{R}^3.$$

In the following we will denote the right hand side by $S_3(x)$ for $x \in \mathbb{R}^d$. Notice that for $d > 3$ such a function will be entirely different from R_3. We will use the notation $\mathbf{1}^k = (1, 1, \ldots, 1) \in \mathbb{R}^k$.

Definition 4.4 *For $x \in \mathbb{R}^N$ and $N > d$ define $S_3(x)$ by*

$$S_3(x) = 72e_3(x) - 4e_1(x) + 4e_1^2(x) - 8e_2(x) + 1$$

and $S_k(x)$ for $k > 3$ by the recursive formula

$$S_k(x) = r_k e_k(x) - S_{k-1}(x),$$

where the constant r_k is determined recursively by $S(k^{-1}\mathbf{1}^k) = 1$. Furthermore we denote $S_d(x)$ by $R_d(x)$ if $x \in \mathbb{R}^d$.

In the above definition the functions S_k are defined for any number of variables. However, the number r_k is determined using the formula for $S_k(x)$ for $x \in \mathbb{R}^k$. Notice that we cannot define R_d without extending the definition of R_{d-1} formally to d variables, which is why we introduce S_k for any number of variables. As an example, we give the explicit formula of R_4:

$$\begin{aligned} R_4(x) = {} & 896x_1x_2x_3x_4 - 72(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\ & + 8(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\ & - 4(x_1 + x_2 + x_3 + x_4)^2 + 4(x_1 + x_2 + x_3 + x_4) - 1. \end{aligned}$$

The definition also shows that

$$R_d(x) = \sum_{k=2}^{d} (-1)^{d-k} r_k e_k(x) + (-1)^{d-3}(4e_1^2(x) - 4e_1(x) + 1)$$

where $r_2 := 8$. For $d \geq 3$, the numbers r_d can be deduced from the following explicit formula ([25])

$$r_d = d \sum_{k=4}^{d} k^{d-3} \binom{d}{k} \left((-1)^k (9k^2 - 32k + 24) + k^2 \right).$$

The first few values of r_d and their prime factorization are list below:

$$\begin{aligned}
r_3 &= 72 = 2^3 \cdot 3^2, \\
r_4 &= 896 = 2^7 \cdot 7, \\
r_5 &= 14400 = 2^6 \cdot 3^2 \cdot 5^2, \\
r_6 &= 283392 = 2^8 \cdot 3^3 \cdot 41, \\
r_7 &= 6598144 = 2^9 \cdot 7^2 \cdot 263, \\
r_8 &= 177373184 = 2^{15} \cdot 5413.
\end{aligned}$$

It turns out that R_d play the role of R_3 as an extremal polynomial on T^d.

Theorem 4.5 *Let $X_1(x) = x_1 \cdots x_{d-1}(1 - x_1 - \cdots - x_{d-1})$. For $d = 3, 4, 5$,*

$$E_{d-1}(x_1 \cdots x_d; T^d) = E_{d-1}(X_1, T^{d-1}) = r_d^{-1} \|R_d\|_{T^{d-1}} = r_d^{-1},$$

and, furthermore,

$$E_{2d-1}(x_1^2 \cdots x_d^2; B^d) = E_{2d-1}(x_1^2 \cdots x_d^2; S^{d-1}) = r_d^{-1} \|R_d\|_{T^{d-1}} = r_d^{-1}.$$

In fact we made the conjecture in [25] that the theorem holds for all $d \geq 3$. Recall that we normally have two steps, one is to find a sharp lower bound for $E_n(f; \Omega)$ and the other is to find an extremal polynomial that attains the lower bound. In our case we have the lower bound:

Theorem 4.6 *For all $d > 3$*

$$E_{d-1}(x_1 \cdots x_d; T^d) = E_{d-1}(X_1, T^{d-1}) \geq r_d^{-1}, \tag{4.6}$$

and, furthermore,

$$E_{2d-1}(x_1^2 \cdots x_d^2; B^d) = E_{2d-1}(x_1^2 \cdots x_d^2; S^{d-1}) \geq r_d^{-1}. \tag{4.7}$$

The proof of this step uses the extremal signature and Theorem 3.1. The definition of R_d shows that

$$R_d(x_1, \ldots, x_{d-1}, 0) = -R_{d-1}(x_1, \ldots, x_{d-1}). \tag{4.8}$$

This equation allows us to identify many points on which $|R_d(x)| = 1$. Such points are invariant under the symmetric group S_d of the coordinates. It turns out that these points are enough for the extremal signature.

By the part 2 of Theorem 3.1, the existence of the extremal signature proves Theorem 4.6. The next step is to find an extremal polynomial that attains the lower bound in (4.6). Clearly, a candidate for such a polynomial is $r_d^{-1} R_d$. What we need is a proof that shows the candidate actually works. Such a proof comes down to showing that $|R_d(x)| \le 1$ on T^d. This is proved in [25] for $d = 3, 4, 5$ by computing the critical points of R_d (with the help of a computer algebra system).

It turns out that for $d = 3, 4, 5$, $|R_d(x)|$ is attained exactly in the set $\mathcal{S}_+$ and $\mathcal{S}_-$ defined above; in particular, $|R_d(x)|$ attains its maximum on the face $T_0^d = \{x \in T^d : x_1 + \ldots + x_d = 1\}$. By the definition of $R_d(x)$, it is easy to see that $\Delta R_d(x) = (-1)^{d-1} 8d$ where Δ denote the usual Laplacian operator. This shows that $(-1)^{d-1} R_d(x)$ is a subharmonic function, which implies that

$$(-1)^{d-1} R_d(x) \le \max_{x \in \partial T^d} (-1)^{d-1} R_d(x)$$

by the maximal principle (cf. [9]). However, this is only half of what we need. Because of the definition of R_d, induction argument might play a role in a potential proof. If the maximal of $|R_d(x)|$ were attained in the boundary, then the equation (4.8) shows that the maximal of R_d are attained on the face T_0^d of the boundary of T^d. Let

$$U_d(x) = R_d(x_1, \ldots, x_{d-1}, 1 - x_1 - \ldots - x_{d-1}), \qquad x \in \mathbb{R}^{d-1}.$$

We then need to prove that $|U_d(x)| \le 1$ on T^{d-1}. The computation in the case of $d = 3, 4, 5$ points out that the $|U_d(x)|$ attains its maximal on the boundary of T^{d-1}. If this were true, then by (4.8), the inequality $|U_d(x)| \le 1$ would follow from the induction since $U_d(x_1, \ldots, x_{d-2}, 0) = U_{d-1}(x_1, \ldots, x_{d-1})$ and

$$U_d(x_1, \ldots, x_{d-2}, 1 - x_1 - \ldots - x_{d-2}) = U_{d-1}(x_1, \ldots, x_{d-2}).$$

The above discussion justifies the following conjecture.

Conjecture 4.7 *For $d \ge 6$, $|R_d(x)|$ attains its maximum on the boundary of T^d and $|U_d(x)|$ attains its maximal on the boundary of T^{d-1}.*

If the conjecture were proved, then the equality would hold in (4.6) with R_d as one least polynomial and equality would hold in (4.7) with $R_d \circ \phi$ as one least polynomial.

4.4 Extremal polynomial for $x_1^2x_2^2x_3^2$ on T^3

The construction in the previous section suggests that a least polynomial for $(x_1x_2x_3)^n$ should agree with the Chebyshev polynomial $T_{n-1}(2x-1)$ on the boundary of T^3. The following result gives the case of $n = 2$.

Proposition 4.8 *Let $X_1(x) = x_1x_2(1-x_1-x_2)$. Define the polynomial*

$$\begin{aligned} U_5(x_1,x_2) =& 27^2 b X_1^2(x_1,x_2) \\ &+ 27b\, X_1(x_1,x_2)\left(3a(x_1^2+x_2^2+(1-x_1-x_2)^2)-c\right) \\ &+ 2\left[-3+4(x_1^2+x_2^2+(1-x_1-x_2)^2)\right]^2 - 1 \end{aligned}$$

where $a = 28.5926243...$, $b = 21.8935834...$ and $c = 32/9 + a + b$. Then

$$E_5(X_2,T^2) = \inf_{p\in\Pi_5^2} \|X_2 - p\|_{C(T^2)} = 27^{-2}b^{-1}\|U_5\|_{C(T^2)} = 27^{-2}b^{-1}.$$

The polynomial U_5 is symmetric in $x_1, x_2, 1-x_1-x_2$ by Theorem 3.2. The form of U_5 is chosen so that it agrees with the Chebyshev polynomial $T_4(2t-1) = 1 - 8(1-2x)^2 + 8(1-2x)^4$ on the boundary of T^2; that is

$$U_5(x,0) = U_5(0,x) = U_5(x,1-x) = T_2(T_2(2x-1)) = T_4(2x-1).$$

The constant c is determined by fixing $U_5(1/3,1/3) = 1$. The values of a and b are chosen so that $|U_5(x)| \le 1$ inside T^2. This is proved by making $1 - U_5(x,y) \ge 0$ and then solving $U_5(x,x) = -1$ and $U_5'(x,x) = 0$, which yields three point in T^2, by symmetry, for which $U_5(x,y) = -1$. It turns out that these three points, together with the maximal and minimal points on the boundary of T^2, are enough for an extremal signature.

From the best approximation to $x_1^2x_2^2(1-x_1-x_2)^2$ on T^2 we should get best approximation to $x_1^2x_2^2x_3^2$ on T^3. One only has to find a least polynomial which satisfies $V_5(x)$ such that $V_5(x_1,x_2,1-x_1-x_2) = U_5(x_1,x_2)$ and $|V_5(x,y)| \le 1$ on T^3. This step is not trivial since an additional multiple of $(x_1+x_2+x_3)^k$ added to any term in $V_5(x)$ does not change the value of the polynomial on the boundary of T^3. Once V_5 is found, then $V_5 \circ \phi$ is an least polynomial for $x_1^4x_2^4x_3^4$ on B^3 and S^2 by Theorem 3.3. The result is stated below.

Theorem 4.9 *Let a and b be as above. Define polynomial V_5 by*

$$\begin{aligned} V_5 =& 27^2 b\, e_3^2 - 1 + 2e_1 - 2e_2^2 + 2\left(1 - 4e_1 + 4e_1^2\right)^2 \\ & - 27e_3\left((32/9 - 2a + b)e_1^2 + 6ae_2\right), \quad x \in \mathbb{R}^3. \end{aligned}$$

Then $27^{-2}b^{-1}V_5(x)$ is a least polynomial for $x_1^2x_2^2x_3^2$ on T^3

$$\begin{aligned} E_5(x_1^2x_2^2x_3^2; T^3) &= E_5(x_1^2x_2^2(1 - x_1 - x_2)^2; T^2) \\ &= 27^{-2}b^{-1}\|V_5\|_{C(T^3)} = 27^{-2}b^{-1}. \end{aligned}$$

Furthermore, $72^{-2}b^{-1}V_5(x_1^2, x_2^2, x_3^2)$ is a least polynomial for $x_1^4x_2^4x_3^4$ on B^3 and S^2,

$$\begin{aligned} E_5(x_1^2x_2^2x_3^2; B^3) &= E_5(x_1^2x_2^2x_3^2; S^2) \\ &= 72^{-2}b^{-1}\left\|V_5(x_1^2, x_2^2, x_3^2)\right\|_{C(S^2)} = 27^{-2}b^{-1}. \end{aligned}$$

What comes out as a surprise is that there does not seem to be a nice formula for $E_5(X_2, T^2) = 27^{-2}b^{-1}$. This suggests that the best approximation to monomials in several variables is much more complicated than that of two variables.

The above result suggests many questions that one may want to explore. Some questions are if least polynomials for other monomials, say $x_1^kx_2^lx_3^j$, on T^3 also attains their maximum and minimum on the boundary of T^3, and if a least polynomial p satisfies that $p(x_1, x_2, 1-x_1-x_2)$ agrees with the Chebyshev polynomials of appropriate degree on the boundary of T^2. One can also ask the question that to what extend a polynomial is determined by its values on the boundary of the triangle.

5 Extremal polynomials in L^2 norm

In this section we report results on the best L^2 approximation on the unit sphere S^{d-1}, the unit ball B^d and the standard simplex T^d. Here the L^2 norm can be taken with respect to the measure $d\mu = W(x)dx$ for some weight function W on the domain Ω. We shall denote the error of best approximation to f by $E_n(x^\alpha; \Omega)_{L^2(W)}$ to indicate the dependence on W. We call the polynomial R_α a least polynomial for x^α if $R_\alpha = x^\alpha - Q_\alpha$, $Q_\alpha \in \Pi^d_{|\alpha|-1}$, and

$$E_n(x^\alpha; \Omega)_{L^2(W)} = \inf_{p \in \Pi^d_{|\alpha|-1}} \|x^\alpha - p\|_{L^2(\Omega, W)} = \|R_\alpha\|_{L^2(\Omega, W)}.$$

According to the standard Hilbert space theory, the extremal polynomial R_α is orthogonal to all polynomials of degree lower than $|\alpha|$ and it is unique. Such a polynomial is called a monomial orthogonal polynomial in [27] and their explicit expression and their L^2 norms are computed explicitly. For earlier study of L^2 norm of these polynomials, we refer to [4] and the references therein. The results in [27] are derived for a family of weight functions on these domains, which includes the classical weight functions (see below). It should be pointed out that many of the monomial orthogonal polynomials are already known in [3] for the classical weight functions.

Throughout this section we use the standard multiindex notation. For $\alpha, \beta \in \mathbb{N}_0^m$ we also write $\alpha! = \alpha_1! \cdots \alpha_m!$ and $(\alpha)_\beta = (\alpha_1)_{\beta_1} \cdots (\alpha_m)_{\beta_m}$, where $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol. Furthermore, for $\alpha \in \mathbb{N}^m$ and $a, b \in \mathbb{R}$, we write $a\alpha + b = (a\alpha_1 + b, \ldots, a\alpha_m + b)$.

5.1 Best L^2 approximation on S^{d-1}

The weight function on S^{d-1} is

$$W_\kappa(x) = c_\kappa \prod_{i=1}^d |x_i|^{2\kappa_i}, \qquad \kappa_i \ge 0,$$

where c_κ is a normalization constant so that the integral of W_κ over S^{d-1} is 1. In the following we write $|\kappa| = \kappa_1 + \ldots + \kappa_d$. Let $\lfloor x \rfloor$ denote the integer part of x. For a multiindex $\alpha \in \mathbb{N}_0^d$ we use $\lfloor \alpha/2 \rfloor$ to denote $(\lfloor \alpha_1/2 \rfloor, \ldots, \lfloor \alpha_d/2 \rfloor)$.

Let F_B be the Lauricella hypergeometric series of type B, which generalizes the hypergeometric function ${}_2F_1$ to several variables (cf. [6]),

$$F_B(\alpha, \beta; c; x) = \sum_\gamma \frac{(\alpha)_\gamma (\beta)_\gamma}{(c)_{|\gamma|} \gamma!} x^\gamma, \quad \alpha, \beta \in \mathbb{N}_0^d, \quad c \in \mathbb{R}, \quad \max_{1 \le i \le d} |x_i| < 1,$$

where the summation is taken over $\gamma \in \mathbb{N}_0^d$. We will also need the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}(t)$ defined by

$$C_n^{(\lambda,\mu)}(x) = c_\mu \int_{-1}^1 C_n^\lambda(xt)(1+t)(1-t^2)^{\mu-1} dt.$$

These polynomials are orthogonal with respect to the weight function

$$w_{\lambda,\mu}(t) = |t|^{2\mu}(1-t^2)^{\lambda-1/2}, \qquad -1 \le t \le 1,$$

and they become Gegenbauer polynomials when $\mu = 0$; that is, $C_n^{(\lambda,0)}(t) = C_n^{\lambda}(t)$. We denote by $k_n^{(\lambda,\mu)}$ the leading coefficient of $C_n^{(\lambda,\mu)}(x)$. The explicit formula of the least polynomial R_α for x^α is given as follows.

Theorem 5.1 *Let $\rho = |\kappa| + (d-2)/2$, $\beta = \alpha - \lfloor \frac{\alpha+1}{2} \rfloor$ and $n = |\alpha|$. Then*

1. $R_\alpha(x) = x^\alpha F_B\left(-\beta, -\alpha+\beta-\kappa+\frac{1}{2}; -n-\rho+1; \frac{\|x\|^2}{x_1^2}, \ldots, \frac{\|x\|^2}{x_d^2}\right)$.
2. *The best approximation to x^α is given by*

$$\inf_{p\in\Pi_{n-1}^d} \int_{S^{d-1}} |x^\alpha - p(x)|^2 W_\kappa(x)d\omega = \int_{S^{d-1}} |R_\alpha(x)|^2 W_\kappa(x)d\omega$$
$$= \frac{2\rho\left(\kappa+\frac{1}{2}\right)_\alpha}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^d \frac{C_{\alpha_i}^{(\frac{1}{2},\kappa_i)}(t)}{k_{\alpha_i}^{(\frac{1}{2},\kappa_i)}} t^{|\alpha|+2\rho-1}dt.$$

In the case of $\kappa_1 = \ldots = \kappa_d = 0$, this is the classical case of L^2 approximation with respect to the Lebesgue measure on S^{d-1}. The polynomial R_α is defined using a generating function. It also satisfies several other properties, including a recurrence relation. The norm of R_α is written as an integral of one variable. This integral, however, is difficult to evaluate. It can be written as an multiple sum of hypergeometric type. For a detail study and several special cases, see [27].

5.2 Best L^2 approximation on B^d

The weight function on B^d is

$$W_\kappa^B(x) = c_\kappa^B \prod_{i=1}^d |x_i|^{2\kappa_i}(1-\|x\|)^{\kappa_{d+1}-1/2}, \qquad x \in B^d, \quad \kappa_i \ge 0,$$

where c_κ^B is a normalization constant so that the integral of W_κ^B over B^d is 1. The explicit formula of the least polynomial R_α^B for x^α is given as follows.

Theorem 5.2 *Let $\rho = |\kappa| + (d-1)/2$. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$,*

1. $R_\alpha^B(x) = x^\alpha F_B\left(-\beta, -\alpha+\beta-\kappa+\frac{1}{2}; -|\alpha|-\rho+1; \frac{1}{x_1^2}, \ldots, \frac{1}{x_d^2}\right)$.

2. The best approximation to x^α is given by

$$\inf_{p\in\Pi_{n-1}^d}\int_{B^d}|x^\alpha-p(x)|^2W_\kappa^B(x)dx=\int_{B^d}|R_\alpha^B(x)|^2W_\kappa^B(x)dx$$
$$=\frac{2\rho\left(\kappa+\frac12\right)_\alpha}{(\rho)_{|\alpha|}}\int_0^1\prod_{i=1}^d\frac{C_{\alpha_i}^{(\frac12,\kappa_i)}(t)}{k_{\alpha_i}^{(\frac12,\kappa_i)}}t^{|\alpha|+2\rho-1}dt.$$

This theorem more or less follows from a close relation between orthogonal polynomials on B^d and those on S^d (see [23, 27]). The polynomials correspond to the special case $W(x)=(1-\|x\|^2)^{\mu-1/2}$ are the classical orthogonal polynomials. Let us mention the special case of Lebesgue measure on B^d. Let $P_n(t)$ denote the Legendre polynomial of degree n. Then

$$\min_{Q\in\Pi_{n-1}^d}\frac{1}{\operatorname{vol}B^d}\int_{B^d}|x^\alpha-Q(x)|^2dx=\frac{d\,\alpha!}{2^n(d/2)_n}\int_0^1\prod_{i=1}^dP_{\alpha_i}(t)t^{n+d-1}dt,\tag{5.1}$$

where $\operatorname{vol}B^d=\pi^{d/2}/\Gamma(d/2+1)$ is the volume of B^d.

5.3 Best L^2 approximation on T^d

The weight function on T^d is

$$W_\kappa^T(x)=c_\kappa^T\prod_{i=1}^d|x_i|^{\kappa_i-1/2}(1-|x|)^{\kappa_{d+1}-1/2},\qquad x\in T^d,\quad \kappa_i\ge0$$

where c_κ^T is a normalization constant so that the integral of W_κ^T over T^d is 1. We need another generalization of the hypergeometric series ${}_2F_1$ to several variables, the Lauricella function of type A, which is defined by (cf. [6])

$$F_A(c,\alpha;\beta;x)=\sum_\gamma\frac{(c)_{|\gamma|}(\alpha)_\gamma}{(\beta)_\gamma\gamma!}x^\gamma,\qquad\alpha,\beta\in\mathbb{N}_0^{d+1},\quad c\in\mathbb{R},$$

where the summation is taken over $\gamma\in\mathbb{N}_0^{d+1}$. For the simplex T^d we work with the homogeneous coordinates $X=(x_1,\dots,x_d,1-x_1-\dots-x_d)$. For $\alpha\in\mathbb{N}_0^{d+1}$ we can derive the polynomial of best approximation for X^α. The explicit formula of the least polynomial R_α^T for X^α is given as follows.

Theorem 5.3 *Let $\rho=|\kappa|+(d-1)/2$. For $\alpha\in\mathbb{N}_0^d$ and $x\in\mathbb{R}^d$,*

1. the least polynomial for X^α in the L^2 norm is

$$R_\alpha^T(x) = X^\alpha F_B\Big(-\alpha, -\alpha-\kappa+\frac{1}{2}; -2|\alpha|-\rho+1; \frac{1}{x_1}, \ldots, \frac{1}{x_{d+1}} \Big)$$
$$= (-1)^n \frac{(\kappa+\mathbf{1})_\alpha}{(n+|\kappa|+d)_n} F_A(|\alpha|+|\kappa|+d, -\alpha; \kappa+\mathbf{1}; X).$$

2. The best approximation to X^α is given by

$$\inf_{p\in\Pi_{n-1}^d} \int_{T^d} |X^\alpha - p(x)|^2 W_\kappa^T(x)dx = \int_{T^d} |R_\alpha^T(x)|^2 W_\kappa^T(x)dx$$
$$= \frac{\rho\,\alpha!\,(\kappa+\frac{1}{2})_\alpha}{(\rho)_{2|\alpha|}} \int_0^1 \prod_{i=1}^{d+1} P_{\alpha_i}^{(0,\kappa_i-1/2)}(2r-1) r^{|\alpha|+\rho-1} dr.$$

This theorem essentially follows from the result on B^d, using either Theorem 3.3 or the close relation between orthogonal polynomials on these two domains. Putting $\alpha_{d+1} = 0$ gives the best approximation to x^α on T^d. It is interesting to note that the additional term $(1-x_1-\ldots-x_d)^k$ causes many problems for the uniform norm but none for the L^2 norm. Let us mention the special case of Lebesgue measure on T^d. Again $P_n(t)$ denotes the Legendre polynomial of degree n. Then for $\alpha \in \mathbb{N}_0^d$ and $n = |\alpha|$,

$$\min_{Q\in\Pi_{n-1}^d} \frac{1}{d!} \int_{T^d} |x^\alpha - Q(x)|^2 dx = \frac{d\,\alpha!^2}{(d)_{2n}} \int_0^1 \prod_{i=1}^{d} P_{\alpha_i}(2r-1) r^{n+d-1} dr. \quad (5.2)$$

The formulas of the L^2 norm of the least polynomials in all three cases look simple; for example, see the formulas (5.1) and (5.2). However, these formulas are not easy to evaluate exactly. On the other hand, they seem to provide a way to obtain sharp estimates for these quantities, which remains to be done.

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References

[1] N. N. Andreev and V. A. Yudin, Polynomials of least deviation from zero, and Chebyshev-type cubature formulas, *Proc. Steklov Inst. Math.* **232** (2001), no. 1, 39–51.

[2] N. N. Andreev and V. A. Yudin, Best approximation of polynomials on the sphere and on the ball, in *Recent progress in multivariate approximation (Witten-Bommerholz, 2000)*, 23–30, Internat. Ser. Numer. Math., 137, Birkhuser, Basel, 2001.

[3] P. Appell and J. K. de Fériet. *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[4] B. D. Bojanov, W. Haussmann, and G. P. Nikolov, Bivariate polynomials of least deviation from zero, *Canad. J. Math.* **53** (2001), 489–505.

[5] L. Bos, On Kergin-interpolation in the disk, *J. Approx. Theory* **37** (1983), 251–261.

[6] H. Exton, *Multiple hypergeometric functions and applications*, Halsted, New York, 1976.

[7] M. I. Ganzburg and S. A. Pichugov, Invariance of the elements of the best approximation and a theorem of Gleaser, *Ukrainian Math. Journal*, **33**, no. 5 (1981), 508–510.

[8] W. B. Gearhart, Some Chebyshev Approximations by polynomials in two variables, *J. Approx. Theory*, **8** (1973), 195-209.

[9] F. John. *Partial Differential Equations*, 4th ed. Springer-Verlag, New York, 1982.

[10] X. Z. Liang, Y. M. Ye, Kergin interpolation at the points which are zeros of the bivariate polynomial of least deviation from zero on the disk, *Northeast. Math. J.* **2** (1986), no. 4, 408–414

[11] V.I. Lebedev Quadrature on a sphere, *USSR Comp. Math. and Math. Phys.* **16** (1977), 10–24.

[12] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2ed ed. Oxford Mathematical Monographs, Clarendon Press, New York, 1995.

[13] U. Maier, On best approximation of the monomials on the unit ball of $\mathbb{R}^r$, *J. Approx. Theory*, **92** (1998), 74–81.

[14] I. P. Mysovskikh, Interpolatory cubature formulas (in Russian), "Nauka", Moscow, 1981.

[15] D. J. Newman and H. S. Shapiro, Some theorems on Cebysev approximation, *Duke Math. J.* **30** (1963), 673–682.

[16] D. J. Newman and Yuan Xu, Tchebycheff polynomials on a triangular region, *Constr. Approx.* **9** (1993), 543–546.

[17] M. Reimer, On multivariate polynomials of least deviation from zero on the unit ball, *Math. Z*, **1**53 (1977), 51-58.

[18] T. J. Rivlin and H. S. Shapiro, A unified approach to certain problems of approximation and minimization, *J. Soc. Indust. Appl. Math.* **9** (1961), 670–699.

[19] H. S. Shapiro, Some theorems on Cebysev approximation II, *J. Math. Anal. Appl.* **17** (1967), 262–268.

[20] S. L. Sobolev Cubature formulas on the sphere invariant under finite groups of rotations, *Sov. Math. Dokl.* **3** (1962), 1307–1310.

[21] A. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[22] J. P. Thiran and C. Detaille, On real and complex valued bivariate Chebyshev polynomials, *J. Approx. Theory*, **59** (1989), p 321–337.

[23] Yuan Xu, Orthogonal polynomials and cubature formulae on spheres and on balls, *SIAM J. Math. Anal.* **29** (1998), 779–793.

[24] Yuan Xu, Orthogonal polynomials and cubature formulae on spheres and on simplices, *Methods Anal. and Appl.* **5** (1998), 169–184.

[25] Yuan Xu, On polynomials of least deviation from zero in several variables, *Experimental Math.*, **13** (2004), p.103 -112.

[26] Yuan Xu, Weighted approximation of functions on the unit sphere, *Const. Approx.* **21** (2005), 1-28.

[27] Yuan Xu, Monomial orthogonal polynomials of several variables, J. Approx. Theory, to appear.
`http://www.math.uoregon.edu/~yuan` (Monomial.ps.gz).

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PART II

CONTRIBUTED PAPERS

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Simultaneous Asymptotic Approximation by Jakimovski-Leviatan-Durrmeyer Operators and Their Linear Combinations

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Abstract. The present paper deals with a Durrmeyer variant of the Jakimovski-Leviatan operators which was recently defined by Prof. Leviatan. As main result we derive the complete asymptotic expansion for the sequence of these operators. Furthermore, using divided differences we define general classes of linear combinations in order to improve the rate of convergence. The results are also proved for simultaneous approximation.

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1 Introduction

In 1969, A. Jakimovski and D. Leviatan [18] introduced a Favard-Szasz type operator, by means of Appell polynomials.

Throughout this paper let $g(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu$ be a function analytic in the disk $|z| < R$ $(R > 1)$ with $g(1) \neq 0$. In order to simplify the denotation

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let us assume that $g(1) = 1$ as a normalization. The Appell polynomials $p_\nu(x)$ $(\nu = 0, 1, \ldots)$ are defined by the equation

$$g(z)\ e^{xz} = \sum_{\nu=0}^{\infty} p_\nu(x)\, z^\nu. \tag{1.1}$$

Let E be the class of all functions f of exponential type on $[0, \infty)$, i.e., $|f(t)| \leq ce^{At}$ $(t \geq 0)$ for some finite constants c, $A > 0$.

The Jakimovski-Leviatan operators P_n $(n = 1, 2, \ldots)$ associate to each function $f \in E$

$$P_n(f; x) = e^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx)\ f\left(\frac{\nu}{n}\right) \qquad (x \geq 0). \tag{1.2}$$

Note that the operators P_n are well-defined, for all sufficiently large n, since the infinite sum in (1.2) is convergent if $n > A/\log R$.

B. Wood [24] proved that the operator P_n is positive if and only if $g_\nu/g(1) = g_\nu \geq 0$ $(\nu = 0, 1, \ldots)$. Throughout this paper we will assume that $g_\nu \geq 0$ $(\nu = 0, 1, \ldots)$.

In the special case $g(z) \equiv 1$ we get back the classical operators S_n of Szász-Mirakjan

$$S_n(f; x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!}\ f\left(\frac{\nu}{n}\right) \qquad (x \geq 0).$$

In [18] Jakimovski and Leviatan obtained several approximation properties of the operators (1.2). They proved that, for all $f \in C[0, \infty) \cap E$,

$$\lim_{n\to\infty} P_n(f; x) = f(x),$$

the convergence being uniform in each compact subset of $[0, \infty)$. Jakimovski and Leviatan [18, Theorem 5] defined also, for locally integrable functions f, a Kantorovich variant

$$K_n(f; x) = ne^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx) \int_{\frac{\nu}{n}}^{\frac{\nu+1}{n}} f(t)\ dt \qquad (x \geq 0), \tag{1.3}$$

of their operators.

Recently, Prof. Leviatan proposed a Durrmeyer variant of his operators given by

$$L_n(f; x) = e^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx) \frac{\int_0^\infty e^{-nt} p_\nu(nt)\, f(t)\ dt}{\int_0^\infty e^{-nt} p_\nu(nt)\ dt} \qquad (x \geq 0), \tag{1.4}$$

and requested to study its approximation properties. Here we will consider a slightly more general definition:

$$\begin{aligned} D_n(f;x) &\equiv D_n(f,g,h;x) \qquad (1.5)\\ &= e^{-nx}\sum_{\nu=0}^{\infty} p_\nu(nx)\,\frac{\int_0^\infty e^{-nt}\ q_\nu(nt)\ f(t)\ dt}{\int_0^\infty e^{-nt}\ q_\nu(nt)\ dt} \qquad (x>0), \end{aligned}$$

with a further function $h(z)=\sum\limits_{\nu=0}^{\infty} h_\nu z^\nu$ being analytic in the disk $|z|<R$ $(R>1)$ with $h(1)\neq 0$ and

$$h(z)\ e^{xz}=\sum_{\nu=0}^{\infty} q_\nu(x)\,z^\nu.$$

When it happens that an integral in the denominator of definition (1.5) or (1.4) vanishes, the corresponding fraction is to be interpreted as zero. Throughout the paper, we assume that $g_\nu, h_\nu \geq 0$ $(\nu=0,1,2,\ldots)$ which assures positivity of the linear operators (1.5). Furthermore, as a normalization let $g(1)=h(1)=1$.

Inserting

$$q_\nu(x)=\sum_{\mu=0}^{\nu} h_{\nu-\mu}\frac{x^\mu}{\mu!}$$

we obtain

$$\int_0^\infty e^{-t}\ q_\nu(t)\ t^r\ dt=\sum_{\mu=0}^{\nu} h_\mu\frac{(r+\nu-\mu)!}{(\nu-\mu)!}. \qquad (1.6)$$

In particular, there holds

$$\int_0^\infty e^{-nt}\ q_\nu(nt)\ dt=n^{-1}\sum_{\mu=0}^{\nu} h_\mu.$$

Therefore, the denominator in (1.5) can vanish only for a finite number of values of ν, because it is equal to $n^{-1}\sum_{\mu=0}^{\nu} h_\mu$, where $h_\mu \geq 0$ $(\mu=0,1,2,\ldots)$ and $\sum_{\mu=0}^{\infty} h_\mu = h(1)=1$. Dropping finitely many terms of the sum (1.5) does not change the limit and the asymptotic behaviour of the operators as we shall see later on. Note that $h_0>0$ ensures that all denominators are positive.

We shall refer to the operators D_n as (generalized) Jakimovski-Leviatan-Durrmeyer operators or briefly JLD operators. It is clear that (1.5) reduces to Leviatan's definition (1.4) in the case $g(z)=h(z)$.

In the special case $h(z) \equiv 1$ we get the operators

$$\begin{aligned} & D_n(f,g,1;x) \\ = \; & ne^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx) \int_0^\infty e^{-nt} \frac{(nt)^\nu}{\nu!} f(t)\, dt \qquad (x \geq 0), \end{aligned} \tag{1.7}$$

(see [12]). In the special case $g(z) = h(z) \equiv 1$ we get back the Szász-Mirakjan-Durrmeyer operators (briefly SMD operators)

$$D_n(f,1,1;x) = ne^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} \int_0^\infty e^{-nt} \frac{(nt)^\nu}{\nu!} f(t)\, dt \qquad (x \geq 0), \tag{1.8}$$

as studied by Heilmann [15]. A modified version was introduced by Mazhar and Totik [19].

The purpose of this paper is to derive the complete asymptotic expansion

$$D_n(f,g,h;x) \sim f(x) + \sum_{k=1}^{\infty} c_k(f,g,h;x)\; n^{-k} \qquad (n \to \infty), \tag{1.9}$$

provided f admits derivatives of sufficiently high order at $x \geq 0$. The coefficients $c_k(f,g,h;x)$ which are dependent on the functions g and h, but are independent of n, will be given in a very concise form. Formula (1.9) means that, for all $r = 1, 2, \ldots$, there holds

$$D_n(f,g,h;x) = f(x) + \sum_{k=1}^{r} c_k(f,g,h;x)\; n^{-k} + o(n^{-r}) \qquad (n \to \infty).$$

Moreover, we study simultaneous approximation. It turns out that, for $r = 0, 1, \ldots$, the complete asymptotic expansion for the "derivated" JLD operators can be obtained by term-by-term differentiation of the series (1.9), i.e.,

$$\left(\frac{d}{dx}\right)^r D_n(f,g,h;x) \sim f^{(r)}(x) + \sum_{k=1}^{\infty} \left(\frac{d}{dx}\right)^r c_k(f,g,h;x)\; n^{-k} \quad (n \to \infty).$$

In the special case $g(z) = h(z) \equiv 1$ we obtain the complete asymptotic expansion of the Szász-Mirakjan-Durrmeyer operators (1.9) and their derivatives.

The analogous problem for the Jakimovski-Leviatan operators P_n and their Kantorovich variant was studied by Abel and Ivan [7, 8]. We mention

that there are further results on asymptotic expansions of approximation operators such as Bernstein-Kantorovich operators [4], the operators of Balázs and Szabados [6], the Meyer-König and Zeller operators [1, 3], the operators of Butzer, Bleimann and Hahn [2, 5], and the Gamma operators [10]. See also [17, 9]. The coefficients of the asymptotic expansions for these operators are sums of terms mostly containing combinatorial numbers such as Stirling numbers. We would like to emphasize that in the case of JLD operators we are able to give a very concise representation of $c_k\left(f,g,h;x\right)$ which is free of any combinatorial numbers.

In order to improve the rate of convergence we consider approximation by linear combinations of JLD operators. Because of the restricted place, we must drop the results on left quasi-interpolants of the JLD operators.

2 Main Results

Let $q \in \mathbb{N}$. For a fixed $x \in [0,\infty)$, let $K^{[q]}(x)$ be the class of all functions $f \in E$ that admit a derivative of order q at x. The next theorem contains as our main result the complete asymptotic expansion for the JLD operators D_n and their derivatives.

Theorem 2.1 *Let $q \in \mathbb{N}$, $x > 0$, and $f \in K^{[2q]}(x)$. The Jakimovski-Leviatan-Durrmeyer operators D_n satisfy the asymptotic relation*

$$D_n\left(f,g,h;x\right) = f(x) + \sum_{k=1}^{q} c_k\left(f,g,h;x\right)\ n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n\to\infty)\,, \tag{2.1}$$

where the coefficients $c_k\left(f,g,h;x\right)$ are given by

$$c_k\left(f,g,h;x\right) = \frac{1}{k!}\left(\frac{\partial}{\partial z}\right)^k \left(z^k f^{(k)}(xz) g\left(z\right) h\left(z^{-1}\right)\right)\Bigg|_{z=1}. \tag{2.2}$$

Moreover, if $f \in K^{[2q+2r]}(x)$ with $r \in \mathbb{N}_0$, the derivatives of the JLD operators satisfy the asymptotic relation

$$\begin{aligned} &\left(\frac{d}{dx}\right)^r D_n\left(f,g,h;x\right) \qquad (2.3)\\ =\ & f^{(r)}(x) + \sum_{k=1}^{q} c_k^{[r]}\left(f,g,h;x\right)\ n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n\to\infty)\,, \end{aligned}$$

where the coefficients $c_k^{[r]}(f,g,h;x)$ *are given by*

$$c_k^{[r]}(f,g,h;x) = \left(\frac{d}{dx}\right)^r c_k(f,g,h;x). \tag{2.4}$$

Remark 2.2 *The continuity of* $f^{(2k+r)}$ *at the point* $x=1$ *implies for the coefficients in* (2.3) *the representation*

$$c_k^{[r]}(f,g,h;x) = \frac{1}{k!}\left(\frac{\partial}{\partial z}\right)^k \left(z^{k+r} f^{(k+r)}(xz) g(z) h\left(z^{-1}\right)\right)\Bigg|_{z=1}. \tag{2.5}$$

Remark 2.3 *If* $f \in K^{[\infty]}(x) = \bigcap_{q=1}^{\infty} K^{[q]}(x)$*, the JLD operators and their derivatives possess the complete asymptotic expansion*

$$\left(\frac{d}{dx}\right)^r D_n(f,g,h;x) \sim f^{(r)}(x) + \sum_{k=1}^{\infty}\left(\frac{d}{dx}\right)^r c_k(f,g,h;x)\ n^{-k} \quad (n\to\infty),$$

where the coefficients $c_k(f,g,h;x)$ *are as given in Theorem* 2.1.

For the convenience of the reader we list explicit expressions of the initial coefficients $c_k^{[r]}(f,g,h;x) = (d/dx)^r c_k(f,g,h;x)$.

$$\begin{aligned}
c_0^{[r]}(f,g,h;x) &= f^{(r)}(x)\\
c_1^{[r]}(f,g,h;x) &= xf^{(r+2)}(x) + \left(r+1+g'(1)-h'(1)\right) f^{(r+1)}(x)\\
c_2^{[r]}(f,g,h;x) &= \frac{1}{2}x^2 f^{(r+4)}(x) + (r+2+g'(1)-h'(1)) x f^{(r+3)}(x)\\
&\quad + f^{(r+2)}(x)\left(\binom{r+2}{2} + (r+2)g'(1) - (r+1)h'(1)\right.\\
&\quad \left. - g'(1)h'(1) + \frac{1}{2}(g''(1)+h''(1))\right)
\end{aligned}$$

In the special case of the Szász-Mirakjan-Durrmeyer operators D_n we get the following corollary.

Corollary 2.4 (Complete asymptotic expansion for the derivatives of the SMD operators). *Let* $q \in \mathbb{N}$, $x>0$, $r \in \mathbb{N}_0$ *and* $f \in K^{[2q+2r]}(x)$. *The derivatives of the Szász-Mirakjan-Durrmeyer operators* D_n *satisfy the asymptotic relation*

$$\left(\frac{d}{dx}\right)^r D_n(f,1,1;x) = f^{(r)}(x) + \sum_{k=1}^{q} d_k^{[r]}(f;x)\ n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n\to\infty),$$

where the coefficients $d_k^{[r]}(f;x)$ *are given by*

$$d_k^{[r]}(f;x) = \frac{x^{-r}}{k!}\left(x^{k+r} f^{(k+r)}(x)\right)^{(k)}.$$

In the special case $q = 1$ Theorem 2.1 reveals the following Voronovskaja-type result.

Corollary 2.5 (Voronovskaja-theorem for the JLD operators). *For* $x > 0$, $r \in \mathbb{N}_0$ *and* $f \in K^{[2r+2]}(x)$, *the Jakimovski-Leviatan-Durrmeyer operators* D_n *and their derivatives satisfy the asymptotic relation*

$$\begin{aligned} &\lim_{n\to\infty} n\left(\left(\frac{d}{dx}\right)^r D_n(f,g,h;x) - f^{(r)}(x)\right) \\ = \;& x f^{(r+2)}(x) + \left(r+1+g'(1)-h'(1)\right) f^{(r+1)}(x). \end{aligned}$$

The specialization of Corollary 2.5 to the Szász-Mirakjan-Durrmeyer operators

$$\lim_{n\to\infty} n\left(\left(\frac{d}{dx}\right)^r D_n(f,1,1;x) - f^{(r)}(x)\right) = x f^{(r+2)}(x) + (r+1) f^{(r+1)}(x)$$

is due to Heilmann [16, Theorem 3.2 with r=1].

3 Auxiliary results

As a first step we calculate the moments of the JLD operators. For $r = 0$, $1, \ldots$, put $e_r(x) = x^r$. For each real x, we define the function ψ_x by $\psi_x(t) = t - x$. For technical reasons, we consider in the following the auxiliary operators $D_{n,m}$ $(m = 0, 1, 2, \ldots)$ given by

$$D_{n,m}(f;x) = e^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx) \frac{\int_0^\infty e^{-nt}\, q_{\nu+m}(nt)\; f(t)\; dt}{\int_0^\infty e^{-nt}\, q_{\nu+m}(nt)\; dt}, \tag{3.1}$$

where the quotient is to be interpreted as zero when the integral in the denominator vanishes. Obviously, the JLD operators come out as the special case $D_n = D_{n,0}$.

Firstly, we give an approximative expression for the operators $D_{n,m}$ when acting on the space of algebraic polynomials.

Lemma 3.1 *Let $x > 0$. For all $r = 0, 1, \ldots$, there exists a constant $c > 0$ such that,*

$$D_{n,m}(e_r; x) = n^{-r} e^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx) \left(\frac{\partial}{\partial y}\right)^r \left[y^{r+\nu+m} h\left(y^{-1}\right)\right]\Big|_{y=1} + \mathrm{O}\left(e^{-cnx}\right)$$

as $n \to \infty$. More precisely, for the constant c we can choose each value in $\left(0, 1 - R^{-1}\right)$.

Lemma 3.2 *Let $x > 0$ and $c \in \left(0, 1 - R^{-1}\right)$. For all $r = 0, 1, \ldots$, the moments of the operators $D_{n,m}$ satisfy, as $n \to \infty$,*

$$D_{n,m}(e_r; x) = \sum_{k=0}^{r} \binom{r}{k} \frac{x^{r-k}}{n^k} \left(\frac{\partial}{\partial z}\right)^k \left(z^{r+m} g(z) h\left(z^{-1}\right)\right)\Big|_{z=1} + \mathrm{O}\left(e^{-cnx}\right).$$

Lemma 3.3 *Let $x > 0$ and $c \in (0,\ 1 - R^{-1})$. For all $s = 0,\ 1,\ \ldots$, the central moments of the JLD operators D_n possess the representation*

$$\begin{aligned} & D_{n,m}\left(\psi_x^s; x\right) \\ = \ & s! \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} \frac{x^{s-k}}{k! n^k} \binom{k}{s-k} \left(\frac{\partial}{\partial z}\right)^{2k-s} \left(z^{k+m} g(z) h\left(z^{-1}\right)\right)\Big|_{z=1} \\ & + \mathrm{O}\left(e^{-cnx}\right). \end{aligned}$$

In particular, we have $D_n\left(\psi_x^s; x\right) = \mathcal{O}\left(n^{-\lfloor (s+1)/2 \rfloor}\right)$ as $n \to \infty$.

In order to show Theorem 2.1 we use a general approximation theorem for positive linear operators due to Sikkema [22, Theorems 1 and 2]. Since it does not apply to functions of exponential growth we shall need the following lemma for the proof of Theorem 2.1.

Lemma 3.4 (Localization Theorem) *Let $x > 0$ be given. Assume that $f \in E$ vanishes in a neighborhood of x. Then for each positive constant q there holds $D_{n,m}(f; x) = \mathcal{O}(n^{-q})$ as $n \to \infty$.*

Finally, for the study of simultaneous approximation we use the following formula for the derivatives of the auxiliary operators $D_{n,m}$.

Lemma 3.5 *For $m = 0, 1, 2, \ldots$, the operators* (3.1) *possess the representation*

$$D_{n,m}^{(r)} = n^r \Delta^r D_{n,m} \qquad (r = 0, 1, \ldots),$$

where Δ *denotes the forward difference operator of step* 1 *with respect to* m*. In particular for the Jakimovski-Leviatan-Durrmeyer operators* D_n *we have*

$$\left(\frac{d}{dx}\right)^r D_n(f;x) = n^r \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} D_{n,m}(f;x). \tag{3.2}$$

4 Proofs

Proof of Lemma 3.1. Eq. (1.4) can be written in the more concise form

$$\begin{aligned} &\int_0^\infty e^{-t} q_\nu(t)\ t^r\ dt \\ = &\left(\frac{\partial}{\partial y}\right)^r \sum_{\mu=0}^{\nu} h_\mu y^{r+\nu-\mu}\Bigg|_{y=1} = \left(\frac{\partial}{\partial y}\right)^r \left[y^{r+\nu} T_\nu\left(h; y^{-1}, 0\right)\right]\Big|_{y=1} \end{aligned}$$

with the Taylor polynomial $T_\nu(h;z,z_0)$ of h around $z_0 = 0$.

Obviously, we have $|h_\mu| \le K\rho^{-\mu}$ $(\mu = 0,1,\ldots)$ with $1 < \rho < R$. As a consequence, there holds

$$\left|\sum_{\mu=\nu+1}^{\infty} h_\mu \mu^{\underline{j}}\right| \le K(h,\rho,j)\,\rho^{-\nu} \qquad (\nu, j = 0,1,\ldots)$$

with a constant $K(h,\rho,j)$ independent of ν, where $z^{\underline{n}}$ $(z \in \mathbb{R})$ denotes the falling factorial defined by $z^{\underline{0}} = 1$ and $z^{\underline{n}} = z(z-1)\ldots(z-n+1)$ $(n \in \mathbb{N})$. Applying the Leibniz rule twice we conclude, that

$$\begin{aligned} \int_0^\infty e^{-t} q_\nu(t)\ t^r\ dt &= \sum_{j=0}^{r} \binom{r}{j} (r+\nu)^{\underline{r-j}} \sum_{\mu=0}^{\nu} h_\mu (-\mu)^{\underline{j}} \\ &= \sum_{j=0}^{r} \binom{r}{j} (r+\nu)^{\underline{r-j}} \sum_{\mu=0}^{\infty} h_\mu (-\mu)^{\underline{j}} + \mathcal{O}\left(\rho^{-\nu}\right) \\ &= \left(\frac{\partial}{\partial y}\right)^r \left[y^{r+\nu} h\left(y^{-1}\right)\right]\Big|_{y=1} + \mathcal{O}\left(\rho^{-\nu}\right). \end{aligned}$$

Finally, we arrive at

$$\begin{aligned}\frac{\int_0^\infty e^{-nt}\ q_\nu(nt)\ t^r\ dt}{\int_0^\infty e^{-nt}\ q_\nu(nt)\ dt} &= n^{-r}\frac{\sum_{\mu=0}^{\nu} h_\mu \cdot (r+\nu-\mu)^{\underline{r}}}{\sum_{\mu=0}^{\nu} h_\mu} \\ &= n^{-r}\frac{\left(\frac{\partial}{\partial y}\right)^r \left[y^{r+\nu}h\left(y^{-1}\right)\right]\Big|_{y=1} + \mathcal{O}\left(\rho^{-\nu}\right)}{1+\mathcal{O}\left(\rho^{-\nu}\right)} \\ &= n^{-r}\left(\frac{\partial}{\partial y}\right)^r \left[y^{r+\nu}h\left(y^{-1}\right)\right]\Big|_{y=1} + \mathcal{O}\left(\rho^{-\nu}\right)\end{aligned}$$

as $\nu \to \infty$, where the first equality is senseful for all sufficiently large values of ν. Put $\nu_0 = \min\{\nu \in \mathbb{N}_0 \mid h_{\nu+m} \neq 0\}$. The condition $h(1) = 1$ and the non-negativity of the coefficients h_ν ensure that ν_0 is a finite integer. Inserting the above formula into (3.1) proves Lemma 3.1, since, for fixed $x > 0$, we have

$$e^{-nx}\sum_{\nu=0}^{\nu_0-1} p_\nu(nx)\, n^{-r}\left(\frac{\partial}{\partial y}\right)^r \left[y^{r+\nu}h\left(y^{-1}\right)\right]\Big|_{y=1} = \mathcal{O}\left(n^{\nu_0-1-r}e^{-nx}\right)$$

and

$$e^{-nx}\sum_{\nu=0}^{\infty} p_\nu(nx)\,\rho^{-\nu} = g\left(\rho^{-1}\right)\ e^{-nx\left(1-\rho^{-1}\right)} \tag{4.1}$$

as $n \to \infty$. ■

Proof of Lemma 3.2. Use of the Cauchy integral formula yields, by Eq. (1.1),

$$p_\nu(x) = \frac{1}{2\pi i}\oint\limits_{|z|=\rho} \frac{g(z)\,e^{xz}}{z^{\nu+1}}\,dz$$

with $1 < \rho < R$. By Lemma 3.1, we obtain

$$\begin{aligned}&D_n(e_r;x) \\ &= \frac{n^{-r}e^{-nx}}{2\pi i}\oint\limits_{|z|=\rho} g(z)\,e^{nxz}\left(\frac{\partial}{\partial y}\right)^r \left[\sum_{\nu=0}^{\infty}\frac{y^{r+\nu+m}}{z^{\nu+1}}h\left(y^{-1}\right)\right]\Bigg|_{y=1} dz + \mathcal{O}\left(e^{-cnx}\right) \\ &= \frac{n^{-r}e^{-nx}}{2\pi i}\left(\frac{\partial}{\partial y}\right)^r y^{r+m}h\left(y^{-1}\right)\oint\limits_{|z|=\rho}\frac{g(z)\,e^{nxz}}{z-y}\,dz\Bigg|_{y=1} + \mathcal{O}\left(e^{-cnx}\right)\end{aligned}$$

as $n \to \infty$. Since $(2\pi i)^{-1} \oint\limits_{|z|=\rho} \frac{g(z)e^{nxz}}{z-y}\,dz = g(y)\,e^{nxy}$, for all y with $|y-1| < \rho - 1$, an application of the Leibniz rule completes the proof of Lemma 3.2. ■

Proof of Lemma 3.3. Application of the binomial formula yields

$$D_{n,m}\left(\psi_x^s; x\right) = \sum_{r=0}^{s} \binom{s}{r} (-x)^{s-r} D_{n,m}\left(e_r; x\right)$$

and, by Lemma 3.2, we obtain

$$D_{n,m}\left(e_r; x\right) = \sum_{k=0}^{r} \binom{r}{k} \frac{x^{r-k}}{n^k} \left(\frac{\partial}{\partial z}\right)^k \left(z^{r+m} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} + \mathcal{O}\left(e^{-cnx}\right).$$

$$\begin{aligned}
& D_{n,m}\left(\psi_x^s; x\right) \\
= & \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \sum_{k=0}^{r} \binom{r}{k} \frac{x^{s-k}}{n^k} \left(\frac{\partial}{\partial z}\right)^k \left(z^{r+m} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} \\
& + \mathcal{O}\left(e^{-cnx}\right) \\
= & \sum_{k=0}^{s} \binom{s}{k} \frac{x^{s-k}}{n^k} \sum_{r=k}^{s} (-1)^{s-r} \binom{s-k}{r-k} \left(\frac{\partial}{\partial z}\right)^k \left(z^{r+m} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} \\
& + \mathcal{O}\left(e^{-cnx}\right) \\
= & \sum_{k=0}^{s} \binom{s}{k} \frac{x^{s-k}}{n^k} \left(\frac{\partial}{\partial z}\right)^k \left(z^{k+m} \left(z-1\right)^{s-k} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} + \mathcal{O}\left(e^{-cnx}\right).
\end{aligned}$$

By Leibniz rule, we obtain

$$\begin{aligned}
& \left(\frac{\partial}{\partial z}\right)^k \left(z^{k+m} \left(z-1\right)^{s-k} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} \\
= & \begin{cases} 0 & (2k < s), \\ \binom{k}{s-k} (s-k)! \left(\frac{\partial}{\partial z}\right)^{2k-s} \left(z^{k+m} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} & (2k \geq s). \end{cases}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& D_{n,m}\left(\psi_x^s; x\right) \\
= & \; s! \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} \binom{k}{s-k} \frac{x^{s-k}}{k! n^k} \left(\frac{\partial}{\partial z}\right)^{2k-s} \left(z^{k+m} g\left(z\right) h\left(z^{-1}\right)\right)\Big|_{z=1} \\
+ & \; \mathcal{O}\left(e^{-cnx}\right)
\end{aligned}$$

as $n \to \infty$, which completes the proof of Lemma 3.3. ■

PROOF OF LEMMA 3.4. We write the operators $D_{n,m}$ in the form

$$D_{n,m}(f;x) = \int_0^\infty K_{n,m}(x,t)\ f(t)\ dt \qquad (x>0),$$

with the kernel function

$$K_{n,m}(x,t) = e^{-nx} \sum_{\nu=0}^{\infty} p_\nu(nx)\, \frac{e^{-nt}\ q_{\nu+m}(nt)}{\int_0^\infty e^{-nu}\ q_{\nu+m}(nu)\ du},$$

where the quotient is to be interpreted as zero when the integral in the denominator vanishes.

Let $|f(t)| \le ce^{At}$ $(t \ge 0)$, for some constants $c, A > 0$. Assume that f vanishes in $(x-\delta, x+\delta)$, where $0 < \delta < x$. Put $s = \lfloor q+1 \rfloor$. Since $\delta^{-1}|t-x| \ge 1$ for all $t \notin (x-\delta, x+\delta)$, there holds

$$|D_{n,m}(f;x)| \le c\delta^{-2s} \left| \int_0^\infty K_{n,m}(x,t)\ e^{At} |t-x|^{2s}\ dt \right|.$$

Application of Schwarz inequality yields

$$\begin{aligned} & |D_{n,m}(f;x)| \qquad (4.2) \\ \le\ & c\delta^{-2s} \left[\int_0^\infty K_{n,m}(x,t)\ e^{2At}\ dt \right]^{1/2} \left[\int_0^\infty K_{n,m}(x,t)\ |t-x|^{4s}\ dt \right]^{1/2}. \end{aligned}$$

Since

$$\sum_{\mu=\nu+m+1}^{\infty} h_\mu \left(\frac{n-2A}{n} \right)^{\mu-\nu-m} = \mathcal{O}\left(\rho^{-\nu}\right) \qquad (\nu \to \infty),$$

we have

$$\begin{aligned} & (n-2A) \int_0^\infty e^{-(n-2A)t} q_{\nu+m}(nt)\ dt \\ =\ & \sum_{\mu=0}^{\nu+m} h_\mu \left(\frac{n-2A}{n} \right)^{\mu-\nu-m} \\ =\ & \left(\frac{n}{n-2A} \right)^{\nu+m} \left(h\left(\frac{n-2A}{n} \right) + \mathcal{O}\left(\rho^{-\nu}\right) \right) \end{aligned}$$

as $\nu \to \infty$. Therefore,

$$\begin{aligned}
&\frac{\int_0^\infty e^{-(n-2A)t} q_{\nu+m}(nt)\ dt}{\int_0^\infty e^{-nt} q_{\nu+m}(nt)\ dt} \\
= \ &\frac{(n-2A)^{-1}\left(\frac{n}{n-2A}\right)^{\nu+m}\left(h\left(\frac{n-2A}{n}\right)+\mathcal{O}\left(\rho^{-\nu}\right)\right)}{n^{-1}\left(h(1)+\mathcal{O}\left(\rho^{-\nu}\right)\right)} \\
= \ &\left(\frac{n}{n-2A}\right)^{\nu+m+1} h\left(\frac{n-2A}{n}\right)+\mathcal{O}\left(\rho^{-\nu}\right) \qquad (\nu\to\infty).
\end{aligned}$$

By Eq. (4.1), this implies

$$\begin{aligned}
&\int_0^\infty K_{n,m}(x,t)\ e^{2At}\ dt \\
= \ &\left(\frac{n}{n-2A}\right)^{m+1} e^{nx\frac{2A}{n-2A}} g\left(\frac{n}{n-2A}\right) h\left(\frac{n-2A}{n}\right)+\mathcal{O}\left(e^{-nx\left(1-\rho^{-1}\right)}\right)
\end{aligned}$$

which tends e^{2Ax} as $n \to \infty$. By Lemma 3.3 the second integral term in Eq. (4.2) is of order $\mathcal{O}(n^{-s})$. This completes the proof of Lemma 3.4. ■

Proof of Lemma 3.5. Using the fact that $p_\nu^{(r)}(x) = p_{\nu-r}(x)$ $(\nu \geq r \geq 0)$ and $p_\nu^{(r)}(x) = 0$ $(r > \nu \geq 0)$ it follows that

$$\left(\frac{d}{dx}\right)^r \left(e^{-nx} p_\nu(nx)\right) = n^r e^{-nx} \sum_{j=0}^{\nu} \binom{r}{j} (-1)^{r-j} p_{\nu-j}(nx).$$

Therefore, we have

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^r D_{n,m}(f;x) \\
= \ &n^r e^{-nx} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sum_{\nu=0}^{\infty} p_\nu(nx) \frac{\int_0^\infty e^{-nt}\ q_{\nu+j+m}(nt)\ f(t)\ dt}{\int_0^\infty e^{-nt}\ q_{\nu+j+m}(nt)\ dt} \\
= \ &n^r e^{-nx} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D_{n,m+j}(f;x),
\end{aligned}$$

which proves the assertion. ■

Proof of Theorem 2.1. First we derive the complete asymptotic expansion of the undifferentiated operators $D_{n,m}$. Let $f \in K^{[2q]}(x)$. We can

assume without loss of generality that f is bounded on $[0,\infty)$ since, by Lemma 3.4, for each $q>0$, $D_{n,m}(f;x)=D_{n,m}\left(\chi_{[x-d,x+d]}f;x\right)+\mathcal{O}\left(n^{-q}\right)$ as $n\to\infty$, where $\chi_{[x-d,x+d]}$ is the characteristic function on the interval $[x-d,x+d]$.

Hence, by Lemma 3.3, the operators $D_{n,m}$ satisfy the assumption of the general approximation theorem [22, Theorems 1 and 2] and we conclude that

$$\begin{aligned}
& D_{n,m}(f;x)\\
= & \sum_{s=0}^{2q} f^{(s)}(x) \sum_{k=\lfloor (s+1)/2\rfloor}^{s} \frac{x^{s-k}}{k!n^k}\binom{k}{s-k}\left(\frac{\partial}{\partial z}\right)^{2k-s}\left(z^{k+m}g(z)\,h\left(z^{-1}\right)\right)\Big|_{z=1}\\
& +o\left(n^{-q}\right)\\
= & \sum_{k=0}^{q}\frac{1}{k!n^k}\sum_{s=k}^{2k}\binom{k}{s-k}x^{s-k}f^{(s)}(x)\left(\frac{\partial}{\partial z}\right)^{2k-s}\left(z^{k+m}g(z)\,h\left(z^{-1}\right)\right)\Big|_{z=1}\\
& +o\left(n^{-q}\right)\\
= & \sum_{k=0}^{q}\frac{1}{k!n^k}\sum_{s=0}^{k}\binom{k}{s}\left(\left(\frac{\partial}{\partial z}\right)^{s} f^{(k)}(zx)\Big|_{z=1}\right)\\
& \times\left(\left(\frac{\partial}{\partial z}\right)^{k-s}\left(z^{k+m}g(z)\,h\left(z^{-1}\right)\right)\Big|_{z=1}\right)+o\left(n^{-q}\right)\\
= & \sum_{k=0}^{q}\frac{1}{k!n^k}\left(\frac{\partial}{\partial z}\right)^{k} f^{(k)}(zx)z^{k+m}g(z)\,h\left(z^{-1}\right)\Big|_{z=1}+o\left(n^{-q}\right)
\end{aligned}$$

as $n\to\infty$. This completes the proof of the first part of Theorem 2.1.

Now we proceed with simultaneous approximation. Applying Eq. (1.4) we obtain, by the first part of the theorem,

$$\begin{aligned}
& \left(\frac{d}{dx}\right)^{r} D_n(f;x)\\
= & n^r\sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m}\left[\sum_{k=0}^{q+r}\frac{1}{k!n^k}\left(\frac{\partial}{\partial z}\right)^{k} f^{(k)}(zx)z^{k+m}g(z)\,h\left(z^{-1}\right)\Big|_{z=1}\right.\\
& \left.+o\left(n^{-q-r}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{q+r} \frac{1}{k! n^{k-r}} \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \left(\frac{\partial}{\partial z}\right)^k f^{(k)}(zx) z^{k+m} g(z)\, h\left(z^{-1}\right)\Big|_{z=1} \\
&\quad + o(n^{-q}) \\
&= \sum_{k=r}^{q+r} \frac{1}{k! n^{k-r}} \left(\frac{\partial}{\partial z}\right)^k f^{(k)}(zx) z^k (z-1)^r\, g(z)\, h\left(z^{-1}\right)\Big|_{z=1} + o\left(n^{-q}\right)
\end{aligned}
$$

as $n \to \infty$, and application of the Leibniz rule proves Eq. (2.5).

The continuity of $f^{(2k+r)}$ at the point $x = 1$ implies Eq. (2.4). This completes the proof of Theorem 2.1. ■

5 Linear combinations of JLD operators

The aim of this section is to study linear combinations of JLD operators having a better degree of approximation than the JLD operators themselves (see [14, 10]). For given $m \in \mathbb{N}$ we fix certain integers $1 \le \beta_0 < \cdots < \beta_m$ and consider special linear combinations of the type

$$
D_n^{\langle m \rangle} := \sum_{j=0}^{m} \alpha_j \; D_{\beta_j n} \tag{5.1}
$$

where in some cases the coefficients α_j may depend on n and m. The α_j are determined by certain conditions on $D_n^{\langle m \rangle}$. Relation (5.1) can be written in terms of divided differences:

$$
D_n^{\langle m \rangle} = [n\,\beta_0, \ldots, n\,\beta_m;\; \varphi(z)\, D_z],
$$

for a certain function φ, hence

$$
\alpha_j = \frac{\varphi(n\beta_j)}{\omega'_{n,m}(n\beta_j)} \qquad (j = 0, \ldots, m)
$$

where $\omega_{n,m}(z) = (z - n\,\beta_0) \ldots (z - n\,\beta_m)$.

By Theorem 2.1, for $x > 0$, $m \in \mathbb{N}_0$, and for each $f \in K^{[2(q+m)]}(x)$, we have the asymptotic expansion

$$
\left(\frac{d}{dx}\right)^r D_n^{\langle m \rangle}(f; x) = f^{(r)}(x) + \sum_{k=1}^{q} \frac{c_k^{[r]}(f; x)}{n^k} \sum_{j=0}^{m} \alpha_j \; \beta_j^{-k} + o\left(n^{-q}\right), \tag{5.2}
$$

i.e.,

$$\begin{aligned}&\left(\frac{d}{dx}\right)^r D_n^{\langle m\rangle}(f;x)\\ =\;& f^{(r)}(x)+\sum_{k=1}^{q} c_k^{[r]}(f;x)\,[n\beta_0,\ldots,n\beta_m;\ z^{-k}\varphi(z)]+o(n^{-q})\end{aligned}$$

as $n\to\infty$, provided

$$[n\beta_0,\ldots,n\beta_m;\ \varphi(z)]=\sum_{j=0}^{m}\alpha_j=1 \text{ and } \alpha_j=\mathcal{O}(1) \tag{5.3}$$

as $n\to\infty$ $(j=0,\ldots,m)$.

In order to improve the order of convergence we are looking for constants $\alpha_j=\alpha_j(m)$ independent of n such that $D_n^{\langle m\rangle}(f;x)-f(x)=\mathcal{O}\left(n^{-m-1}\right)$ as $n\to\infty$, for all functions $f\in E$ sufficiently smooth in x. To this end we put $\varphi(z)=z^m$, i.e.,

$$D_n^{\langle m\rangle}:=[n\beta_0,\ldots,n\beta_m;\ z^m\ D_z].$$

Obviously the coefficients

$$\alpha_j(m)=\frac{(n\beta_j)^m}{\omega'_{n,m}(n\beta_j)}=\prod_{\substack{\nu=0\\ \nu\neq j}}^{m}(1-\beta_\nu/\beta_j)^{-1}\qquad(j=0,\ldots,m)$$

are independent of n and $[n\beta_0,\ldots,n\beta_m;\ z^m]=1$. By Eq. (5.3), we obtain

$$\begin{aligned}&\left(\frac{d}{dx}\right)^r D_n^{\langle m\rangle}(f;x)\\ =\;&\sum_{k=0}^{q} c_k^{[r]}(f;x)\,[n\beta_0,\ldots,n\beta_m;\ e_{m-k}]+o\left(n^{-q}\right)\\ =\;&\sum_{k=0}^{q}\frac{c_k^{[r]}(f;x)}{n^k}[\beta_0,\ldots,\beta_m;\ e_{m-k}]+o\left(n^{-q}\right)\\ =\;& f^{(r)}(x)+\sum_{k=m+1}^{q}\frac{c_k^{[r]}(f;x)}{n^k}\,[\beta_0,\ldots,\beta_m;e_{m-k}]+o\left(n^{-q}\right)\qquad(n\to\infty).\end{aligned}$$

In particular, if $q=m+1$, we have the Voronovskaja type formula

$$\lim_{n\to\infty} n^{m+1}\left(\left(\frac{d}{dx}\right)^r D_n^{\langle m\rangle}(f;x)-f^{(r)}(x)\right)=(-1)^m\left(\prod_{\nu=0}^{m}\beta_\nu^{-1}\right)c_{m+1}^{[r]}(f;x),$$

where we used the well-known formula $[\beta_0,\ldots,\beta_m;e_{-1}]=(-1)^m\prod_{\nu=0}^{m}\beta_\nu^{-1}$.

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References

[1] U. Abel, The moments for the Meyer-König and Zeller operators, *J. Approx. Theory* **82** (1995), 352–361.

[2] U. Abel, On the asymptotic approximation with operators of Bleimann, Butzer and Hahn, *Indag. Math., (N.S.),* **7** (1996), no. 1, 1–9.

[3] U. Abel, The complete asymptotic expansion for Meyer-König and Zeller operators, *J. Math. Anal. Appl.,* **208** (1997), 109–119.

[4] U. Abel, Asymptotic approximation with Kantorovich polynomials, *Approx. Theory Appl. (N.S.)* **14** (1998), no. 3, 106–116.

[5] U. Abel, On the asymptotic approximation with bivariate operators of Bleimann, Butzer and Hahn, J. Approx. Theory **97** (1999), 181–198.

[6] U. Abel and B. Della Vecchia, Asymptotic approximation by the operators of Balázs and Szabados, *Acta Sci. Math. (Szeged)* **66** (2000), 137–145.

[7] U. Abel and M. Ivan, Asymptotic expansion of the Jakimovski-Leviatan operators and their derivatives, *in: Functions, Series, Operators,* L. Leindler, F. Schipp, J. Szabados (eds.), Budapest, 2002, pp. 103–119.

[8] U. Abel and M. Ivan, The asymptotic expansion for approximation operators of Favard-Szász type, *Friedberger Hochschulschriften* **2** (2000), 1–13.

[9] U. Abel and M. Ivan, Asymptotic approximation with a sequence of positive linear operators, *J. Comp. Anal. Appl.* **3** (2001), 331-341.

[10] U. Abel and M. Ivan, Asymptotic approximation of functions and their derivatives by Müller's Gamma operators, *Result. Math.* **43**(2003), 1–12.

[11] U. Abel and M. Ivan, The complete asymptotic expansion for the left quasi-interpolants of Müller's Gamma operators, *Numer. Funct. Anal. Optim.* **24** (2003), 427–436.

[12] A. Ciupa, A class of integral Favard-Szasz type operators, *Studia Univ. Babeş-Bolyai, Mathematica* **XL** (1995), no. 1, 39–47.

[13] J. Favard, Sur les multiplicateurs d'interpolation, *J. Math. Pures Appl.,* **23** (1944), 219–247.

[14] I. Gavrea, Linear combinations of linear operators, *Conference on Analysis, Functional Equations, Approximation and Convexity in Honour of Professor Elena Popoviciu,* Cluj-Napoca (1999), 71–76.

[15] M. Heilmann, Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren, *Habilitationsschrift,* Universität Dortmund, 1991.

[16] M. Heilmann, Explicit Voronovskaja-type results for linear combinations of BDT-operators, *Approx. Theory Appl. (N.S.)* **11** (1995), no. 1, 54–61.

[17] M. Ivan and I. Raşa, A sequence of positive linear operators, *Rev. Anal. Numér. Théor. Approx.* **24** (1995), no. 1–2, 159–164.

[18] A. Jakimovski and D. Leviatan, Generalized Szász operators for the approximation in the infinite interval, *Mathematica (Cluj)* **34**(1969), 97–103.

[19] S. M. Mazhar and V. Totik, Approximation by modified Szász operators, *Acta Sci. Math. (Szeged)* **49** (1985), no. 1-4, 257–269.

[20] G. M. Mirakjan, Approximation of continuous functions with the aid of polynomials (Russian), *Dokl. Akad. Nauk SSSR* **31** (1941), 201–205.

[21] P. C. Sikkema, On some linear positive operators, *Indag. Math.* **32** (1970), 327–337.

[22] P. C. Sikkema, On the asymptotic approximation with operators of Meyer-König and Zeller, *Indag. Math.* **32** (1970), 428–440.

[23] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Research Nat. Bur. Standards* **45** (1950), 239–245.

[24] B. Wood, Generalized Szasz operators for the approximation in the complex domain, *SIAM J. Appl. Math.* **17** (1969), no. 4, 790–801.

[25] B. Wood, Graphic behaviour of positive linear operators, *SIAM J. Appl. Math.* **20** (1971), no. 3, 329–335.

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Rate of convergence of a class of Bézier type operators for functions of bounded variation

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Abstract. By using probability methods we introduce a general class of Bézier type linear operators. The aim of the present paper is to estimate the rate of pointwise convergence of this class for functions of bounded variation defined on an interval J. Two cases are analyzed: $Int(J) = (0, \infty)$ and $Int(J) = (0, 1)$. In a particular case, our operators turn into the Kantorovich-Bézier operators. Also some examples are delivered.

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1 Introduction

Recently, in [1] we introduced Λ_n, $n \in \mathbb{N}$, a general class of linear operators acting on a function real space $\mathcal{S}$, $\mathcal{S} \subset \mathbb{R}^J$, J a real interval. Using a Lipschitz-type maximal function, the Peetre functional K_2 and the Hardy-Littlewood maximal function, has been estimated approximation order in L_p-spaces for smooth functions.

The first aim of this paper is to present the Bézier variant $\Lambda_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \geq 1$, of the above operators. Section 2 contains both this construction and some direct properties of these integral operators. Our main goal is to estimate the rate of pointwise convergence for functions f at those

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points x at which the one-sided limits $f(x+)$ and $f(x-)$ exist, $x \in Int(J)$. Actually, in the last two decades it has been intensively considered functions of bounded variation. All discontinuities of a such function are only of first kind, consequently the proposed study is well raised.

We recall: the *total variation* of a function $f \in \mathcal{S}$ on $[a,b] \subset J$ is defined as the upper bound of the numbers $v(f;\Delta_n) := \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$, for any $n \in \mathbb{N}$ and all meshes $\Delta_n (a = x_0 < x_1 < \cdots < x_n = b)$. Setting $\bigvee_a^b (f) := \sup_{\Delta_n} v(f;\Delta_n) \in [0,\infty]$, whenever this quantity is finite we shall say that f is of *bounded variation* on $[a,b]$. Throughout the paper, BV stands for the class of all functions of bounded variation on every compact subinterval of $\mathbb{R}$.

In Section 3 we gather all notations which will be used for enunciation our main result. We also prove several preliminary results. Mainly these are estimates of the quantities in which we split the expression

$$|(\Lambda_{n,\alpha} f)(x) - q^{-\alpha} f(x+) - (1 - q^{-\alpha}) f(x-)|, \tag{1.1}$$

where $q > 1$ plays an important role in the structure of the initial operators Λ_n.

Section 4 is devoted to give an upper pointwise bound of the mentioned deviation under some additional conditions imposed to f. We consider here both the case J unbounded and J bounded. Considerations concerning the convergence of our sequence of operators are delivered. Also particular cases are analyzed. We reobtain the Bézier variant of some Kantorovich type operators.

At the end of this section we mention that the rate of convergence of some operators of functions with bounded variation is usually given in terms of the arithmetic means of the sequence of total variation. A pioneer work in this direction is due to R. Bojanic and M. Vuilleumier [6], they deepening a technique later often used in many papers.

In time, for $q = 2$ and $\alpha = -1$ the deviation (1.1) was intensively studied by a large number of mathematicians. Best of our knowledge, we mention some of their papers in connection with functions of bounded variation. F. Cheng [9] established the rate of convergence for Bernstein operators. His results have been extended by S. S. Guo and M. K. Khan [10] the approximation of functions of bounded variation on $\mathbb{R}$ has been achieved and several classical operators (Bernstein, Szász, Baskakov, Gamma, Weierstrass) have been discussed as examples. At the same time M. K. Khan

[15] investigated Bernstein power series operators and later, in a joint paper with R. Bojanic [7], functions with derivative of bounded variation on $\mathbb{R}$ have been considered. For the same values, $q = 2$, $\alpha = -1$, we quote [17], [14], [13], in which the authors gave estimates of (1.1) for modified Szász and Baskakov operators in Durrmeyer sense, respectively. Actually, Vijay Gupta has studied this problem in several papers for different classes of discrete and integral operators. The pointwise convergence of Meyer-König and Zeller operators for bounded functions was investigated by X.-M. Zeng and J.-N. Zhao [21].

2 The class $(\Lambda_{n,\alpha})$

Let J be a given interval of the real line. Let I_n, $n \in \mathbb{N}$, be sets of indices such that $I_n \subset I_{n+1}$ holds true. We have in mind the variants: I_n finite thus as a model can be chosen $\{0, 1, 2, \ldots, s_n\}$, or I_n is infinite thus our model can be considered $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each integer $n \geq 1$ we consider a net on J namely $(kn^{-\beta})_{k \in I_n}$, where $\beta > 0$ is a fixed real number. We start from a sequence $(L_n)_n$ of linear positive operators of discrete type having the form $(L_n f)(x) = \sum_{k \in I_n} a_{n,k}(x) f(k/n^{\beta})$, $x \in J$, where $a_{n,k} \in C(J)$, $a_{n,k} \geq 0$, for every $(n, k) \in \mathbb{N} \times I_n$ and f belongs to a vectorial subspace of $\mathbb{R}^J$ such that the operators are well defined. Setting e_j the j-th monomial, $e_j(t) = t^j$, $j \in \mathbb{N}_0$, for every $n \in \mathbb{N}$ we require the following conditions to be fulfilled

$$\begin{cases} L_n e_0 = e_0,\ L_n e_1 = e_1,\ L_n e_2 = e_2 + \varphi_n, \\ a_{n,k}(x) \leq \phi_n(x),\ k \in I_n,\ x \in Int(J), \end{cases} \tag{2.1}$$

where $\varphi_n \in C(J)$, $\phi_n \in \mathbb{R}_+^J$ are certain functions. These requirements imply that each operator L_n has the degree of exactness 1.

Next, let X be a non constant real random variable on a probability space $(\Omega, \mathcal{F}, P)$. Denoting by ψ its probability density function, we assume that $\psi \in L_2(\mathbb{R})$ and $\mathrm{supp}(\psi) \subset [-\mu, \mu] \cap J$, $\mu > 0$. A bounded compactly supported $\psi \in L_2(\mathbb{R})$ is automatically in $L_1(\mathbb{R})$. Also, one has $\psi \geq 0$ and

$$\|\psi\|_1 = \int_{\mathbb{R}} \psi(t) dt = 1. \tag{2.2}$$

We set $E(X) := e$, $Var(X) := \sigma^2$, the expectation and the variance of X respectively. Starting from X we generate the random variables $X_{n,k}$ defined by

$$X_{n,k} := \frac{1}{n^{\beta}}(X + k - e), \quad (n, k) \in \mathbb{N} \times I_n. \tag{2.3}$$

Consequently, $P_{X_{n,k}}$, the distribution function of $X_{n,k}$, satisfies $dP_{X_{n,k}} = n^{\beta}\psi(n^{\beta}\cdot -k+e)$ and one has $E(X_{n,k}) = k/n^{\beta}$, representing exactly the mesh of the L_n operator.

Letting $\mathcal{S} := \{f \in \mathbb{R}^{\mathbb{R}} : \ E(|f \circ X_{n,k}|) < \infty \text{ for every } (n,k) \in \mathbb{N} \times I_n\}$, we introduce the operators $\Lambda_n : \mathcal{S} \to C(J)$, $n \in \mathbb{N}$, as follows

$$\Lambda_n f = \sum_{k \in I_n} a_{n,k} E(f \circ X_{n,k}) = \sum_{k \in I_n} a_{n,k} \int\limits_{\Omega} f \circ X_{n,k} dP, \tag{2.4}$$

this meaning $(\Lambda_n f)(x) = n^{\beta} \sum\limits_{k \in I_n} a_{n,k}(x) \int\limits_{\mathbb{R}} f(t)\psi(n^{\beta}t - k + e)dt$, $x \in J$.

Following Altomare and Campiti monograph [3; §*5.2*] this is a positive approximation process generated by a random scheme on J. Since X is non-constant, by examining (2.3) we deduce that for any $(k_1, k_2) \in I_n \times I_n$ the variables X_{n,k_1}, X_{n,k_2} are not independent. All these variables represent scaled versions of the same variable X, they being obtained from it by contractions $(n^{-\beta},\ n \in \mathbb{N})$ and by translations $((k-e)n^{-\beta},\ k \in I_n)$. Moreover, by using (2.1) a simple computation shows us that the operators Λ_n keep the degree of exactness 1 and $\Lambda_n e_2 = e_2 + \varphi_n + \sigma^2/n^{2\beta}$.

The next step is to define the Bézier variant of Λ_n operators. Let α be a real number, $\alpha \geq 1$. We consider the operators $\Lambda_{n,\alpha}$, $n \in \mathbb{N}$, given as follows

$$\begin{aligned}(\Lambda_{n,\alpha} f)(x) &= n^{\beta} \sum_{k \in I_n} Q_{n,k}^{(\alpha)}(x) \int\limits_{\mathbb{R}} f(t)\psi(n^{\beta}t - k + e)dt \\ &= \sum_{k \in I_n} Q_{n,k}^{(\alpha)}(x) \int\limits_{\mathrm{supp}(\psi)} \psi(t) f\Big(\frac{t+k-e}{n^{\beta}}\Big)dt,\end{aligned} \tag{2.5}$$

where

$$Q_{n,k}^{(\alpha)}(x) := S_{n,k}^{\alpha}(x) - S_{n,k+1}^{\alpha}(x), \quad S_{n,k}(x) := \sum_{\substack{j \geq k \\ j \in I_n}} a_{n,j}(x), \tag{2.6}$$

for every $x \in J$ and $k \in I_n$.

Based on (2.2) and knowing that $\sum\limits_{k \in I_n} a_{n,k} = 1$, we deduce $S_{n,k} = 1$ for $k \leq inf(I_n)$. If $k > \sup(I_n)$, we agree to take $S_{n,k} = 0$.

In the last years the operators of Bézier-type have also been studied. By using probabilistic tools, X. M. Zeng and W. Chen [19] estimated (1.1) for

Durrmeyer-Bézier operators (instead of q^α appeared as valid $(\alpha+1)^{-1}$). X.-M. Zeng and V. Gupta [20], [11] approached the Bézier variants of Baskakov and Baskakov-Kantorovich operators, respectively. As regards the Bézier variant of Szász-Durrmeyer operators a fruitful investigation has been carried out by U. Abel and V. Gupta [12]. Also a general class of Durrmeyer-Bézier type operators was recently presented in [2]. A similar technique as in [2] will be applied in this study. However, the results aim at different classes of operators and arise from distinct hypotheses.

Remark 2.1 *The operator $\Lambda_{n,\alpha}$ is a linear positive one and it can be written as a singular integral of the type*

$$(\Lambda_{n,\alpha}f)(x) = \int_{\mathbb{R}} K_{n,\alpha}(x,t)f(t)dt, \quad x \in J,$$

with the kernel $K_{n,\alpha}(x,t) := n^\beta \sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x)\psi(n^\beta t - k + e)$, $(x,t) \in J \times \mathbb{R}$. Clearly, $\Lambda_{n,1}$ becomes just Λ_n defined by (2.4).

Remark 2.2 *For all $k \in I_n$, $x \in J$ and $\alpha \geq 1$, the quantities $Q_{n,k}^{(\alpha)}$, $S_{n,k}$, $K_{n,\alpha}$ verify the following direct properties useful in the sequel.*

$$S_{n,k}(x) - S_{n,k+1}(x) = a_{n,k}(x), \quad 0 \leq S_{n,k}(x) \leq 1, \tag{2.7}$$

$$\sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x) = 1, \quad \int_{\mathbb{R}} K_{n,\alpha}(x,t)dt = 1, \tag{2.8}$$

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha a_{n,k}(x). \tag{2.9}$$

Relations (2.7) and (2.8) are implied by (2.6) combined with (2.2). Relation (2.8) means that $\Lambda_{n,\alpha}$, $\alpha \geq 1$, reproduces the constants, in other words $\Lambda_{n,\alpha}e_0 = e_0$. Further on, if $a, b \in [0,1]$ and $\alpha \geq 1$ then $c_{a,b}$ between a, b exists such that $|b^\alpha - a^\alpha| = \alpha|b-a|c_{a,b}^{\alpha-1} \leq \alpha|b-a|$. Based on this inequality, (2.9) is implied by (2.6) and (2.7).

In what follows, as regards the family $(a_{n,k})_k$, we make an additional assumption considering that a real number q, $q > 1$, exists such that

$$\Big| \sum_{k>n^\beta x} a_{n,k}(x) - \frac{1}{q} \Big| \leq \widetilde{a}(n,x), \quad x \in Int(J), \tag{2.10}$$

where $\widetilde{a}(n,\cdot)$ is continuous on $Int(J)$ for each $n \in \mathbb{N}$ and $\widetilde{a}(n,x) = o(1)$ $(n \to \infty)$. Here $o(\cdot)$ represents, as usual, the Landau symbol. In order to justify that this is not an unusual request we deliver the following examples.

Let $J = [0,\infty)$, $I_n = \mathbb{N}_0$ and $\beta = 1$. We consider the Banach lattice

$$E_2 := \left\{ f \in C([0,\infty)) |\ \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}$$

endowed with the norm $\|\cdot\|_*$, $\|f\|_* := \sup_{x\geq 0}(1+x^2)^{-1}|f(x)|$.

Example 2.3 *Choosing $a_{n,k}(x) = e^{-nx}(nx)^k/k!$, L_n become Mirakjan-Favard-Szász operators, $n \in \mathbb{N}$. The domain $\mathcal{S}$ can be considered E_2. For these operators X. M. Zeng [18; Lemma 2] has proved that (2.10) is fulfilled with $q = 2$ and $\widetilde{a}(n,x) = \sqrt{1+3x}/\sqrt{nx}$.*

Example 2.4 *Choosing $a_{n,k}(x) = (1+x)^{-n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k$, L_n become Baskakov operators, $n \in \mathbb{N}$, with $\mathcal{S} = E_2$. Again (2.10) holds true with $q = 2$ and $\widetilde{a}(n,x) = (3x+C)/\sqrt{nx(1+x)}$ in concordance with [20; Lemma 5]. In the above C indicates a constant.*

As a matter of fact, following (2.1), for these examples we specify that $\varphi_n(x) = x/n$ and $\varphi_n(x) = (x+x^2)/n$, respectively.

Note that a similar relation as (2.10) was proved for other discrete operators, like as Meyer-König and Zeller, see [21; *Lemma 6*].

Remark 2.5 *We try to construct a true "bridge" between our assumption (2.10) and the approximation processes presenting both a particular approximation process for which (2.10) is not valid and some classes of approximation processes which verify (2.10) for an arbitrary number $q > 1$.*

2.5.1. We consider Lupaş operator [16] defined by

$$(S_n f)(x) = \frac{1}{n}\sum_{k=0}^{n}\left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |t-x|\right]_t f\left(\frac{k}{n}\right),$$

$x \in [0,1]$, $f \in C([0,1])$, where, for mutually distinct numbers a, b, c, we denote by $[a,b,c; f(t,x)]_t$ the fact that the divided difference is applied on the variable t.

Actually, $S_n f$ is the piecewise linear continuous function that interpolates f at the points k/n, $k = \overline{0,n}$. Among the properties of S_n

operator, we mention: it is linear positive, $S_n e_i = e_i$ for $i = 0$ and $i = 1$, $S_n e_2 = e_2 + z_n(1 - z_n)/n^2$ where $z_n(x) := nx - [nx]$, $[\cdot]$ indicating the integral part function. In harmony with (2.10), we have $a_{n,k}(x) := n^{-1}[(k-1)/n, k/n, (k+1)/n; |t-x|]_t$ and further on

$$\sum_{k>nx} a_{n,k}(x) = \frac{1}{2} \sum_{k>nx} \{|k+1-nx| + |k-1-nx| - 2|k-nx|\} = z_n(x).$$

Clearly, for any $q > 1$ the relation $|z_n(x) - q^{-1}| = o(1)$ $(n \to \infty)$ is false. Consequently, the operators S_n, $n \in \mathbb{N}$, satisfy (2.1) but (2.10) is not fulfilled.

2.5.2. Following [8] we recall the notion of *bell-shaped function*. A non negative function b belonging to $L_1(\mathbb{R})$ is named bell-shaped if a real number a exists such that b takes a global maximum in $x = a$, b is non-decreasing on $(-\infty, a)$ and non-increasing on $[a, \infty)$. Here a will be called the *center* of the bell-shaped function. The function b may have jump discontinuities. In what follows we consider that supp$b$ is contained in the interval $[-T, T]$, $T > 0$, and the center of b is $a = 0$. Letting $I^* := \int_{-T}^{T} b(t)dt > 0$, the univariate Cardaliaguet-Euvrard neural network operators are given by the formula

$$(F_n f)(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I^* n^\alpha} b(n^{1-\alpha}(x - k/n)), \quad x \in \mathbb{R},$$

where $0 < \alpha < 1$ and f is continuous and bounded on $\mathbb{R}$.

Based on [4, *Theorem 2.1*], we obtain that (F_n) is a positive approximation process on $L_1(\mathbb{R}) \cap C_B(\mathbb{R})$ with respect to $L_1(\mathbb{R})$, in the sense of Altomare and Campiti [3; *page 264*].

Letting $a_{n,k}(x) := n^{-\alpha} b(n^{1-\alpha}(x - k/n))/I^*$ and following [4; *Eq.(2.10)*] we define $S_n^1(x) := \sum_{k \geq [nx]+1} a_{n,k}(x)$. It was proved, see [4; *Lemma 2.1*] that $S_n^1(x) \to (I^*)^{-1} \int_{-T}^{0} b(t)dt$ as $n \to \infty$. Comparing this result with relation (2.10), it is enough to choose $q = I^* \Big(\int_{-T}^{0} b(t)dt \Big)^{-1}$. Consequently, for a given $q > 1$ we can define a bell-shaped function b such that the above identity holds true. Actually there are an infinity of such functions b, an easy construction being piecewise constant functions.

3 Notation and preliminary results

Since an affine substitution maps (a,b), $-\infty \leq a < b \leq \infty$ onto $(0,1)$, $(0,\infty)$ or $\mathbb{R}$, in general is enough to consider these intervals as being $Int(J)$. For the first two situations, we are going to present the rate of pointwise convergence of $\Lambda_{n,\alpha}$ operators for functions $f \in BV$ and satisfying the growth condition $|f(t)| \leq M_f(1+|t|^N)$, $t \in \mathbb{R}$, where M_f is a positive constant depending on f. We denote briefly this class by BV_N.

For any real numbers $a < b$ set $J^-(a) := (-\infty, a)$, $J(a,b) := [a,b]$ and $J^+(b) := (b,\infty)$.

Throughout the paper we will use the quantities $u_{n,x} := x - xn^{-\beta}$, $v_{n,x} := x + xn^{-\beta}$, $w_{n,x} := x + (1-x)n^{-\beta}$. Next we define the real functions g_x, $\varepsilon_{x,q}$, $\widehat{g}_{1,x}$, $\widehat{g}_{2,x}$, δ_x as follows

$$g_x(t) = \begin{cases} f(t) - f(x-), & t < x, \\ 0, & t = x, \\ f(t) - f(x+), & t > x, \end{cases} \qquad \varepsilon_{x,q}(t) = \begin{cases} -1, & t < x, \\ 0, & t = x, \\ q^\alpha - 1, & t > x, \end{cases}$$

$$\widehat{g}_{1,x}(t) = \begin{cases} g_x(t), & t \geq 0, \\ g_x(0), & t < 0, \end{cases} \qquad \widehat{g}_{2,x}(t) = \begin{cases} g_x(t), & t \leq 2x, \\ g_x(2x), & t > 2x, \end{cases}$$

$$\delta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x. \end{cases}$$

The size of the saltus of f at $x \in Int(J)$ will be denoted by $s_f(x)$, that is $s_f(x) := f(x+) - f(x-)$. For any integer $s \geq 0$ we introduce the s-order central moment of the operator $\Lambda_{n,\alpha}$, $\mu_{n,s}^{\langle\alpha\rangle} := \Lambda_{n,\alpha}\psi_{x,s}$ where $\psi_{x,s}(t) := (t-x)^s$. If $\alpha = 1$ then these moments will be simply denote by $\mu_{n,s}$. For every $(n,s) \in \mathbb{N} \times \mathbb{N}$ and $\alpha \geq 1$ one has

$$\mu_{n,2s}^{\langle\alpha\rangle} \leq \alpha \mu_{n,2s}. \tag{3.1}$$

The proof runs as follows. By using the known inequality $b^\alpha - a^\alpha \leq \alpha(b-a)$, $0 \leq a, b \leq 1$, $\alpha \geq 1$, and taking into account (2.6) we get

$$Q_{n,k}^{(\alpha)}(x) = S_{n,k}^\alpha(x) - S_{n,k+1}^\alpha(x) \leq \alpha(S_{n,k}(x) - S_{n,k+1}(x)) = \alpha Q_{n,k}^{(1)}(x).$$

Since $\Lambda_{n,\alpha}$ are linear positive operators, for every $\psi_{x,2s} \geq 0$ relation (3.1) holds true.

Some inequalities involving the kernel $K_{n,\alpha}$ will be read as follows.

Lemma 3.1 *If $x \in Int(J)$ then the following relations hold true*
(i) for each $y < x$,

$$\int\limits_{J^-(y)} K_{n,\alpha}(x,t)dt \leq \frac{\alpha\mu_{n,2}(x)}{(x-y)^2}, \tag{3.2}$$

(ii) for each $z > x$,

$$\int\limits_{J^+(z)} K_{n,\alpha}(x,t)dt \leq \frac{\alpha\mu_{n,2}(x)}{(z-x)^2}. \tag{3.3}$$

Proof. Let $x \in Int(J)$.
If $y < x$, then $1 \leq (t-x)^2(x-y)^{-2}$ for each $t \in J^-(y)$.
If $z > x$, then $1 \leq (t-x)^2(z-x)^{-2}$ for each $t \in J^+(z)$.
These inequalities combined with (3.1) lead us to the claimed result. □

Further on, we associate with the kernel $K_{n,\alpha}$ the following mapping

$$\lambda_{n,\alpha}(x,t) := \int\limits_{J^-(t)} K_{n,\alpha}(x,u)du, \quad x \in J. \tag{3.4}$$

We are now in position to enunciate some technical results involving the decomposition of the integral $\int\limits_{\mathbb{R}} g_x(t)K_{n,\alpha}(x,t)dt$.

Lemma 3.2 *If*

$$A_{n,\alpha}(x) := \int\limits_{J^-(u_{n,x})} \widehat{g}_{1,x}(t)K_{n,\alpha}(x,t)dt$$

and

$$D_{n,\alpha}(x) := \int\limits_{J^+(v_{n,x})} \widehat{g}_{2,x}(t)K_{n,\alpha}(x,t)dt,$$

then the following statements hold true

$$|A_{n,\alpha}(x)| \leq \frac{\alpha\mu_{n,2}(x)}{x^2}\Big(\bigvee_0^x(g_x) + \sum_{k=1}^{n-1}((k+1)^{2\beta}-k^{2\beta})\bigvee_{u_{k,x}}^x(g_x)\Big), \quad n \geq 2, \tag{3.5}$$

$$|D_{n,\alpha}(x)| \leq \frac{\alpha\mu_{n,2}(x)}{x^2}\Big(\bigvee_x^{2x}(g_x) + \sum_{k=1}^{n-1}((k+1)^{2\beta}-k^{2\beta})\bigvee_x^{v_{k,x}}(g_x)\Big), \quad n \geq 2. \tag{3.6}$$

Proof. Recalling (3.4), integrating by parts and using the definition of $\widehat{g}_{1,x}$ we have

$$A_{n,\alpha}(x) = \widehat{g}_{1,x}(t)\lambda_{n,\alpha}(x,t)\Big|_{t\in J^-(u_{n,x})} + \int\limits_{J(u_{n,x})} \lambda_{n,\alpha}(x,t)d_t(-\widehat{g}_{1,x}(t))$$
$$= g_x(u_{n,x}-)\lambda_{n,\alpha}(x,u_{n,x}-) + \int_0^{u_{n,x}} \lambda_{n,\alpha}(x,t)d_t(-g_x(t)). \quad (3.7)$$

We can write $|g_x(u_{n,x}-)| = |g_x(u_{n,x}-)-g_x(x)| \leq \bigvee\limits_{u_{n,x}}^{x}(g_x)$. For $t < u_{n,x} < x$, applying (3.2) we get

$$\lambda_{n,\alpha}(x,u_{n,x}-) \leq \frac{\alpha\mu_{n,2}(x)}{x^2}n^{2\beta}, \quad \lambda_{n,\alpha}(x,t) \leq \frac{\alpha\mu_{n,2}(x)}{(x-t)^2}.$$

Since the mappings $t \mapsto \bigvee\limits_{t}^{x}(g_x) \pm g_x(t)$ are decreasing for $t < x$, we have $|d_t(-g_x(t))| \leq d_t\Big(-\bigvee\limits_{t}^{x}(g_x)\Big)$. Gathering these relations, identity (3.7) implies

$$|A_{n,\alpha}(x)| \leq \alpha\mu_{n,2}(x)\Big(n^{2\beta}x^{-2}\bigvee_{u_{n,x}}^{x}(g_x) - \int_0^{u_{n,x}}(x-t)^{-2}d_t\big(\bigvee_{t}^{x}(g_x)\big)\Big)$$
$$= \alpha\mu_{n,2}(x)\Big(x^{-2}\bigvee_{0}^{x}(g_x) + 2\int_0^{u_{n,x}}(x-t)^{-3}\bigvee_{t}^{x}(g_x)dt\Big). \quad (3.8)$$

In the last integral making the change $t = x - x/y^\beta$, one gets $1 \leq y < n$ ($n \geq 2$) and it becomes

$$\frac{\beta}{x^2}\int_1^n y^{2\beta-1}\bigvee_{x-xy^{-\beta}}^{x}(g_x)dy = \frac{\beta}{x^2}\sum_{k=1}^{n-1}\int_k^{k+1}\bigvee_{x-xy^{-\beta}}^{x}(g_x)y^{2\beta-1}dy$$

$$\leq \frac{\beta}{x^2}\sum_{k=1}^{n-1}\int_k^{k+1}\bigvee_{u_{k,x}}^{x}(g_x)y^{2\beta-1}dy = \frac{1}{2x^2}\sum_{k=1}^{n-1}((k+1)^{2\beta}-k^{2\beta})\bigvee_{u_{k,x}}^{x}(g_x).$$

We took into account that $y \in [k,k+1)$ implies $[x-xy^{-\beta},x) \subset [u_{k,x},x)$. Returning to (3.8) we obtain inequality (3.5).

In a similar manner (3.6) can be proved, consequently we omit it. □

Lemma 3.3 *Let $f \in BV_N$ and $Int(J) = (0, \infty)$. If*

$$B_{n,\alpha}(x) := \int\limits_{J^-(0)} (g_x(t) - g_x(0)) K_{n,\alpha}(x,t)dt$$

and

$$E_{n,\alpha}(x) := \int\limits_{J^+(2x)} (g_x(t) - g_x(2x)) K_{n,\alpha}(x,t)dt,$$

then the following statements hold true

$$|B_{n,\alpha}(x)| \leq \alpha M_f \left(x^{-2}\mu_{n,2}(x) + \sqrt{\alpha^{-1}\mu_{n,2N}(x)} \right), \tag{3.9}$$

$$|E_{n,\alpha}(x)| \leq \alpha M_f \left((2 + (2x)^N) x^{-2}\mu_{n,2}(x) + 2^N \sqrt{\alpha^{-1}\mu_{n,2N}(x)} \right). \tag{3.10}$$

Proof. Clearly, both proofs follow the same line. We put the second one in detail. Because of $t > 2x$ and $f \in BV_N$ we obtain

$$|g_x(t) - g_x(2x)| = |f(t) - f(2x)| \leq M_f((1 + t^N) + (1 + 2^N x^N)).$$

Further on,

$$|E_{n,\alpha}(x)| \leq M_f \Big((2 + 2^N x^N) \int\limits_{J^+(2x)} K_{n,\alpha}(x,t)dt + \int\limits_{J^+(2x)} t^N K_{n,\alpha}(x,t)dt \Big).$$

For the first integral we apply (3.3) and for the second we use Schwarz inequality

$$\int\limits_{J^+(2x)} t^N K_{n,\alpha}(x,t)dt \leq 2^N \int\limits_{J^+(2x)} (t-x)^N K_{n,\alpha}(x,t)dt$$

$$\leq 2^N \Big\{ \int\limits_{J^+(2x)} (t-x)^{2N} K_{n,\alpha}(x,t)dt \Big\}^{1/2} \Big\{ \int\limits_{J^+(2x)} K_{n,\alpha}(x,t)dt \Big\}^{1/2}$$

$$\leq 2^N \sqrt{\mu^{\langle\alpha\rangle}_{n,2N}(x)},$$

because of $t > 2x$, $J^+(2x) \subset J$ and (2.8). Relation (3.1) finishes the proof. □

Lemma 3.4 *If* $C_{n,\alpha}(x) := \int\limits_{J(u_{n,x},v_{n,x})} g_x(t)K_{n,\alpha}(x,t)dt$, *then one has*

$$|C_{n,\alpha}(x)| \leq \bigvee_{u_{n,x}}^{v_{n,x}} (g_x). \tag{3.11}$$

The same relation is true if we substitute $v_{n,x}$ *by* $w_{n,x}$.

Proof. For $t \in J(u_{n,x}, v_{n,x})$ one has $|g_x(t)| = |g_x(t) - g_x(x)| \leq \bigvee_{u_{n,x}}^{v_{n,x}} (g_x)$ and knowing that $0 \leq \int\limits_{J(u_{n,x},v_{n,x})} K_{n,\alpha}(x,t)dt \leq 1$, inequality (3.11) is proved. The last assertion of our lemma is evident. □

Lemma 3.5 *Under the hypothesis (2.10) one has*

$$|(\Lambda_{n,\alpha}\varepsilon_{x,q})(x)| \leq C_{\alpha,q,\psi}(\widetilde{a}(n,x) + \phi_n(x)), \quad x \in Int(J), \tag{3.12}$$

where $C_{\alpha,q,\psi}$ *depends on* α, q *and the length of the set* $\mathrm{supp}(\psi)$.

Proof. We split I_n into three subsets. $I_{n,1} := \{k \in I_n : \ k < n^\beta x + e - t$ for every $t \in \mathrm{supp}(\psi)\}$, $I_{n,2} := \{k \in I_n : \ k > n^\beta x + e - t$ for every $t \in \mathrm{supp}(\psi)\}$ and $I_{n,3} := I_n \setminus (I_{n,1} \cup I_{n,2})$. Taking into account (2.5), (2.2) and (2.8) we obtain

$$\Lambda_{n,\alpha}\varepsilon_{x,q} = -\sum_{k\in I_{n,1}} Q_{n,k}^{(\alpha)} + (q^\alpha - 1)\sum_{k\in I_{n,2}} Q_{n,k}^{(\alpha)}$$

$$+ \sum_{k\in I_{n,3}} Q_{n,k}^{(\alpha)} \int\limits_{\mathrm{supp}(\psi)} \psi(t)\varepsilon_{x,q}\left(\frac{t+k-e}{n^\beta}\right)dt$$

$$= q^\alpha \sum_{k\in I_{n,2}} Q_{n,k}^{(\alpha)} - 1 + \sum_{k\in I_{n,3}} Q_{n,k}^{(\alpha)}\Big(1 + \int\limits_{\mathrm{supp}(\psi)} \psi(t)\varepsilon_{x,q}\left(\frac{t+k-e}{n^\beta}\right)dt\Big).$$

Letting $k_0 := \min(I_{n,2})$, it is evident that any $k > k_0$, $k \in I_n$, will belong to $I_{n,2}$. Taking also into account (2.6), we get $\sum\limits_{k\in I_{n,2}} Q_{n,k}^{(\alpha)} = S_{n,k_0}^{\alpha}$. Using

again the inequality $|a^{\alpha}-1|\leq \alpha|a-1|$, $\alpha\geq 1$, $a\in[0,1]$, and observing that $|\varepsilon_{x,q}|<q^{\alpha}$, with the help of (2.6) and (2.9) we can write

$$|\Lambda_{n,\alpha}\varepsilon_{x,q}|\leq \alpha q\Big|\sum_{\substack{j\geq k_0\\ j\in I_n}} a_{n,j}-\frac{1}{q}\Big|+\alpha(1+q^{\alpha})\sum_{k\in I_{n,3}} a_{n,k}. \tag{3.13}$$

Let $\widetilde{l}_{\psi}$ be the length of the bounded interval $\mathrm{supp}(\psi)$. The construction of $I_{n,3}$ implies $Card(I_{n,3})\leq \lceil \widetilde{l}_{\psi}\rceil$, where $\lceil s\rceil$ indicates the ceiling of the number s. If $j\geq k_0$, then $j>n^{\beta}x+e-t$ for every $t\in\mathrm{supp}(\psi)$. At the same time, for every $t\in\mathrm{supp}(\psi)$ one has $|e-t|=\Big|\int\limits_{\mathrm{supp}(\psi)}(u-t)\psi(u)du\Big|\leq \widetilde{l}_{\psi}$. All these facts combined with (2.10) and (2.1) allow us to get from (3.13) to (3.12). □

4 Main results

The focus of this section is to present the rate of pointwise convergence of $\Lambda_{n,\alpha}$ operators for functions of bounded variation. At first we discuss the case $Int(J)=(0,\infty)$.

Theorem 4.1 *Let $Int(J)=(0,\infty)$. Let a function f belong to BV_N and $\Lambda_{n,\alpha}$ be defined by (2.5) such that (2.10) is fulfilled. For every real number $\beta>0$, $x>0$ and integer $n\geq 2$, the following inequality*

$$|(\Lambda_{n,\alpha}f)(x)-q^{-\alpha}f(x+)-(1-q^{-\alpha})f(x-)|\leq \frac{\alpha\mu_{n,2}(x)}{x^2}\Delta_n(\beta,f;x)$$

$$+\bigvee_{x-x/n^{\beta}}^{x+x/n^{\beta}}(g_x)+M_{\alpha,f}\mu_{n,2N}^{1/2}(x)+C_{\alpha,q,\psi}|s_f(x)|(\widetilde{a}(n,x)+\phi_n(x))$$

holds true, where

$$\Delta_n(\beta,f;x)=\bigvee_{0}^{2x}(g_x)+M_f(3+(2x)^N)+\sum_{k=1}^{n-1}((k+1)^{2\beta}-k^{2\beta})\bigvee_{x-x/k^{\beta}}^{x+x/k^{\beta}}(g_x)$$

and $M_f, M_{\alpha,f}, C_{\alpha,q,\psi}$ are positive constants depending only on the quantities which appear as their indices.

Proof. Having in mind the notation introduced in Section 3, for each $t \in \mathbb{R}$ we can write

$$f(t) = q^{-\alpha} f(x+) + (1 - q^{-\alpha}) f(x-) + g_x(t) + q^{-\alpha} s_f(x) \varepsilon_{x,q}(t)$$

$$+\delta_x(t)(f(t) - q^{-\alpha} f(x+) - (1 - q^{-\alpha}) f(x-)).$$

In the above we apply the linear operator $\Lambda_{n,\alpha}$. Since $\Lambda_{n,\alpha}$ reproduces the constants and $\Lambda_{n,\alpha}\delta_x$ is null, we get

$$|(\Lambda_{n,\alpha} f)(x) - q^{-\alpha} f(x+) - (1 - q^{-\alpha}) f(x-)|$$

$$\leq |(\Lambda_{n,\alpha} g_x)(x)| + q^{-\alpha} |s_f(x)| |(\Lambda_{n,\alpha} \varepsilon_{x,q})(x)|. \tag{4.1}$$

Further on, examining Lemma 3.2, Lemma 3.3 and Lemma 3.4 as well, we can write $\Lambda_{n,\alpha} g_x = A_{n,\alpha} + B_{n,\alpha} + C_{n,\alpha} + D_{n,\alpha} + E_{n,\alpha}$.

Using the statements of these mentioned lemmas together with Lemma 3.5, after some arrangements, from (4.1) we arrive at the claimed result. □

In order to discuss the case $Int(J) = (0, 1)$ we need the function $\widehat{g}_{3,x}$ defined as follows

$$\widehat{g}_{3,x}(t) = \begin{cases} g_x(t), & t \leq 1, \\ g_x(1), & t > 1. \end{cases}$$

Lemma 4.2 *Let $f \in BV_N$ and $Int(J) = (0, 1)$. If*

$$F_{n,\alpha}(x) := \int\limits_{J^+(w_{n,x})} \widehat{g}_{3,x}(t) K_{n,\alpha}(x, t) dt$$

and

$$G_{n,\alpha}(x) := \int\limits_{J^+(1)} (g_x(t) - g_x(1)) K_{n,\alpha}(x, t) dt,$$

then the following statements hold true

$$|F_{n,\alpha}(x)| \leq \frac{\alpha \mu_{n,2}(x)}{(1-x)^2} \Big(\bigvee_x^1 (g_x) + \sum_{k=1}^{n-1} ((k+1)^{2\beta} - k^{2\beta}) \bigvee_x^{w_{k,x}} (g_x) \Big), \quad n \geq 2, \tag{4.2}$$

$$|G_{n,\alpha}(x)| \leq \alpha M_f \left((3 + 2^{N-1} x^N)(1-x)^{-2} \mu_{n,2}(x) + 2^{N-1} \sqrt{\alpha^{-1} \mu_{n,2N}(x)} \right). \tag{4.3}$$

Proof. At first step, integrating by parts, we have

$$F_{n,\alpha}(x) = \widehat{g}_{3,x}(t)\lambda_{n,\alpha}(x,t)\Big|_{t\in J^+(w_{n,x})} - \int\limits_{J^+(w_{n,x})} \lambda_{n,\alpha}(x,t)d_t(\widehat{g}_{3,x}(t))$$

$$= g_x(1) - g_x(w_{n,x}+)\lambda_{n,\alpha}(x,w_{n,x}+) - \int_{w_{n,x}}^{1} \lambda_{n,\alpha}(x,t)d_t(g_x(t)).$$

Since

$$g_x(1) = g_x(w_{n,x}+) + \int_{w_{n,x}}^{1} d_t(g_x(t)) \text{ and } 1-\lambda_{n,\alpha}(x,z) = \int\limits_{J^+(z)} K_{n,\alpha}(x,u)du$$

for any real z, we get

$$F_{n,\alpha}(x) = g_x(w_{n,x}+) \int\limits_{J^+(w_{n,x})} K_{n,\alpha}(x,u)du$$

$$+ \int_{w_{n,x}}^{1} \Big(\int\limits_{J^+(t)} K_{n,\alpha}(x,u)du\Big) d_t(g_x(t)).$$

Taking into account the following inequalities

$$|g_x(w_{n,x}+)| = |g_x(w_{n,x}+) - g_x(x)| \leq \bigvee_{x}^{w_{n,x}}(g_x), \quad |d_t(g_x(t))| \leq d_t\Big(\bigvee_{x}^{t}(g_x)\Big)$$

and the relation (3.3) as well, one obtains

$$|F_{n,\alpha}(x)| \leq \alpha\mu_{n,2}(x)\Big\{\frac{1}{(w_{n,x}-x)^2}\bigvee_{x}^{w_{n,x}}(g_x) + \int_{w_{n,x}}^{1}\frac{1}{(t-x)^2}d_t\Big(\bigvee_{x}^{t}(g_x)\Big)\Big\}$$

$$= \alpha\mu_{n,2}(x)\Big\{\frac{1}{(1-x)^2}\bigvee_{x}^{1}(g_x) + 2\int_{w_{n,x}}^{1}(t-x)^{-3}\bigvee_{x}^{t}(g_x)dt\Big\}.$$

In the integral, replacing $t = x+(1-x)z^{-\beta}$ and using $[x, x+(1-x)z^{-\beta}] \subset [x, w_{k,x}]$ for any $z \in [k, k+1]$, $k = \overline{1, n-1}$, $n \geq 2$, we have

$$\int_{w_{n,x}}^{1}\bigvee_{x}^{t}(g_x)\frac{dt}{(t-x)^3} \leq \frac{1}{2(1-x)^2}\sum_{k=1}^{n-1}((k+1)^{2\beta}-k^{2\beta})\bigvee_{x}^{w_{k,x}}(g_x).$$

Assembling all relations, we obtain (4.2).

The proof of the second statement follows the same line as the proof of (3.10). There is, though, a slight modification. This time, for $x < 1 < t$, we use the following inequality $t^N \leq 2^{N-1}((t-x)^N + x^N)$. □

As regards the case $Int(J) = (0,1)$, we decompose $\Lambda_{n,\alpha} g_x = A_{n,\alpha} + B_{n,\alpha} + C_{n,\alpha} + F_{n,\alpha} + G_{n,\alpha}$, $n \geq 2$, and using relations (3.5), (3.9), (3.11), (4.2), (4.3) we obtain the following result.

Theorem 4.3 *Let $Int(J) = (0,1)$. Let a function f belong to BV_N and $\Lambda_{n,\alpha}$ be defined by (2.5) such that (2.10) is fulfilled. For every real number $\beta > 0$, $x \in (0,1)$ and integer $n \geq 2$, the following inequality*

$$|(\Lambda_{n,\alpha} f)(x) - q^{-\alpha} f(x+) - (1-q^{-\alpha}) f(x-)| \leq \alpha \mu_{n,2}(x) \tau(x) \Delta_n(\beta, f; x)$$

$$+ \bigvee_{x-x/n^\beta}^{x+(1-x)/n^\beta} (g_x) + M_{\alpha,f} \mu_{n,2N}^{1/2}(x) + C_{\alpha,q,\psi} |s_f(x)| (\widetilde{a}(n,x) + \phi_n(x))$$

holds true, where

$$\Delta_n(\beta, f; x) = \bigvee_0^1 (g_x) + M_f(4 + 2^{N-1} x^N) + \sum_{k=1}^{n-1} ((k+1)^{2\beta} - k^{2\beta}) \bigvee_{x-x/k^\beta}^{x+(1-x)/k^\beta} (g_x),$$

$\tau(x) = \max\{x^{-2}, (1-x)^{-2}\}$ *and $M_f, M_{\alpha,f}, C_{\alpha,q,\psi}$ are positive constants depending only on the quantities which appear as their indices.*

Remark 4.4 *From the point of view of Approximation Theory, we are interested in those sequences $(\Lambda_{n,\alpha})_n$ which form an approximation process, see [3; page 264]. In this respect, it is natural to assume that $(\Lambda_n)_n$ enjoys this property. On the other hand, continuity of g_x at x implies that $\bigvee_{x-z_1}^{x+z_2} (g_x) \to 0$ as $z_1, z_2 \to 0^+$. By the virtue of Theorem 4.1 these facts allow us to state the following.*

If

(i) $\phi_n(x) = o(1)$ $(n \to \infty)$ and

(ii) $\mu_{n,2}(x) \sum_{k=1}^{n-1} ((k+1)^{2\beta} - k^{2\beta}) \bigvee_{x-x/k^\beta}^{x+x/k^\beta} (g_x) = o(1)$ $(n \to \infty)$

then $\lim_{n\to\infty} (\Lambda_{n,\alpha} f)(x) = \dfrac{f(x+) + (q^\alpha - 1) f(x-)}{q^\alpha}$ for every $f \in BV_N$, where $x \in (0, \infty)$.

A similar statement is true for the case $Int(J) = (0,1)$.

Remark 4.5 *If x is a continuity point of f then Theorems 4.1 and 4.3 establish an upper bound of the expression $|(\Lambda_{n,\alpha}f)(x) - f(x)|$. This time $s_f(x) = 0$ and the proved inequalities have a simpler form.*

Particular case. In what follows we consider that the random variable X is uniformly distributed in the interval $I_\theta := (0, 2\theta)$, $\theta > 0$. Since $\psi(t) = (2\theta)^{-1}$ for $t \in I_\theta$ and $\psi(t) = 0$ otherwise, we find $E(X) = \theta$. In this special case, denoting by $\Lambda^*_{n,\alpha}$ the operators defined at (2.5), we get

$$(\Lambda^*_{n,\alpha}f)(x) = \frac{n^\beta}{2\theta} \sum_{k\in I_n} Q^{(\alpha)}_{n,k}(x) \int_{I_{k,n,\theta}} f(y)dy, \quad x \in J, \tag{4.4}$$

where $I_{k,n,\theta} := [(k-\theta)n^{-\beta}, (k+\theta)n^{-\beta}]$, $k \in I_n$.

For $\alpha = 1$, $(\Lambda^*_{n,1})_n$ represents a known class of the Kantorovich type operators. It is self-evident that as discrete operators L_n, see (2.1), should be chosen Bernstein, Szasz or Baskakov operators. For a similar sequence, the rate of pointwise convergence for locally bounded functions f measurable on an interval I was studied by G. Aniol. It is fair to notice that those operators [5; *Eq. (2)*] are constructed by using more general subintervals as ours. However, the established results are not the same because of a different approach. For $\theta = 1/2$ and $\beta = 1$, the operators defined by (4.4) represent the Bézier variant of the classical generalized Kantorovich operators, in a slight modified form.

Consequently, for $\Lambda^*_{n,\alpha}$ our main results given at this section can be applied.

Final remark. I am thankful to the anonymous referee for his pertinent suggestions. Moreover, he indicated a probabilistic look of the operators $\Lambda_{n,\alpha}$ as follows.

$$(\Lambda_{n,\alpha}f)(x) = Ef\left(\frac{X + Z^{(\alpha)}_n(x) - e}{n^\beta}\right),$$

where $Z_n(x)$ is a variable independent of X verifying

$$P(Z^{(\alpha)}_n(x) = k) = P^\alpha(Z_n(x) \geq k) - P^\alpha(Z_n(x) \geq k+1)$$

and $P(Z_n(x) = k) = a_{n,k}(x)$. Certainly, manipulating probabilistic tools, similar results as ours can be given in terms of the numerical characteristics of the above random variables.

References

[1] Agratini, O., Linear operators generated by a probability density function, pp. 1-12, In: *Advances Constructive Approximation: Vanderbilt 2003*, M. Neamtu and E. B. Saff (eds.), Nashboro Press, Brentwood, TN, 2004.

[2] Agratini, O., On the rate of convergence of some integral operators for functions of bounded variation, *Studia Sci. Math. Hungarica,* vol. **42** (in print).

[3] Altomare, F., Campiti, M., *Korovkin-type Approximation Theory and its Applications*, de Gruyter Studies in Mathematics, vol. **17**, Walter de Gruyter, Berlin, 1994.

[4] Anastassiou, G., *Quantitative Approximations*, Chapman & Hall/ CRC, Boca Raton, 2001.

[5] Aniol, G., On the rate of pointwise convergence of the Kantorovich-type operators, *Fasciculi Mathematici*, **29**(1999), 5-15.

[6] Bojanic, R., Vuilleumier, M., On the rate of convergence of Fourier-Legendre series of functions of bounded variation, *J. Approx. Theory,* **31**(1981), 67-79.

[7] Bojanic, R., Khan, M. K., Rate of convergence of some operators of functions with derivatives of bounded variation, *Atti. Sem. Mat. Fis. Univ. Modena*, **29**(1991), 158-170.

[8] Cardaliaguet, P., Euvrard, G., Approximation of a function and its derivative with a neural network, *Neural Networks*, **5**(1992), 207-220.

[9] Cheng, F., On the rate of convergence of Bernstein polynomials of functions of bounded variation, *J. Approx. Theory*, **39**(1983), 259-274.

[10] Guo, S. S., Khan, M. K., On the rate of convergence of some operators of functions of bounded variation, *J. Approx. Theory*, **58**(1989), 90-101.

[11] Gupta, V., The Bézier variant of Kantorovich operators, *Computers and Mathematics with Applications,* **47**(2004), 227-232.

[12] Gupta, V., Abel, U., On the rate of convergence of Bézier variant of Szász-Durrmeyer operators, *Analysis in Theory and Applications,* **19**(2003), 1, 81-88.

[13] Gupta, V., Arya, K. V., On the rate of pointwise convergence of modified Baskakov type operators for functions of bounded variation, *Kyungpook Math. J.*, **38**(1998), 283-291.

[14] Gupta, V., Pant, R. P., Rate of convergence for the modified Szász-Mirakyan operators on functions of bounded variation, *J. Math. Anal. Appl.,* **233**(1999), 476-483.

[15] Khan, M. K., On the rate of convergence of Bernstein power series for functions of bounded variation, *J. Approx. Theory*, **57**(1989), 90-103.

[16] Lupaş, A., *Contributions to the theory of approximation by linear operators* (in Romanian), Ph. D. Thesis, Babeş-Bolyai University, Cluj-Napoca, 1975.

[17] Sahai, A., Prasad, G., On the rate of convergence for modified Szász-Mirakyan operators on functions of bounded variation, *Publications de l'Institut Mathématique*, Beograd, **53**(**67**)(1993), 73-80.

[18] Zeng, X.-M., On the rate of convergence of the generalized Szász type operators for functions of bounded variation, *J. Math. Anal. Appl.,* **226**(1998), 309-325.

[19] Zeng, X.-M., Chen, W., On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation, *J. Approx. Theory,* **102**(2000), 1-12.

[20] Zeng, X.-M., Gupta, V., Rate of convergence of Baskakov-Bézier type operators for locally bounded functions, *Computers and Mathematics with Applications,* **44**(2002), 1445-1453.

[21] Zeng, X.-M., Zhao, J.-N., Pointwise approximation by Meyer-König and Zeller operators, *Annales Polonici Mathematici*, **73**(2000), 2, 185-196.

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Fractional and Other Approximation of Csiszar's f-Divergence

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Abstract. Here are established various sharp and nearly optimal probabilistic inequalities that give best or nearly best estimates for the Csiszar's f-divergence. These involve fractional and ordinary derivatives of the directing function f. Also are given lower bounds for the Csiszar's distance. The Csiszar's discrimination is the most essential and general measure for the comparison between two probability measures.

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1 Preliminaries

Throughout this paper we use the following.

I) Let f be a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Let $(X, \mathcal{A}, \lambda)$ be a measure space, where λ is a finite or a σ-finite measure on $(X, \mathcal{A})$. And let μ_1, μ_2 be two probability measures on $(X, \mathcal{A})$ such that $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$. Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon-Nikodym derivatives of μ_1, μ_2 with respect to λ. Here we assume that

$$0 < a \leq \frac{p}{q} \leq b, \quad \text{a.e. on } X$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

and $a \leq 1 \leq b$.

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \tag{1.1}$$

was introduced by I. Csiszar in 1967, see [5], and is called *f-divergence* of the probability measures μ_1 and μ_2. By Lemma 1.1 of [5], the integral (1.1) is well-defined and $\Gamma_f(\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. In [5] the author without proof mentions that $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of λ. We would like to give a proof of that fact in our setting.

Lemma 1.1 *Call*

$$p = \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \quad q = \frac{d\mu_2}{d(\mu_1 + \mu_2)}, \quad p^* = \frac{d\mu_1}{d\lambda}, \quad q^* = \frac{d\mu_2}{d\lambda}.$$

Then

$$\int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d(\mu_1 + \mu_2) = \int_X q^*(x) f\left(\frac{p^*(x)}{q^*(x)}\right) d\lambda. \tag{1.2}$$

Proof. Since $f\colon (0, +\infty) \to \mathbb{R}$ is a convex function then is continuous there and Borel measurable, making $f\left(\frac{p(x)}{q(x)}\right)$ to be X-measurable.

In general let μ, ν be σ-finite measures on X such that $\nu \ll \mu$ and g-measurable function on X, then it is known that

$$\int_X g_\pm \, d\nu = \int_X g_\pm \left(\frac{d\nu}{d\mu}\right) d\mu,$$

respectively, where $g = g_+ - g_-$, $g_\pm \geq 0$. Finally we get

$$\int_X g \, d\nu = \int_X g \left(\frac{d\nu}{d\mu}\right) d\mu. \tag{1.3}$$

Next let $E \in \mathcal{A}$ be such that $\lambda(E) = 0$, then $\mu_1(E) = \mu_2(E) = 0$ and $(\mu_1 + \mu_2)(E) = 0$, i.e. $\mu_1 + \mu_2 \ll \lambda$. Clearly here we have that

$$\mu_1 \ll \mu_1 + \mu_2 \ll \lambda, \tag{1.4}$$

$$\mu_2 \ll \mu_1 + \mu_2 \ll \lambda. \tag{1.5}$$

Then it is known that

$$\frac{d\mu_1}{d\lambda} = \frac{d\mu_1}{d(\mu_1 + \mu_2)} \frac{d(\mu_1 + \mu_2)}{d\lambda}, \tag{1.6}$$

and

$$\frac{d\mu_2}{d\lambda} = \frac{d\mu_2}{d(\mu_1+\mu_2)} \frac{d(\mu_1+\mu_2)}{d\lambda}. \tag{1.7}$$

That is

$$p^* = p\frac{d(\mu_1+\mu_2)}{d\lambda} \tag{1.8}$$

and

$$q^* = q\frac{d(\mu_1+\mu_2)}{d\lambda}. \tag{1.9}$$

Here by assumption

$$0 < a \leq \frac{p}{q}, \ \frac{p^*}{q^*} \leq b, \quad \text{a.e. on } X,$$

i.e. $q, q^* > 0$ a.e. on X. That is $\frac{d(\mu_1+\mu_2)}{d\lambda} > 0$ a.e. on X. Therefore from the above we get

$$\frac{p^*}{q^*} = \frac{p}{q} \quad \text{a.e. on } X. \tag{1.10}$$

Consequently it holds

$$\int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d(\mu_1+\mu_2) = \int_X f\left(\frac{p^*(x)}{q^*(x)}\right) d\mu_2 = \int_X q^*(x) f\left(\frac{p^*(x)}{q^*(x)}\right) d\lambda.$$

■

The concept of f-divergence was introduced first in [4] as a generalization of Kullback's "*information for discrimination*" or *I-divergence* (*generalized entropy*) [8], [9] and of Rényi's "*information gain*" (*I-divergence of order* α) [10]. In fact the *I-divergence of order* 1 equals

$$\Gamma_{u\log_2 u}(\mu_1, \mu_2).$$

The choice $f(u) = (u-1)^2$ produces again a known measure of difference of distributions that is called χ^2-*divergence*, of course the *total variation* distance $|\mu_1 - \mu_2| = \int_X |p(x) - q(x)|\, d\lambda(x)$ equals $\Gamma_{|u-1|}(\mu_1, \mu_2)$.

Here by assuming $f(1) = 0$ we can consider $\Gamma_f(\mu_1, \mu_2)$ as a measure of the difference between the probability measures μ_1, μ_2. The f-divergence is in general asymmetric in μ_1 and μ_2. But since f is convex and strictly convex at 1 (see Lemma 2 next) so is

$$f^*(u) = uf\left(\frac{1}{u}\right) \tag{1.11}$$

and as in [5] we get

$$\Gamma_f(\mu_2, \mu_1) = \Gamma_{f^*}(\mu_1, \mu_2). \tag{1.12}$$

Lemma 1.2 *Let $f\colon (0,+\infty)\to\mathbb{R}$ be a strictly convex function at 1. Then so is $f^*(u)=uf\left(\frac{1}{u}\right)$, $u\in(0,+\infty)$.*

Proof. Let $0<u_1<1<u_2$, then $\frac{1}{u_2}<1<\frac{1}{u_1}$. Hence

$$f\left(\frac{2}{u_1+u_2}\right)=f\left(\left(\frac{u_1}{u_1+u_2}\right)\frac{1}{u_1}+\left(\frac{u_2}{u_1+u_2}\right)\frac{1}{u_2}\right)$$
$$<\left(\frac{u_1}{u_1+u_2}\right)f\left(\frac{1}{u_1}\right)+\left(\frac{u_2}{u_1+u_2}\right)f\left(\frac{1}{u_2}\right).$$

Clearly we get

$$\left(\frac{u_1+u_2}{2}\right)f\left(\frac{1}{\left(\frac{u_1+u_2}{2}\right)}\right)<\frac{u_1f\left(\frac{1}{u_1}\right)+u_2f\left(\frac{1}{u_2}\right)}{2}.$$

Thus we have

$$f^*\left(\frac{u_1+u_2}{2}\right)<\frac{f^*(u_1)+f^*(u_2)}{2},$$

proving our claim. ■

In Information Theory and Statistics many other concrete divergences are used which are special cases of the above general Csiszar f-divergence, e.g. Hellinger distance D_H, α-divergence D_α, Bhattacharyya distance D_B, Harmonic distance D_{Ha}, Jeffrey's distance D_J, triangular discrimination D_Δ, for all these see, e.g. [2], [6]. The problem of finding and estimating the *proper distance* (*or difference or discrimination*) of two probability distributions is one of the major ones in Probability Theory.

The above f-divergence measures in their various forms have been also applied to Anthropology, Genetics, Finance, Economics, Political Science, Biology, Approximation of Probability distributions, Signal Processing and Pattern Recognition. A great inspiration for this article has been the very important monograph on the topic by S. Dragomir [6].

II) In the next we follow [3]. Let $g\in C([0,1])$. Let ν be a positive number, $n:=[\nu]$ and $\alpha:=\nu-n$ $(0<\alpha<1)$. Define

$$(J_\nu g)(x):=\frac{1}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}g(t)\,dt,\quad 0\le x\le 1, \tag{1.13}$$

the *Riemannian-Liouville integral*, where Γ is the gamma function. We define the subspace $C^\nu([0,1])$ or $C^n([0,1])$ as follows:

$$C^\nu([0,1]):=\{g\in C^n([0,1])\colon J_{1-\alpha}D^ng\in C^1([0,1])\},$$

where $D := \frac{d}{dx}$. So for $g \in C^\nu([0,1])$, we define the *ν-fractional derivative* of g as

$$D^\nu g := DJ_{1-\alpha}D^n g. \tag{1.14}$$

When $\nu \geq 1$ we have the Taylor's formula

$$g(t) = g(0) + g'(0)t + g''(0)\frac{t^2}{2!} + \cdots + g^{(n-1)}(0)\frac{t^{n-1}}{(n-1)!} + (J_\nu D^\nu g)(t), \quad \text{for all } t \in [0,1]. \tag{1.15}$$

When $0 < \nu < 1$ we find

$$g(t) = (J_\nu D^\nu g)(t), \quad \text{for all } t \in [0,1]. \tag{1.16}$$

Next we transfer above notions over to arbitrary $[a,b] \subseteq \mathbb{R}$ (see [1]). Let $x, x_0 \in [a,b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a,b])$ and define

$$(J_\nu^{x_0} f)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} f(t)\, dt, \quad x_0 \leq x \leq b, \tag{1.17}$$

the *generalized Riemann-Liouville integral*. We define the subspace $C_{x_0}^\nu([a,b])$ of $C^n([a,b])$:

$$C_{x_0}^\nu([a,b]) := \{f \in C^n([a,b])\colon J_{1-\alpha}^{x_0} D^n f \in C^1([x_0,b])\}.$$

For $f \in C_{x_0}^\nu([a,b])$, we define the *generalized ν-fractional derivative of f over* $[x_0,b]$ as

$$D_{x_0}^\nu f := DJ_{1-\alpha}^{x_0} f^{(n)} \quad (f^{(n)} := D^n f). \tag{1.18}$$

Observe that

$$(J_{1-\alpha}^{x_0} f^{(n)})(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f^{(n)}(t)\, dt$$

exists for $f \in C_{x_0}^\nu([a,b])$.

We recall the following generalization of Taylor's formula (see [1], [3]).

Theorem 1.3 *Let $f \in C_{x_0}^\nu([a,b])$, $x_0 \in [a,b]$, fixed.*

(i) *If $\nu \geq 1$ then*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \cdots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + (J_\nu^{x_0} D_{x_0}^\nu f)(x), \quad \textit{for all } x \in [a,b]\colon x \geq x_0. \tag{1.19}$$

(ii) *If* $0 < \nu < 1$ *then*

$$f(x) = (J_\nu^{x_0} D_{x_0}^\nu f)(x), \quad \text{for all } x \in [a,b] \colon x \geq x_0. \tag{1.20}$$

We make

Remark 1.4 1) $(D_{x_0}^n f) = f^{(n)}$, $n \in \mathbb{N}$.

2) *Let* $f \in C_{x_0}^\nu([a,b])$, $\nu \geq 1$ *and* $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$; $n := [\nu]$. *Then by (1.19)*

$$f(x) = (J_\nu^{x_0} D_{x_0}^\nu f)(x).$$

I.e.

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} (D_{x_0}^\nu f)(t)\, dt, \tag{1.21}$$

for all $x \in [a,b]$ *with* $x \geq x_0$. *Notice that (1.21) is true, also when* $0 < \nu < 1$.

We also make

Remark 1.5 *Let* $\nu, \gamma \geq 1$ *such that* $\nu - \gamma \geq 1$, *so that* $\gamma < \nu$. *Call* $n := [\nu]$, $\alpha := \nu - n$; $m := [\gamma]$, $\rho := \gamma - m$. *Note that* $\nu - m \geq 1$ *and* $n - m \geq 1$. *Let* $f \in C_{x_0}^\nu([a,b])$ *be such that* $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$. *Hence by (1.19)*

$$f(x) = (J_\nu^{x_0} D_{x_0}^\nu f)(x), \quad \text{for all } x \in [a,b] \colon x \geq x_0.$$

Therefore by Leibnitz's formula and $\Gamma(p+1) = p\Gamma(p)$, $p > 0$, *we get that*

$$f^{(m)}(x) = (J_{\nu-m}^{x_0} D_{x_0}^\nu f)(x), \quad \text{for all } x \geq x_0.$$

It follows that $f \in C_{x_0}^\gamma([a,b])$ *and thus* $(D_{x_0}^\gamma f)(x) := (D J_{1-\rho}^{x_0} f^{(m)})(x)$ *exists for all* $x \geq x_0$.

Easily we obtain

$$\begin{aligned} (D_{x_0}^\gamma f)(x) &= D((J_{1-\rho}^{x_0} f^{(m)})(x)) \\ &= \frac{1}{\Gamma(\nu-\gamma)} \int_{x_0}^x (x-t)^{(\nu-\gamma)-1} (D_{x_0}^\nu f)(t)\, dt, \end{aligned} \tag{1.22}$$

and thus

$$(D_{x_0}^\gamma f)(x) = (J_{\nu-\gamma}^{x_0} (D_{x_0}^\nu f))(x)$$

and is continuous in x on $[x_0, b]$.

2 Main Results

Again f and the whole setting is as in Section 1 (I). Other notations as in Section 1 (II). We present the following

Theorem 2.1 *Let $a < b$, $1 \leq \nu < 2$, $f \in C_a^{\nu}([a,b])$, and $a \leq \frac{p(x)}{q(x)} \leq b$, a.e. on X. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \int_X (q(x))^{1-\nu}(p(x)-aq(x))^{\nu}\, d\lambda(x). \tag{2.1}$$

Proof. Here $n := [\nu] = 1$ and from (1.21) we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-w)^{\nu-1}(D_a^{\nu} f)(w)\, dw, \quad \text{all } a \leq x \leq b. \tag{2.2}$$

Then

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \int_a^x (x-w)^{\nu-1} |(D_a^{\nu} f)(w)|\, dw \\ &\leq \frac{\|D_a^{\nu} f\|_{\infty,[1,b]}}{\Gamma(\nu)} \int_a^x (x-w)^{\nu-1}\, dw \\ &= \frac{\|D_a^{\nu} f\|_{\infty,[1,b]}}{\Gamma(\nu+1)} (x-a)^{\nu}. \end{aligned} \tag{2.3}$$

I.e. we get

$$|f(x)| \leq \frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} (x-a)^{\nu}, \quad \text{for all } a \leq x \leq b. \tag{2.4}$$

Consequently we obtain

$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) &= \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right)\right| d\lambda \\ &\leq \left(\frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)}\right) \int_X q(x) \left(\frac{p(x)}{q(x)} - a\right)^{\nu} d\lambda \\ &= \frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \int_X (q(x))^{1-\nu}(p(x)-aq(x))^{\nu}\, d\lambda(x). \end{aligned} \tag{2.5}$$

That is proving (2.1). ∎

The counterpart of previous result follows.

Theorem 2.2 *Let $a < b$, $\nu \geq 2$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$, and $a \leq \frac{p(x)}{q(x)} \leq b$, a.e. on X. Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\|D_a^\nu f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \int_X (q(x))^{1-\nu} (p(x) - aq(x))^\nu \, d\lambda(x). \tag{2.6}$$

Proof. As in the proof of Theorem 2.1. ■

Next we give an L_α estimate.

Theorem 2.3 *Let $a < b$, $\nu \geq 1$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$, and $a \leq \frac{p(x)}{q(x)} \leq b$, a.e. on X. Let $\alpha, \beta > 1$: $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then*

$$\begin{aligned} \Gamma_f(\mu_1, \mu_2) \leq{}& \frac{\|D_a^\nu f\|_{\alpha,[a,b]}}{\Gamma(\nu)(\beta(\nu-1)+1)^{1/\beta}} \\ & \cdot \int_X (q(x))^{2-\nu-\frac{1}{\beta}} (p(x) - aq(x))^{\nu-1+\frac{1}{\beta}} \, d\lambda(x). \end{aligned} \tag{2.7}$$

Proof. From (1.21) we have

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-w)^{\nu-1} (D_a^\nu f)(w) \, dw, \quad \text{for all } a \leq x \leq b. \tag{2.8}$$

Also $(D_a^\nu f)(w) \in C([a,b])$. Hence

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \int_a^x (x-w)^{\nu-1} |(D_a^\nu f)(w)| \, dw \\ &\leq \frac{1}{\Gamma(\nu)} \left(\int_a^x (x-w)^{\beta(\nu-1)} \, dw \right)^{1/\beta} \left(\int_a^x |(D_a^\nu f)(w)|^\alpha \, dw \right)^{1/\alpha} \\ &\leq \frac{\|D_a^\nu f\|_{\alpha,[a,b]}}{\Gamma(\nu)} \left(\int_a^x (x-w)^{\beta(\nu-1)} \, dw \right)^{1/\beta} \\ &= \frac{\|D_a^\nu f\|_{\alpha,[a,b]}}{\Gamma(\nu)} \left(\frac{(x-a)^{\beta(\nu-1)+1}}{\beta(\nu-1)+1} \right)^{1/\beta}. \end{aligned} \tag{2.9}$$

I.e. we get that

$$|f(x)| \leq \frac{\|D_a^\nu f\|_{\alpha,[a,b]}}{\Gamma(\nu)(\beta(\nu-1)+1)^{1/\beta}} (x-a)^{\left(\nu-1+\frac{1}{\beta}\right)}. \tag{2.10}$$

Hence

$$\Gamma_f(\mu_1,\mu_2) \le \int_X q\left|f\left(\frac{p}{q}\right)\right| d\lambda$$
$$\le \frac{\|D_a^\nu f\|_{\alpha,[a,b]}}{\Gamma(\nu)(\beta(\nu-1)+1)^{1/\beta}} \int_X q\left(\frac{p}{q}-a\right)^{\nu-1+\frac{1}{\beta}} d\lambda \qquad (2.11)$$
$$= \frac{\|D_a^\nu f\|_{\alpha,[a,b]}}{\Gamma(\nu)(\beta(\nu-1)+1)^{1/\beta}} \int_X q^{2-\nu-\frac{1}{\beta}}(p-aq)^{\nu-1+\frac{1}{\beta}}\, d\lambda,$$

proving (2.7). ■

It follows an L_∞ estimate

Theorem 2.4 *Let $a<b$, $\nu \ge 1$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a)=0$, $i=0,1,\ldots,n-1$, and $a \le \frac{p(x)}{q(x)} \le b$, a.e. on X. Then*

$$\Gamma_f(\mu_1,\mu_2) \le \frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)} \int_X q(x)^{2-\nu}(p(x)-aq(x))^{\nu-1}\, d\lambda(x). \qquad (2.12)$$

Proof. Again from (1.21) and (2.8) we have

$$|f(x)| \le \frac{1}{\Gamma(\nu)}\int_a^x (x-w)^{\nu-1}|D_a^\nu f(w)|\, dw$$
$$\le \frac{1}{\Gamma(\nu)}(x-a)^{\nu-1}\int_a^x |D_a^\nu f(w)|\, dw$$
$$\le \frac{1}{\Gamma(\nu)}(x-a)^{\nu-1}\int_a^b |D_a^\nu f(w)|\, dw$$
$$= \frac{1}{\Gamma(\nu)}(x-a)^{\nu-1}\|D_a^\nu f\|_{1,[a,b]}. \qquad (2.13)$$

I.e.

$$|f(x)| \le \frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)}(x-a)^{\nu-1}, \qquad (2.14)$$

for all x in $[a,b]$. Therefore

$$\Gamma_f(\mu_1,\mu_2) \le \int_X q\left|f\left(\frac{p}{q}\right)\right| d\lambda \le \left(\int_X q\left(\frac{p}{q}-a\right)^{\nu-1} d\lambda\right)\frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)}$$
$$= \frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)}\left(\int_X q^{2-\nu}(p-aq)^{\nu-1}\, d\lambda\right), \qquad (2.15)$$

proving (2.12). ■

Remark 2.5 *Let $f\colon [a,b] \to \mathbb{R}$ convex,*

$$f \in C^1([a,b]), \quad x_0 \in [a,b], \quad a \neq b.$$

Then

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0), \quad \text{for all } x \in [a,b]. \tag{2.16}$$

Then easily one can prove that

$$\Delta(x_0) := \frac{1}{b-a}\int_a^b f(x)\,dx - f(x_0) \geq f'(x_0)\left[\frac{a+b}{2} - x_0\right] \tag{2.17}$$

see also S. Dragomir and C.E.M. Pearce, [7, p. 9, Theorem 18]. In this paper's setting

$$0 < a \leq \frac{p(x)}{q(x)} \leq b, \quad \text{a.e. on } X,$$

and by calling

$$M := \frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.18}$$

we get

$$M - f\left(\frac{p(x)}{q(x)}\right) \geq f'\left(\frac{p(x)}{q(x)}\right)\left[\frac{a+b}{2} - \frac{p(x)}{q(x)}\right], \quad \text{a.e. on } X. \tag{2.19}$$

Then

$$q(x)M - q(x)f\left(\frac{p(x)}{q(x)}\right) \geq f'\left(\frac{p(x)}{q(x)}\right)\left[q(x)\frac{(a+b)}{2} - p(x)\right], \quad \text{a.e. on } X. \tag{2.20}$$

Here $q(x) > 0$ a.e. on X. Therefore by integrating (2.20) against λ we obtain

$$\begin{aligned} M - \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x) \\ \geq \int_X f'\left(\frac{p(x)}{q(x)}\right)\left(\frac{q(x)(a+b)}{2} - p(x)\right) d\lambda(x) \end{aligned} \tag{2.21}$$

We have established

Theorem 2.6 *Let all elements as in Section 1 (I) hold and additionally assume that $f \in C^1([a,b])$, $a \neq b$. Then*

$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ - \int_X f'\left(\frac{p(x)}{q(x)}\right)\left(\frac{q(x)(a+b)}{2} - p(x)\right) d\lambda. \end{aligned} \tag{2.22}$$

Remark 2.7 *Let all elements as in Section 1 (I) hold and $f \in C^1([a,b])$. Then by (2.16) we get*

$$f(x) \geq f'(1)(x-1) \tag{2.23}$$

and

$$f\left(\frac{p(x)}{q(x)}\right) \geq f'(1)\left(\frac{p(x)}{q(x)} - 1\right), \quad \text{a.e. on } X. \tag{2.24}$$

Thus

$$q(x) f\left(\frac{p(x)}{q(x)}\right) \geq f'(1)(p(x) - q(x)), \quad \text{a.e. on } X. \tag{2.25}$$

Integrating (2.25) against λ over X we get

$$\Gamma_f(\mu_1, \mu_2) \geq 0 \tag{2.26}$$

which is known and mentioned in [5].

Remark 2.8 *Let $n \in \mathbb{N}$, $f \in C^{n+1}([a,b])$, $[a,b] \subset \mathbb{R}$, such that $f^{(n+1)} \geq 0$ (≤ 0), then by Taylor's formula we get that*

$$f(x) \geq (\leq) \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(x-x_0)^i}{i!}, \tag{2.27}$$

respectively, for any $x, x_0 \in [a,b]$: $x \geq x_0$. Inequalities (2.27) are valid also when n is odd and $x \leq x_0$. Take $0 < a \leq 1 \leq b$, $a \leq \frac{p}{q} \leq b$, a.e. on X and $f(1) = 0$, along with $f \in C^{n+1}([a,b])$, n odd, such that $f^{(n+1)} \geq 0$ (≤ 0), then

$$f(x) \geq (\leq) \sum_{i=1}^{n} f^{(i)}(1) \frac{(x-1)^i}{i!}, \tag{2.28}$$

for all x in $[a,b]$. Hence

$$qf\left(\frac{p}{q}\right) \geq (\leq) \sum_{i=1}^{n} f^{(i)}(1) \frac{q\left(\frac{p}{q} - 1\right)^i}{i!}. \tag{2.29}$$

And finally we obtain

$$\Gamma_f(\mu_1, \mu_2) \geq (\leq) \sum_{i=2}^{n} \frac{f^{(i)}(1)}{i!} \int_X q^{1-i}(p-q)^i \, d\lambda, \quad n \text{ odd}. \tag{2.30}$$

Remark 2.9 *Let n odd, $f \in C^{n+1}([a,b])$, $[a,b] \subset \mathbb{R}$, such that $f^{(n+1)} \geq 0$ (≤ 0), $x, x_0 \in [a,b]$, then as before we have*

$$f(x) - f(x_0) \geq (\leq) \sum_{i=1}^{n} f^{(i)}(x_0) \frac{(x-x_0)^i}{i!}. \tag{2.31}$$

Then

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,dx - f(x_0) &= \frac{1}{b-a}\int_a^b (f(x) - f(x_0))\,dx \\
&\geq (\leq) \frac{1}{b-a}\int_a^b \sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i\,dx \\
&= \frac{1}{b-a}\sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{i!}\int_a^b (x-x_0)^i\,dx \\
&= \frac{1}{b-a}\sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{(i+1)!}\left((x-x_0)^{i+1}\Big|_a^b\right) \\
&= \frac{1}{b-a}\sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{(i+1)!}\left[(b-x_0)^{i+1} - (a-x_0)^{i+1}\right].
\end{aligned} \tag{2.32}$$

We obtained

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(x_0) \geq (\leq) \sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{(i+1)!}\left[\sum_{k=0}^{i}(b-x_0)^{i-k}(a-x_0)^k\right], \tag{2.33}$$

where n is odd.

Next let $0 < a \leq \frac{p}{q} \leq b$, a.e. on X. Then from (2.33) we get

$$\begin{aligned}
&\frac{q}{b-a}\int_a^b f(x)\,dx - qf\left(\frac{p}{q}\right) \\
&\quad \geq (\leq) \sum_{i=1}^{n} \frac{f^{(i)}\left(\frac{p}{q}\right)}{(i+1)!}\left[\sum_{k=0}^{i} q\left(b-\frac{p}{q}\right)^{i-k}\left(a-\frac{p}{q}\right)^k\right] \\
&\quad = \sum_{i=1}^{n} \frac{f^{(i)}\left(\frac{p}{q}\right)}{(i+1)!}\left[\sum_{k=0}^{i} q^{1-i}(bq-p)^{i-k}(aq-p)^k\right].
\end{aligned} \tag{2.34}$$

So when n is odd we get

$$\frac{1}{b-a}\int_a^b f(x)\,dx - \Gamma_f(\mu_1,\mu_2) \tag{2.35}$$
$$\geq\ (\leq)\ \sum_{i=1}^{n}\frac{1}{(i+1)!}\left(\sum_{k=0}^{i}\int_X f^{(i)}\left(\frac{p}{q}\right)q^{1-i}(bq-p)^{(i-k)}(aq-p)^k d\lambda\right).$$

We have proved

Theorem 2.10 *Let n odd and $f \in C^{n+1}([a,b])$, such that $f^{(n+1)} \geq 0$ (≤ 0), $0 < a \leq \frac{p}{q} \leq b$, a.e. on X. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq (\geq)\ \frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.36}$$
$$-\sum_{i=1}^{n}\frac{1}{(i+1)!}\left(\sum_{k=0}^{i}\int_X f^{(i)}\left(\frac{p}{q}\right)q^{1-i}(bq-p)^{(i-k)}(aq-p)^k d\lambda\right).$$

Remark 2.11 *Let n odd, $f \in C^{n+1}([a,b])$, $a,b \in \mathbb{R}$, such that $f^{(n+1)} \geq 0$ (≤ 0), $x, x_0 \in [a,b]$ and μ be any probability measure on $[a,b]$, then by (2.31) we get*

$$\int_{[a,b]} f(x)d\mu - f(x_0) \geq (\leq)\ \sum_{i=1}^{n}\frac{f^{(i)}(x_0)}{i!}\int_{[a,b]}(x-x_0)^i d\mu. \tag{2.37}$$

Comment 1. Assume all singletons of X belong to $\mathcal{A}$. Let the Dirac measure $\delta_{x_0} \ll \lambda$, $x_0 \in X$, then there exists density $\hat{f} \geq 0$ such that

$$\delta_{x_0}(E) = \int_E \hat{f}\,d\lambda, \quad \text{for all } E \in \mathcal{A},$$

(here $\hat{f}$ is the Radon-Nikodym derivative of δ_{x_0} with respect to λ). Clearly

$$1 = \delta_{x_0}(\{x_0\}) = \int_{\{x_0\}} \hat{f}\,d\lambda = \hat{f}(x_0)\lambda(\{x_0\}). \tag{2.38}$$

Since $\hat{f}$ is real valued we must have $\lambda(\{x_0\}) \neq 0$ and

$$\hat{f}(x_0) = \frac{1}{\lambda(\{x_0\})} > 0. \tag{2.39}$$

Also

$$0 = \delta_{x_0}(X - \{x_0\}) = \int_{X-\{x_0\}} \hat{f}\, d\lambda.$$

Therefore

$$\hat{f}\big|_{X-\{x_0\}} = 0, \quad \text{a.e. on } X. \tag{2.40}$$

So let $\mu_2 = \delta_{x_0}$, then $q = \hat{f}$ and

$$\Gamma_f(\mu_1, \delta_{x_0}) = \int_X \hat{f}\, f\left(\frac{p}{\hat{f}}\right) d\lambda = f(\lambda(\{x_0\})p(x_0)), \tag{2.41}$$

trivial and of no interest for further study.

Finally we give

Comment 2. Let $f \in C^{\nu}_{x_0}([a,b])$, $x_0 \in [a,b] \subset \mathbb{R}$ and $\nu \geq 1$, the remainder of fractional Taylor formula (see Theorem 1.3, (1.19)) is

$$\left(J^{x_0}_{\nu}(D^{\nu}_{x_0} f)\right)(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} (D^{\nu}_{x_0} f)(t)\, dt, \tag{2.42}$$

for all $x_0 \leq x \leq b$. If $(D^{\nu}_{x_0} f)(t) \begin{matrix} \geq 0 \\ (\leq 0) \end{matrix}$ over $[x_0, b]$ then $\left(J^{x_0}_{\nu}(D^{\nu}_{x_0} f)\right)(x) \begin{matrix} \geq 0 \\ (\leq 0) \end{matrix}$ over $[x_0, b]$. The last implies (by (1.19)) that

$$f(x) \geq (\leq) \sum_{i=0}^{n-1} f^{(i)}(x_0) \frac{(x-x_0)^i}{i!}, \tag{2.43}$$

where $n := [\nu]$, for all $x_0 \leq x \leq b$. According to Section 1 (I) we take here f to be convex from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Also we take $0 < a \leq \frac{p}{q} \leq b$ a.e. on X, with $a \leq 1 \leq b$.

Let $\nu \geq 1$ and additionally assume that $f \in C^{\nu}_{a}([a,b])$ such that $(D^{\nu}_{a} f)(t) \geq 0$ (≤ 0) over $[a,b]$, then we get

$$f(x) \geq (\leq) \sum_{i=0}^{n-1} f^{(i)}(a) \frac{(x-a)^i}{i!}, \tag{2.44}$$

for all $a \leq x \leq b$, $n := [\nu]$. Then

$$qf\left(\frac{p}{q}\right) \geq (\leq) \sum_{i=0}^{n-1} f^{(i)}(a) \frac{q\left(\frac{p}{q} - a\right)^i}{i!}, \tag{2.45}$$

a.e. on X. Consequently we obtain

$$\Gamma_f(\mu_1,\mu_2) \geq (\leq) \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} \int_X q^{1-i}(p-qa)^i \, d\lambda, \tag{2.46}$$

where $n := [\nu]$, $\nu \geq 1$.

References

[1] George A. Anastassiou, Opial type inequalities involving fractional derivatives of functions, *Nonlinear Studies*, **6**, no. 2 (1999), 207–230.

[2] N. S. Barnett, P. Cerone, S. S. Dragomir, and A. Sofo, Approximating Csiszar's f-divergence by the use of Taylor's formula with integral remainder, (paper #10, pp. 16), in *Inequalities for Csiszar f-Divergence in Information Theory*, S. S. Dragomir (ed.), Victoria University, Melbourne, Australia, 2000. On line: `http://rgmia.vu.edu.au`

[3] José A. Canavati, The Riemann-Liouville integral, *Nieuw Archief Voor Wiskunde*, **5**, no. 1 (1987), 53–75.

[4] I. Csiszar, Eine Informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten, *Magyar Trud. Akad. Mat. Kutato Int. Közl.* **8** (1963), 85–108.

[5] I. Csiszar, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica* **2** (1967), 299–318.

[6] S. S. Dragomir (ed.), *Inequalities for Csiszar f-Divergence in Information Theory*, Victoria University, Melbourne, Australia, 2000. On line: `http://rgmia.vu.edu.au`

[7] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Monograph, 2000, Victoria University, Melbourne, Adelaide, Australia, on line `http://rgmia.vu.edu.au`

[8] S. Kullback and R. Leibler, On information and sufficiency, *Ann. Math. Statist.* **22** (1951), 79–86.

[9] S. Kullback, *Information Theory and Statistics*, Wiley, New York, 1959.

[10] A. Rényi, On measures of entropy and information, in *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, I, Berkeley, CA, 1960, 547–561.

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Ultra Non-vanishing Function Spaces and Operating Functions

Eggert Briem

Abstract. Associated with a space B of continuous real-valued functions vanishing at infinity on a locally compact Hausdorff space X and an infinite set Λ, is the space $\ell^\infty_\Lambda(B)$ of all bounded Λ-nets of functions from B. This space has a natural representation as a space of functions on a locally compact space. Separation conditions for $\ell^\infty_\Lambda(B)$ are translated into separation conditions for B. The concept of an ultra non-vanishing space of functions is introduced and results about functions which operate by composition on such a space are obtained.

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1 Separation conditions

Let X be a locally compact Haussdorff space and let Λ be an infinte set. For a subspace B of $C_0(X,\mathbb{R})$, the space of all continuous real-valued functions vanishing at infinity on X, we define the "sequence" space $\ell^\infty_\Lambda(B)$ as

$$\ell^\infty_\Lambda(B) = \{\tilde{b} = (b_\lambda)_{\lambda\in\Lambda} : b_\lambda \in B \quad \text{and} \quad \|\tilde{b}\|_\infty := \sup_{\lambda\in\Lambda} \|b_\lambda\|_\infty < \infty\}\,.$$

We represent $\ell^\infty_\Lambda(B)$ as a space of functions on a locally compact Hausdorff space: The sequence space $\ell^\infty_\Lambda(C_0(X,\mathbb{C}))$, defined analogously to

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$\ell^\infty_\Lambda(B)$ with $C_0(X,\mathbb{C})$ replacing B, is a C^*-algebra, with multiplication and conjugation defined coordinatewise. Thus $\ell^\infty_\Lambda(C_0(X,\mathbb{C}))$ can be identified with $C_0(\gamma(\Lambda\times X),\mathbb{C})$, where $\gamma(\Lambda\times X)$ denotes the locally compact maximal ideal space of $\ell^\infty_\Lambda(C_0(X,\mathbb{C}))$, and $\Lambda\times X$ is identified with a dense subset of $\gamma(\Lambda\times X)$ via $\tilde{f}(\rho,x)=f_\rho(x)$, where $\tilde{f}=(f_\lambda)_{\lambda\in\Lambda}\in\ell^\infty_\Lambda(C_0(X,\mathbb{C}))$, for $(\rho,x)\in\Lambda\times X$. Thus $\ell^\infty_\Lambda(B)$ is a subspace of $C_0(\gamma(\Lambda\times X),\mathbb{R})$.

We are interested in which implications separation properties for $\ell^\infty_\Lambda(B)$ have for the space B, especially which implications the condition that $\ell^\infty_\Lambda(B)$ does not vanish at any point of $\gamma(\Lambda\times X)$ has for B. When X is compact $\gamma(\Lambda\times X)=\beta(\Lambda\times X)$, the Stone-Cech compactification of $\Lambda\times X$ (Λ is given the discrete topology). If B contains the constant functions, then so does $\ell^\infty_\Lambda(B)$ and thus saying that $\ell^\infty_\Lambda(B)$ separates the points of $\beta(\Lambda\times X)$ is equivalent to saying that $\mathrm{lat}(\ell^\infty_\Lambda(\mathrm{B}))$, the lattice generated by $\ell^\infty_\Lambda(B)$, is dense in $C(\beta(\Lambda\times X),\mathbb{R})$. This of course implies that B separates the points of X and thus in particular that lat(B) is dense in $C_0(X,\mathbb{R})$. But actually a much stronger statement holds:
There is a number $M>0$ and a natural number n such that, given any $f\in C(X,\mathbb{R})$ with $\|f\|_\infty\le 1$ there are $b_i,c_i\in B$, with $\|b_i\|_\infty,\|c_i\|_\infty\le M$, $1\le i\le n$, such that

$$\|f-\wedge_{i=1}^n b_i-\wedge_{i=1}^n c_i\|_\infty<1/2\,.$$

This is proved in [5] for $\Lambda=\mathbb{N}$, but the proof goes for any infinite set Λ. Now, the above intrinsic condition for B is easily seen to imply that $\ell^\infty_\Lambda(B)$ separates the points of $\beta(\Lambda\times X)$ for any Λ.

The sequence spaces were introduced by A. Bernard in [1] for the case $\Lambda=\mathbb{N}$. He called spaces B for which $\ell^\infty_{\mathbb{N}}(B)$ separates the points of $\beta(\mathbb{N}\times X)$ *ultraseparating* on X.

In the locally compact case where the constant functions are absent, point separation by a subspace need not imply density of the lattice generated by the subspace. For that to hold a stronger separation condition is needed: for any pair of points a function must exist in the space whose values at the points have *opposite signs*.

In the locally compact case the following conditions are equivalent:

1. *There is an infinite set Λ such that, given a pair $\xi,\eta\in\gamma(\Lambda\times X)$ there is $\tilde{b}\in\ell^\infty_\Lambda(B)$ with $\tilde{b}(\xi)<0$ and $\tilde{b}(\eta)>0$.*
2. *$\mathrm{lat}(\ell^\infty_\Lambda(B))$ is dense in $C_0(\gamma(\Lambda\times X),\mathbb{R})$, i.e. given any $\tilde{f}\in C_0(\gamma(\Lambda\times X),\mathbb{R})$ there is $n\in\mathbb{N}$ and $\tilde{b}_i,\tilde{c}_i\in\ell^\infty_\Lambda(B)$, $1\le i\le n$, such that*

$$\|\tilde{f}-\wedge_{i=1}^n\tilde{b}_i-\wedge_{i=1}^n\tilde{c}_i\|_\infty<1/2\,.$$

3. *There is a number $M > 0$ and a natural number n such that, given any $\tilde{f} \in C_0(\gamma(\Lambda \times X), \mathbb{R})$ with $\|\tilde{f}\|_\infty \leq 1$ there are $\tilde{b}_i, \tilde{c}_i \in \ell^\infty_\Lambda(B)$ with $\|\tilde{b}_i\|_\infty, \|\tilde{c}_i\|_\infty \leq M$, $1 \leq i \leq n$, such that*

$$\|\tilde{f} - \wedge_{i=1}^n \tilde{b}_i - \wedge_{i=1}^n \tilde{c}_i\|_\infty < 1/2\,.$$

4. *There is a number $M > 0$ and a natural number n such that given any $f \in C_0(X, \mathbb{R})$ with $\|f\|_\infty \leq 1$ there are $b_i, c_i \in B$ with $\|b_i\|_\infty, \|c_i\|_\infty \leq M$, $1 \leq i \leq n$, such that*

$$\|f - \wedge_{i=1}^n \tilde{b}_i - \wedge_{i=1}^n c_i\|_\infty < 1/2\,.$$

5. *Condition 1. holds for any infinite Λ.*

The proof of the equivalence of these conditions is similar to the proof of the equivalence of related conditions in [5], Thm. 6.

If the above conditions hold B is said to be *ultraseparating* on X.

Remark 1.1 *The equivalence of Conditions 1, 2 and 3 shows that point separation (with opposite signs) for $\ell^\infty_\Lambda(B)$ is much stronger than point separation for B, it implies a bound on the number of functions from $\ell^\infty_\Lambda(B)$ and their norms, needed to approximate any continuous function of norm less than or equal to 1.*

The following example due to O. Hatori [6] shows that even for compact X and for B ultraseparating on X, the space of functions in B which vanish at some x_0 in X need not to be ultraseparating on $X \setminus \{x_0\}$.

Example 1.2 *Let $X = \{\pm 1/n : n \in \mathbb{N}\}$ and let $X_1 = X \cup \{0\}$. The space B_1, consisting of the constant functions and those continuous functions f on X for which $f(1/n) = 1/2\, f(-1/n)$ for all n, is ultraseparating on X_1. However, the space B consisting of those functions f on X for which $f(1/n) = 1/2\, f(-1/n)$ for all n and $\lim_{n\to\infty} f(1/n) = 0$ is not ultraseparating on X.*

We now come to a much weaker separation condition. Let X be locally compact and let B a subspace of $C_0(X, \mathbb{R})$ that does not vanish at any point of X, i.e. for any x in X there is b in B with $b(x) \neq 0$. Further let Λ be any infinite set.

Suppose $\ell^\infty_\Lambda(B)$ does not vanish at any point of $\gamma(\Lambda \times X)$. Which consequences does this have for B? To answer this question we first note that the condition that $\ell^\infty_\Lambda(B)$ does not vanish at any point of $\gamma(\Lambda \times X)$

is equivalent to the condition that there are, for any compact subset $\mathcal{K}$ of $\gamma(\Lambda \times X)$, elements $\tilde{b}_1, \ldots, \tilde{b}_n$ in $\ell^\infty_\Lambda(B)$ such that $\tilde{b}_1 \vee \cdots \vee \tilde{b}_n > 1$ on $\mathcal{K}$. Just as was the case with ultraseparation this is seen to imply that there is, independent of $\mathcal{K}$, a bound on the number n, of functions needed, and on their norms.

Proposition 1.3 *The following conditions are equivalent:*

1. *There is an infinite set Λ such that $\ell^\infty_\Lambda(B)$ does not vanish at any point of $\gamma(\Lambda \times X)$.*
2. *For any compact subset $\mathcal{K}$ of $\gamma(\Lambda \times X)$ there is $n \in \mathbb{N}$ and $\tilde{b}_i \in \ell^\infty_\Lambda(B)$, $1 \leq i \leq n$, such that*
$$\tilde{b}_1 \vee \cdots \vee \tilde{b}_n > 1 \text{ on } \mathcal{K}\,.$$
3. *There is a number $M > 0$ and a natural number n, such that for any compact subset $\mathcal{K}$ of $\gamma(\Lambda \times X)$ there are $\tilde{b}_i \in \ell^\infty_\Lambda(B)$ with $\|\tilde{b}_i\|_\infty \leq M$, $1 \leq i \leq n$, such that*
$$\tilde{b}_1 \vee \cdots \vee \tilde{b}_n > 1 \text{ on } \mathcal{K}\,.$$
4. *There is a number $M > 0$ and a natural number n such that for any compact subset K of X there are $b_i \in B$ with $\|b_i\|_\infty \leq M$, $1 \leq i \leq n$, such that*
$$b_1 \vee \cdots \vee b_n > 1 \text{ on } K\,.$$
5. *There is a number $M > 0$ and a natural number n such that for any $f \in C_0(X, \mathbb{R})$ with $\|f\|_\infty \leq 1$ there are $b_i \in B$ with $\|b_i\| \leq M$, $1 \leq i \leq n$, such that*
$$\|f - f \wedge (b_1 \vee \cdots \vee b_n)\|_\infty < 1/2\,.$$
6. *For any infinite set Λ, $\ell^\infty_\Lambda(B)$ does not vanish at any point of $\gamma(\Lambda \times X)$.*

Proof. Clearly Condition 1 implies Condition 2. Given Condition 2 suppose Condition 4 does not hold. Then there is a sequence (K_n) of compact subsets of X such that if $b_1, \ldots, b_k$ are in B and $b_1 \vee \cdots \vee b_k > 1$ on K_n then, either $k > n$ or $\|b_i\|_\infty > n$ for some i with $1 \leq i \leq n$. We may suppose that $\mathbb{N}$ is a subset of Λ. Let K be a fixed compact of X and $K_\lambda = K$ for $\lambda \in \Lambda \backslash \mathbb{N}$. There is for each $\lambda \in \Lambda$ a function $f_\lambda \in C_0(X, \mathbb{R})$ with $\|f_\lambda\|_\infty \leq 1$, such that $f_\lambda = 1$ on K_λ. Let $\tilde{f} = (f_\lambda)_{\lambda \in \Lambda}$ and let $\mathcal{K} = \{\xi \in \gamma(\Lambda \times X) : \tilde{f}(\xi) = 1\}$, a compact subset of $\gamma(\Lambda \times X)$. By Condition 2 there are $\tilde{b}_1, \ldots, \tilde{b}_n$ in $\ell^\infty_\Lambda(B)$

such that $\tilde{b}_1 \vee \cdots \vee \tilde{b}_n > 1$ on $\mathcal{K}$. Since $\|\tilde{b}_i\|_\infty$, $1 \leq i \leq n$, are finite quantities, looking at the inequality $\tilde{b}_1 \vee \cdots \vee \tilde{b}_n > 1$ on $\{\lambda\} \times K_\lambda$ for each $\lambda \in \Lambda$ gives a contradiction.

It is fairly obvious that Condition 4 implies Condition 3 and 5. Also, since Condition 5 clearly implies that there are for any $\tilde{f} \in \ell^\infty_\Lambda(C_0(X, \mathbb{R}))$ elements $\tilde{b}_1, \ldots \tilde{b}_n$ in $\ell^\infty_\Lambda(B)$ such that $\|\tilde{f} - \tilde{f} \wedge (\tilde{b}_1 \vee \cdots \vee \tilde{b}_n)\|_\infty \leq 1/2$ it follows that Condition 5 implies Condition 6.

If any of the conditions above are satisfied we say that B is *ultra non-vanishing* on X.

Remark 1.4 *If a subspace B of $C_0(X)$ does not vanish at any point of X then for K compact there are $b_1, \ldots, b_n \in B$ such that $b_1 \vee \cdots \vee b_n > 1$ on K. In general the number n of functions needed as well as a bound on their norms will depend on the compact set K. The equivalence of Conditions 1, 2 and 3 shows that for the sequence spaces these quantities are independent of the compact set.*

Example 1.5

1. *Let X and B be as in Example 1. Then, given any compact subset K of X there is $b \in B$ such that $b \geq 1$ on K and $\|b\|_\infty = 2$, so that Condition 3 of Proposition 1 is satisfied with $n = 1$ and $M = 2$ and B is ultra non-vanishing on X.*
2. *Let again X be as in Example 1 and let B consist of the continuous odd functions on X. Again Condition 3 of Proposition 1 is satisfied, this time with $n = 2$ and $M = 1$.*
3. *As an example of a space not satisfying the conditions of Proposition we can take any subspace B of $C_0(X, \mathbb{R})$ with X containing a line segment of the dual space B^*.*

2 Operating functions

If B is a vector space of real-valued functions on a set X, a real-valued function φ defined on an interval I of the real line is said to *operate* on B if the composite function $\varphi \circ b$ is in B whenever b is in B and the composition is defined. Affine functions, functions of the form $\varphi(t) = \alpha t + \beta$ ($\beta = 0$, if B does not contain the constant functions), operate on B and in general these may be the only functions which operate on B.

Suppose X is a locally compact Hausdorff space and B a subspace of $C_0(X,\mathbb{R})$, the space of all continuous real-valued functions vanishing at infinity on X. Operating functions for B are assumed to be continuous.

If X is compact and B is a uniformly closed subspace of $C(X,\mathbb{R})$ which contains the constant functions and separates the points of X, a theorem of de Leeuw and Y. Katznelson, ([4], Theorem 4.21), says that if a non-affine function operates on B, then $B = C(X,\mathbb{R})$. This is a generalization of the Stone-Weierstrass theorem which in this context says that if $\varphi(t) = t^2$ operates on B, then $B = C(X,\mathbb{R})$.

When B contains the constant functions, $B = C(X,\mathbb{R})$ if and only if B contains every f in $C(X,\mathbb{R})$ which can be approximated from B on every pair of points in X. This latter condition is not equivalent to the condition $B = C_0(X,\mathbb{R})$ if B does not contain the constant functions. It might happen for some real number α and some pair x_1, x_2 of points in X that $b(x_1) = \alpha\, b(x_2)$ for all b in B.

It is natural to ask whether the following is true.
Let X be locally compact and let B be a uniformly closed subspace of $C_0(X,\mathbb{R})$ which separates the points of X. If there is an operating function for B, defined in a neighbourhood of 0, and non-affine in every neighbourhood of 0, then B contains every f in $C_0(X,\mathbb{R})$ which can be approximated from B on every pair of points in X.

This result is known to be true with some additional conditions on the operating function φ, such as $|\varphi(t)| \le k|t|$ for all t and some positive number k or, $\lim_{t\to o} \varphi(t)/t = \infty$, [2] and [3].

If φ is a continuous, non-affine real-valued function defined on an interval I of the real line, it is easy to see that there is an interior point t_0 of I and real numbers δ, k, with $\delta > 0$, such that the function

$$\psi(t) = k\Big(\varphi(t+t_0) - \varphi(-t+t_0) - 2\varphi(t_0)\Big)$$

is defined on the interval $(-\delta,\delta)$ and satisfies

$$\psi(t) > \psi(0) = 0 \quad \text{for} \quad t \in (-\delta,\delta),\, t \neq 0\,.$$

Thus, if B contains the constant functions and if there is a non-affine function which operates on B then there is a function, operating on B, which is defined in a neighbourhood of 0, vanishes at 0, and is strictly positive otherwise.

With this condition on the operating function and the conditions from Proposition 1 we obtain the following version of the Stone-Weierstrass Theorem.

Theorem 2.1 *Let B be uniformly closed subspace of $C_0(X,\mathbb{R})$, which is ultra non-vanishing on X. If there is an operating function φ for B, defined in a neighbourhood of 0 and satisfying $\varphi(t) > \varphi(0) = 0$ for $t \neq 0$, then B contains every function in $C_0(X,\mathbb{R})$ that can be approximated from B on every pair of points in X.*

Proof. If there is a number α, with $0 < \alpha < 1$, such that $\varphi(\alpha t) = \alpha\varphi(t)$ for t in some neighbourhood of 0 then one can easily show that there is a number $k > 0$ such that $|\varphi(t)| \leq k|t|$ for all t. In this case the result is already known, [2]. So we assume there is no such number α as above.

Let Λ denote the collection of all compact subsets of X, and let $\gamma(\Lambda \times X)$ denote the maximal ideal space of $\ell^\infty_\Lambda(C_0(X,\mathbb{R}))$. There is a natural embedding of $\Lambda \times X$ into $\gamma(\Lambda \times X)$ given by $(\lambda, x) \to \varphi_{\lambda,x}$, where $\varphi_{\lambda,x}(\tilde{f}) = f_\lambda(x)$.

Let $K_\lambda = \lambda$ for each $\lambda \in \Lambda$ and let $\mathcal{K}$ denote the closure of $\cup_{\lambda\in\Lambda}\{\lambda\}\times K_\lambda$ in $\gamma(\Lambda \times K)$. The set $\mathcal{K}$ is a compact subset of $\gamma(\Lambda \times K)$, since there is an element $\tilde{g} \in \ell^\infty_\Lambda(C_0(X,\mathbb{R}))$ with $\tilde{g} = 1$ on $\mathcal{K}$; we let $g_\lambda = 1$ on K_λ, with $\|g_\lambda\|_\infty = 1$ and put $\tilde{g} = (g_\lambda)$.

The conditions on B show that there is for each $\xi \in \mathcal{K}$ a function $\tilde{b} \in \ell^\infty_\Lambda(B)$ such that $\tilde{b}(\xi) \neq 0$ and hence $(\varphi \circ r\tilde{b})(\xi) > 0$, if r is small so that $\varphi \circ r\tilde{b}$ is defined. Since $\mathcal{K}$ is compact we conclude that there is $\tilde{c} \in \ell^\infty_\Lambda(B)$ such that $\tilde{c} > 0$ on $\mathcal{K}$.

Let us show that given any pair ξ, η of points in $\mathcal{K}$, either $\ell^\infty_\Lambda(B)$ does not separate ξ and η or, there is $\tilde{b} \in \ell^\infty_\Lambda(B)$ such that $\tilde{b}(\xi) = 0$ and $\tilde{b}(\eta) = 1$. If $\{(\tilde{b}(\xi), \tilde{b}(\eta)) : \tilde{b} \in \ell^\infty_\Lambda(B)\}$ is a 1-dimensional supbspace of $\mathbb{R}^2$, then for some number α, $\tilde{b}(\xi) = \alpha\tilde{b}(\eta)$ for all $\tilde{b} \in \ell^\infty_\Lambda(B)$. Suppose $\alpha \neq 1$. Since $\varphi \circ (r\tilde{b}) \geq 0$, α is positive. We can assume that $\alpha < 1$. Otherwise we write $\tilde{b}(\eta) = \alpha^{-1}\tilde{b}(\xi)$. Suppose $\tilde{b}(\eta) \neq 0$. For all sufficiently small t, we have $(\varphi \circ t\tilde{b})(\xi) = \alpha(\varphi \circ t\tilde{b})(\eta)$ and thus $\varphi(\alpha ts) = \alpha\varphi(ts)$, where $s = \tilde{b}(\eta)$, for all sufficiently small t, contradicting the assumptions on φ. Thus, if the subspace of $\mathbb{R}^2$ defined above is 1-dimensional then $\ell^\infty_\Lambda(B)$ does not separate ξ and η.

For $\tilde{b} \in \ell^\infty_\Lambda(B)$, for a C^∞_0-function ψ with support in a sufficiently small neighbourhood of 0 and for real numbers r, t with $|r|$ and $|t|$ small,

$$\int \varphi \circ (r\tilde{c} + t\tilde{b} - s\tilde{c})\psi(s)ds \in \ell^\infty_\Lambda(B)\,.$$

Let

$$\tilde{d} = \int \varphi \circ (r\tilde{c} - s\tilde{c})\psi''(s)ds\,.$$

If we differentiate the first expression above twice w.r.t. t and put $t = 0$ we find that $\tilde{b}^2\tilde{d}/\tilde{c}^2$ can be approximated uniformly on $\mathcal{K}$ by elements of $\ell^\infty_\Lambda(B)$, and hence the functions $\tilde{a}\tilde{b}\tilde{d}/\tilde{c}^2$, where $\tilde{a}, \tilde{b} \in \ell^\infty_\Lambda(B)$ can be approximated uniformly on $\mathcal{K}$ by elements of $\ell^\infty_\Lambda(B)$.

Let $\mathcal{M}$ denote the set of those $\tilde{f} \in C(\mathcal{K}, \mathbb{R})$ for which $\tilde{f}\tilde{b}$ can be approximated uniformly on $\mathcal{K}$ by elements of $\ell^\infty_\Lambda(B)$. Then $\mathcal{M}$ is a uniformly closed subalgebra of $C(\mathcal{K}, \mathbb{R})$ containing the constant functions and separating any two points of X that are separated by $\ell^\infty_\Lambda(B)$. It follows that for any $\tilde{b} \in \ell^\infty_\Lambda(B)$, $\tilde{b}^2$ can be approximated uniformly on $\mathcal{K}$ by elements of $\ell^\infty_\Lambda(B)$.

Let $b \in B$ and let $b_\lambda = b$ for all $\lambda \in \Lambda$ and $\tilde{b} = (b_\lambda)$. For any $\varepsilon > 0$ there is $\tilde{a} \in \ell^\infty_\Lambda(B)$ such that $|\tilde{b}^2 - \tilde{a}| < \varepsilon$ on $\mathcal{K}$ and thus there is for any $\varepsilon > 0$ a number $M > 0$ such that there is for any compact subset K of X a function $a_K \in B$, with $\|a_K\|_\infty \leq M$ such that $|b^2 - a_K| < \varepsilon$ on K. By standard approximation arguments we have $b^2 \in B$. It follows that $B = C_0(X, \mathbb{R})$.

References

[1] A. Bernard, Espaces des parties relles des Iments d'une algebre de Banach de fonctions, *J. Funct. Anal.* **10** 387-409, 1972.

[2] E. Briem, Approximations from Subspaces of $C_0(X)$, *J. Approx. Theory* **112** , 279-294, 2001.

[3] E. Briem, Symbolic calculus for subspaces of $C_0(X)$, *Math. Scand.* **76**, 98-104, 1995.

[4] R.B. Burckel, Characterizations of $C(X)$ among its subalgebras (Lecture Notes in *Pure and Appl. Math.* **6**). Marcel Dekker, New York 1972.

[5] A.J. Ellis, Separation and ultraseparation properties for continuous function spaces, *J. London Math. Soc.* **29**(2), 521-532, 1984.

[6] O. Hatori, Separation properties and operating functions on a space of continuous functions, *Intern. J. Math.* **4**, 551-600, 1993.

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q-Variation and Commutators for Fourier Multipliers

Joan Cerdà and Joaquim Martín

Abstract. If T_μ is a Fourier multiplier such that μ is any (possibly unbounded) symbol with uniformly bounded q-variation on dyadic coronas, we prove that the commutator $[T, T_\mu] = TT_\mu - T_\mu T$ is bounded on the Besov space $B_p^{\sigma,r}(\mathbf{R}^n)$, if T is any bounded linear operator on a couple of Besov spaces $B_p^{\sigma_j,r_j}(\mathbf{R}^n)$ ($j = 0, 1$, and $0 < \sigma_1 < \sigma < \sigma_0$).

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1 Introduction

In [11] Rochberg and Weiss developed the study of the commutators $[T, \Omega]$ of bounded linear operators T and certain operators, Ω, generaly unbounded and nonlinear, associated with the complex interpolation method. A similar analysis was carried out for the real method by Jawerth, Rochberg and Weiss in [8], where they noticed that, although there are strong analogies between the two cases, the details are very different.

Finally, it was shown in [3] how some common general principles of these abstract interpolation methods can be used to prove the boundedness of commutators. The purpose of that paper was to set up a unified

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method of both theories with an analysis that leads to a simple approach to commutator theorems, giving the precise rôle that cancelation plays in the theory.

A typical application using complex interpolation is the one given in [11], where the boundedness of $[T, b]$ on $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ was proved when T is a Calderón-Zygmund operator and b the multiplication by a BMO function. Another application of these ideas was given in [4] using real interpolation. For Besov spaces of periodic functions $B_p^{\sigma,r} = B_p^{\sigma,r}(\mathbf{T})$, it was shown that the commutator

$$[T, T_\mu] : B_p^{\sigma,r}(\mathbf{T}) \to B_p^{\sigma,r}(\mathbf{T})$$

is bounded for a wide class of operators T and symbols μ. Here, T_μ is the Fourier multiplier defined by $\widehat{T_\mu f} = \mu \widehat{f}$, where $\widehat{f}$ is the Fourier transform of f.

This result has been extended in [5] to the case $B_p^{\sigma,r} = B_p^{\sigma,r}(\mathbf{R}^n)$ of non periodic functions of several variables with a direct proof, but still using cancellation properties combined with the real interpolation method. The admissible symbols μ may be unbounded, but their variations on closed dyadic coronas need to be uniformly bounded. This condition allows to use the Macinkiewicz multiplier theorem (and its extension to n variables) for symbols that are admissible and bounded.

In this note, we briefly review the main facts needed for the proof of this commutator theorem and we show how the class of admissible multipliers can be slightly enlarged by considering q-variations on closed dyadic coronas. In the spacial case $q = 2$, the commutator theorem holds for any $1 < p < \infty$, and this is an improvement of the main result in [5]. Moreover, as in [4], we give a new proof based on the abstract method of [3], or [8].

Concerning notation, if A and B are two Banach spaces, we write $T : A \longrightarrow B$ to mean that T is a bounded linear operator between A and B. Finally, $P \lesssim Q$ means that $P \leq cQ$ for some constant $c > 0$ independent of the variables involved, while by $P \simeq Q$ we mean that $P \lesssim Q$ and $Q \lesssim P$.

We always assume $0 < \theta < 1$, $1 \leq r, q < \infty$, $1 < p < \infty$, $\sigma > 0$, and function classes and Banach spaces will be included in $L_0 = L_0(\mathbf{R}^n)$, the space of measurable functions on $\mathbf{R}^n$. If X is a Banach space, $\|f\|_X = \infty$ will mean that $f \notin X$.

For undefined notation and standard definitions concerning interpolation theory we refer to [1].

2 Preliminaries

Let us recall some facts about Besov spaces and real interpolation theory (see [1] or [2] for more details concerning interpolation theory, and [6] and [9] for Besov spaces).

If X is a given Banach space, we say that $\{V(t)\}_{t>0}$ is an approximation family if $V(s) \subset V(t)$ when $s < t$, $-V(t) = V(t)$, and $V(s)+V(t) \subset V(s+t)$. The corresponding distance function is

$$E(t,f) := \inf_{g \in V(t)} \|f-g\|_X \simeq \|f - f_t\|_X$$

if $f_t \in V(t)$ and $\|f - f_t\|_X \le cE(t,f)$ for some constant $c > 1$, and then we say that f_t is almost optimal for this distance function.

The associated approximation spaces $E^{r,q}$ are defined by the condition

$$\|f\|_{E^{r,q}} = \|t^{1/r} E(t,f)\|_{L^q(dt/t)} < \infty.$$

Example 2.1 *$X = L^\infty$, $V(t) = \{f;\, |\operatorname{supp} f| \le t\}$ ($|\operatorname{supp} f|$ denotes the measure of support of f); then $E(t,f) = f^*(t)$ and $E^{r,q} = L^{r,q}$, a Lorentz space. See [10].*

The Besov space $B_p^{\sigma,q} = B_p^{\sigma,q}(\mathbf{R}^n)$ is the approximation space $E^{1/\sigma,q}$ when $X = L^p(\mathbf{R}^n)$ and $V(t) = \{g \in \mathcal{S}'(\mathbf{R}^n);\, \operatorname{supp} \widehat{g} \subset [-t,t]^n\}$. As usual, $\mathcal{S}'(\mathbf{R}^n)$ denotes the class of all tempered distributions. Hence,

$$\|f\|_{B_p^{\sigma,q}} = \|t^\sigma E(t,f)\|_{L^q(dt/t)} = \left(\int_0^\infty [t^\sigma E(t,f)]^q \, \frac{dt}{t} \right)^{1/q}.$$

As a special case, $B_2^{\sigma,2} = H^{\sigma,2}$, a Sobolev space.

Let us also recall that, for a given Banach couple $\bar{B} = (B_0, B_1)$, the corresponding interpolation Banach space $\bar{B}_{\theta,q}$ is defined by the condition

$$\|f\|_{\theta,q} = \left\| \frac{K(t,f)}{t^\theta} \right\|_{L^q(\frac{dt}{t})} < \infty,$$

where

$$K(t,f) = K(t,f;\bar{B}) := \inf_{f=f_0+f_1} \|f_0\|_0 + t\|f_1\|_1$$

is the Peetre's K–functional (we write $\|\cdot\|_j$ instead of $\|\cdot\|_{B_j}$).

In the case of Besov spaces,

$$(B_p^{\sigma_0,q_0}, B_p^{\sigma_1,q_1})_{\theta,q} = B_p^{\sigma,q}, \quad (\sigma = (1-\theta)\sigma_0 + \theta\sigma_1).$$

If $\bar{A}$, $\bar{B}$ are two Banach couples, we denote by $\mathcal{L}(\bar{A};\bar{B})$ the set of all linear operators $T : \Sigma(\bar{A}) \longrightarrow \Sigma(\bar{B})$ such that $T(A_j) \subset B_j$ $(j = 0, 1)$ and $\|T\| = \max(\|T\|_{A_0,B_0};\, \|T\|_{A_1,B_1}) < \infty$. If $T \in \mathcal{L}(\bar{A};\bar{B})$, then by interpolation, $T \in \mathcal{L}(\bar{A}_{\theta,q};\bar{B}_{\theta,q})$.

The following proposition is a weak type interpolation result as defined in [1] for Lorentz spaces. For the sake of an easy reading we include its short proof.

Proposition 2.2 *If*

$$Sf(t) := \int_0^t f(s)\frac{ds}{s} + t\int_t^\infty f(s)\frac{ds}{s^2}$$

is the Calderón operator and $T : \Sigma(\bar{A}) \to \Sigma(\bar{B})$ *is such that*

$$K(t, Tf;\bar{B}) \le CS(K(\cdot, f;\bar{A}))(t),$$

then $T : \bar{A}_{\theta,q} \longrightarrow \bar{B}_{\theta,q}$ *with* $\|Tf\|_{\theta,q} \le \frac{C}{\theta(1-\theta)}\|f\|_{\theta,q}$.

Proof. If in Hardy's inequalities for averages,

$$\Big\|t^{-\theta}\int_0^t g(s)\,ds\Big\|_{L^q(\frac{dt}{t})} \le \frac{1}{\theta}\|t^{1-\theta}g(t)\|_{L^q(\frac{dt}{t})} \quad (\theta > 0)$$

and

$$\Big\|t^{1-\theta}\int_t^\infty g(s)\,\frac{ds}{s}\Big\|_{L^q(\frac{dt}{t})} \le \frac{1}{1-\theta}\|t^{1-\theta}g(t)\|_{L^q(\frac{dt}{t})}\,(\theta < 1),$$

we take $g(s) = K(s, f)/s$, then

$$\begin{aligned}
\|Tf\|_{\theta,q} &= \|t^{-\theta}K(t, Tf)\|_{L^q(\frac{dt}{t})} \le C\|t^{-\theta}S(K(\cdot, f))(t)\|_{L^q(\frac{dt}{t})} \\
&\le C\Big\|t^{-\theta}\int_0^t K(s, f)\,\frac{ds}{s}\Big\|_{L^q(\frac{dt}{t})} + C\Big\|t^{-\theta}t\int_t^\infty \frac{K(s, f)}{s}\,\frac{ds}{s}\Big\|_{L^q(\frac{dt}{t})} \\
&\le M\Big\|t^{1-\theta}\frac{K(t, f)}{t}\Big\|_{L^q(\frac{dt}{t})} = M\|f\|_{\theta,q}
\end{aligned}$$

and the constant is $M = \frac{C}{\theta(1-\theta)}$. □

We set

$$\sigma(\bar{A}) := \big\{x \in \Sigma(\bar{A});\, \|x\|_{\sigma(\bar{A})} := S(K(\cdot, x))(1) < \infty\big\},$$

where S is the Calderón operator. Observe that $\sigma(\bar{A})$ is a linear subspace of $\Sigma(\bar{A})$ which contains all real interpolation spaces $\bar{A}_{\theta,q}$ and moreover

$\|Tx\|_{\sigma(\bar{B})} \leq \|T\|\|x\|_{\sigma(\bar{A})}$, $(T \in \mathcal{L}(\bar{A};\bar{B}))$. Thus $\sigma(\cdot)$ is an interpolation method.

The following theorem, whose proof easily follows by combining Theorem 3 and Corollary 1 of [4], may be considered as a special instance of the general commutator method given in [3] and can also be obtained from the work by Jawerth, Rochberg and Weiss in [8]. It will be useful in order to proof our main result.

Theorem 2.3 *Let $\{t_j\}_{j\in Z} \subset (0,\infty)$ be an increasing sequence such that $t_j \leq 1$ if $j < 0$ and $t_j \geq 1$ if $j > 1$, $t_j \uparrow \infty$ as $j \uparrow \infty$ and $t_j \downarrow 0$ as $j \downarrow -\infty$, and let $\{\mu_k\}_{k\in Z}$ be any sequence of complex numbers such that*

$$\sup_{n\in\mathbf{Z}} \sum_{t_k\in[2^{n-1},2^n)} |\mu_{k+1} - \mu_k| < \infty.$$

For a given Banach couple $\bar{B}$ and for every t_j let $x = x_0(t_j) + x_1(t_j)$ be a **quasi-optimal decomposition** *for the K-functional, i.e.*

$$\|x_0(t_j)\|_0 + t_j \|x_1(t_j)\|_1 \leq cK(t_j, x) \qquad \left(x \in \sigma(\bar{B})\right)$$

where $c > 1$ is a fixed constant.

Let us define

$$T_\mu x := \sum_{k=1}^{\infty} \mu_k \left(x_0(t_k) - x_0(t_{k-1})\right).$$

Then, if $T_\mu : \sigma(\bar{B}) \to \sum(\bar{B})$ is bounded, there exists a constant $C > 0$ such that

$$K(t, [T, T_\mu](x)) \leq C \|T\| S(K(\cdot, x))(t)$$

for any $x \in \sigma(\bar{B})$ and $T \in \mathcal{L}(\bar{B};\bar{B})$.

In order to apply this theorem we shall need a quasi-optimal decomposition for the K-functional for a couple of Besov spaces, which is obtained in the following proposition.

Proposition 2.4 *Let $\sigma_0 > \sigma_1$ and $\delta := \sigma_0 - \sigma_1$. Then, for a couple $(B_p^{\sigma_0,q_0}(\mathbf{R}^n), B_p^{\sigma_1,q_1}(\mathbf{R}^n))$ of Besov spaces,*

$$K(t^\delta, f) \simeq \|P_t f\|_0 + t^\delta \|f - P_t f\|_1,$$

where $P_t = T_\mu$ and $\mu = \chi_{[-t,t]^n}$.

Proof. Since P_t is admissible and bounded, with $\|P_t\|_{p,p} \le c_p$ and $P_t(L^p) \subset V(t)$, it follows that $\|f - P_t f\|_p \le \|f - f_t\|_p + \|P_t(f_t - f)\|_p \le cE(t,f)$ and $P_t f$ is almost optimal. Then the Holmstedt type formula

$$K(t^\delta, f) \simeq \|P_t f\|_0 + t^\delta \|f - P_t f\|_1$$

holds (cf. [4, Theorem 4]). □

3 Multipliers

A basic tool for the proof of our result is Xu's sufficient condition for a function to be a Fourier multiplier of $L^p(\mathbf{R}^n)$ via its q-variation on dyadic intervals (cf. [12]).

Let us denote $Q_j = [-2^j, 2^j]^n$, $C_0 = Q_0$, $C_j = Q_j \setminus Q_{j-1}$ $(j > 0)$. To define the q-variation $V_{\bar{C}}^q(\mu)$ of μ on a closed set $\bar{C}$ we only consider rectangles R with sides parallel to the axes and with vertices in $\bar{C}$ and, with the notation of [12],

$$\Delta_R(\mu) := \Delta_R^1 \Delta_R^2 \cdots \Delta_R^n \mu(a), \quad a = (a_1, \ldots, a_n)$$

where, if $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $x = (x_1, \ldots, x_n)$, Δ_R^k is the difference operator in the k-th variable

$$\Delta_R^k \mu(x) = f(x_1, \ldots, x_{k-1}, b_k, x_{k+1} \ldots, x_n) - f(x_1, \ldots, x_{k-1}, a_k, x_{k+1} \ldots, x_n)$$

(if $n = 2$, $\Delta_R^k \mu = \mu(b_1, b_2) - \mu(b_1, a_2) - \mu(a_1, b_2) + \mu(a_1, a_2)$). Then

$$V_{\bar{C}}^q(\mu) := \sup_{\mathcal{R}} \left(\sum_{R \in \mathcal{R}} |\Delta_R(\mu)|^q \right)^{1/q},$$

where the supremum is taken over all finite families $\mathcal{R}$ of rectangles as above, with disjoint interior.

In the case $n = 1$, $V_{\bar{C}}^q(\mu) = \sup(\sum_{k \ge 1} (|\mu(t_k) - \mu(t_{k-1})|^q)^{1/q}$, where the supremum runs over all finite increasing sequences $\{t_k\}_{k \ge 0} \subset \bar{C}$. We say that μ is q-admissible if $V^q(\mu) := \sup V_{\bar{C}_j}^q(\mu) < \infty$.

If $n > 1$, we say that μ is q-admissible if the following three conditions are satisfied:
(1) For each $1 \le k < n$, $S_k := \sup_j V_C^q(\mu_{k,j}) < \infty$, where $\mu_{k,j}(x_1, \ldots, x_k) = \mu(x_1, \ldots, x_k, 2^j, \ldots, 2^j)$ is a function of k variables and $C = [-2^j, 2^j]^k$.
(2) The condition $S_{\pi,k} < \infty$ analogous to (1) is valid for each permutation π of the n variables.

(3) $S_n := \sup_{j\geq 0} V^q_{\tilde{C}_j}(\mu) < \infty$.

We define $V^q(\mu)$ as the sum of all the quantities S appearing in the above conditions. Thus, μ is admissible if $V^q(\mu) < \infty$.

It follows from the main result in [12] that, if μ is q-admissible and bounded, then $T_\mu : L^p(\mathbf{R}^n) \longrightarrow L^p(\mathbf{R}^n)$ for every $1 < p < \infty$ satisfying $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$. This extends to n variables the improvement of the Marcinkiewicz multiplier theorem due to Coifman, Rubio de Francia and Semmes.

A **dyadic multiplier** will be a function

$$\mu = \{\mu_j\}_{j\geq 0} := \sum_{j=0}^{\infty} \mu_j \chi_{C_j},$$

which is constant on every corona C_j.

For dyadic multipliers and for any $1 \leq q < \infty$ the admissibility condition is simply

$$V(\mu) = \sup_{j\geq 0} |\mu_j - \mu_{j-1}| < \infty \qquad (\mu_{-1} = 0)$$

and, for any admissible multiplier μ,

$$\mu^{(d)} := \sum_{j=0}^{\infty} \mu(2^j, \cdots, 2^j)\chi_{C_j} = \sum_{j=0}^{\infty} \mu_j \chi_{C_j}.$$

defines an admissible dyadic multiplier with

$$V(\mu^{(d)}) \leq V(\mu).$$

Observe that χ_{Q_0}, $\log^+(\max\{|x_1|, \cdots, |x_n|\})$ and $\{k\}_{k=0}^{\infty}$ are q-admissible, while $\chi_{\{\|x\|<1\}}$ for $n > 1$ and $\chi_{\{x_1>0\}}(x) \log^+(\max\{|x_1|, \cdots, |x_n|\})$ are never q-admissible.

4 The Main Result

In order to prove the main theorem, let us start with a reduction to dyadic multipliers.

Lemma 4.1 *Let μ be a q-admissible multiplier, $\mu^{(d)}$ its admissible dyadic multiplier, i.e.*

$$\mu^{(d)} := \sum_{j=0}^{\infty} \mu(2^j, \cdots, 2^j)\chi_{C_j} = \sum_{j=0}^{\infty} \mu_j \chi_{C_j},$$

and $|1/p - 1/2| < 1/q$. *If* T *is any bounded linear operator on* $B_p^{\sigma,r}(\mathbf{R}^n)$, *then*

$$[T, T_\mu] : B_p^{\sigma,r}(\mathbf{R}^n) \longrightarrow B_p^{\sigma,r}(\mathbf{R}^n)$$

if and only if

$$[T, T_{\mu^{(d)}}] : B_p^{\sigma,r}(\mathbf{R}^n) \longrightarrow B_p^{\sigma,r}(\mathbf{R}^n).$$

Proof. As observed in [5, Lemma 2], $[T, T_\mu] - [T, T_{\mu^{(d)}}] = [T, T_{\mu-\mu^{(d)}}]$, where $\mu - \mu^{(d)}$ is bounded and $V^q(\mu - \mu^{(d)}) < \infty$, and it follows from Xu's result that $T_{\mu-\mu^{(d)}} : L^p \to L^p$ if $|1/p - 1/2| < 1/q$. But then $T_{\mu-\mu^{(d)}} : B_p^{\sigma,r} \to B_p^{\sigma,r}$ for any $\sigma > 0$ and $1 \le r < \infty$, and $[T, T_\mu] - [T, T_{\mu^{(d)}}] : B_p^{\sigma,r} \to B_p^{\sigma,r}$. □

Lemma 4.2 *Let us consider a couple* $\bar{B} = (B_p^{\sigma_0,q_0}(\mathbf{R}^n), B_p^{\sigma_1,q_1}(\mathbf{R}^n))$ *of Besov spaces with* $\delta := \sigma_0 - \sigma_1 > 0$. *Then* $Lf := \sum_{j\ge 0}(f - P_{2^j} f)$ *defines a bounded operator* $L : \sigma(\bar{B}) \longrightarrow \Sigma(\bar{B})$.

Proof. We have $\|Lf\|_{\Sigma(\bar{B})} \le \sum_{j\ge 0} \|f - P_{2^j} f\|_1$, where, by Proposition 2.4,

$$\begin{aligned} \sum_{j\ge 0} \|f - P_{2^j} f\|_1 &\lesssim \sum_{j\ge 0} \frac{K(2^{\delta j}, f; \bar{A})}{2^{\delta j}} \\ &\le \frac{2^\delta}{\delta \log 2} \int_1^\infty \frac{K(s, f; \bar{A})}{s} \frac{ds}{s} \le \frac{2^\delta}{\delta \log 2} \|f\|_{\sigma(\bar{B})}. \end{aligned}$$ □

Theorem 4.3 *Let* $\sigma_0 > \sigma_1 > 0$, $1 \le r, q_0, q_1 < \infty$, $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$ *(*$0 < \theta < 1$*),* $T \in \mathcal{L}((B_p^{\sigma_0,q_0}(\mathbf{R}^n), B_p^{\sigma_1,q_1}(\mathbf{R}^n)); (B_p^{\sigma_0,q_0}(\mathbf{R}^n), B_p^{\sigma_1,q_1}(\mathbf{R}^n)))$ *and* μ *a* q*-admissible multiplier. Then*

$$[T, T_\mu] : B_p^{\sigma,r}(\mathbf{R}^n) \longrightarrow B_p^{\sigma,r}(\mathbf{R}^n),$$

if $1 < p < \infty$ *and* $|1/p - 1/2| < 1/q$.

Proof. By Lemma 4.1, we may assume that μ is a dyadic admissible multiplier, i.e. $\mu = \sum_{k=0}^\infty \mu_k \chi_{C_k}$, with $\mu_0 = 0$ and $\sup_k |\mu_{k+1} - \mu_k| = C < \infty$. Then

$$T_\mu f = \sum_{k=1}^\infty \mu_k (P_{2^k} f - P_{2^{k-1}} f),$$

since $\mu_k (P_{2^k} f - P_{2^{k-1}} f)\widehat{\ } = \mu_k (\chi_{Q_k} - \chi_{Q_{k-1}})\widehat{f} = \mu_k \chi_{C_k} \widehat{f}$.

Now, by denoting

$$\lambda_0 = \mu_1 - \mu_0 = \mu_1,\ \lambda_1 = \mu_2 - \mu_1, \ldots, \lambda_k = \mu_{k+1} - \mu_k, \ldots$$

we get $\|\lambda\|_\infty = \sup_k |\lambda_k| = \sup_k |\mu_{k+1} - \mu_k| < \infty$, and $\lambda_0 = \mu_1$, $\lambda_0 + \lambda_1 = \mu_2, \dots, \sum_{j=0}^{k} \lambda_j = \mu_{k+1}$.

Then, by Lemma 4.2, $T_\mu f = \sum_{j=0}^{\infty} \lambda_j (f - P_{2^j} f)$ and

$$T_\mu : \sigma(\bar{B}) \longrightarrow \Sigma(\bar{B}), \quad \text{if } \bar{B} = (B_p^{\sigma_0, r_0}(\mathbf{R}^n); B_p^{\sigma_1, r_1}(\mathbf{R}^n)),$$

with $\|T_\mu f\|_{\Sigma(\bar{B})} \le \|\lambda\|_\infty \|Lf\|_{\Sigma(\bar{B})} \lesssim \|f\|_{\sigma(\bar{B})}$.

Let $\delta = \sigma_0 - \sigma_1$, an define $t_k = 2^{\delta k}$ $(k \ge 0)$. By Proposition 2,

$$\|P_{2^k} f\|_0 + t_k \|f - P_{2^k} f\|_1 \le cK(t_k, f).$$

Thus, if we define

$$x_0(t_k) = P_{2^k} f,$$

we can write

$$T_\mu f = \sum_{k=1}^{\infty} \mu_k \left(x_0(t_k) - x_0(t_{k-1}) \right).$$

On the other hand, since $2^\delta > 1$, we have that

$$\begin{aligned} \sup_{n \in \mathbf{Z}} \sum_{t_k \in [2^{n-1}, 2^n)} |\mu_{k+1} - \mu_k| &\simeq \sup_{n \in \mathbf{Z}} \sum_{t_k \in [2^{\delta(n-1)}, 2^{\delta n})} |\mu_{k+1} - \mu_k| \\ &= \sup_k |\mu_{k+1} - \mu_k| = C < \infty \end{aligned}$$

(see [7]), and Theorem 2.3 applies. □

Remark. Of course, with the obvious changes, Theorem 4.3 may be adapted to obtain

$$[T, T_\mu] : B_p^{\sigma, r}(\mathbf{R}^n) \longrightarrow B_p^{\sigma', r}(\mathbf{R}^n)$$

for

$$T : (B_p^{\sigma_0, q_0}(\mathbf{R}^n), B_p^{\sigma_1, q_1}(\mathbf{R}^n)) \longrightarrow (B_p^{\sigma_0', q_0'}(\mathbf{R}^n), B_p^{\sigma_1', q_1'}(\mathbf{R}^n)),$$

if μ is q-admissible, $\sigma_0 - \sigma_1 = \sigma_0' - \sigma_1'$ and $1 < p < \infty$ and $|1/p - 1/2| < 1/q$.

Remark. The interested reader can find in [5] a direct proof that does not use the abstract methods of Theorem 1. Only the last part of the proof of Theorem 4.3 needs to be adapted.

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References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators,* Academic Press, 1988.

[2] Yu.A. Brudnyiĭ and N.Ya. Krugljak, *Interpolation Functors and Interpolation Spaces,* North Holland, 1991.

[3] M.J. Carro, J. Cerdà and J. Soria, Commutators and interpolation methods, *Ark. Mat.* **33** (1995), 199-216.

[4] J. Cerdà, N.Ya. Krugljak and J. Martín, Commutators for approximation spaces and Marcinkiewicz–type multipliers, *J. Approx. Theory* **100** (1999), 251-265.

[5] J. Cerdà and J. Martín, Commutators for Fourier multipliers on Besov spaces, *J. Approx. Theory* **129** (2004), 119-128.

[6] R.A. DeVore and G.G. Lorentz, *Constructive Approximation,* Springer–Verlag, 1993.

[7] R.E. Edwards and G.I. Gaudry, *Littlewood-Paley Multipliers Theory,* Springer-Verlag, 1977.

[8] B. Jawerth, R. Rochberg and G. Weiss, Commutators and other second order estimates in real interpolation theory, *Ark. Mat.* **24** (1986), 191–219.

[9] J. Peetre, *New thoughts on Besov spaces,* Duke University Mathematics Series, 1976.

[10] J. Peetre and G. Sparr, Interpolation of normed abelian groups, *Ann. Math. Pura Appl.* **92** (1972), 217–262.

[11] R. Rochberg and G. Weiss, Derivatives of analytic families of Banach spaces, *Ann. of Math.* **118** (1983), 315–347.

[12] Q. Xu, Fourier multipliers for $L^p(R^n)$ via q–variation, *Pacific. J. Math.* **176** (1996), 287–296.

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Uniform Integrability, Rearrangements and Embeddings

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Abstract. A theorem of de la Vallée Poussin characterizes a uniformly integrable subset of $L_1(X, \Sigma, \mu)$, where (X, Σ, μ) is a probability space, as a set that is bounded in some Orlicz space L_Φ, with $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. In this paper we investigate the possibility of replacing the Orlicz space in this result by other types of Banach function spaces. The fact that uniform integrability can be charcterized in terms of rearrangements of functions enables such rearrangement invariant spaces to be constructed, and for the results to be generalized to non-commutative function spaces.

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1 Introduction

If (X, Σ, μ) is a probability space, $1 < p \leq \infty$ and K is a bounded subset of $L_p(X, \Sigma, \mu)$, then K is a uniformly integrable set. Not all uniformly integrable subsets of $L_1(X, \Sigma, \mu)$ arise in this fashion, however; for this purpose the family $\{L_p(X, \Sigma, \mu) : p > 1\}$ is not large enough. A classical theorem of de la Vallée Poussin (see, for example, [10], Theorem I.2.2) essentially says that the family of Orlicz spaces $L_\Phi(X, \Sigma, \mu)$ for which $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ is adequate for this purpose. The aim of this paper is to investigate whether there are other adequate families of Banach function spaces, and to attempt to characterize such spaces.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In the first section we give an overview of the various characterizations of uniform integrability and related concepts. This is done as far as possible for σ-finite measure spaces, with specialization to probability spaces where appropriate. Although most of the results are not new, we attempt to bring together ideas scattered throughout the literature and indicate the useful role of the decreasing rearrangement of a measurable function in this context. Adequate families of Banach function spaces are introduced in the second section and examples of such families of rearrangement invariant Banach function space constructed. This leads to a new proof of an improved version of de la Vallée Poussin's theorem. The fact that a set is uniformly integrable if and only if the same is true of the set of its rearrangements makes it possible to extend these results to non-commutative Banach function spaces, as is shown in the final section of the paper.

2 Uniformly integrable sets

In what follows (X, Σ, μ) will be a σ-finite measure space. The space of all real-valued measurable functions on X, with the usual identification of functions that are equal almost everywhere, will be denoted by $L_0(X, \Sigma, \mu)$. As usual, $L_1(X, \Sigma, \mu)$ and $L_\infty(X, \Sigma, \mu)$ stand for the space of integrable and essentially bounded real-valued functions respectively. We will frequently use the abbreviated notation $L_0(X)$, $L_1(X)$ and $L_\infty(X)$. The norm of $L_1(X)$ is given by $||f||_1 = \int_X |f|\, d\mu \quad (f \in L_1(X)$. The positive part f^+ of $f \in L_0(X)$ is given by $f^+(x) = f(x)$ if $f(x) \geq 0$ and $f^+(x) = 0$ otherwise. The cone $\{f \in L_0(X) : f = f^+\}$ is denoted by $L_0^+(X)$; with the partial order induced by this cone, $L_0(X)$ is a vector lattice with $L_1(X)$ as a vector sublattice.

The set of measurable sets with finite measure will be denoted by Σ_f, i.e. $\Sigma_f = \{A \in \Sigma : \mu(A) < \infty\}$. If $f \in L_0(X)$ and $t > 0$, $[|f| > t]$ stands for the set $\{x \in X : |f(x)| > t\}$. For a sequence (A_n) in Σ, we write $A_n \downarrow \emptyset$ when (A_n) is decreasing and $\mu(\cap_{n=1}^\infty A_n) = 0$ (equivalently, $\mu(B \cap A_n) \to 0$ as $n \to \infty$ for every $B \in \Sigma_f$).

Definition 2.1 *A subset K of $L_1(X)$ is* uniformly absolutely continuous *if and only if for every $\epsilon > 0$ there is a $\delta > 0$ and an $X_\epsilon \in \Sigma_f$ such that $\sup_{f \in K} \int_A |f|\, d\mu < \epsilon$ whenever $A \in \Sigma$ and $\mu(A \cap X_\epsilon) < \delta$.*

Note that if $\mu(X) < \infty$, K is uniformly absolutely continuous if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sup_{f \in K} \int_A |f|\, d\mu < \epsilon$

whenever $A \in \Sigma$ and $\mu(A) < \delta$. For future reference, we list some known or easily checked equivalent forms of the definition.

Proposition 2.2 *Let $K \subset L_1(X)$. The following statements are equivalent:*

1. *K is uniformly absolutely continuous.*
2. *For every $\epsilon > 0$ there is a $\delta > 0$ and an $X_\epsilon \in \Sigma_f$ such that whenever $A \in \Sigma$ and $\mu(A \cap X_\epsilon) < \delta$ then $\sup_{f\in K} |\int_A f\, d\mu| < \epsilon$.*
3. *For every $\epsilon > 0$ there is a $\delta > 0$ and an $X_\epsilon \in \Sigma_f$ such that $\sup_{f\in K} \int_{X\setminus X_\epsilon} |f|\, d\mu < \epsilon$ and $\sup_{f\in K} \int_A |f|\, d\mu < \epsilon$ whenever $A \in \Sigma$ and $\mu(A) < \delta$.*
4. *$\sup_{f\in K} |\int_{A_n} f\, d\mu| \to 0$ as $n \to \infty$ whenever $A_n \downarrow \emptyset$ in Σ.*
5. *$\sup_{f\in K} |\int_{A_n} f\, d\mu| \to 0$ as $n \to \infty$ whenever (A_n) is a pairwise disjoint sequence in Σ.*

Proposition 2.3 *Let (X, Σ, μ) be non-atomic and $K \subset L_1(X)$ be uniformly absolutely continuous. Then K is bounded.*

Proof. Choose $A_1 \in \Sigma_f, 0 < \delta < \mu(A_1)$ such that if $A \in \Sigma, \mu(A \cap A_1) < \delta$, then $\sup_{f\in K} \int_A |f|\, d\mu < 1$. Since A_1 is non-atomic, there are disjoint sets $A_2, A_3 \subset A$ such that $A_2 \cup A_3 = A_1$ and $\mu(A_2) = \frac{1}{2}\delta$. Let $A = (X \setminus A_1) \cup A_2$, then $\mu(A \cap A_1) = \mu(A_2) < \delta$, so $\sup_{f\in K} \int_A |f|\, d\mu \leq 1$. Choose $n \in \mathbb{N}$ large enough to ensure that $\frac{1}{n}\mu(A_3) < \delta$. Since A_3 is non-atomic, there are disjoint sets $B_1, \ldots, B_n \in \Sigma_f$ such that $\cup_{i=1}^n B_i = A_3$ and $\mu(B_i) = \frac{1}{n}\mu(A_3)$. It follows that if $f \in K$, then $\int_{A_3} |f|\, du = \sum_{i=1}^n \int_{B_i} |f|\, d\mu \leq n$. Hence

$$\sup_{f\in K} \int_X |f|\, du = \sup_{f\in K} \int_A |f|\, du + \sup_{f\in K} \int_{A_3} |f|\, du \leq 1 + n.$$

□

Corollary 2.4 *Every uniformly absolutely continuous subset of $L_1(X)$ is bounded if and only if X contains no atoms of finite positive measure.*

Proof. If $B_0 \in \Sigma$ is an atom such that $0 < \mu(B_0) < \infty$, then the set $\{n\chi_{B_0} : n \in \mathbb{N}\}$ is uniformly absolutely continuous but unbounded.

Conversely, suppose X contains no atoms of finite positive measure, and write $X = X_1 \cup X_2$, where X_1 contains no atoms and X_2 is a disjoint union of atoms of infinite measure. Now let $K \subset L_1(X)$ be uniformly absolutely

continuous. Then for every $f \in K$ and every atom A in X_2, f is 0 on A. Thus we may assume $K \subset L_1(X)$. It follows from the proposition that K is bounded. □

The fact that, in general, a uniformly absolutely continuous set need not be bounded leads to:

Definition 2.5 *A subset K of $L_1(X)$ will be called* uniformly integrable *if and only if it is bounded in $L_1(X)$ and uniformly absolutely continuous.*

If $u \in L_1(X)$ then $\{u\}$ is uniformly integrable. From this it follows easily that the *order interval* $[-u, u] = \{f \in L_1(X) : |f| \leq u\}$ is uniformly integrable. This example can be stretched somewhat:

Definition 2.6 *Let $B_\epsilon = \{f \in L_1(X) : ||f||_1 < \epsilon\}$. A subset K of $L_1(X)$ is* order precompact *or* almost order bounded *if for every $\epsilon > 0$ there is a $u_\epsilon \in L_1^+(X)$ such that $K \subset [-u_\epsilon, u_\epsilon] + B_\epsilon$, or, equivalently, such that $\sup_{f\in K} ||(|f| - u_\epsilon)^+||_1 = \sup_{f\in K} || \, |f| - |f| \wedge u_\epsilon||_1 < \epsilon$.*

The notion of order precompactness makes sense in any normed lattice (or even a lattice with a locally solid topology). The reader is referred to [7] and [12], §122 for further information in this regard.

It is easy to check that an order precompact set is uniformly integrable. The converse also holds:

Proposition 2.7 *A subset K of $L_1(X)$ is uniformly integrable if and only if it is order precompact.*

Proof. If K is uniformly integrable, it is bounded, say $\sup_{f\in K} ||f||_1 = C$. If $\epsilon > 0$, we can find a $\delta > 0$ and an $X_\epsilon \in \Sigma_f$ such that $\sup_{f\in K} \int_{X\setminus X_\epsilon} |f| \, d\mu < \frac{1}{2}\epsilon$ and $\sup_{f\in K} \int_A |f| \, d\mu < \frac{1}{2}\epsilon$ whenever $A \in \Sigma$ and $\mu(A) < \delta$. Choose $m > 0$ large enough to ensure that $\frac{C}{m} < \delta$. Since $m\mu([|f| > m]) \leq ||f||_1 \leq C$, for every $f \in K$, $\mu([|f| > m]) \leq \frac{C}{m} < \delta$. Put $u_\epsilon = m\chi_{X_\epsilon}$. Then $u_\epsilon \in L_1(X)^+$ and if $f \in K$,

$$|| \, |f| - |f| \wedge u_\epsilon||_1 \leq \int_{[|f|>m]} |f| \, d\mu + \int_{X\setminus X_\epsilon} |f| \, d\mu < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon,$$

showing that K is order precompact. □

Proposition 2.8 *If $K \subset L_1(X)$ is uniformly integrable, $\sup\{\int_{[|f|>c]} |f| \, d\mu : f \in K\} \to 0$ as $c \to \infty$. When (X, Σ, μ) is non-atomic,* uniformly integrable *may be replaced by* uniformly absolutely continuous.

Proof. Since K is bounded, $C = \sup\{||f||_1 : f \in K\} < \infty$. For $c > 0$ and $f \in K$, $c\mu([|f| > c]) \leq ||f||_1 \leq C$. This shows that $\mu([|f| > c]) \leq \frac{k}{c} \to 0$ as $c \to \infty$. The first result now follows easily from the fact that K is uniformly absolutely continuous; the second is then immediate from Proposition 2.3. □

If K is uniformly absolutely continuous, it need not be the case that $\sup\{\int_{[|f|>c]} |f|\, d\mu : f \in K\} \to 0$ as $c \to \infty$, as shown by the the set $\{n\chi_A : n \in \mathbb{N}\}$, where A is an atom with $\mu)(A) > 0$. The set $K = \{\chi_{[0,n]} : n \in \mathbb{N}\}$ in $L_1([0,\infty)$ satisfies $\sup\{\int_{[|f|>c]} |f|\, d\mu : f \in K\} \to 0$ as $c \to \infty$, but is neither bounded, nor uniformly absolutely continuous.

If $\mu(X) < \infty$ and $\sup_{f\in K} \int_{[|f|>c]} |f|\, d\mu \to 0$ as $c \to \infty$, then K is bounded. This is clear from the inequality

$$\int_X |f|\, d\mu = \int_{[|f|>c]} |f|\, d\mu + \int_{[|f|\leq c]} |f|\, d\mu \leq \int_{[|f|>c]} |f|\, d\mu + c\mu(X).$$

It follows from this observation that our definition of uniform integrability agrees with the usual one in finite measure spaces:

Corollary 2.9 *If $\mu(X) < \infty$, a subset K of $L_1(X)$ is uniformly integrable if and only if $\sup_{f\in K} \int_{[|f|>c]} |f|\, d\mu \to 0$ as $c \to \infty$.*

We note in passing that the condition $\sup\{\int_{[|f|>c]} |f|\, d\mu : f \in K\} \to 0$ as $c \to \infty$ is not equivalent to $\sup\{|\int_{[|f|>c]} f\, d\mu| : f \in K\} \to 0$ as $c \to \infty$. The first condition clearly implies the second. To see that the converse does not hold, let $f_n(t) = 2^n\chi_{[0,2^{-n-1}]}(t) - 2^n\chi_{[2^{-n-1},2^{-n}]}(t)$ and $K = \{f_n : n \in \mathbb{N}\}$. Since $||f_n||_1 = 1$ for every n, K is bounded and $\sup\{|\int_{[|f_n|>c]} f_n\, d\mu| : n \in \mathbb{N}\} = 0$ for every $c > 0$, but $\sup\{\int_{[|f_n|>c]} |f_n|\, d\mu : n \in \mathbb{N}\} = 1$ for every $c > 0$.

If $f \in L_0(X)$, its *decreasing rearrangement* is the function f^* on $[0,\infty)$ defined by

$$f^*(t) = \inf\{s : \mu([|f| > s]) \leq t\}.$$

For $f \in L_1(X)$ we have $f^*(t) < \infty$ for every $t > 0$. More information on decreasing rearrangements can be found in [2] and [9].

The following result is a slight improvement of [3]], Theorem 2.1:

Proposition 2.10 *Let $K \subset L_1(X)$. Then the following are equivalent:*

1. $\sup_{f\in K} \int_{[|f|>c]} |f|\, d\mu \to 0$ *as* $c \to \infty$
2. $\sup_{f\in K} \int_X (|f| - c)^+\, d\mu \to 0$ *as* $c \to \infty$

3. $\sup_{f\in K}\int_0^t f^*(s)\,ds \to 0$ *as* $t \to 0$.

Proof. The inequality

$$\int_X (|f|-c)^+\,d\mu = \int_{[|f|>c]} (|f|-c)^+\,d\mu \le \int_{[|f|>c]} |f|\,d\mu$$

shows that (1) implies (2).
Suppose (2) holds. By considering the cases $f^*(t) > c$ and $f^*(t) \le c$ separately, it can be shown that

$$\int_0^t f^*(s)\,ds \le \int (|f|-c)^+\,d\mu + ct$$

for all $t > 0, c > 0$ and $f \in K$. Given $\epsilon > 0$, choose c large enough to ensure that $\sup_{f\in K|}\int_X (|f|-c)^+\,d\mu < \frac{1}{2}\epsilon$, and then choose t small enough to ensure that $2ct < \epsilon$. Then for every $f \in K$,

$$\int_0^t f^*(s)\,ds \le \int (|f|-c)^+\,d\mu + ct = \int_{[|f|>c]} |f|\,d\mu + ct < \epsilon,$$

so that (3) holds.

Finally, suppose (3) holds and $\epsilon > 0$. We can choose $t > 0$ such that $\sup_{f\in K}\int_0^t f^*(s)\,ds < \epsilon$. Then it follows as in the proof of [3], Theorem 2.1 that there is a $c_t > 0$ such that $\sup_{f\in K} f^*(t) = c_t$. Then $\lambda([f^* > c_t]) \le \lambda([0,t]) = t$, where λ denotes Lebesgue measure on $[0,\infty)$. Hence for every $f \in K$,

$$\int_{[|f|>c]} |f|\,d\mu = \int_0^{\mu([f^*>c_t])} f^*(s)\,ds \le \int_0^t f^*(s)\,ds < \epsilon.$$

□

From Corollary 2.9 we immediately get

Corollary 2.11 *If* $\mu(X) < \infty$*, a set* K *in* $L_1(X)$ *is uniformly integrable if and only it satisfies any one of the above three conditions.*

The maximal function f^{**} of f^* is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \qquad (t > 0).$$

Let (X_1, Σ_1, μ_1) be a (possibly different) σ-finite measure space (with $\mu(X) = \mu_1(X_1)$ if the measure spaces are finite) and $f \in L_0(X), g \in L_0(X_1)$. If

$f^{**}(t) \leq g^{**}(t)$ for every $t \in (0,\infty)$, we say f is submajorized by g and write this as $f \prec\prec g$. For fixed $g \in L_1(X_1)$ we write

$$\Omega(g) = \{f \in L_1(X) : f \prec\prec g\}.$$

It follows from the definition that if $K \subset \Omega(g)$ then $\sup_{f\in K} \int_0^t f^*(s)\,ds \to 0$ as $t \to 0$ and K is bounded in $L_1(X)$. The converse is also true; this result (for the case $\mu(X) < \infty$) is essentially given in [3], Lemma 4.1. We include a proof since the techniques used will be useful in the next section.

Proposition 2.12 *For a subset K of $L_1(X)$ there is a $g \in L_1^+[0,\infty)$ such that $K \subset \Omega(g)$ if and only if K is bounded in $L_1(X)$ and $\sup_{f\in K} \int_0^t f^*(s)\,ds \to 0$ as $t \to 0$.*

Proof. Suppose $||f||_1 \leq C$ for every $f \in K$. We define the function $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \sup_{f\in K} \int_0^t f^*(s)\,ds.$$

It follows immediately from this definition that $\psi(0) = 0$, $\psi(t) > 0$ if $t > 0$ (unless $K = \{0\}$) and that ψ is increasing. Also $\psi(t) = \int_0^t f^*(s)\,ds \leq \int_0^\infty f^*(s)\,ds = \int_X |f|\,d\mu \leq C$ for all $t \geq 0$. For $t > 0$,

$$\frac{\psi(t)}{t} = \sup_{f\in K} \frac{1}{t} \int_0^t f^*(s)\,ds = \sup_{f\in K} f^{**}(t).$$

Since each f^{**} is decreasing ([2], Proposition 2.3.2), so is $\frac{\psi(t)}{t}$. This shows that ψ is a quasi-concave function. It follows that if $\widetilde{\psi}$ is the least concave majorant of ψ, we have $\frac{1}{2}\widetilde{\psi}(t) \leq \psi(t) \leq \widetilde{\psi}(t)$ for every $t \in [0,\infty)$ ([2], Proposition 2.5.10). Since $\widetilde{\psi}$ is concave, it is differentiable almost everywhere. We define the function $g : [0,\infty) \to \mathbb{R}$ such that it is right continuous and $g(t) = \widetilde{\psi}'(t)$ for those t at which $\widetilde{\psi}$ is differentiable. Then g is positive, decreasing and right continuous on $[0,\infty)$. Also

$$0 \leq \frac{1}{2}\widetilde{\psi}(0^+) \leq \psi(0^+) = \lim_{t\downarrow 0} \sup\{\int_0^t f^*(s)\,ds : f \in K\} = 0,$$

and so $\int_0^\infty g(s)\,ds = \int_0^\infty \widetilde{\psi}'(s)\,ds = \lim_{t\to\infty} \widetilde{\psi}(t) - \widetilde{\psi}(0^+) \leq C$, showing that $g \in L_1[0,\infty)$. Thus if $f \in K$ and $0 < t < \infty$,

$$\int_0^t f^*(s)\,ds \leq \psi(t) \leq \widetilde{\psi}(t) = \int_0^t \widetilde{\psi}'(s)\,ds = \int_0^t g(s)\,ds.$$

Hence $f \prec\prec g$, for every $f \in K$. □

Corollary 2.13 *If $\mu(X) = a < \infty$, then $K \subset L_1(X)$ is uniformly integrable if and only if there is a $g \in L_1([0,a])$ such that $K \subset \Omega(g)$.*

Note that if $\mu(X) = \infty$, $\Omega(g)$ need not be uniformly absolutely continuous. As an example, let $g(t) = \chi_{[0,1]}(t)$, $f_n(t) = \chi_{[n-1,n]}(t)$, $t > 0$ and $K = \{f_n : n \in \mathbb{N}\}$. Then $K \subset \Omega(g)$, but K is not uniformly absolutely continuous, as can be seen by considering the sequence (A_m) in Σ, with $A_m = [m, \infty)$ for which we have $A_m \downarrow \emptyset$ but $\sup_n \int_{A_m} |f_n|\, du = 1$ for every m.

3 Adequate families

We shall call a Banach space E (with norm $||\cdot||_E$) contained in $L_0(X)$ a *Banach function space* if it is an order ideal in $L_0(X)$ (i.e if $f \in L_0(X), g \in E$ and $|f| \leq |g|$ implies $f \in E$ and $||f||_E \leq ||g||_E$). The Köthe dual of E is the order ideal $E^\times$ of $L_0(X)$ defined by

$$E^\times = \{g \in L_0(X) : \int_X |fg|\, d\mu < \infty \text{ for every } f \in E\}.$$

Then for every $g \in E^\times$ the functional on E defined by $\varphi_g(f) = \int_X fg\, d\mu$ is in the continuous dual E^* of E and

$$||\varphi_g||_{E^*} = ||g||_{E^\times} \equiv \sup\{\int_X |fg|\, d\mu : f \in E, ||f||_E \leq 1\}.$$

With the norm $||\cdot||_{E^\times}$ $E^\times$ is itself a Banach function space. Further information may be found in [2], Chapter 1 and [11], Chapter 15.

The closed linear subspace of a Banach function space E generated by the set of all characteristic functions of sets in Σ_f will be denoted by E_b. We say that $f \in E$ has absolutely continuous norm if $||f\chi_{A_n}||_E \to 0$ as $n \to \infty$ whenever (A_n) is a sequence in Σ such that $A_n \downarrow \emptyset$. The set of all $f \in E$ with absolutely continuous norm is denoted by E_a. We have $E_a \subset E_b$.

Definition 3.1 *We call a family $\mathcal{E}$ of Banach function spaces* adequate *if*
(a) $E \subset L_1(X)$ for every $E \in \mathcal{E}$;
(b) a subset K of $L_1(X)$ is uniformly integrable if and only if there is an $E \in \mathcal{E}$ such that K is a bounded subset of E.

It is immediate from the definition that $\cup\{E : E \in \mathcal{E}\} = L_1(X)$. A necessary condition for a family to be adequate is that any norm bounded

subset of a member of the family must be uniformly integrable. Our first result narrows down the field.

Proposition 3.2 *Let E be a Banach function space such that $E \subset L_1(X)$. Then every norm bounded subset of E is uniformly integrable if and only if $L_\infty(X) \subset E_a^\times$. If $\mu(X) < \infty$, this is equivalent to $E_b^\times = E_a^\times$.*

Proof. It follows from $E \subset L_1(X)$ that $L_\infty(X) \subset E^\times$. Suppose that $L_\infty(X) \subset E_a^\times$ and that $K \subset E$ is bounded in E, say $||f||_E \leq C$ for all $f \in K$. For any $A \in \Sigma$,

$$\sup_{f\in K} \int_A |f|\, d\mu = \sup_{f\in K} \int_X |f|\chi_A\, d\mu \leq \sup_{f\in K} ||f||_E ||\chi_A||_{E^\times} \leq C||\chi_A||_{E^\times}.$$

Taking $A = X$ shows at once that K is bounded in $L_1(X)$. Let (A_n) be a sequence in Σ such that $A_n \downarrow \emptyset$. Since $\chi_X \in E_a^\times$, it follows from the inequality above that $\sup_{f\in K} \int_{A_n} |f|\, d\mu \leq C||\chi_X \chi_{A_n}||_{E^\times} \to 0$ as $n \to \infty$. It follows that K is uniformly integrable.

Conversely, suppose that $B = \{f \in E : ||f||_E \leq 1\}$ is uniformly integrable and $g \in L_\infty(X)$. For $A \in \Sigma$

$$||g\chi_A||_{E^\times} = \sup_{f\in B} \int_X |fg\chi_A|\, d\mu \leq ||g||_\infty \sup_{f\in B} \int_A |f|\, d\mu.$$

Taking $A = X$ shows that $g \in E^\times$; the same inequality shows that if $A_n \downarrow \emptyset$ then $||g\chi_{A_n}||_{E^\times} \to 0$ as $n \to \infty$, and hence $L_\infty(X) \subset E_a^\times$.

The last statement follows from [2], I.3.13. □

A Banach function space E is called *rearrangement invariant* if $f \in L_0(X), g \in E$ and $f^* = g^*$ implies $f \in E$ and $||f||_E = ||g||_E$. If $f, g \in E$ and $f \prec\prec g$ implies $||f||_E \leq ||g||_E$, then E is *symmetric*. The reader is referred to [2], Chapter II and [9], Chapter II for further information.

Proposition 3.3 *Let E be a rearrangement invariant Banach function space over the σ-finite infinite measure space (X, Σ, μ). Then we cannot have $L_\infty(X) \subset E_a$.*

Proof. There is a sequence (X_n) in Σ_f such that $X_n \uparrow X$. Then $A_n = X \setminus X_n \downarrow \emptyset$ and $\mu(A_n) = \infty$ for every n. If $L_\infty(X) \subset E_a$, then $\chi_X \in E_a$ and hence we must have $||\chi_X \chi_{A_n}||_E = ||\chi_{A_n}||_E \to 0$ as $n \to \infty$. But $\chi_{A_n}^* = \chi_X^*$ and so $||\chi_{A_n}||_E = ||\chi_X||_E > 0$ for every n, a contradiction. □

If (X, Σ, μ) is either non-atomic or consists of a countable number of atoms of equal measure and E is rearrangement invariant, then so is $E^\times$ ([2], Corollary II.4.4). The above proposition shows that there is no hope of finding adequate families of rearrangement invariant spaces over infinite measure spaces of this kind. In the rest of this section we shall therefore assume that $\mu(X) < \infty$, and without loss of generality we may assume that (X, Σ, μ) is a probability space.

Definition 3.4 *Let $E \subset L_0(X)$ be a rearrangement invariant Banach function space. The fundamental function φ_E of E is defined by*

$$\varphi_E(t) = \sup\{||\chi_A||_E : A \in \Sigma, \mu(A) \leq t\} \qquad t \in [0, 1].$$

If t is in the range of μ, then it follows that for every $A \in \Sigma$ such that $\mu(A) = t$, we have $\varphi_E(t) = ||\chi_A||_E$.

The proof of [2], Theorem II.5.5 can be adapted to show that if the measure space (X, Σ, μ) has the property that $\inf\{\mu(A) : A \in \Sigma, \mu(A) > 0\} = 0$ (in particular, if the measure space is non-atomic), then $E_a = E_b$ if and only if $\varphi_{E^\times}(0^+) = 0$. If $\inf\{\mu(A) : A \in \Sigma, \mu(A) > 0\} = a > 0$, then $\varphi_E(t) = 0$ for $0 \leq t < a$ and $E_a = E_b$. Combining Proposition 3.2 with these observations we obtain:

Corollary 3.5 *Let $E \subset L_1(X)$ be a rearrangement invariant Banach function space. Then every norm bounded subset of E is uniformly integrable if and only if $\varphi_{E^\times}(0^+) = 0$.*

It follows from Proposition 2.13 that for a family $\mathcal{E}$ to be adequate, we need to be able to find, for each $g \in L_1([0, 1])$, an $E \in \mathcal{E}$ such that $\Omega(g)$ is bounded in E. This suggests a way of constructing an adequate family. For each $g \in L_1^+([0, 1])$ we construct a Banach function space M_g such that $\Omega(g)$ is the closed unit ball of M_g. Put

$$\psi_g(t) = \int_0^t g^*(s)\, ds \text{ for } t \in [0, 1] \text{ and } \varphi_g(0) = 0, \varphi_g(t) = \frac{t}{\psi_g(t)} \text{ for } t \in (0, 1].$$

It is then easy to check that φ_g is a quasi-concave function on $[0, 1]$. It follows that

$$||f||_{M_g} = \inf\{\lambda > 0 : f \in \lambda\Omega(g)\} = \sup\{\varphi_g(t) f^{**}(t) : t \in (0, 1]\}$$

is a rearrangement invariant norm on the subspace

$$M_g(X) = \cup\{n\Omega(g) : n \in \mathbb{N}\} = \{f \in L_1(X) : ||f||_{M_g} < \infty\}$$

of $L_1(X)$, and that the fundamental function of M_g is φ_g (cf. [2], Proposition II.5.8). This construction immediately gives:

Proposition 3.6 *A subset K of $L_1(X)$ is uniformly integrable if and only if there is a $g \in L_1^+([0,1])$ such that K is bounded in $M_g(X)$. The family $\mathcal{E} = \{M_g(X) : g \in L_1^+([0,1])\}$ is therefore adequate.*

Corollary 3.7 *A family $\mathcal{E}$ is adequate if and only if*

1. *for every $g \in L_1([0,1])$, there is an $E \in \mathcal{E}$ such that $M_g \hookrightarrow E$ and*
2. *for every $E \in \mathcal{E}$, there is a $g \in L_1([0,1])$ such that $E \hookrightarrow M_g$,*

where $\hookrightarrow$ denotes a continuous embedding.

We are now ready to construct another important adequate family. Let $h \in L_1^+([0,1])$ and put

$$\Lambda_h(X) = \{f \in L_0(X) : \int_0^1 f^*(t)h^*(t)\,dt < \infty\}$$

and

$$||f||_{\Lambda_h} = \int_0^1 f^*(t)h^*(t)\,dt.$$

Then $\Lambda_h(X)$ is a rearrangement invariant Banach function space when equipped with the norm $||\cdot||_{\Lambda_h}$, and its fundamental function is given by $\varphi_{\Lambda_h}(t) = \int_0^t h^*(t)\,dt$ (cf. [2], II.5.12). Note that if $h \in L_\infty([0,1])$, then $L_1(X) \subset \Lambda_h(X)$.

Proposition 3.8 *Let (X, Σ, μ) be non-atomic. A subset K of $L_1(X)$ is uniformly integrable if and only if there is a $h \in L_1^+([0,1]) \setminus L_\infty([0,1])$ such that K is bounded in $\Lambda_h(X)$. The family $\mathcal{E} = \{\Lambda_h(X) : h \in L_1([0,1]) \setminus L_\infty([0,1])\}$ is therefore adequate.*

Proof. Let K be uniformly integrable. Then it follows from Proposition 3.6 that there is a $g \in L_1^+([0,1])$ such that K is a bounded subset of $M_g(X)$, say $K \subset C_1\Omega(g)$. Since $M_g([0,1]) \subset L_1([0,1])$ but $M_g([0,1]) \neq L_1([0,1])$, we have $L_\infty([0,1]) \subset M_g([0,1])^\times$ but $L_\infty([0,1]) \neq M_g([0,1])^\times$. Hence there is an $h \in M_g([0,1])^\times \setminus L_\infty([0,1])$ such that $\int_0^1 g^*(t)h^*(t)\,dt = C_2 < \infty$ ([2], Corollary II.4.4). If $f \in K$, then $f \prec\prec C_1 g$ and therefore

$$||f||_{\Lambda_h} = \int_0^1 f^*(t)h^*(t)\,dt \leq C_1 \int_0^1 g^*(t)h^*(t)\,dt = C_1C_2,$$

by Hardy's Lemma ([2], II.3.6). Hence K is bounded in $\Lambda_h(X)$.

Conversely, suppose K is bounded in $\Lambda_h(X)$, for some $h \in L_1^+([0,1]) \setminus L_\infty([0,1])$. The fundamental function of $\Lambda_h(X)^\times$ is given by

$$\varphi_{\Lambda_h(X)^\times}(t) = \frac{t}{\int_0^t h^*(s)\,ds} = \frac{1}{h^{**}(t)},$$

by [2], II.5.2. Since $h \notin L_\infty([0,1])$ and therefore $\lim_{t\to 0} h^{**}(t) = \infty$, it follows that $\varphi_{\Lambda_h(X)^\times}(t) \to 0$ as $t \to 0$. It follows from Proposition 3.5 that K is uniformly integrable. □

Let (X, Σ, μ) be non-atomic and $E \subset L_0(X)$ be a rearrangement invariant Banach function space. Let φ_E be the least concave majorant of the fundamental function of E. The rearrangement invariant Banach function spaces $M(E)$ and $\Lambda(E)$ are defined by putting

$$||f||_{M(E)} = \sup_{0<t\leq 1} \varphi_E(t) f^{**}(t), \quad M(E) = \{f \in L_0(X) : ||f||_{M(E)} < \infty\}$$

and

$$||f||_{\Lambda(E)} = \int_0^1 f^*(t) d\varphi_E(t)\,dt, \quad \Lambda(E) = \{f \in L_0(X) : ||f||_{\Lambda(E)} < \infty\}.$$

Each of these spaces has fundamental function φ_E. We have the continuous norm 1 embeddings

$$\Lambda(E) \hookrightarrow E \hookrightarrow M(E)$$

([2], Theorem II.5.13). If $\varphi_E(0^+) = a > 0$, it follows from $E \hookrightarrow M(E)$ that $||f||_\infty = \sup_{0<t\leq 1} f^{**}(t) \leq a^{-1}||f||_E$ for all $f \in E$. Since $L_\infty(X) \subset E$ ([2], II.6.6), this implies that $E = L_\infty(X)$ in this case.

The definitions in fact make sense with any non-negative concave function φ on $[0,\infty)$ which is 0 only at 0 in the place of φ_E. The spaces corresponding to $M(E)$ and $\Lambda(E)$ are known as Marcinkiewicz and Lorentz spaces respectively and it can be shown that these spaces are symmetric as well.

Theorem 3.9 *Let $\mathcal{E}$ be a family of rearrangement invariant Banach function spaces over a non-atomic probability space and put $\Phi = \{\varphi_E : E \in \mathcal{E}\}$, where for each E, φ_E denotes the least concave majorant of the fundamental function of E.*

1. *If $\mathcal{E}$ is an adequate family, then*
   (a) $\lim_{t\downarrow 0} \frac{t}{\varphi(t)} = 0$ *for every $\varphi \in \Phi$ and*

   (b) for every $g \in L_1([0,1])$ *there is a* $\varphi \in \Phi$ *such that* $\sup_{0<t\leq 1} g^{**}(t)\varphi(t) < \infty$.

2. *The family* $\mathcal{E}$ *is adequate if*
   (a) $\lim_{t\downarrow 0} \frac{t}{\varphi(t)} = 0$ *for every* $\varphi \in \Phi$ *and*
   (b) for every $g \in L_1([0,1])$ *there is a* $\varphi \in \Phi$ *such that* $\int_0^1 g^*(t)\, d\varphi(t) < \infty$.

Proof.

1. If $\mathcal{E}$ is an adequate family and $E \in \mathcal{E}$, then $\lim_{t\downarrow 0} \frac{t}{\varphi_E(t)} = \varphi_{E^\times}(0^+) = 0$, by Corollary 3.5. If $g \in L_1([0,1])$,then $\Omega(g)$ is uniformly integrable, so there is an $E \in \mathcal{E}$ such that $\Omega(g)$ is bounded in E. Since $E \hookrightarrow M(E)$, it follows that $\Omega(g)$ is bounded in $M(E)$. This implies that $\sup_{f\in\Omega(g)} \sup_{0<t\leq 1} \varphi_E(t) f^{**}(t) = \sup_{0<t\leq 1} \varphi_E(t) g^{**}(t) < \infty$.
2. The first condition ensures that for every $E \in \mathcal{E}$, bounded subsets of E are uniformly integrable (Corollary 3.5). If $K \subset L_1(X)$ is uniformly integrable, then there is a $g \in L_1([0,1])$ such that $K \subset \Omega(g)$. By (b) there is an $E \in \mathcal{E}$ such that $||g||_\infty \varphi_E(0^+) + \int_0^1 g^*(t)\varphi'_E(t)\, dt = \int_0^1 g^*(t)\, d\varphi_E(t) < \infty$. Hence if $\varphi_E(0^+) = 0$ and $f \in \Omega(g)$,

$$\int_0^1 g^*(t)\, d\varphi_E(t) = \int_0^1 f^*(t)\varphi'_E(t)\, dt \leq \int_0^1 g^*(t)\varphi'_E(t)\, dt < \infty,$$

   showing that K is bounded in $\Lambda(E)$. If $\varphi_E(0^+) > 0$, it follows from the remarks above that $\Lambda(E) = E = L_\infty(X)$ and $g \in L_\infty(X)$. Then $f \in \Omega(g)$ implies that $f^{**}(t) \leq ||g||_\infty$ for every $0 < t \leq 1$, and hence K is bounded in $L_\infty(X) = \Lambda(E)$.

□

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ which is convex, continuous at 0 and 0 only at 0 is called a Young's functions. The *Orlicz space* $L_\Phi(X)$ consists of all $f \in L_0(X)$ such that there is a $\lambda > 0$ such that $\int_X \Phi(|\lambda f|)\, d\mu < \infty$. When equipped with the Luxemburg norm

$$||f|| = \inf\{\lambda^{-1} : \int_X \Phi(|\lambda f|)\, d\mu \leq 1\},$$

$L_\Phi(X)$ becomes a symmetric, rearrangement invariant Banach function space. More information on Orlicz spaces can be found in [2], IV.8, [8] and [10].

Let Φ be a Young's function with right-inverse Φ^{-1} given by

$$\Phi^{-1}(t) = \sup\{s : \Phi(s) \leq t\}.$$

The fundamental function φ_{L_Φ} of the Orlicz space $L_\Phi(X)$ is given by $\varphi_{L_\Phi}(t) = [\Phi^{-1}(\frac{1}{t})]^{-1}$ ([2], IV.8.17). The fundamental function $\varphi_{L_\Phi^\times}$ of the Kőthe dual of L_Φ is therefore given by $\varphi_{L_\Phi^\times}(t) = \frac{t}{\varphi_{L_\Phi}(t)} = t\Phi^{-1}(\frac{1}{t})$ ([2], II.5.2). Hence

$$\lim_{t\downarrow 0} \varphi_{L_\Phi^\times}(t) = \lim_{t\downarrow 0} t\Phi^{-1}(\frac{1}{t}) = \lim_{s\to\infty} \frac{\Phi^{-1}(s)}{s} = \lim_{s\to\infty} \frac{s}{\Phi(s)}.$$

It follows that $\lim_{t\downarrow 0} \varphi_{L_\Phi^\times}(t) = 0$ if and only if $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. In conjunction with Corollary 3.5 this gives

Proposition 3.10 *Let (X, Σ, μ) be a non-atomic probability space and Φ be a Young's function. The bounded subsets of the Orlicz space $L_\Phi(X)$ are uniformly integrable if and only if* $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

The proposition gives a necessary condition for a family of Orlicz spaces to be adequate. We show that it is also sufficient, thus recovering de la Vallée Poussin's theorem in the process.

Theorem 3.11 *Let (X, Σ, μ) be a non-atomic probability space. The family*

$$\mathcal{E} = \{L_\Phi(X) : \Phi \text{ is a Young's function, } \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty\}$$

is adequate.

Proof. In the light of the last proposition it will suffice to show that every uniformly integrable set $K \subset L_1(X)$ is bounded in some member of the family. By Proposition 3.8 we can find an $h \in L_1([0,1]) \setminus L_\infty([0,1])$ such that K is bounded in $\Lambda_h(X)$. It will suffice to find an $L_\Phi(X) \in \mathcal{E}$ with the same fundamental function as $\Lambda_h(X)$, since we have a continuous embedding $\Lambda_h(X) \hookrightarrow L_\Phi(X)$ in this case. We know that the fundamental function φ of $\Lambda(X)$ is given by $\varphi(t) = \int_0^t h^*(s)\, ds$, and that φ is concave and strictly increasing. It follows that it has an inverse φ^{-1} which is a convex function on the range of φ. Let $k = \varphi(1)$ and $\alpha = k[h^*(1)]^{-1}$. Then $\alpha \geq 1$. Define

$$\Phi(t) = \begin{cases} (kt)^\alpha & \text{for } 0 \leq t \leq \frac{1}{k} \\ (\varphi^{-1}(\frac{1}{t}))^{-1} & \text{for } t > \frac{1}{k}. \end{cases}$$

The choice of α ensures that Φ is continuous and differentiable on $[0,\infty)$. Since φ^{-1} and the function $t \mapsto \frac{1}{t}$ are both convex and positive functions on $[k^{-1},\infty)$, and $t \mapsto (kt)^\alpha$ is positive and convex on $[0,k^{-1}]$, the same is true of Φ on $[0,\infty)$. We also have

$$\lim_{t\to\infty}\frac{\Phi(t)}{t} = \lim_{t\to\infty}\left[t\varphi^{-1}\left(\frac{1}{t}\right)\right]^{-1} = \lim_{s\to 0}\frac{\varphi(s)}{s} = \lim_{s\to 0} h^{**}(s) = \infty,$$

since $h \notin L_\infty([0,1])$. It follows from this that $L_\Phi(X)$ is in the family $\mathcal{E}$.

The fundamental function ψ of $L_\Phi(X)$ is given by $\psi(t) = [\Phi^{-1}(\frac{1}{t})]^{-1}$. It can be checked that $\psi(1) = \sup_{0<t\leq 1}\psi(t) = k$. If $0 < t \leq 1$ and $s = \psi(t)$, then $\Phi(s^{-1}) = t^{-1}$, and since $0 < s \leq k$, $t = [\Phi(s^{-1})]^{-1} = \varphi^{-1}(s)$, so that $\psi(t) = s = \varphi(t)$, as required. □

It is possible to refine the above construction to show that the family of Young's functions with the Δ_2 property is still adequate. Recall that a Young's function Φ is said to satisfy the Δ_2 condition if there are constants $t_0 > 0$ and $k > 0$ such that $\Phi(2t) \leq k\Phi(t)$ for $t \geq t_0$. If this is the case we write $\Phi \in \Delta_2$.

Suppose $K \subset L_1(X)$ is uniformly integrable and $g \in L_1([0,1])$ is such that $K \subset \Omega(g)$. Let ψ_g be defined as in the discussion preceding Proposition 3.6 and for fixed $0 < p < 1$ put

$$g_p(t) = \frac{1}{[\psi_g(t)]^p}, \quad 0 < t \leq 1.$$

Since g_p is decreasing on $(0,1]$, $g_p^* = g_p$ and since $\psi_p(0^+) = 0, g_p(0^+) = \infty$, so $g_p \notin L_\infty([0,1])$. Furthermore

$$||g_p||_{\Lambda_g} = \int_0^1 g_p^*(t)\psi_g'(t)\,dt = \int_0^1 \frac{\psi_g'(t)}{[\psi_g(t)]^p}\,dt = \frac{[\psi_g(1)]^{1-p}}{1-p} < \infty,$$

so $g_p \in \Lambda_g$. If $f \in K$ then $f \prec\prec g$. Hence

$$\int_0^1 f^*(t)g_p^*(t)\,dt \leq \int_0^1 g^*(t)g_p^*(t)\,dt = ||g_p||_{\Lambda_g},$$

showing that K is bounded in Λ_{g_p}. The fundamental function of Λ_{g_p} is given by $\varphi_p(t) = \int_0^t g_p^*(s)\,ds$, and $\varphi_p'(t) = g_p(t)$ almost everywhere in $[0,1]$. Since φ_p is positive, concave and strictly increasing, φ_p^{-1} exists and is positive and convex on the range of φ_p. Let $k = \varphi_p(1)$ and $\alpha = k[g_p^*(1)]^{-1}$; then $\alpha \geq 1$. Put

$$\Phi_p(t) = \begin{cases} (kt)^\alpha & \text{for } 0 \leq t \leq \frac{1}{k} \\ (\varphi_p^{-1}(\frac{1}{t}))^{-1} & \text{for } t > \frac{1}{k}. \end{cases}$$

Then it follows as before that Φ_p is convex and $\lim_{t\to\infty} \frac{\Phi_p(t)}{t} = \infty$, and that the fundamental function of L_{Φ_p} is φ_p. It remains to show that $\Phi_p \in \Delta_2$. Let $u > 0, 0 < r \leq 1$. Then it is easy to check that $\psi_g(ru) \geq r\psi_g(u)$, and hence that $g_p(ru) \leq r^{-p} g_p(u)$. From this it follows that $\varphi_p(ru) \leq r^{1-p}\varphi_p(u)$. Now let $t > k^{-1}$ and put $u = \varphi_p^{-1}(\frac{1}{t})$, so that $\varphi_p(u) = \frac{1}{t}$. Choose r such that $0 < r < 1$ and $r^{1-p} \leq \frac{1}{2}$. Then

$$r\varphi_p^{-1}\left(\frac{1}{t}\right) = ru \leq \varphi_p^{-1}\left(r^{1-p}\varphi_p(u)\right) \leq \varphi_p^{-1}\left(r^{1-p}\frac{1}{t}\right) \leq \varphi_p^{-1}\left(\frac{1}{2}\frac{1}{t}\right),$$

and hence $\Phi_p(2t) \leq r^{-1}\Phi_p(t)$. We have shown:

Theorem 3.12 *Let (X, Σ, μ) be a non-atomic probability space. The family*

$$\mathcal{E} = \{L_\Phi(X) : \Phi \text{ is a Young's function, } \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty,\ \Phi \in \Delta_2\}$$

is adequate.

We note that de la Vallée Poussin's theorem has also been generalized by Alexopoulos in [1], Theorem 2.5.

4 Uniform integrability in non-commutative spaces

A commutative von Neumann algebra can be represented as an L_∞-space over a suitable measure space, and the corresponding L_1-space is a predual for the von Neumann algebra. This motivates the interpretation of von Neumann algebras as non-commutative L_∞-spaces, and their preduals as non-commutative L_1-spaces. We look briefly at uniformly integrable sets in this context.

To have at hand an analogue of a measure, we restrict attention to a semi-finite von Neumann algebra $\mathcal{M}$ equipped with a faithful semi-finite normal trace τ. We assume that $\mathcal{M}$ is a subalgebra of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H and denote the inner product on H by $(\cdot, \cdot)$. The set of projections in $\mathcal{M}$ will be denoted by $\mathcal{M}_p$.

A closed densely defined operator x on H is affiliated with $\mathcal{M}$ if $u^* x u = x$ for all unitary operators u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. For such an operator x and $s \geq 0$ we denote by $\chi_{(s,\infty)}(|x|)$ the projection in $\mathcal{M}$ corresponding to the characteristic function $\chi_{(s,\infty)}$ via the functional calculus for $|x|$. We call an operator x affiliated with $\mathcal{M}$ *τ-measurable* if there is an

$s \geq 0$ such that $\tau(\chi_{(s,\infty)}(|x|)) < \infty$. The set of all τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. It becomes a $*$-algebra when sums and products are defined as the closures of respectively the algebraic sum and algebraic product. For $x \in \widetilde{\mathcal{M}}$ we write $x \geq 0$ if $(x\xi, \xi) \geq 0$ for all ξ in the domain of x and we put $\widetilde{\mathcal{M}}^+ = \{x \in \widetilde{\mathcal{M}} : x \geq 0\}$. The cone $\widetilde{\mathcal{M}}^+$ defines a partial order on the self-adjoint elements of $\widetilde{\mathcal{M}}$.

For $x \in \widetilde{\mathcal{M}}$, the *decreasing rearrangement* $\mu_\cdot(x)$ of x is the real-valued decreasing right-continuous function defined on $[0, \infty)$ by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}) \leq t\}, \qquad t \geq 0.$$

The relation $\prec\prec$ is defined on $\widetilde{\mathcal{M}}$ by

$$x \prec\prec y \quad \text{if and only if } \int_0^s \mu_t(x)\, dt \leq \int_0^s \mu_t(y)\, dt \text{ for every } s \geq 0,$$

where $x, y \in \widetilde{\mathcal{M}}$.

A normed vector subspace E of $\widetilde{\mathcal{M}}$ is called *rearrangement invariant* if and only if $x \in \widetilde{\mathcal{M}}$, $y \in E$ and $\mu(x) \leq \mu(y)$ implies $x \in E$ and $||x||_E \leq ||y||_E$ and *symmetric* if and only if $x, y \in E$ and $x \prec\prec y$ implies $||x||_E \leq ||y||_E$.

Since $L^\infty((0,\infty))$ may be regarded as an abelian von Neumann algebra and the Lebesgue integral as a trace on it, we may apply the above definitions in this context. Measurable operators correspond to almost everywhere finite measurable functions on $(0,\infty)$ that are bounded except on a set of finite measure. If $E((0,\infty))$ is a symmetric rearrangement invariant space of such functions, we put

$$E(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E((0,\infty))\}, \qquad ||x||_{E(\mathcal{M})} = ||\mu(x)||_{E((0,\infty))}.$$

Then $E(\mathcal{M})$ is a rearrangement invariant symmetric Banach space. This applies in particular to the spaces $L^1(\mathcal{M})$ and $L^\infty(\mathcal{M})$, and it can be shown that $L^\infty(\mathcal{M}) = \mathcal{M}$ and $L^1(\mathcal{M}) = \mathcal{M}_*$, where $\mathcal{M}_*$ stands for the predual of $\mathcal{M}$. This implies that if $x \in \widetilde{\mathcal{M}}$, then $x \in L_1(\mathcal{M})$ if and only if $\mu(x) \in L_1([0,1])$ and then $||x||_1 = \tau(|x|) = \int_0^1 \mu_t(x)\, dt$.

For more information on non-commutative Banach function spaces the reader is referred to [4], [5] and [6].

For the rest of this section we assume that $\tau(1) = 1$, where 1 denotes the identity of $\mathcal{M}$.

Definition 4.1 *A subset K of $L_1(\mathcal{M})$ is uniformly integrable if*

$$\lim_{t\to\infty} \sup_{x\in K} \tau(|x|\chi_{(t,\infty)}(|x|)) = 0.$$

Every uniformly integrable set K is bounded in $L_1(\mathcal{M})$. To see this, choose $t > 0$ such that $\sup_{x\in K} \tau(|x|\chi_{(t,\infty)}(|x|)) \leq 1$. If $x \in K$, then

$$\begin{aligned} ||x||_1 = \tau(|x|) &= \tau(|x|\chi_{(t,\infty)}(|x|)) + \tau(|x|\chi_{[0,t]}(|x|)) \\ &\leq 1 + \tau(t\chi_{[0,t]}(|x|)) \\ &\leq 1 + t\tau(1) = 1 + t. \end{aligned}$$

Proposition 4.2 *A subset K of $L_1(\mathcal{M},\tau)$ is uniformly integrable iff it is bounded in $L_1(\mathcal{M},\tau)$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sup_{x\in K} \tau(p|x|p) < \epsilon$ for all $p \in \mathcal{M}^p$ with $\tau(p) < \delta$.*

Proof. Let $\epsilon > 0$ and choose $t > 0$ such that $\sup_{x\in K} \tau(|x|\chi_{(t,\infty)}(|x|)) < \frac{1}{2}\epsilon$. Let $\delta = \frac{\epsilon}{2t}$. If $x \in K$ and $p \in \mathcal{M}^p, \tau(p) < \delta$, then using the inequality $|x|\chi_{[0,t[}(|x|) \leq t\chi_{[0,t[}(|x|)$,

$$\begin{aligned} \tau(p|x|p) &= \tau(p|x|\chi_{(t,\infty)}(|x|)p) + \tau(p|x|\chi_{[0,t]}(|x|)p) \\ &\leq ||p||\tau(|x|\chi_{(t,\infty)}(|x|))||p|| + \tau(pt\chi_{[0,t]}(|x|)p) \\ &\leq \frac{1}{2}\epsilon + t||p\chi_{[0,t]}(|x|)||\tau(p) \\ &\leq \frac{1}{2}\epsilon + t\tau(p) = \epsilon \end{aligned}$$

Suppose, conversely, that K is bounded and satisfies the condition. Put $C = \sup_{x\in K} ||x||_1$. Given $\epsilon > 0$, choose $\delta > 0$ as in the condition and t_0 large enough to ensure that $\frac{C}{t_0} < \delta$. Then if $t \geq t_0$ and $x \in K$, we have $t\chi_{(t,\infty)}(|x|) \leq |x|\chi_{(t,\infty)}(|x|)$ and hence

$$t\tau(\chi_{(t,\infty)}(|x|)) \leq \tau(|x|\chi_{(t,\infty)}(|x|)) \leq ||x||_1 \leq C.$$

It follows that $\tau(\chi_{(t,\infty)}(|x|)) \leq \frac{C}{t_0} < \delta$, and hence $\tau(|x|\chi_{(t,\infty)}(|x|)) = \tau(\chi_{(t,\infty)}(|x|)|x|\chi_{(t,\infty)}(|x|)) < \epsilon$. □

Proposition 4.3 *A subset K of $L_1(\mathcal{M})$ is uniformly integrable iff*

$$\lim_{s\to 0} \sup_{x\in K} \int_0^s \mu_t(x)\, dt = 0 \qquad (*)$$

Proof. A set K satisfying $(*)$ is bounded in $L_1(\mathcal{M})$. To see this, choose $s > 0$ such that $\sup_{x\in K} \int_0^s \mu_t(x)\, dt \leq 1$. If $x \in K$, then $\mu_s(x)s \leq \int_0^s \mu_t(x)\, dt \leq 1$, and hence $\mu_s(x) \leq \frac{1}{s}$. Therefore

$$\tau(|x|) \leq \int_0^s \mu_t(x)\, dt + \int_s^1 \mu_t(x)\, dt \leq 1 + \mu_s(x)(1-s) \leq 1 + \frac{1-s}{s} = C.$$

For every $t > 0$, $\tau(\chi_{(t,\infty)|(|x|)}) = \lambda(\{s > 0 : \mu_s(x) > t\})$, where λ denotes Lebesgue measure. Hence

$$\begin{aligned} t\tau(\chi_{(t,\infty)|(|x|)}) &= t\lambda(\{s > 0 : \mu_s(x) > t\}) \\ &\leq \int_{[\mu(x)>t]} \mu_s(x)\, ds \\ &\leq \int_0^1 \mu_s(x)\, ds \\ &= \tau(|x|) \leq C, \end{aligned}$$

and so $\tau(\chi_{(t,\infty)(|x|)}) \leq \frac{C}{t}$ for every $t > 0$. Let $\epsilon > 0$. First use $(*)$ to choose $s > 0$ such that $\sup_{x\in K} \int_0^s \mu_u(x)\, du < \epsilon$ and then choose $t_0 > 0$ such that $\frac{C}{t_0} \leq s$. If $t \geq t_0$, then $\tau(\chi_{(t,\infty)|(|x|)}) \leq \frac{C}{t_0} < s$, and so if $x \in K$,

$$\begin{aligned} \tau(|x|\chi_{(t,\infty)}(|x|)) &= \int_0^1 \mu_r(x\chi_{(t,\infty)}(|x|))\, dr \\ &= \int_{[\mu(x)>t]} \mu_r(x)\, dr \\ &\leq \int_0^s \mu_r(x)\, dr < \epsilon. \end{aligned}$$

Hence K is uniformly integrable.

Conversely, suppose K is uniformly integrable, $\epsilon > 0$ and choose $t_0 > 0$ such that $\sup_{x\in K} \tau(|x|\chi_{(t,\infty)}(|x|)) < \frac{\epsilon}{2}$ if $t \geq t_0$. If $s < \frac{\epsilon}{2t_0}$ and $x \in K$, then

$$\begin{aligned} \int_0^s \mu_r(x)\, dr &= \int_0^s \mu_r(x)\chi_{[\mu(x)>t_0]}\, dr + \int_0^s \mu_r(x)\chi_{[\mu(x)\leq t_0]}\, dr \\ &\leq \int_0^1 \mu_r(x\chi_{(t_0,\infty)})\, dr + t_0 s \\ &< \tau(|x|\chi_{(t_0,\infty)}) + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

□

Corollary 4.4 *A subset K of $L_1(\mathcal{M})$ is uniformly integrable if and only if the subset $\{\mu(x) : x \in K\}$ of $L_1([0,1])$ is uniformly integrable.*

We note that it follows from [5], Proposition 2.10, the classical Dunford-Pettis theorem and the corollary above that a subset of $L_1(\mathcal{M})$ is relatively $\sigma(L_1(\mathcal{M}), \mathcal{M})$-compact if and only if it is uniformly integrable.

The corollary and the fact that a subset K of $L_1(\mathcal{M})$ is bounded in $E(\mathcal{M})$ if and only if the set $\{\mu(x) : x \in K\}$ is bounded in E enables us to generalize the results of section 2 to the non-commutative setting. Since the Marcinkiewicz, Lorentz and Orlicz spaces used there are all symmetric rearrangement invariant Banach function spaces, they have non-commutative counterparts. As an example of the results that can be obtained we give a non-commutative version of Theorem 3.12.

Theorem 4.5 *Let $\mathcal{M}$ be a semi-finite von Neumann algebra. A subset K of $L_1(\mathcal{M})$ is uniformly integrable if and only if there is a Young's function Φ with $\Phi \in \Delta_2$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ such that K is bounded in $L_\Phi(\mathcal{M})$.*

References

[1] J. Alexoupolos, De La Vallee Poussin's theorem and weakly compact sets in Orlicz spaces, *Quaestiones Mathematicae* **17** (1994), 231–248.

[2] C. Bennet and R. Sharpley, *Interpolation of Operators,* Academic Press, 1988.

[3] K.M. Chong, Spectral orders, uniform integrability and Lebesgue's dominated convergence theorem, *Trans. Amer. Math. Soc.* **191** (1974), 395–404.

[4] P.G. Dodds,T.K. Dodds and B. de Pagter, Non-commutative Banach function spaces, *Math. Z.* **201** (1989), 583–597.

[5] P.G. Dodds,T.K. Dodds and B. de Pagter, Weakly compact sets of symmetric operator spaces, *Math. Proc. Camb. Phil. Soc.* **111** (1992), 355–368.

[6] P.G. Dodds,T.K. Dodds and B. de Pagter, Non-commutative Köthe duality, *Trans. Amer. Math. Soc.* **339** (1993), 717–750.

[7] M. Duhoux, Order precompactness in topological Riesz spaces, *J. London Math. Soc.* **23** (1981), 509–522.

[8] M.A. Krasnoselskii and Ya.B. Rutickii, *Convex functions and Orlicz spaces,* Noordhoff 1961.

[9] S.G. Krein, Ju.I. Petunin and E.M. Seminov, *Interpolation of linear operators,* Translations of Mathematical Monographs, *Amer. Math. Soc.* **54,** 1982.

[10] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces,* Marcel Dekker 1991.

[11] A.C. Zaanen, *Integration,* North Holland 1967.

[12] A.C. Zaanen, *Riesz spaces II,* North Holland 1983.

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New Automatic non Interpolatory Quadrature and Cubature Formulas

Francesco Costabile and Maria Italia Gualtieri

Abstract. Given a real function $f \in C^{2k}[0,1]$, $k \geq 1$ Costabile, Gualtieri, Serra, in 1996 [7], proposed an asymptotic expansion formula for the corresponding Bernstein polynomials $B_n(f,x)$ in terms of $h = 1/n$. Previously this problem had been studied from the point of view of the search of an expansion formula in terms of the independent variable x.

This idea was generalized to multivariate case by several authors ([1], [11], [15], [24], [25]).

In the present paper we apply these results to the numerical integration problem. In this way we can build some scheme of adaptive non interpolatory quadrature for univariate and bivariate functions over rectangular and triangular domains. The theoretical results are validate by numerical tests.

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1 Introduction

There are several well-known useful quadrature and cubature schemes which are not derived from interpolatory procedures: Romberg's quadrature, based upon extrapolation technique for the error; adaptive quadrature methods, useful when no properties of integrand are known; Sard's optimal quadratures obtained by minimizing the L_2 norm of the Peano kernel of

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the error functional; Euler-McLaurin quadrature; these are someone of the more known examples. In particular the Romberg scheme, which evaluates $\int_a^b f(x)dx$ by means of trapezoidal rule with step size $h_i = (b-a)/2^i$, is an important example of application of Richardson extrapolation method.

The aim of this paper is to propose new schemes of non interpolatory quadrature and cubature, obtained as possible applications of some recent results about the acceleration of the convergence for the sequence of Bernstein operators.

The outline of the paper is the following:

in section 2 we give a survey over the asymptotic expansions of Bernstein polynomials in the univariate and bivariate case; in section 3 we derive the quadrature and cubature formulas by Bernstein polynomials with asymptotic expansion of the error; in section 4 we formulate the new scheme of non interpolatory quadrature and cubature by means of the Richardson extrapolation; finally in section 5 we consider some numerical tests which prove that the behaviour of the new schemes is comparable with Romberg's ones.

2 Bernstein polynomials and asymptotic expansion

The Bernstein operators for a function f, denoted by $B_n(f,x)$, are a well known tool to approximate real functions, with very good qualitative features of approximation, but in spite of these properties, they are not useful from a computational point of view, because of a very slow rate of convergence.

Many authors worked on the possibility of improve the order of convergence, that is exactly n^{-1}, if n is the degree of the operator and $f \in C^2$.

In general the question is: can the convergence of a given sequence be accelerated? The focus has been on the way to obtain the acceleration of the convergence by a suitable extrapolation algorithm. So the attention has been shifted on the conditions which allow to use the extrapolation schemes. To obtain these conditions, the most widespread criterion is the existence of an asymptotic expansion for the sequence to be accelerated.

In the following some results, obtained in the last years by several researchers, concerning the existence of an asymptotic expansion for Bernstein polynomials under suitable conditions of regularity for the function to be approximated, are showed.

2.1 The univariate case

The general Bernstein polynomial of a function f within the interval $[0,1]$ is defined as [2]

$$B_n(f,x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) p_{n,\nu}(x) \tag{2.1}$$

where $p_{n,\nu}(x) = \begin{pmatrix} n \\ \nu \end{pmatrix} x^{\nu}(1-x)^{n-\nu}$.

It's well known that $B_n(f,x)$ approaches f uniformly on $[0,1]$ for $f \in C[0,1]$; unfortunately, the classical result of Voronowskaja [23]

$$\lim_{n\to\infty} n\left[B_n(f,x) - f(x)\right] = \frac{x(1-x)}{2} f^{(2)}(x), \tag{2.2}$$

prove that the order of approximation to $f(x)$ by $B_n(f,x)$ is exactly $O(1/n)$, if $f^{(2)}(x) \neq 0$, that is the convergence is very slow. This make the approximation by means of Bernstein polynomials useless from a numerical point of view.

By assuming sufficiently conditions of regularity for f, Costabile et al., generalized the result of Voronowskaja, in fact in [7] the following theorem was proved

Theorem 2.1 *[7] Let $f \in C^{2k}[0,1]$, $k \geq 1$, then the following expansion holds*

$$B_n(f,x) = f(x) + \sum_{i=1}^{k} h^i S_i(f,x) + h^k E_h(f,x) \tag{2.3}$$

where $h = 1/n$, the coefficients $S_i(f,x)$, $i = 1,\ldots,k$, are independent of h and $E_h(f,x)$ vanishes as $h \to 0$.

The asymptotic expansion (2.3) in terms of $h = 1/n$ allowed to apply some extrapolation algorithms in order to speed up the rate of the convergence of $B_n(f)$ to f and in the following allow us to obtain an asymptotic expansion of the error for quadrature formulas too.

2.2 The bivariate case: squared domains

The definition of the Bernstein polynomials can be spread to the bivariate case, over a rectangular domain, preserving all properties of approximation of the univariate case.

In detail, let $D \equiv [0,1] \times [0,1]$ a squared domain, for $f : D \to \mathbb{R}$ Bernstein polynomials are defined as

$$B_n^m(f;x,y) = \sum_{\nu=0}^{n} \sum_{\mu=0}^{m} f\left(\frac{\nu}{n}, \frac{\mu}{m}\right) p_{n,\nu}^{m,\mu}(x,y)$$

wher $p_{n,\nu}^{m,\mu}(x,y) = p_{n,\nu}(x)\, p_{m,\mu}(y)$.

Let us consider the case $n = m$, the order of approximation for the bivariate Bernstein polynomials, does not exceed $1/n$ also. In fact a Voronowskaja type result [12] holds

$$\lim_{n\to\infty} n\left[B_n^n(f;x,y) - f(x,y)\right] = \frac{x(1-x)}{2}\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{y(1-y)}{2}\frac{\partial^2 f(x,y)}{\partial y^2} \tag{2.4}$$

and hence the convergence is very slow, but the existence of an asymptotic expansion in terms of $h = 1/n$ is proved in this case also.

With the same tools of the monodimensional case [7], we can prove the following result of asymptotical expansion

Theorem 2.2 *Let $f \in C^{2k}(D)$, $n \geq 1$, then the following expansion holds*

$$B_n^n(f;x,y) = f(x,y) + \sum_{i=1}^{k} h^i S_i(f;x,y) + h^k E_h(f,x) \tag{2.5}$$

where $h = 1/n$, the coefficients $S_i(f,x)$, $i = 1,\ldots,k$, are independent of h and $E_h(f,x)$ vanishes as $h \to 0$.

Sketch of proof. The proof is based on the remark that the bivariate Bernstein polynomials on the square are built in terms of the univariate ones. If we set the polynomials

$$T_{n,s}^m(x,y) = \sum_{k=0}^{n} (k-nx)^s\, p_{n,k}^{m,h}(x,y),$$

$$T_n^{m,p}(x,y) = \sum_{h=0}^{p} (h-my)^p\, p_{n,k}^{m,h}(x,y),$$

$$T_{n,s}^{m,p}(x,y) = \sum_{k=0}^{n}\sum_{h=0}^{n} (k-nx)^s (h-my)^p\, p_{n,k}^{m,h}(x,y) = T_{n,s}(x)\, T_{m,p}(y),$$

where $T_{n,s}(x) \sum_{k=0}^{n} (k-nx)^s\, p_{n,k}(x)$, the proof follows easily in a very similar way as the monodimensional case [7]. □

In this case also, the existence of the asymptotic expansion allow us to build some extrapolation schemes in order to have a superlinear convergence to f.

2.3 The bivariate case: triangular domain

In the previous section, we have taken into account a tool to approximate a function f over a rectangular domain in $\mathbb{R}^2$, by means of Bernstein polynomials. Such polynomials show good properties of approximation over domains differently shaped also, for example over triangular domains.

Several authors studied about the existence of an asymptotic expansion for bivariate, and more generally for multivariate, Bernstein operators over a simplex ([1], [11], [15], [24], [25]). Someone obtained the looked-for results by using a different system of coordinates: the so-called barycentric coordinates [24].

For our aims, it seems to be more useful to give an expression of the expansion in terms of the classical system of cartesian coordinates.

Let us define the Bernstein operators for a bivariate function defined over a triangular domain. Let

$$\Delta = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le x, y \le 1, x+y \le 1\right\}$$

be a simplex in $\mathbb{R}^2$, the Bernstein polynomials approximating f over Δ is defined as

$$B_n^{\Delta}(f;x,y) = \sum_{\substack{0\le i,j\le n \\ i+j\le n}} f\left(\frac{i}{n},\frac{j}{n}\right) B_{n,(i,j)}(x,y)$$

where

$$B_{n,(i,j)}(x,y) = \left(\begin{array}{c} n \\ i,j \end{array}\right) x^i y^j (1-x-y)^{n-i-j},$$

and $\left(\begin{array}{c} n \\ i,j \end{array}\right) = \left(\begin{array}{c} n \\ i \end{array}\right)\left(\begin{array}{c} n-i \\ j \end{array}\right)$.

For such polynomials we can give the expression for an asymptotic expansion formula

Theorem 2.3 *[1] If $f \in C^{2k}(\Delta)$, $k > 0$, then the following expansion*

holds

$$B_n^{\Delta}(f;x,y) = f(x,y) + \sum_{|i|\le k}\sum_{\mu=0}^{i_1}\sum_{\nu=\mu}^{\mu+i_2} h^{\nu} S_{\nu}(f;x,y) + \\ + O\left[\frac{\rho(1-\rho)}{n^k}\left(\rho(1-\rho)+\frac{1}{n}\right)^{k-1}\omega\left(f^{(2k)},\sqrt{\frac{\rho(1-\rho)}{n}}+\frac{1}{n}\right)\right] \tag{2.6}$$

where $h = 1/n$, $\omega(f,\delta)$ *is the modulus of continuity of* f, *the coefficients* $S_\nu(f;x,y)$ *are independent of* h *for all* ν *and* $\rho = \rho(x,y) = \sqrt{x^2+y^2}$.

In (2.6) we set $i = (i_1, i_2)$ *and* $|i| = i_1 + i_2$.

For details on the proof of the previous theorem [1], [11], [15], [25].

Even in this case we can apply some extrapolation algorithms which lead to convergent schemes.

3 Quadrature and cubature formulas based on Bernstein polynomials

The aim of this section is to use the previous results to obtain some convergent non interpolatory quadrature and cubature formulas by means of the Bernstein approximation.

3.1 The univariate case

The quadrature formula based on the integration of the Bernstein polynomials $B_n(f,x)$, is well known [10]

$$\int_0^1 f(x)dx \approx \int_0^1 B_n(f,x) = \frac{1}{n+1}\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right).$$

It has equal weights and it is exact for $f(x) = 1, x$. By putting

$$T_n(f) = \int_0^1 B_n(f,x), \qquad T(f) = \int_0^1 f(x)dx, \tag{3.1}$$

we can formulate the following theorem

Theorem 3.1 *[10] Lef* $f \in C[0,1]$, *then*

$$\lim_{n\to\infty} T_n(f) = T(f), \tag{3.2}$$

and, if $f \in C^{2k}[0,1]$, $k \geq 1$, *then*

$$T_n(f) = T(f) + \sum_{i=1}^{k} h^i \gamma_i(f) + h^k \psi(h), \tag{3.3}$$

where $h = 1/n$, *the coefficients* $\gamma_i(f)$ *are independent of* n *and* $\psi(h)$ *is a bounded function.*

Proof. (3.2) follows on the convergence of $B_n(f,x)$ to f, while we can be deduce (3.3) by integrating (2.3). □

3.2 The bivariate case: squared domains

For the integration of bivariate functions, some cubature formulas by means of Bernstein approximation, can be derived also. In fact the integration of Bernstein polynomials on $D \equiv [0,1]\times[0,1]$ leads to the following convergent cubature rule

$$\iint_D f(x,y)\,dxdy \approx \iint_D B_n^n(f;x,y)\,dxdy = \frac{1}{(n+1)^2} \sum_{\nu,\mu=0}^{n} f\left(\frac{\nu}{n}, \frac{\mu}{n}\right), \tag{3.4}$$

which has equal weights and is exact for $f(x,y) = 1, x, y$. By putting

$$T_n^D(f) = \iint_D B_n^n(f;x,y)\,dxdy, \qquad T^D(f) = \iint_D f(x,y)\,dxdy, \tag{3.5}$$

then the following theorem holds

Theorem 3.2 *If* $f \in C(D)$, *then*

$$\lim_{n\to\infty} T_n^D(f) = T^D(f). \tag{3.6}$$

Furthermore, if $f \in C^{2k}(D)$, $k \geq 1$ *and* $h = 1/n$, *we can find coefficients* $\theta_i(f)$ *independent of* n *and a bounded function* $\Psi(h)$ *such that*

$$T_n^D(f) = T^D(f) + \sum_{i=1}^{k} h^i \theta_i(f) + h^k \Psi(h). \tag{3.7}$$

Proof. The proof is similar to the proof of theorem 3.1. In fact (3.6) follows from the convergence of Bernstein polynomials trivially, then we can obtain (3.7) by integrating (3.5). □

3.3 The bivariate case: triangular domain

The Bernstein polynomials can be integrated and used to obtain a cubature formula over the simplex also. In fact, by integrating $B_n^{\Delta}(f;x,y)$, we obtain

$$\iint_{\Delta} f(x,y)\,dx\,dy \approx \iint_{\Delta} B_n^{\Delta}(f;x,y)\,dx\,dy = \sum_{\substack{0\le i,j\le n\\ i+j\le n}} \sum_{p=0}^{n-i-j} \sum_{q=0}^{n-i+1} c_{n,i,j}^{p,q} \frac{(-1)^{2n-2i-j-p-q+1}}{(n-i-p+1)(n-q+2)} f\left(\frac{i}{n},\frac{j}{n}\right) \tag{3.8}$$

with $c_{n,i,j}^{p,q} = \begin{pmatrix} n \\ i,j \end{pmatrix} \begin{pmatrix} n-i-j \\ p \end{pmatrix} \begin{pmatrix} n-i+1 \\ q \end{pmatrix}$.

By setting

$$T_n^{\Delta}(f) = \iint_{\Delta} B_n^{\Delta}(f;x,y)\,dx\,dy, \qquad T^{\Delta}(f) = \iint_{\Delta} f(x,y)\,dx\,dy,$$

we can formulate a theorem on the convergence of quadrature formula (3.8) and the related asymptotic expansion, in similar way as in theorem 3.1 and 3.2.

Theorem 3.3 *If $f \in C(\Delta)$, then*

$$\lim_{n\to\infty} T_n^{\Delta}(f) = T^{\Delta}(f). \tag{3.9}$$

If $f \in C^{2k}(\Delta)$, $k>0$, $h=1/n$ then we can find coefficients $\zeta_i(f)$ independent of n and a bounded function $Z(h)$ such that

$$T_n^{\Delta}(f) = T^{\Delta}(f) + \sum_{i=1}^{k} h^i \zeta_i(f) + h^k Z(h). \tag{3.10}$$

Proof. The proof comes along in a similar way as in the previous cases. □

4 A new non interpolatory automatic quadrature scheme

Now we are ready to use the previous results to obtain some non interpolatory procedures for the numerical computation of a defined integral, with a satisfying order of approximation. The expansions (3.3), (3.7)

and (3.10), proved in the former section, about the quadrature and cubature Bernstein formulas, allows us to apply the Richardson extrapolation: if $f \in C^{2k}[0,1]$, we chose an increasing sequence of positive integers $\{n_0 = n < n_1 < n_2 < \cdots\}$ and we set $h_i = n_i^{-1}$; now we can define a sequence of formulas $\{X_q^{(i)}\}_{q \in \mathbf{N}}$ in the following way

$$\begin{cases} X_0^{(i)} & := & X_0^{(i)}(f,x) & & \\ X_q^{(i)} & := & X_q^{(i)}(f,x)) & = & \frac{h_{i+q} X_{q-1}^{(i)} - h_i X_{q-1}^{(i+1)}}{h_{i+q} - h_i}, \quad q = 1,2,\ldots,k-1. \end{cases} \tag{4.1}$$

where

$$X_0^{(i)}(f) := \begin{cases} T_{n_i}(f) & \textit{for univariate functions;} \\ T_{n_i}^{D}(f) & \textit{for bivariate functions, squared domain;} \\ T_{n_i}^{\Delta}(f) & \textit{for bivariate functions, triangular domain.} \end{cases}$$

The terms $X_q^{(i)}$ are usually arranged in a triangular scheme as follows

$$\begin{array}{cccccc} X_0^{(0)} & X_1^{(0)} & X_2^{(0)} & \cdots & \cdots & X_{k-1}^{(0)} \\ X_0^{(1)} & X_1^{(1)} & \vdots & \vdots & \vdots & \\ X_0^{(2)} & X_1^{(2)} & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & X_1^{(k-2)} & & & & \\ X_0^{(k-1)} & & & & & \end{array}$$

The properties of convergence to the exact solution, along the columns and the diagonals, under suitable conditions on the sequence n_i, are well known [4]. The general theorem on the extrapolation processes assures the the previous scheme is convergent under the hypothesis $\frac{h_i}{hi-1} < 1$ and gives an explicit representation of the error

$$X_q^{(i)} - \Gamma(f) = h_i h_{i+1} \ldots h_q (-1) \left(\sigma_{q+1}(f) + O(h_i)\right),$$

where $\Gamma(f) = T(f)$, $T^D(f)$, $T^{\Delta}(f)$ and $\sigma_{q+1}(f) = \gamma_{q+1}(f)$, $\theta_{q+1}(f)$, $\zeta_{q+1}(f)$, if we have a monovariate, a bivariate on a square or a bivariate on a triangle function respectively.

In the following in particular, for the sequence n_i, we consider the classical Romberg sequence $n_i = \rho^i$, with $\rho \geq 2$ positive integer.

5 The algorithm and some numerical results

In order to obtain an efficient algorithm, we need an auxiliary routine, $CalcPol(n_i, f)$, to generate the first column of the triangular scheme. This routine is able to calculate the integral on the interval $[0, 1]$, on the square $[0, 1] \times [0, 1]$ or on the simplex Δ, depending on the problem to be resolved.

Now we propose a possible procedure: it is totally automatic and give the approximate value of the integral with a tolerance given in input.

Let us set $h_i = 1/n_i$ and let us choose an upper limit n_{max} for the dimension of the scheme: this limit determine the condition of numerical non-convergence.

init
  $eps, n_{max}, f, h_i \; i = 0, \ldots, n_{max}$, *(input)*
  $i = 0$
  $X_{i,0} = CalcPol(n_i, f)$
  $err = 2 * eps$;
while i< n_{max} **and** err> eps
  $i = i + 1$
  $X_{i,0} = CalcPol(n_i, f)$
  for $k = 1, i$
    $j = i - k + 1$
    $t(j, k) = \frac{h_{j+k} X_{j,k-1} - h_j X_{j+1,k-1}}{h_{j+k} - h_j}$
  end for
  $err = |t(1, i) - t(2, i - 1)|$;
end while

The previous procedure has been performed on a large set of numerical tests, with the option $n_i = \rho^i, \quad \rho = 2$ for the sequence n_i, that is the Romberg sequence.

The numerical results prove that the showed formulas work as fine as the classical ones obtained by applying the extrapolation procedures to the trapezoidal formula. In some cases the new formulas work better than the classical ones.

In the following tables we report the test functions performed and the number of function evaluations necessary to obtain the requested tolerance for extrapolated Bernstein formula (N_B) and extrapolated trapezoidal formula (N_T). We used a tolerance $\varepsilon = 10^{-5}$.

Univariate case

$f(x)$	N_B	N_T
$x^{11/2}$	5	6
e^{-t^2}	5	5
e^{t^2}	6	6
$\cos \pi \frac{t^2}{2}$	6	6
$\cos t^2$	5	6
$\sin \pi \frac{t^2}{2}$	6	6
$\sin(t^2)$	5	6
$J_0(x)$	4	5
$I_0(x)$	4	4

Bivariate case

	$[0,1] \times [0,1]$		Δ	
$f(x,y)$	N_B	N_T	N_B	N_T
$2x - 5y + 4$	2	2	2	2
$x^2 + y$	3	4	3	4
$e^x + e^y$	5	5	5	6
$x + y$	2	2	2	2
$\sin y^2$	5	6	5	5
$x + 1$	2	2	2	2
x	2	2	2	2
$\frac{y^2}{1+x^2+y^2}$	5	6	4	6
$1 + x^2 + y^2$	3	4	3	4
y^2	3	4	3	4
xy	2	2	3	4
$2x^2 + 3y$	3	4	3	4
$x - 2y$	2	2	2	2
$x + y$	2	2	2	2
$x + 2y$	2	2	2	2
$x \sin \left\lvert x^2 - y \right\rvert$	7	7	5	7
x^2	3	4	3	4
$\sin x + \cos y$	5	5	5	5
$\sin x \cos y$	5	5	4	5
$x^2 + y^2$	3	4	3	4
$\sqrt{x^2 + y^2}$	6	5	5	6
$\frac{\sqrt{64-81((x-0.5)^2+(y-0.5)^2)}}{9} - 0.5$	5	5	5	5

References

[1] Abel, Ulrich; Ivan, Mircea, Asymptotic expansion of the multivariate Bernstein polynomials on a simplex, *Approx. Theory Appl. (N.S.)* **16** (2000), no. 3, 85–93.

[2] S. N. Bernstein, Complétement á l'article de E. Voronowskaja, *C. R. (Dokl.) Acad. Sci. URSS A* (1932), no. 4, 86–92.

[3] Bauer, F. L.; Rutishauser, H.; Stiefel, E., New aspects in numerical quadrature, *Proc. Sympos. Appl. Math.* **15** (1963), 199–218.

[4] R. Bulirsch, J. Stoer, Numerical quadrature by extrapolation, *Numer. Math.* **9** (1967), 271–278.

[5] Cools, Ronald, Constructing cubature formulae: the science behind the art, *Acta Numerica* **6** (1997), 1–54.

[6] Costabile, F.; Luceri, R.; Gualtieri, M.I., An operator of Bernstein type with applications to the indefinite integral and differential Cauchy problem, *Rend. Circ. Mat. Palermo (2) Suppl.* **40** (1996), 109–119. MR1407084 (97f:41024)

[7] Costabile, F.; Gualtieri, M. I.; Serra, S., Asymptotic expansion and extrapolation for Bernstein polynomials with applications, *BIT* **36** (1996), no. 4, 676–687. MR1420271 (97j:65022)

[8] Cranley, R.; Patterson, T. N. L., On the automatic numerical evaluation of definite integrals, *Comput. J.* **14** (1971), 189–198.

[9] de Doncker, Elise, New Euler-Mclaurin expansions and their applications to quadrature over the s-dimensional simplex, *Math. Comp.* **33** (1979), no. 147, 1003–1018.

[10] Engels, H., *Numerical Quadrature and Cubature,* Computational Mathematics and Applications, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1980.

[11] Feng, Yu Yu; Kozak, Jernej, Asymptotic expansion formula for Bernstein polynomials defined on a simplex, *Constr. Approx.* **8** (1992), no. 1, 49–58.

[12] Ipatov, A. F., Estimation of the error and order of approximation of functions of two variables by Bernstein polynomials, *Uč. Zap. Petrozavodsk. Gos. Univ.* **4** (1955), no. 4, 31–48.

[13] Joyce, D. C., Survey of extrapolation processes in numerical analysis, *SIAM Rev.* **13** (1971), no. 4, 435–490.

[14] Kingsley, Edward H., Bernstein polynomials for functions of two variables of class $C^{(k)}$, *Proc. Am. Math. Soc.* **2** (1951), 64–71.

[15] Lai, Ming Jun, Asymptotic formulae of multivariate Bernstein approximation, *J. Approx. Theory* **70** (1992), no. 2, 229–242.

[16] Laurent, Pierre-Jean, Formules de quadrature approchée sur domaines rectangulaires convergentes pour toute fonction intégrable Riemann, *C. R. Acad. Sci. Paris* **258** (1964), 798–801.

[17] Lyness, J. N., Quadrature over a simplex. I. A representation for the integrand function, *SIAM J. Numer. Anal.* **15** (1978), no. 1, 122–133.

[18] Lyness, J. N., Quadrature over a simplex. II. A representation for the error functional, *SIAM J. Numer. Anal.* **15** (1978), no. 5, 870–887.

[19] Lyness, J. N.; Cools, Ronald, A survey of numerical cubature over triangles,, *Proc. Sympos. Appl. Math.* **48** (1994), 127–150.

[20] Lorentz, G. G., *Bernstein Polynomials,* Second edition, Chelsea Publishing Co., New York, 1986.

[21] O'Hara, H.; Smith, Francis J., The evaluation of definite integrals by interval subdivision, *Comput. J.* **12** (1969/1970), 179–182.

[22] Stoer, J.; Bulirsch, R., *Introduction to Numerical Analysis, Third edition* Texts in Applied Mathematics **12,** Springer-Verlag, New York, 2002.

[23] E. Voronowskaja, Détermination de la forme asymptotique d'approximation des fonctions par les polynomes de M. Bernstein, *C. R. (Dokl.) Acad. Sci. URSS A* (1932), no.4, 79–85.

[24] Walz, Guido, Asymptotic expansions for multivariate polynomial approximation, *J. Comput. Appl. Math.* **122** (2000), no. 1-2, 317–328.

[25] Zhang, Renjiang, An asymptotic expansion formula for Bernstein polynomials on a triangle, *Approx. Theory Appl. (N.S.)* **14** (1998), no. 1, 49–56.

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Modified Bernstein Polynomials and Jacobi Polynomials in q-Calculus

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Abstract. We introduce here a generalization of the modified Bernstein polynomials for Jacobi weights using the q-Bernstein basis proposed by G.M. Phillips to generalize classical Bernstein polynomials. The function is evaluated at points which are in geometric progression in $]0,1[$. Numerous properties of the modified Bernstein polynomials are extended to their q-analogues: simultaneous approximation, pointwise convergence even for unbounded functions, shape-preserving property, Voronovskaya theorem, self-adjointness. Some properties of the eigenvectors, which are q-extensions of Jacobi polynomials, are given.

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1 Introduction

G.M.Phillips has proposed a generalization of Bernstein polynomials based on the q-integers (cf. [9]). We introduce here a q-analogue of the operators which are often called Bernstein-Durrmeyer polynomials and denoted $M_{n,1}^{\alpha,\beta}$ (cf. [3],[2]).

Throughout the paper, we shall assume that $q \in]0,1[$ and, part 5 excepted, $\alpha, \beta > -1$.

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For any $n \in \mathbb{N}$, and a function f defined on $]0,1[$ we set

$$M_{n,q}^{\alpha,\beta} f(x) = \sum_{k=0}^{n} f_{n,k,q}^{\alpha,\beta}\ b_{n,k,q}(x) \tag{1.1}$$

where each $f_{n,k,q}^{\alpha,\beta}$ is a mean of f defined by Jackson integrals.

The polynomials $b_{n,k,q}(x)$ are q-analogues of the Bernstein basis polynomials and are defined by
$b_{n,k,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}$, with $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$ (q-binomial coefficients), $k = 0, \ldots, n$. They verify $\sum_{k=0}^{n} b_{n,k,q}(x) = 1$, (cf. [9]).

We follow the definitions and notations of ([7], p. 10,47,79). For any real a and x, $[a]_q = (1-q^a)/(1-q)$, $(1-x)_q^a = \prod_{j=0}^{\infty}(1-q^j x)\,/(1-q^{j+a}x)$, $\Gamma_q(a+1) = (1-q)_q^a/(1-q)^a$, consequently $(1-x)_q^{a+b} = (1-x)_q^a(1-q^a x)_q^b$ holds for any b. For the integer $a \geq 1$, the q-integer is $[a]_q = 1+q+\cdots+q^{a-1}$, $\Gamma_q(a+1) = ([a]_q)!$ and $(1-x)_q^a = \prod_{j=0}^{a-1}(1-q^j x)$.

The q-derivative of a function f is $D_q f(x) = \dfrac{f(qx)-f(x)}{(q-1)x}$ for any $x \neq 0$. It is extended by continuity at $x = 0$ when it is possible. For example, if $f'(0)$ exists, then $D_q f(0) = f'(0)$.
So, for any real p , $D_q x^p = [p]_q\, x^{p-1}$ when $x \neq 0$ and $[D_q x^p]_{x=0} = 0$ if $p > 1$, (respectively $= 1$ if $p = 1$)).

The notations will be simplified as much as possible, the superscripts α, β and the index q when q is fixed, will be suppressed in some proofs.

We introduce the positive bilinear form :

$$\langle f, g\rangle_q^{\alpha,\beta} = q^{(\alpha+1)(\beta+1)}(1-q)\sum_{i=0}^{\infty} q^i q^{i\alpha}(1-q^{i+1})_q^{\beta} f(q^{i+\beta+1}) g(q^{i+\beta+1}), \tag{1.2}$$

whenever it is defined. It can be written under the form of two definite q-integrals

$$\begin{aligned} \langle f, g\rangle_q^{\alpha,\beta} &= \int_0^{q^{\beta+1}} t^{\alpha}(1-q^{-\beta}t)_q^{\beta} f(t) g(t)\, d_q t \\ \text{and } \langle f, g\rangle_q^{\alpha,\beta} &= q^{(\alpha+1)(\beta+1)} \int_0^1 t^{\alpha}(1-qt)_q^{\beta} f(q^{\beta+1}t) g(q^{\beta+1}t)\, d_q t. \end{aligned}$$

(the definite q-integral of a function f is $\int_0^a f(x)\, d_q x = a(1-q)\sum_{i=0}^{\infty} q^i f(q^i a)$ (cf. [7], p. 69))

Definition 1.1 *We set in formula (1.1) :*

$$f_{n,k,q}^{\alpha,\beta} = \frac{\langle b_{n,k,q}, f\rangle_q^{\alpha,\beta}}{\langle b_{n,k,q}, 1\rangle_q^{\alpha,\beta}} = \frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ f(q^{\beta+1}t)\ d_qt}{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ d_qt},\ k=0,...,n, \tag{1.3}$$

to define

$$M_{n,q}^{\alpha,\beta} f(x) = \sum_{k=0}^{n} \frac{\langle b_{n,k,q}, f\rangle_q^{\alpha,\beta}}{\langle b_{n,k,q}, 1\rangle_q^{\alpha,\beta}} b_{n,k,q}(x). \tag{1.4}$$

We see that the polynomial $M_{n,q}^{\alpha,\beta} f$ is well defined if there exists $\gamma \geq 0$ such that U_f^γ defined by $U_f^\gamma(x) = x^\gamma f(x)$ is bounded on $]0, A]$ for some $A \in]0,1[$ and $\alpha > \gamma - 1$. Indeed, under this assumption, U_f^γ is q-integrable with respect to the weight $w_q^{\alpha,\beta}(x) = x^\alpha (1-qx)_q^\beta$. We will say, in this case, that f satisfies the condition $C(\alpha)$. Also $\langle f, g\rangle_q^{\alpha,\beta}$ is well defined if the product fg satisfies $C(\alpha)$, particularly if f^2 and g^2 do it.

In many cases, the limit of $M_{n,q}^{\alpha,\beta} f(x)$ when q tends to 1 is

$$M_{n,1}^{\alpha,\beta} f(x) = \sum_{k=0}^{n} \frac{\int_0^1 t^{k+\alpha}\ (1-t)^{n-k+\beta}\ f(t)\ dt}{\int_0^1 t^{k+\alpha}\ (1-t)^{n-k+\beta}\ dt} b_{n,k}(x)$$

with $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Numerous properties of the operator $M_{n,1}^{\alpha,\beta}$ (cf. [2],[4]) will be extended to $M_{n,q}^{\alpha,\beta}$ in this paper.

2 First properties

For any $n \in \mathbb{N}$, the operator $M_{n,q}^{\alpha,\beta}$ has the following properties.

- It is linear, positive and it preserves the constants so it is non-expansive ($\sup_{x\in[0,1]} \left| M_{n,q}^{\alpha,\beta} f(x) \right| \leq \sup_{x\in]0,1[} |f(x)|$).
- It is self-adjoint: $\langle M_{n,q}^{\alpha,\beta} f, g\rangle_q^{\alpha,\beta} = \langle f, M_{n,q}^{\alpha,\beta} g\rangle_q^{\alpha,\beta}$.
- It preserves the degrees of the polynomials of degree $\leq n$.

The first two properties are consequences of the definition. The last one follows after the following proposition.

Proposition 2.1 *If f verifies the condition $C(\alpha)$, we have, for any $n \geq 1$, and $x \in [0,1]$:*

$$D_q M_{n,q}^{\alpha,\beta} f(x) = \frac{[n]_q}{[n+\alpha+\beta+2]_q} q^{\alpha+\beta+2} M_{n-1,q}^{\alpha+1,\beta+1} \left(D_q f\left(\frac{\cdot}{q}\right)\right)(qx). \tag{2.1}$$

Proof. We compute $Db_{n,0}(x) = -[n]\, b_{n-1,0}(qx)$, if $1 \leq k \leq n-1$,

$Db_{n,k}(x) = [n]\,(b_{n-1,k-1}(qx)/q^{k-1} - b_{n-1,k}(qx)/q^k)$ and

$Db_{n,n}(x) = [n]\, b_{n-1,n-1}(qx)/q^{n-1}$ to get

$DM_n^{\alpha,\beta} f(x) = [n] \sum_{k=0}^{n-1} b_{n-1,k}(qx)(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})/q^k.$

We denote $\psi_{n,k}^{\alpha,\beta}(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$, $k = 0, \ldots, n$.

Recall that the q-Beta functions are $B_q(u,v) = \int_0^1 t^{u-1}(1-qt)_q^{v-1} d_q t = \Gamma_q(u)\Gamma_q(v)\,/\Gamma_q(u+v)$ (cf. [7], p. 78). The rule for the q-derivative of $g_1 g_2$ is $(D_q(g_1 g_2))(t) = (D_q g_1)(t) g_2(qt) + g_1(t)\,(D_q g_2)(t)$. So, if $g_1 g_2$ is defined on $[0,1]$ and continuous at $t = 0$ and if the series defining the two following q-integrals converge, we have the q-integration by parts

$\int_0^1 (D_q g_1)(t) g_2(qt) d_q t = [(g_1 g_2)(t)]_0^1 - \int_0^1 g_1(t)\,(D_q g_2)(t) d_q t.$

The function $t \longmapsto \psi_{n-1,k}^{\alpha+1,\beta+1}(\frac{t}{q}) f(q^{\beta+1}t)$, $t \in]0,1[$, extended by 0 in 0 is continuous at 0. Since f verifies the condition $C(\alpha)$, the function $D_q f$ verifies the condition $C(\alpha+1)$. Hence we may use a q-integration by parts to write, for $k = 0, \ldots, n-1$:

$B_q(k+\alpha+2, n-k+\beta+1)\,[n+\alpha+\beta+2]\,(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})$

$= -q^{k+\alpha} \int_0^1 (D\psi_{n-1,k}^{\alpha+1,\beta+1})(\frac{t}{q}) f(q^{\beta+1}t) d_q t$

$= \int_0^1 q^{k+\alpha+\beta+2} \psi_{n-1,k}^{\alpha+1,\beta+1}(t)(Df)(q^{\beta+1}t) d_q t - \left[\psi_{n-1,k}^{\alpha+1,\beta+1}(\frac{t}{q}) f(q^{\beta+1}t)\right]_0^1.$

Hence

$$(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta}) = \frac{q^{\alpha+\beta+2+k}}{[n+\alpha+\beta+2]} \frac{\int_0^1 t^{k+\alpha+1}(1-qt)^{n-k+\beta}(Df)(q^{\beta+1}t) d_q t}{B_q(k+\alpha+2, n-k+\beta+1)} \text{ and}$$

$$DM_n^{\alpha,\beta} f(x) = \frac{[n]\, q^{\alpha+\beta+2}}{[n+\alpha+\beta+2]} \sum_{k=0}^{n-1} \frac{\langle b_{n-1,k}, Df(\frac{\cdot}{q})\rangle_q^{\alpha+1,\beta+1}}{B_q(k+\alpha+2, n-k+\beta+1)} b_{n-1,k}(x). \qquad \square$$

Remark 2.2 *When f is derivable on $]0,1[$ and f' has an extension continuous on $[0,1]$, the limit of formula (2.1) is, when q tends to* 1,
$\left(M_{n,1}^{\alpha,\beta} f\right)'(x) = n\,(n+\alpha+\beta+2)^{-1}\, M_{n-1,1}^{\alpha+1,\beta+1}\,(f')\,(x)$ *(cf. [4]).*

Theorem 2.3 *The following equality holds for any function f such that $M_{n,q}^{\alpha,\beta} f$ exists and any $x \in [0,1]$:*

$$\begin{aligned} M_{n,q}^{\alpha,\beta} f(x) &= \sum_{j=0}^{\infty} \Phi_{j,n,q}^{\alpha,\beta}(x) f(q^{j+\beta+1}), \text{ where,} \qquad (2.2) \\ \Phi_{j,n,q}^{\alpha,\beta}(x) &= u_j \sum_{k=0}^{n} v_k b_{n,k,q}(q^{j+\beta+1}) b_{n,k,q}(x), \\ u_j &= (1-q) q^{j(\alpha+1)} (1-q^{j+1})_q^{\beta}, \quad j \in \mathbb{N}, \\ v_k^{-1} &= q^{k(\beta+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q B_q(k+\alpha+1, n-k+\beta+1), \quad k = 0, \ldots, n. \end{aligned}$$

Moreover, for any $r \in \mathbb{N}$, the sequence $\Phi_{r,n,q}^{\alpha,\beta}, \Phi_{r-1,n,q}^{\alpha,\beta}, \ldots, \Phi_{0,n,q}^{\alpha,\beta}$ is totally positive on $[0,1]$, that is to say, for any family (x_i), $0 \le x_1 < \ldots < x_m \le 1$, the collocation matrix $\left(\Phi_{r-j,n,q}^{\alpha,\beta}(x_i)\right)_{i=1,\ldots,m, j=0,\ldots,r}$ is totally positive .

Proof. We set $\Phi_j = \Phi_{j,n,q}^{\alpha,\beta}$, $b_{n,k,q} = b_k, c = q^{\beta+1}$. The formula (2.2) comes by writing the definite q-integrals $\langle b_k, f \rangle_q^{\alpha,\beta}$ as discrete sums in (1.4) and from $\langle b_k, 1 \rangle_q^{\alpha,\beta} = q^{(k+\alpha+1)(\beta+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q B_q(k+\alpha+1, n-k+\beta+1)$, $k = 0, \ldots, n,$.

To obtain the total positivity of the Φ_j, we have to prove that, for any $m \in \mathbb{N}$ and any two families $(x_i)_{i=1,\ldots,m}, (j_l)_{l=1,\ldots,m}$, with $0 \le x_1 \le \ldots \le x_m \le 1$, $0 \le j_m \le \ldots \le j_1$, the determinant $\det(\Phi_{j_l}(x_i))_J$, where $J = \{(i,j) \mid i = 1, \ldots, m, l = 1, \ldots, m\}$, is non negative. From the multilinearity of the determinants, there is a basic composition formula for the discrete sums (cf. [8], p. 17). We have $\det(\Phi_{j_l}(x_i))_J = \prod_{l=1}^m u_{j_l} E$ and
$E = \det(\sum_{k=0}^n v_k b_k(cq^{j_l}) b_k(x_i))_J$
$= \sum_{k_1=0}^n \cdots \sum_{k_m=0}^n v_{k_1} \ldots v_{k_m} \det(b_{k_i}(cq^{j_l}) b_{k_i}(x_i))_J$
$= \sum_{k_1=0}^n \cdots \sum_{k_m=0}^n v_{k_1} \ldots v_{k_m} b_{k_1}(x_1) \ldots b_{k_m}(x_m) \det(b_{k_i}(cq^{j_l}))_J$
$= \sum_{0 \le k_1 \le \ldots \le k_m \le n} v_{k_1} \ldots v_{k_m} \det(b_{k_l}(x_i))_J \det(b_{k_i}(cq^{j_l}))_J.$
We know that the q-Bernstein basis is totally positive (cf. [6]). Hence we have $\det(b_{k_l}(x_i))_J \ge 0$ and also $\det(b_{k_i}(cq^{j_l}))_J \ge 0$, since $cq^{j_1} \le cq^{j_2} \le \ldots \le cq^{j_m}$. So E is non negative and the result follows. □

Corollary 2.4 *The number of sign changes of the polynomial $M_{n,q}^{\alpha,\beta} f$ on $[0,1]$ is not greater than the number of sign changes of the function f.*

Proof. For any $r \in \mathbb{N}$, the sequence $\Phi_r, \Phi_{r-1}, \ldots, \Phi_0$ is totally positive. We deduce that the number of sign changes of the polynomial $\sum_{j=0}^{r} \Phi_j(x) f(q^{j+\beta+1})$

is not greater than the number of sign changes of the sequence $f(q^{j+\beta+1})$, $j = 0, \ldots, r$, hence not greater than the number of sign changes of the function f in $]0,1[$ (cf. [5]). When r tends to infinity this property is preserved hence is true for $M_n f$. □

Corollary 2.5 *Let f be a function satisfying the condition $C(\alpha)$.*

1. *If f is increasing (respectively decreasing), then the function $M_{n,q}^{\alpha,\beta} f$ is increasing (respectively decreasing).*
2. *If f is convex, then the function $M_{n,q}^{\alpha,\beta} f$ is convex.*

Proof. **1)** If f is monotone, for any $s \in \mathbb{R}$ the function $f - s$ has at most one sign change. Hence $M_n^{\alpha,\beta}(f-s) = M_n^{\alpha,\beta} f - s$ has at most one sign change and $M_n^{\alpha,\beta} f$ is monotone. If f is increasing, $Df(\frac{\cdot}{q})$ is positive on $]0,q[$. Since the operators $M_n^{\alpha,\beta}$ are positive, we obtain for $x \in [0,1]$, $M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx) \geq 0$ if $n \geq 1$ and, using (2.1), $DM_n^{\alpha,\beta} f(x) \geq 0$. So the function $M_n^{\alpha,\beta} f$ is increasing if $n \geq 1$.

2) Let suppose the function f is convex. Since $M_n^{\alpha,\beta}$ preserves the degree of the polynomials, for any real numbers γ_1, γ_2 there exist δ_1, δ_2 and a function g such that $g(x) = f(x) - \delta_1 x - \delta_2$ and $M_n^{\alpha,\beta} f(x) - \gamma_1 x - \gamma_2 = M_n^{\alpha,\beta} g(x)$. The number of sign changes of g being at most two, it is the same for $M_n^{\alpha,\beta} g$. Hence $M_n^{\alpha,\beta} f$ is convex or concave.

Moreover, if a function φ is convex (respectively concave),

$D^2\varphi(x) = \frac{q^3}{(q-1)^2 x^2}(\varphi(q^2x) - [2]\,\varphi(qx) + q\varphi(x))$ is ≥ 0 (respectively ≤ 0). Hence, if $n \geq 2$, $M_{n-2}^{\alpha+2,\beta+2}\left(D^2 f\left(\frac{\cdot}{q}\right)\right)(q^2x) \geq 0$. Using (2.1) two times we obtain $D^2 M_n^{\alpha,\beta} f(x) \geq 0$ and $M_n^{\alpha,\beta} f$ is convex, if $n \geq 2$. □

3 Convergence properties

Theorem 3.1 *If f is continuous on $]0,1[$ and has a limit on the right at 0 denoted $f(0)$ and a limit on the left at 1 denoted $f(1)$, we have for $n \geq 1$:*

$$\left\| M_{n,q}^{\alpha,\beta} f - f \right\|_\infty \leq C_{\alpha,\beta}\ \omega\left(f, \frac{1}{\sqrt{[n]_q}}\right),$$

where $\|f\|_\infty$ is the uniform norm of f on $[0,1]$ and $\omega(f,.)$ is the usual modulus of continuity of f, the constant $C_{\alpha,\beta}$ being independent of n, q, f.

Proof. As $M_n^{\alpha,\beta}$ is positive, O. Shisha and B. Mond theorem can be applied. It is sufficient to prove that the order of approximation of f by $M_n^{\alpha,\beta} f$ is $O(\frac{1}{[n]})$ for the functions $f_i(x) = x^i$, $i = 0, 1, 2$. We compute the polynomials $M_n^{\alpha,\beta} f_i$, $i = 1$ for $n \geq 1$ and $i = 2$ for $n \geq 2$ with the help of (2.1) by q-integrations (and directly for $n = 0$ and 1) to obtain :

$[n+\alpha+\beta+2]\,(M_n^{\alpha,\beta} f_1(x) - x) = q^{\beta+1}\,[\alpha+1] - x\,[\alpha+\beta+2]$ and

$[n+\alpha+\beta+2]\,[n+\alpha+\beta+3]\,(M_n^{\alpha,\beta} f_2(x) - x^2) = q^{2\beta+2}\,[\alpha+2]\,[\alpha+1]$

$+[n]\,[2]\,x(q^{\alpha+2\beta+3}\,[\alpha+2] - q^{\alpha+\beta+2}\,[\alpha+\beta+3]\,x) - [\alpha+\beta+3]\,[\alpha+\beta+2]\,x^2.$

The result follows since $0 < q < 1$ and $0 \leq [a] \leq \max(a, 1)$ if $a \geq 0$. □

Remark 3.2 *In order to have uniform convergence for all continuous functions on* $[0, 1]$, *it is sufficient to have* $\lim_{n\to\infty} M_{n,q}^{\alpha,\beta} f_i = f_i$ *for* $i = 1, 2$, *hence* $\lim_{n\to\infty} 1/\,[n]_q = 0$. *This is realized if and only if* $q = q_n$ *and* $\lim_{n\to\infty} q_n = 1$. *Indeed, for every* $n \in N$, *in both cases* $q^n < 1/2$ *and* $q^n \geq 1/2$, *we have* $1 - q < 1/\,[n]_q \leq 2\max(1 - q, \ln 2/n)$. *To maximize the order of approximation by the operator* $M_{n,q_n}^{\alpha,\beta}$ *we are interested to have* $[n]_{q_n}$ *of the same order as* n, *that is to say to have* $\rho n < [n]_{q_n} \leq n$, *for some* $\rho > 0$ *and, since* $[n]_{q_n} \leq n$ *holds in any cases, this is equivalent with the following property* S *for* (q_n).

Definition 3.3 *The sequence* $(q_n)_{n\in\mathbb{N}}$ *has the property* S *if and only if there exists* $N \in \mathbb{N}$ *and* $c > 0$ *such that, for any* $n > N$, $1 - q_n < c/n$.

Lemma 3.4

The property S *holds if and only if the property* P_1 *(respectively* P_2*) holds where :*

P_1 *is:*

"There exist $N_1 \in \mathbb{N}$ *and* $c_1 > 0$ *such that, for any* $n > N_1$, $[n]_{q_n} \geq c_1 n$",

P_2 *is:*

"There exist $N_2 \in \mathbb{N}$ *and* $c_2 > 0$ *such that, for any* $n > N_2$, $q_n^n \geq c_2$".

Proof. For any $n \in \mathbb{N}$, the function $\xi(x) = (1 - x^n)/(1 - x)$ is increasing on $[0, 1[$. If S holds, we have, for any $n > N_1 = N$, $[n]_{q_n} = \xi(q_n) \geq \xi(1 - c/n) \geq n(1 - e^{-c})/c$ and P_1 follows. If P_1 holds, we have, for any $n > N = N_1$, $1/(1 - q_n) \geq [n]_{q_n} \geq c_1 n$ and S follows. If P_2 holds, we have, for any $n > N = N_2$, $n(1 - q_n) \leq -n \ln q_n < -\ln c_2$ and S follows. If S holds, there exists $N_2 > N$ such that, if $n > N_2$, $1 - q_n < 1/2$ hence $q_n^n > e^{-2n(1-q_n)} > e^{-2c}$ and P_2 follows. □

Theorem 3.5 *If the function f is continuous at the point $x \in]0,1[$, then,*

$$\lim_{n\to\infty} M_{n,q_n}^{\alpha,\beta} f(x) = f(x) \tag{3.1}$$

in the two following cases :

1. *f is bounded on $]0,1[$ and the sequence (q_n) is such that $\lim_{n\to\infty} q_n = 1$,*
2. *the function $U_f^{\alpha',\beta'}$ defined by $U_f^{\alpha',\beta'}(x) = x^{\alpha'}(1-x)^{\beta'} f(x)$ is bounded on $]0,1[$ for some real numbers $\alpha', \beta' \geq 0$ such that $\alpha' < \alpha+1$, $\beta' < \beta+1$ and the sequence (q_n) has the property S.*

Theorem 3.6 *If the function f admits a second derivative at the point $x \in]0,1[$, then, in the cases 1 and 2 of Theorem 3.5, we have the Voronovskaya-type limit :*

$$\lim_{n\to\infty} [n]_{q_n} \left(M_{n,q_n}^{\alpha,\beta} f(x) - f(x)\right) = \frac{d}{dx}\left(x^{\alpha+1}(1-x)^{\beta+1} f'(x)\right) \Big/ x^{\alpha}(1-x)^{\beta} . \tag{3.2}$$

(The limit operator is the Jacobi differential operator for the weight $x^{\alpha}(1-x)^{\beta}$)

Remark 3.7 *1) We see that the best order of approximation in (3.2) is in $1/[n]_{q_n}$. If $1-q_n = 1/n^{\gamma}$ with $0 < \gamma < 1$, then $\lim_{n\to\infty} [n]_{q_n}/n^{\gamma} = 1$, hence $[n]_{q_n}$ can be replaced by n^{γ} in (3.2). If $1-q_n = 1/n\log n$ or $1/n^{\gamma}$ with $\gamma > 1$, then $\lim_{n\to\infty} [n]_{q_n}/n = 1$, $[n]_{q_n}$ can be replaced by n and we refound exactly the Voronovskaya-limit property of $M_{n,1}^{\alpha,\beta}(x)$.*

2) In the case 2, the Theorems 3.5 and 3.6 are valid for $M_{n,1}^{\alpha,\beta}$, if wf is Lebesgue integrable on $[0,1]$, and this result is new. (In the proof the Jackson integrals have to be replaced by Lebesgue integrals)

3) As example, for the function $f(x) = x^{-1/2}$, we have, for $q_n = 1-1/n$, $\lim_{n\to\infty} M_{n,q_n}^{0,0} f(x) = x^{-1/2}$ and $\lim_{n\to\infty} n(M_{n,q_n}^{0,0} f(x) - x^{-1/2}) = cx^{-1/2}(1-3x)$ for any $x \in]0,1]$, where $c \sim 0.4$.

4) If f is extendable by continuity at the point $x = 0$ (respectively $x = 1$) the Theorems 3.5 and 3.6 are still valid at the point $x = 0$ (respectively $x = 1$) in the case 1 and in the case 2 which becomes : $U_f^{0,\beta'}$ is bounded on $]0,1[$ and $\beta' < \beta+1$ (respectively $U_f^{\alpha',0}$ is bounded on $]0,1[$ and $\alpha' < \alpha+1$).

For the proofs of Theorems 3.5 and 3.6 we need some preparation.

Proposition 3.8 *We set, for any $n, m \in \mathbb{N}$ and $x \in [0,1]$, $q \in [1/2, 1[$,*

$$T_{n,m,q}(x) = \sum_{k=0}^{n} b_{n,k,q}(x) \frac{\displaystyle\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ (x-t)^m d_q t}{\displaystyle\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta} d_q t}. \qquad (3.3)$$

For any m, there exists a constant $K_m > 0$, independent of n and q, such that :

$$\sup_{x\in[0,1]} |T_{n,m,q}(x)| \leq \begin{cases} K_m / [n]_q^{m/2} & \text{if } m \text{ is even,} \\ \\ K_m / [n]_q^{(m+1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

To prove this proposition we consider the Lemmas 3.9 and 3.10.

Lemma 3.9 *We set, for any $n, m \in \mathbb{N}$ and $x \in [0,1]$,*

$$T^1_{n,m,q}(x) = \sum_{k=0}^{n} b_{n,k,q}(x) \frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ (x-t)_q^m d_q t}{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta} d_q t}.$$

The following recursion formula holds for any $q \in [1/2, 1[$ and $m \geq 2$:

$$[n+m+\alpha+\beta+2]_q\, q^{-\alpha-2m-1} T^1_{n,m+1,q}(x) = -x(1-x) D_q T^1_{n,m,q}(x)$$
$$+T^1_{n,m,q}(x) \left[p_{1,m}(x) + x(1-q)\,[n+\alpha+\beta]_q\,[m+1]_q\, q^{1-\alpha-m} \right]$$
$$+T^1_{n,m-1,q}(x) p_{2,m}(x) + T^1_{n,m-2,q}(x) p_{3,m}(x)(1-q),$$

where the polynomials $p_{i,m}(x)$, $i = 1, 2$ and 3 are uniformly bounded with regard to n and q.

Proof. 1) We set $T^1_{n,m,q} = T^1_m$ and, for $k = 0, \ldots, n$,
$\psi_k(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$, $l_k(x) = b_{n,k}(x) \Big/ \int_0^1 \psi_k(t) d_q t$.

We compute
$x(1-x) D_q T^1_m(x) = x(1-x)\,[m] \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(x-t)_q^{m-1} d_q t$
$+ \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^m ([k] - [n]\,x) d_q t = A + B.$

We have $A = x(1-x)\,[m]\, T^1_{m-1}(x)$. We write, for $k = 0, \ldots, n$,
$[k] - [n]\,x = [-\alpha] + q^{-\alpha}([k+\alpha] - qt\,[n+\alpha+\beta])$
$-q^{1-m-\alpha}\,[n+\alpha+\beta]\,(qx - q^m t) + x(q^{2-m-\alpha}\,[n+\alpha+\beta] - [n])$, and
$t(1-qt)\,(D_q \psi_k)\,(t) = t^{k+\alpha}(1-qt)^{n-k+\beta}([k+\alpha] - qt\,[n+\alpha+\beta])$

to obtain
$B = q^{-\alpha}\sum_{k=0}^{n} l_k(x)\int_0^1 (D_q\psi_k)(t)\, t(1-qt)(qx-t)_q^m d_qt$
$-q^{1-\alpha-m}\,[n+\alpha+\beta]\sum_{k=0}^{n} l_k(x)\int_0^1 \psi_k(t)(qx-t)_q^{m+1}d_qt$
$+(x([n+\alpha+\beta]\,q^{2-\alpha-m}-[n])+[-\alpha])\sum_{k=0}^{n} l_k(x)\int_0^1 \psi_k(t)(qx-t)_q^m d_qt$
$= B_1 + B_2 + B_3.$
We q-integrate by parts, setting $\sigma(t) = \frac{t}{q}(1-t)\left(qx-\frac{t}{q}\right)_q^m$.
The q-integrals in B_1 become
$\int_0^1 D_q\psi_k(t)\, t(1-qt)(qx-t)_q^m d_qt$
$= [\psi_k(t)\sigma(t)]_0^1 - \int_0^1 \psi_k(t)(D_q\sigma)(t)d_qt$ for each $k = 0,\dots,n$.
We expand
$\sigma(t) = -q^{-2m}(x-\frac{t}{q})_q^{m+2} + q^{-2m}\left\{([3]-q^{m+2})x - q^m\right\}(x-\frac{t}{q})_q^{m+1}$
$+q^{-2m+1}x\left\{q^{m-1}+[m]\,(1-q)q^m - x(1+[2]\,q(1-q)\,[m]\right\}(x-\frac{t}{q})_q^m$
$-q^{-2m+3}x^2\,[m]\,(1-q)(q^{m-2}-x)(x-\frac{t}{q})_q^{m-1}.$
We obtain
$B_1 = -q^{-\alpha-2m-1}\left\{[m+2]\,T_{m+1}^1(x) - [m+1]\,([3]\,x - q^{m+2}x - q^m)\right\}T_m^1(x)$
$-q^{-\alpha-2m}\,[m]\,x\left\{q^{m-1}+(1-q)\,[m]\,q^m - x(1+q(1-q)\,[2]\,[m]\right\}T_{m-1}^1(x)$
$+q^{-\alpha+2-2m}\,[m-1]\,[m]\,x^2(q^{m-2}-x)(1-q)T_{m-2}^1(x).$
Moreover, we have
$B_2 = -q^{1-\alpha-m}\,[n+\alpha+\beta]\,(T_{m+1}^1(x) - (1-q)\,[m+1]\,xT_m^1(x)),$
$B_3 = \{x(q^n\,[\beta-m+2]-[2-\alpha-m])+[-\alpha]\}\left[T_m^1(x)-(1-q)\,[m]\,xT_{m-1}^1(x)\right].$
□

Lemma 3.10 *For any $m \in \mathbb{N}$, $q \in [1/2,1[$, $x \in [0,1]$, the expansion of $(x-t)^m$ on the Newton basis at the points x/q^i, $i = 0,\dots,m-1$ is:*

$$(x-t)^m = \sum_{k=1}^{m} d_{m,k}(1-q)^{m-k}(x-t)_q^k, \tag{3.4}$$

where the coefficients $d_{m,k}$, verify $|d_{m,k}| \le d_m$, $k = 1,\dots,m$, and d_m does not depend on x,t,q.

Proof. For $m = 1$, it is obvious. If for some $m \ge 1$, the relation (3.4) holds, we write $x-t = q^{-k}((x-q^kt)-(1-q)\,[k]\,x)$ for $k = 1,\dots,m$ and we obtain
$(x-t)^{m+1} = \sum_{k=1}^{m+1} d_{m+1,k}(1-q)^{m+1-k}(x-t)_q^k$
with $d_{m+1,k} = q^{-k}(qd_{m,k-1} - [k]\,xd_{m,k})$ if $k = 2,\dots,m$
and $d_{m+1,1} = -q^{-1}d_{m,1}x$, $d_{m+1,m+1} = q^{-m}d_{m,m}$. Since $|d_{m,k}| \le d_m$ we have
$|d_{m+1,k}| \le d_{m+1} = 2^m(m+1)d_m$, $k = 1,\dots,m+1$. □

Proof of Proposition 3.8.

At first we prove that, for any x, $\left|T_m^1(x)\right| \leq H_m/\left[n\right]^{m/2}$ if m is even (respectively $\leq H_m/\left[n\right]^{(m+1)/2}$ if m is odd), where H_m does not depend on n, q, x. We have $T_0^1(x) = 1$ and the formulae for $M_n^{\alpha,\beta} f_i, i = 1, 2$, established in the proof of Theorem 3.1, give the result for $m = 1$ and 2. The product $[n+\alpha+\beta]\,(1-q) = 1 - q^{n+\alpha+\beta}$ is positive and bounded by $\max(1, \left|1 - 2^{-(\alpha+\beta)}\right|)$ and $(1-q) < 1/\left[n\right]$. If the result is true for some $p \geq 2$, p odd (respectively even) and any $m \leq p$, the result for $p+1$ follows from the recursion formula of Lemma 3.9.

Then, we write, for any $n, m \in \mathbb{N}$ and $x \in [0,1]$, using Lemma 3.10,
$|T_{n,m,q}(x)| \leq d_m \sum_{k=1}^m (1-q)^{m-k} \left|T_k^1(x)\right|$
$\leq d_m(\left|T_m^1(x)\right| + \sum_{k=1}^{m-1} \left[n\right]^{-m+k} H_k \left[n\right]^{-k/2})$
$\leq d_m \left|T_m^1(x)\right| + \sum_{k=1}^{m-1} H_k \left[n\right]^{-(m+1)/2})$ and the result follows. □

Now the following Lemma is the key.

Lemma 3.11 *Let (q_n) be a sequence possessing the property S. Let α, β, α', β' be real numbers such that $\alpha', \beta' \geq 0$, $\alpha' < \alpha + 1$, $\beta' < \beta + 1$. We set $\varphi(t) = t^{-\alpha'}(1-t)^{-\beta'}$, $t \in]0,1[$ and $I_{x,\delta}(t) = 1$ if $|t-x| > \delta$, $I_{x,\delta}(t) = 0$ elsewhere. We define the sequence*

$$E_n(x,\delta) = \sum_{k=0}^{n} b_{n,k,q_n}(x) \frac{\displaystyle\int_0^1 t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta} \varphi(q_n^{\beta+1} t) I_{x,\delta}(t) d_{q_n} t}{\displaystyle\int_0^1 t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta} d_{q_n} t}.$$

It verifies $\lim_{n\to\infty} nE_n(x,\delta) = 0$ for any x and δ such that $0 < \delta < x < 1 - \delta$.

Proof. Let $\overline{\alpha}$ (respectively $\overline{\beta}$) be the smallest integer such that $\overline{\alpha} \geq \alpha$ (respectively $\overline{\beta} \geq \beta$) and τ (respectively τ') be a real number such that $\tau > \dfrac{\overline{\alpha}+\overline{\beta}+2}{\alpha-\alpha'+1}$ (respectively $\tau' > \dfrac{\overline{\alpha}+\overline{\beta}+2}{\beta-\beta'+1}$).

We set $I_{x,\delta}^-(t) = 1$ if $0 < t < x - \delta$ and $I_{x,\delta}^+(t) = 1$ if $x + \delta < t < 1$, $I_{x,\delta}^-(t) = I_{x,\delta}^+(t) = 0$ elsewhere.

We split the interval $(0,1)$ introducing $e_n = 1/n^\tau, e_n' = 1 - 1/n^{\tau'}, n \in \mathbb{N}$, and we define, using again $\psi_{n,k,q}^{\alpha,\beta}(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$ and $l_{n,k,q}(x) = b_{n,k,q}(x) \Big/ \int_0^1 \psi_{n,k,q}^{\alpha,\beta}(t) d_q t$,

$A_n^1 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^{e_n} t^{k+\alpha-\alpha'}(1-q_n t)_{q_n}^{n-k+\beta} d_{q_n} t$,

$A_n^2 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_{e_n}^1 t^{k+\alpha-\alpha'}(1-q_n t)_{q_n}^{n-k+\beta} I_{x,\delta}^-(t) d_{q_n} t$,

$A_n^3 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^{e'_n} t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta} \Big/ (1-q_n^{\beta+1}t)^{\beta'} \; I_{x,\delta}^+(t) d_{q_n}t,$

$A_n^4 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_{e'_n}^1 t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta} \Big/ (1-q_n^{\beta+1}t)^{\beta'} \; d_{q_n}t.$

If $t > x$, (respectively if $t < x$) then $t^{-\alpha'} < x^{-\alpha'}$ (respectively $(1-q_n^{\beta+1}t)^{\beta'}$ $> (1-q_n^{\beta+1}x)^{\beta'} \geq (1-x)^{\beta'}$).

Hence, we have, if $q_n \geq 1/2$,

$E_n(x,\delta) \leq (1/2)^{-(\beta+1)\alpha'}((1-x)^{-\beta'}(A_n^1 + A_n^2) + x^{-\alpha'}(A_n^3 + A_n^4))$
and it is sufficient to prove that $\lim\limits_{n\to\infty} nA_n^i = 0$ for $i = 1,2,3,4$.

If $q_n^n \geq c$ and $e_n < 1/2$, we have $\begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \int_0^{e_n} t^{k+\alpha-\alpha'}(1-q_nt)_{q_n}^{n-k+\beta} d_{q_n}t =$ $q_n^{-k(\beta+1)} \int_0^{e_n} b_{n,k,q}(q_n^{\beta+1}t) t^{\alpha-\alpha'}(1-q_nt)_{q_n}^{\beta} d_{q_n}t \leq \gamma_1 e_n^{\alpha-\alpha'+1}$, for $k = 0,\ldots,n$, where γ_1 does not depend on k,n,x, since $q_n^{(\beta+1)k} \geq c^{\beta+1}$, $0 \leq b_{n,k,q_n}(q_n^{\beta+1}t)$ ≤ 1, and, $(1-q_nt)_{q_n}^{\beta} \leq 1$ if $\beta \geq 0$ and $t \in [0,1]$, (respectively $(1-q_nt)_{q_n}^{\beta}$ $\leq (1-e_n)^{-1} \leq 2$ if $\beta < 0$ and $t \in [0,e_n]$).

Moreover, for any $k = 0,\ldots,n$, we have

$\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta} d_qt \geq \int_0^1 t^{k+\overline{\alpha}}(1-qt)_q^{n-k+\overline{\beta}} d_qt = \frac{[k+\overline{\alpha}]![n-k+\overline{\beta}]!}{[n+\overline{\alpha}+\overline{\beta}+1]!}$

$\geq \begin{bmatrix} n \\ k \end{bmatrix}^{-1} (n+\overline{\alpha}+\overline{\beta}+1)^{-(\overline{\alpha}+\overline{\beta}+1)})$.

Hence, we have $A_n^1 \leq \gamma_1(n+\overline{\alpha}+\overline{\beta}+1)^{\overline{\alpha}+\overline{\beta}+1} n^{-\tau(\alpha-\alpha'+1)}$. The choice of τ and the property S on (q_n) give $\lim\limits_{n\to\infty} nA_n^1 = 0$.

We choose $m \in \mathbb{N}$ such that $m > \tau\alpha' + 1$ and we write

$A_n^2 \leq n^{\tau\alpha'}\delta^{-2m} \sum_{k=0}^n l_{n,k,q_n}(x) \int_{e_n}^1 t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta}(x-t)^{2m} d_{q_n}t$

$\leq n^{\tau\alpha'}\delta^{-2m} T_{n,2m,q_n}(x) \leq K_{2m}\delta^{-2m} n^{\tau\alpha'-m}$,

hence $\lim\limits_{n\to\infty} nA_n^2 = 0$ by the choice of m.

Now we have, if $t < e'_n$, $(1-q_n^{\beta+1}t)^{\beta'} > (1-e'_n)^{\beta'} = n^{-\tau'\beta'}$, hence $A_n^3 \leq n^{\tau'\beta'} \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^{e'_n} t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta} I_{x,\delta}^+(t) d_{q_n}t$. We choose $m' \in \mathbb{N}$ such that $m' > \tau'\beta' + 1$ to have

$A_n^3 \leq n^{\tau'\beta'}\delta^{-2m'} \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^1 t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta}(x-t)^{2m'} d_{q_n}t$

$\leq n^{\tau'\beta'}\delta^{-2m'} T_{n,2m',q_n} \leq K_{2m'}\delta^{-2m'} n^{\tau'\beta'-m'}$,

hence $\lim\limits_{n\to\infty} nA_n^3 = 0$ by the choice of m'.

To finish, we need to prove that $(1-q_n^{-\beta'}u)_{q_n}^{\beta'} \leq (1-u)^{\beta'}$ for any $u \in [0, q^{\beta+1}]$.

If $\beta' \in \mathbb{N} - \{0\}$, we have $(1-q^{-\beta'}u)^{\beta'} = \prod_{j=0}^{\beta'-1}(1-q^{-\beta'+j}u) \leq (1-u)^{\beta'}$.

If $0 < \beta' < 1$, we use the q-binomial Theorem (cf. [1],p. 488) to write $(1-q^{-\beta'}u)_{q_n}^{\beta'} = \sum_{k=0}^{\infty}(1-q^{-\beta'})_{q_n}^k \Big/ (1-q)_{q_n}^k \; u^k$

$= 1 + \sum_{k=1}^{\infty} \prod_{j=1}^k \left[-\beta'+j-1\right]_{q_n} \Big/ [j]_{q_n} \; u^k$.

Then the inequalities $\left[-\beta'\right]_{q_n} \leq -\beta' < 0$ and $\left[-\beta'+j-1\right]_{q_n} / \left[j\right]_{q_n} \geq \left(-\beta'+j-1\right)/j > 0$ for any integer $j \geq 2$ give the result.

In the other cases, if l is the integer such that $l \leq \beta' < l+1$, we use the rules of product of q-binomials to write
$(1-q^{-\beta'}u)_{q_n}^{\beta'} = (1-q_n^{-\beta'}u)_{q_n}^{l}(1-q_n^{-\beta'+l}t)_{q_n}^{\beta'-l}$ and the result follows from the two first cases.

Then, with the same rules, for any $k = 0, \ldots, n$ and $t \in [0,1]$, setting $q_n^{\beta+1}t = u$, we write
$(1-q_nt)_{q_n}^{n-k+\beta} = (1-q_nt)_{q_n}^{\beta-\beta'}(1-q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'}(1-q_n^{\beta+1}t)_{q_n}^{n-k}$
$\leq (1-q_n^{\beta+1}t)^{\beta'}(1-q_nt)_{q_n}^{\beta-\beta'}(1-q_n^{\beta+1}t)_{q_n}^{n-k}$.

We deduce if $q_n^n \geq c$ and $e'_n > 1/2$,
$A_n^4 \leq \sum_{k=0}^n q_n^{-(\beta+1)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n}^{-1} l_{n,k,q_n}(x) \int_{e'_n}^1 t^\alpha (1-q_nt)_{q_n}^{\beta-\beta'} b_{n,k,q_n}(q_n^{\beta+1}t) d_{q_n}t$
$\leq \gamma_2 (n+\overline{\alpha}+\overline{\beta}+1)^{\overline{\alpha}+\overline{\beta}+1}(1-e'_n)^{\beta-\beta'+1}$ where γ_2 does not depend on k, n, x. The choice of e'_n and τ' gives $\lim\limits_{n\to\infty} nA_n^4 = 0$. □

Proof of Theorem 3.5.

Suppose f is continuous at $x \in]0,1[$. Let $\varepsilon > 0$ be an arbitrary real number. There exists $\delta' > 0$ such that $|f(x)-f(t)| < \varepsilon$ for any $t \in]0,1[$ such that $|x-t| < \delta'$. Let $\delta = \delta'/2$ and $N' \in \mathbb{N}$ such that $(1-q_n^{\beta+1})x < \delta$ for $n > N'$. Then we have, if $|x-t| < \delta$ and $n > N'$, the inequalities $-\delta < -q_n^{\beta+1}\delta < x - q_n^{\beta+1}t = q_n^{\beta+1}(x-t) + (1-q_n^{\beta+1})x < 2\delta$ and $\left|f(x) - f(q_n^{\beta+1}t)\right| < \varepsilon$.

Hence, we have, for any $t \in]0,1]$,
if $|f|$ is bounded by κ, $\left|f(x) - f(q_n^{\beta+1}t)\right| < \varepsilon + 2\kappa I_{x,\delta}(t)$
and, in the case 2, if $\left|U_f^{\alpha',\beta'}\right|$ is bounded by κ',
$\left|f(x) - f(q_n^{\beta+1}t)\right| < \varepsilon + (|f(x)| + \kappa'(q_n^{\beta+1}t)^{-\alpha'}(1-q_n^{\beta+1}t)^{-\beta'})I_{x,\delta}(t)$.

We apply the operator $M_{n,q_n}^{\alpha,\beta}$ at the function $h_x(t) = f(t) - f(x)$.
We have $\left|M_{n,q_n}^{\alpha,\beta}f(x) - f(x)\right| = \left|M_{n,q_n}^{\alpha,\beta}h_x(x)\right| \leq \left(M_{n,q_n}^{\alpha,\beta}|h_x|\right)(x)$
$$\leq \begin{cases} \varepsilon + 2\kappa T_{n,2,q_n}(x)/\delta^2, \text{ in the case 1,} \\ \varepsilon + |f(x)|\, T_{n,2,q_n}(x)/\delta^2 + \kappa' E_n(x,\delta), \text{ in the case 2.} \end{cases}$$

The second terms of the right hand side vanish when $[n]_{q_n}$ tends to infinity hence when n tends to infinity in both cases (remark 1). The result follows from Lemma 3.11. □

Proof of Theorem 3.6.

We write Taylor formula at the point x,
$f(t) = f(x) + (t-x)f'(x) + (t-x)^2 f''(x)/2 + (t-x)^2\varepsilon(t-x)$
where $\lim_{u\to 0}\varepsilon(u) = 0$.
We apply the operator $M_{n,q_n}^{\alpha,\beta}$ at the function f of the variable t to obtain
$M_{n,q_n}^{\alpha,\beta} f(x) - f(x) = -f'(x)T_{n,1,q_n}(x) + \dfrac{f''(x)}{2}T_{n,2,q_n}(x) + R_n(x)$ where
$R_n(x) = M_{n,q_n}^{\alpha,\beta}\zeta_x(x)$ with $\zeta_x(t) = (t-x)^2\varepsilon(t-x)$. We use $\lim_{q\to 1}[a]_q = a$ for any $a \in \mathbb{R}$ and we verify, with the help of the formulae established in the proof of Theorem 3.1, that $\lim_{n\to\infty}[n]_{q_n} T_{n,1,q_n}(x) = (\alpha+\beta+2)x - \alpha - 1$ and $\lim_{n\to\infty}[n]_{q_n} T_{n,2,q_n}(x) = 2x(1-x)$. So, to obtain the result, we have to prove that $\lim_{n\to\infty}[n]_{q_n} R_n(x) = 0$. We proceed in the same manner as in the proof of Theorem 3.5. For any arbitrary $\eta > 0$, we can find $\delta > 0$ such that, for n large enough, $|\varepsilon(x-t)| < \eta$ if $\left|x - q_n^{\beta+1}t\right| < \delta$.
We obtain the inequality $|\zeta_x(t)| \le \eta(x-t)^2 + (\rho_x + |f(t)|)I_{x,\delta}(q^{-(\beta+1)}t)$
for any $t \in]0,1[$, where ρ_x is independent of t and δ. We deduce

$$[n]_{q_n}|R_n(x)| \le \begin{cases} [n]_{q_n}(\eta T_{n,2,q_n}(x) + (\rho_x+\kappa)T_{n,4,q_n}(x)/\delta^4), & \text{in the case 1,} \\ [n]_{q_n}(\eta T_{n,2,q_n}(x) + \rho_x T_{n,4,q_n}(x)/\delta^4) + \kappa' n E_n(x,\delta), & \\ & \text{in the case 2.} \end{cases}$$

The right hand side tends to $2\eta x(1-x)$ when n (hence $[n]_{q_n}$) tends to infinity and is as small as wanted. □

Theorem 3.12 *If the function f has a derivative f' continuous on $]0,1[$ which is extendable by continuity at 0 and 1, we have, for $q > 1/2$ and $n \ge 2$:*

$$\begin{aligned} &\left\| D_q(M_{n,q}^{\alpha,\beta} f) - f' \right\|_\infty \\ &\le \quad C'_{\alpha,\beta}\left(\omega\left(f', \frac{1}{\sqrt{[n]_q}}\right) + \omega\left(f', 1-q\right)\right) + \frac{[\alpha+\beta+2]_q}{[n]_q}\left\|f'\right\|_\infty, \end{aligned}$$

where $C'_{\alpha,\beta}$ is a constant independent of n, q, f.
Hence, if $\lim_{n\to\infty} q_n = 1$, then $\lim_{n\to\infty} D_{q_n}(M_{n,q_n}^{\alpha,\beta} f) = f'$ uniformly on $[0,1]$.

Proof. We write, using (2.1), for any $x \in [0,1]$,
$DM_n^{\alpha,\beta} f(x) - f'(x)$
$= \frac{[n]}{[n+\alpha+\beta+2]} q^{\alpha+\beta+2}\left(M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx) - Df(x) + Df(x) - f'(x)\right)$

$+\left(\frac{[n]}{[n+\alpha+\beta+2]}q^{\alpha+\beta+2}-1\right)f'(x)$.

Since $0<\frac{[n]}{[n+\alpha+\beta+2]}q^{\alpha+\beta+2}<1$, we have

$\left|D(M_n^{\alpha,\beta}f(x))-f'(x)\right|$

$\leq \left|M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx)-Df(x)\right|+|Df(x)-f'(x)|+\frac{[\alpha+\beta+2]}{[n]}\,|f'(x)|\,.$

The Theorem (3.1) for the function $Df\left(\frac{\cdot}{q}\right)$ gives

$\left|M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx)-Df(\frac{\cdot}{q})(qx)\right|\leq C_{\alpha+1,\beta+1}\omega\left(Df(\frac{\cdot}{q}),\frac{1}{\sqrt{[n-1]}}\right).$

Moreover $|Df(x)-f'(x)|=|f'(y)-f'(x)|$ for some y with $qx\leq y\leq x$ hence $|y-x|\leq 1-q$ and $|Df(x)-f'(x)|\leq\omega(f',1-q)$. The modulus of continuity of $Df\left(\frac{\cdot}{q}\right)$ is linked with the modulus of continuity of f'. Indeed, for any $y_i\in[0,1]$ and $i=1,2$, there exists z_i, such that , $y_i\leq z_i\leq y_i/q$ and $Df(\frac{\cdot}{q})(y_i)=f'(z_i)$.

As $|z_1-z_2|\leq|y_1-y_2|/q+(1-q)/q$ we get

$\omega\left(Df(\frac{\cdot}{q}),t\right)=\sup_{|y_1-y_2|\leq t}\left|Df(\frac{\cdot}{q})(y_1)-Df(\frac{\cdot}{q})(y_2)\right|$

$\leq\sup_{|y_1-y_2|\leq t}(|f'(z_1)-f'(z_2)|\leq 2(\omega(f',t)+\omega(f',1-q))$

and the result follows. □

Corollary 3.13 *Under the hypothesis of Theorem 3.12, we have, if* $1-q_n=o(1/n^4)$ *:*

$$\lim_{n\to\infty}\left(M_{n,q_n}^{\alpha,\beta}f\right)'(x)=f'(x)\ \textit{uniformly on}\ [0,1].$$

Proof. For any $x\in[0,1]$, there exists $u\in(q_nx,x)$ such that $D_{q_n}(M_{n,q_n}^{\alpha,\beta}f)(x)=(M_{n,q_n}^{\alpha,\beta}f)'(u)$ and $(x-u)\leq 1-q_n$. Hence $\left|D_{q_n}(M_{n,q_n}^{\alpha,\beta}f)(x)-(M_{n,q_n}^{\alpha,\beta}f)'(x)\right|\leq(1-q_n)\left|(M_{n,q_n}^{\alpha,\beta}f)''(v)\right|$ for some v and $\left\|D_{q_n}(M_{n,q_n}^{\alpha,\beta}f)-(M_{n,q_n}^{\alpha,\beta}f)'\right\|_\infty\leq n^4(1-q_n)\left\|M_{n,q_n}^{\alpha,\beta}f\right\|_\infty\leq n^4(1-q_n)\,\|f\|_\infty\,,$ via Markov inequality. □

4 Self-adjointness properties

In this part $q\in]0,1[$ is independent of n.

On the space of polynomials $\langle.,.\rangle_q^{\alpha,\beta}$ is an inner product. Let $\left(P_{r,q}^{\alpha,\beta}\right)_{r\in\mathbb{N}}$ be the sequence of the orthogonal polynomials for $\langle.,.\rangle_q^{\alpha,\beta}$ such that degree

of $P_{r,q}^{\alpha,\beta} = r$ and $P_{r,q}^{\alpha,\beta}(0) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix}_q = \dfrac{[\alpha+r]_q \dots [\alpha+1]_q}{[r]_q!}$.
We set $\nu_r = \left(\langle P_{r,q}^{\alpha,\beta}, P_{r,q}^{\alpha,\beta} \rangle_q^{\alpha,\beta} \right)^{1/2}$.

We define $U_q^{\alpha,\beta}$ which is a q-analogue of the Jacobi differential operator by:

$$U_q^{\alpha,\beta} f(x) = D_q \left(x^{\alpha+1} (1 - q^{-\beta-1}x)_q^{\beta+1} D_q f(\frac{x}{q}) \right) \Big/ x^{\alpha} (1 - q^{-\beta}x)_q^{\beta} . \quad (4.1)$$

Proposition 4.1 *The operator $U_q^{\alpha,\beta}$ is self-adjoint with respect to $\langle ., . \rangle_q^{\alpha,\beta}$. It preserves the space of polynomials of degree $\leq r$, $r \in \mathbb{N}$. Consequently, for any $r \in \mathbb{N}$, $P_{r,q}^{\alpha,\beta}$ is eigenvector of $U_q^{\alpha,\beta}$ for the eigenvalue*
$\mu_r^{\alpha,\beta} = -q^{-\beta-r} [r]_q [r+\alpha+\beta+1]_q .$

Proof. $U^{\alpha,\beta}$ is a q-differential operator of order 2. We compute with q-binomial relations,
$U^{\alpha,\beta} f(x) =$
$\left\{ -q^{\alpha-\beta} [\beta+1] x + [\alpha+1] (1 - q^{-\beta-1}x \right\} Df(x) + x(1 - q^{-\beta-1}x)/qD^2 f(\frac{x}{q})$,
hence the operator U preserves the degree of polynomials. If f and g are polynomials $\langle Uf, g \rangle$ is well defined. We write, since the q-integration by parts is valid,

$$\langle U^{\alpha,\beta} f, g \rangle = \left[t^{\alpha+1} (1 - q^{-\beta-1}t)_q^{\beta+1} Df(\tfrac{t}{q}) g(t) \right]_0^{q^{\beta+1}} - \int_0^{q^{\beta+1}} (qt)^{\alpha+1} (1 - q^{-\beta}t)_q^{\beta+1} Df(t) Dg(t) d_q t$$

and the first term vanishes.

We compute $U^{\alpha,\beta} f(x)$ for $f(x) = x^r$ to obtain
$U^{\alpha,\beta} f(x) = q^{1-r} [r] \left([\alpha+r] x^{r-1} (1 - x) - q^{-\beta-1} [\beta+1] x^r \right.$ where the coefficient of x^r is the eigenvalue $\mu_r^{\alpha,\beta}$ if $r \geq 1$. □

Proposition 4.2 *The eigenvectors of the operators $M_{n,q}^{\alpha,\beta}, n \in \mathbb{N}$, are the polynomials $P_{r,q}^{\alpha,\beta}, r \in \mathbb{N}$ and, if f satisfies $C(\alpha)$, we have*
$M_{n,q}^{\alpha,\beta} f = \sum_{r=0}^{n} \lambda_{n,r}^{\alpha,\beta} \langle f, P_{r,q}^{\alpha,\beta} \rangle_q^{\alpha,\beta} P_{r,q}^{\alpha,\beta} / \nu_r^2$ *with the eigenvalues*

$$\lambda_{n,r}^{\alpha,\beta} = q^{r(r+\alpha+\beta+1)} \frac{[n]_q!}{[n-r]_q!} \frac{\Gamma_q(n+\alpha+\beta+2)}{\Gamma_q(n+r+\alpha+\beta+2)} \text{ if } r \leq n,$$

$$\lambda_{n,r}^{\alpha,\beta} = 0 \text{ otherwise.}$$

Proof. Since M_n is self adjoint and preserve the degree of polynomials, the orthogonal polynomials P_r are eigenvectors. The eigenvalue $\lambda_{n,r}^{\alpha,\beta}$ is obtained by computing $D^r M_n f(x)$ for $f(x) = x^r$. We use (2.1) r times to get $D^r M_n f(x) = \dfrac{q^{r(\alpha+\beta+r+1)}\,[r]!\,[n]\dots[n-r+1]}{[n+\alpha+\beta+2]\dots[n+r+\alpha+\beta+1]} = \lambda_{n,r}^{\alpha,\beta}\,[r]!$. □

Corollary 4.3 *1. For any $n, m \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $M_{m,q}^{\alpha,\beta}$ commute on the space of functions satisfying $C(\alpha)$.*

2. For any $n \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $U_q^{\alpha,\beta}$ commute on the space of functions f such that f' is defined in a neighbourhood of 0 and is extendable by continuity at the point 0.

Proof. 2) For any $r \in \mathbb{N}$ the q-integrals $\langle f, UP_r\rangle$ and $\langle Uf, P_r\rangle$ are well defined if f' is defined in a neighbourhood of 0 and is extendable by continuity at the point 0. We go from one to the other by two q-integrations by parts which are valid because $\lim_{x\to 0} Df(\frac{x}{q}) = f'(0)$. Then we write

$$UM_n f = \sum_{r=0}^{n} \lambda_{n,r}\langle f, P_r\rangle \mu_r P_r/\nu_r^2 = \sum_{r=0}^{n} \lambda_{n,r}\langle Uf, P_r\rangle P_r/\nu_r^2 = M_n U f. \qquad \square$$

Remark 4.4 *This proposition and its corollary open a field to study $\lim_{n\to\infty} M_{n,q}^{\alpha,\beta} f$ for q fixed. Formally we have*

$$\lim_{n\to\infty} M_{n,q}^{\alpha,\beta} f = S_q^{\alpha,\beta} f = \sum_{r=0}^{\infty} q^{r(r+\alpha+\beta+1)}\langle f, P_{r,q}^{\alpha,\beta}\rangle_q^{\alpha,\beta} P_{r,q}^{\alpha,\beta}/\nu_r^2$$

and, if Q is a polynomial,

$$\lim_{n\to\infty} M_{n,q}^{\alpha,\beta} Q = \sum_{r=0}^{\deg Q} q^{r(r+\alpha+\beta+1)}\langle Q, P_{r,q}^{\alpha,\beta}\rangle_q^{\alpha,\beta} P_{r,q}^{\alpha,\beta}(x)/\nu_r^2.$$

So $\lim_{n\to\infty} M_{n,q}^{\alpha,\beta} f = f$, if and only if f is a constant.

Moreover, $\lambda_{n-1,r}^{\alpha,\beta} - \lambda_{n,r}^{\alpha,\beta} = \dfrac{q^{n+\beta}\lambda_{n,r}^{\alpha,\beta}\mu_r^{\alpha,\beta}}{[n]_q\,[n+\alpha+\beta+1]_q}$, $r \in \mathbb{N}$, hence

$$M_{n-1}^{\alpha,\beta} f - M_n^{\alpha,\beta} f = \frac{q^{n+\beta}}{[n]_q\,[n+\alpha+\beta+1]_q} U^{\alpha,\beta} M_n^{\alpha,\beta} f$$

and it is easy to prove (cf. [2]) that, if f' is defined in a neighbourhood of 0 and is extendable by continuity at the point 0,

$$\left\| M_{n,q}^{\alpha,\beta} f - S_q^{\alpha,\beta} f \right\|_\infty \le \gamma_n \sup_{x\in]0,q^{\beta+2}]} \left| U_q^{\alpha,\beta} f(x) \right|,$$

where $\gamma_n = \sum_{k=n+1}^{\infty} \dfrac{q^{k+\beta}}{[k]_q\,[k+\alpha+\beta+1]_q} \sim \dfrac{q^{n+\beta+1}}{[n]_q}$.

Proposition 4.5 *The polynomials $P_{r,q}^{\alpha,\beta}$ are q-extensions of Jacobi polynomials with respect to the weight $x^\alpha(1-x)^\beta$ denoted $P_r^{\alpha,\beta}, r \in \mathbb{N}$. They have the following properties which are the q-analogues of the well-known properties of Jacobi polynomials.*

1. *For any $r \in \mathbb{N}$, $\lim_{q\to 1} P_{r,q}^{\alpha,\beta} = P_r^{\alpha,\beta}$,*
2. *For any $r \in \mathbb{N}$, the polynomial $P_{r,q}^{\alpha,\beta}$ is a q-hypergeometric function (cf. [7], p. 43) :*

$$P_{r,q}^{\alpha,\beta}(x) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix}_q {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{r+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix} ; q, \quad q^{-\beta}x \right]$$

   So we have $P_{r,q}^{\alpha,\beta}(x) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix}_q p_r(q^{-\beta-1}x; q^\alpha, q^\beta : q)$,
   where $p_r(x; u, v : q)$ is the shifted little q-Jacobi polynomial of degree r (cf. [1], p. 592).
3. *They verify a q-analogue of Rodrigues formula :*

$$P_{r,q}^{\alpha,\beta}(x) = \frac{1}{[r]_q!} \frac{D_q^r \left(x^{\alpha+r}(1-q^{-\beta-r}x)_q^{\beta+r} \right)}{x^\alpha (1-q^{-\beta}x)_q^\beta}.$$

4. *We have the relation for the q-derivative $r \geq 1$:*

$$D_q P_{r,q}^{\alpha,\beta}\left(\frac{\cdot}{q}\right) = -q^{-\beta-r}\,[r+\alpha+\beta+1]_q\, P_{r-1,q}^{\alpha+1,\beta+1}.$$

Proof.
2) We look for the analytic solutions of the equation
$U_q^{\alpha,\beta} f - \mu_{r,q}^{\alpha,\beta} f = 0$.

We write $f(x) = \sum_{k=0}^{\infty} a_k x^k$, so

$$U_q^{\alpha,\beta} f(x) - \mu_{r,q}^{\alpha,\beta} f(x) = [\alpha+1]\, a_1 - \mu_{r,q}^{\alpha,\beta} a_0$$
$$+ \sum_{k=1}^{\infty} \left([k+1]\,[k+\alpha+1]\, q^\beta a_{k+1} - ([k]\,[k+\alpha+\beta+1] - \mu_{r,q}^{\alpha,\beta} q^{k+\beta}) a_k \right) q^{-k-\beta} x^k.$$

We obtain $\frac{a_{k+1}}{a_k} = -q^{-r-\beta} \frac{([r]-[k])\,[k+r+\alpha+\beta+1]}{[k+1]\,[k+\alpha+1]} = q^{-\beta} R(q^k)$, for any $k \in \mathbb{N}$, with $R(t) = \frac{(q^{-r}-t^{-1})(q^{r+\alpha+\beta+1}-t^{-1})}{(q-t^{-1})(q^{\alpha+1}-t^{-1})}$ and the result follows.

3) For any polynomial Q of degree $< r$, we verify, with the help of q-integrations by parts that $\langle \frac{D_q^r \left(x^{\alpha+1}(1-q^{-\beta-r}x)_q^{\beta+r} \right)}{x^\alpha (1-q^{-\beta}x)_q^\beta}, Q \rangle_q^{\alpha,\beta} = 0$.

We compute $P_{r,q}^{\alpha,\beta}(0)$ by using a q-extension of Leibniz formula:
$(D_q^r(g_1g_2))(x) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (D_q^k g_1)(q^{r-k}x)(D_q^{r-k}g_2)(x)$.

We write $D_q^r \left(x^{\alpha+r}(1-q^{-\beta-r}x)_q^{\beta+r} \right)$
$= \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+k+1)\Gamma_q(\beta+r-k+1)} x^{\alpha+k}(1-q^{-\beta}x)_q^{\beta+r-k} q^{c_k}$, where
$c_k = k(k+\alpha-\beta-r+(k-1)/2)$ and
$A(x) = D_q^r \left(x^{\alpha+1}(1-q^{-\beta-r}x)_q^{\beta+r} \right) \Big/ x^{\alpha}(1-q^{-\beta}x)_q^{\beta}$
$= \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+k+1)\Gamma_q(\beta+r-k+1)} x^k(1-q^{-\beta-r+k}x)_q^{r-k} q^{c_k}$.

We obtain $A(0) = \Gamma_q(\alpha+r+1)/\Gamma_q(\alpha+1) = [r]! \begin{bmatrix} r+\alpha \\ r \end{bmatrix}$ hence
$P_{r,q}^{\alpha,\beta}(x) = A(x)/[r]!$.

1) We take $\lim_{q\to 1} A(x) = \sum_{k=0}^{r} \binom{r}{k} \frac{\Gamma(\alpha+r+1)\Gamma(\beta+r+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+r-k+1)} x^k(1-x)^{r-k}$.
It is $r!P_r^{\alpha,\beta}(x)$ (Rodrigues formula).

4) We use (2.1) to prove that $D_q P_{r,q}^{\alpha,\beta}(\frac{\cdot}{q})$ is eigenvector of $M_{r-1,q}^{\alpha+1,\beta+1}$. Hence, it is equal to $P_{r-1,q}^{\alpha+1,\beta+1}$ up to a constant.

We compute $D_q P_{r,q}^{\alpha,\beta}(0) = a_1 = \mu_{r,q}^{\alpha,\beta} a_0$, hence
$a_1 = -\begin{bmatrix} r+\alpha \\ r \end{bmatrix} q^{-\beta-r}[r][r+\alpha+\beta+1]/[\alpha+1]$
$= -q^{-\beta-r}[r+\alpha+\beta+1] \begin{bmatrix} r+\alpha \\ r-1 \end{bmatrix}$ and the result follows. □

5 The case $\alpha = \beta = -1$

In this part, we study the operators $M_{n,q}^{-1,-1}, n \in \mathbb{N}$. They are built for functions defined on the closed interval $[0,1]$. They are obtained from the operators $M_{n-1,q}^{0,0}$ as Kantorovich operators are obtained from Bernstein operators with the help of Proposition 5.3, formula 5.1.

Definition 5.1

For any $n \in \mathbb{N} - \{0\}$ and a function f defined on $[0,1]$ the polynomial $M_{n,q}^{-1,-1}$ is defined by replacing $\alpha = \beta$ by –1 in formula 1.1. It is :

$$M_{n,q}^{-1,-1} f(x) = \sum_{k=0}^{n} f_{n,k,q}^{-1,-1} b_{n,k,q}(x)$$

with $f_{n,0,q}^{-1,-1} = f(0)$, $f_{n,n,q}^{-1,-1} = f(1)$ *and the coefficients* $f_{n,k,q}^{-1,-1}$, *for* $k = 1, \dots, n-1$, *are given by (1.3) taking* $\alpha = \beta = -1$.
The bilinear form is $\langle f, g\rangle_q^{-1,-1} = \int_0^1 \frac{f(t)g(t)}{t(1-t)} d_q t$.

The polynomial $M_{n,q}^{-1,-1} f$ is well defined for any function f bounded in a neighbourhood of 0 (condition $C(-1)$). It verifies $M_{n,q}^{-1,-1} f(0) = f(0)$ and $M_{n,q}^{-1,-1} f(1) = f(1)$, hence it preserves the affine functions.

Proposition 5.2 *If the function* f *is continuous on* $[0,1]$, *then* $\lim_{\alpha \to -1} M_{n,q}^{\alpha,\alpha} f(x) = M_{n,q}^{-1,-1} f(x)$ *for any* $x \in [0,1]$.

Proof. The q-binomial coefficients $b_{n,k,q}(x)$ are positive and form a partition of the unity. Hence it is sufficient to prove that $\lim_{\alpha \to -1} f_{n,k,q}^{\alpha,\alpha} = f_{n,k,q}^{-1,-1}$ for any k. For $k = 1, \dots, n-1$, we compute
$f_{n,k,q}^{\alpha,\alpha} - f_{n,k,q}^{-1,-1}$
$= \int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\alpha}\ \left(f(q^{\alpha+1}t) - f(t)\right) d_q t \,/ B_q(k+\alpha+1, n-k+\alpha+1)$
$\quad + \int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\alpha} f(t) d_q t \,/ B_q(k+\alpha+1, n-k+\alpha+1)$
$\quad - \int_0^1 t^{k-1}\ (1-qt)_q^{n-k-1} f(t) d_q t \,/ B_q(k, n-k)$
$\quad = F_1 + F_2 + F_3$.

We consider $I_k = \dfrac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\alpha}\ \left(f(q^{\alpha+1}t) - \widetilde{f}_k(t)\right) d_q t}{B_q(k+\alpha+1, n-k+\alpha+1)}$, with $\widetilde{f}_0(t) = f(0)$, $\widetilde{f}_n(t) = f(1)$ and $\widetilde{f}_k(t) = f(t)$, $k = 1, \dots, n-1$ and prove that $\lim_{\alpha \to -1} I_k = 0$, for $k = 0, \dots, n$. We use the additivity of the modulus of continuity of f, q-Beta integrals and we set $\delta = \frac{[\alpha+1]}{[n+2+2]}$.
We have $\left|f(q^{\alpha+1}t) - f(0)\right| \le \omega(f,t) \le \omega(f,\delta)(1 + t/\delta)$, hence
$|I_0| \le \omega(f,\delta)(1 + \frac{1}{\delta} \int_0^1 t^{\alpha+1}\ (1-qt)_q^{n+\alpha} d_q t \Big/ \int_0^1 t^{\alpha}\ (1-qt)_q^{n+\alpha}\ d_q t)$
$\quad \le 2\omega(f, \frac{[\alpha+1]}{[n+2\alpha+2]})$.

For $k = n$, we have $\left|f(q^{\alpha+1}t) - f(1)\right| \le \omega(f, 1 - q^{\alpha+1}t)$, hence
$|I_n| \le \omega(f,\delta)(1 + \int_0^1 t^{\alpha+n}\ (1-qt)_q^{\alpha+1} d_q t \Big/ (\delta \int_0^1 t^{\alpha+n}\ (1-qt)_q^{\alpha}\ d_q t))$
$\quad \le 2\omega(f, \frac{[\alpha+1]}{[n+2\alpha+2]})$.

As $f_{n,0,q}^{\alpha,\alpha} - f(0) = I_0$ and $f_{n,n,q}^{\alpha,\beta} - f(1) = I_n$, the result follows for $k = 0$ and n.

For $k = 1, \dots, n-1$, we have $\left|f(q^{\alpha+1}t) - f(t)\right| \le \omega(f, 1 - q^{\alpha+1})$ and $|I_k| \le \omega(f, 1 - q^{\alpha+1})$. Hence, $F_1 = I_k$ vanishes when α tends to -1.

In F_2 the q-integral is $(1-q)\sum_{j=0}^{n} q^{j(k+\alpha+1)}(1-q^{j+1})^{n-k+\alpha} f(q^j)$. This series is uniformly convergent, hence its limit when α tends to -1 is the q-integral of F_3. At last the q-Beta function in F_2 tends to the q-Beta function in F_3 because Γ_q is continuous. □

Numerous properties shown in the case $\alpha, \beta > -1$ are still true if $\alpha = \beta = -1$. Some of them are given in the following.

Proposition 5.3 *If the function f is continuous at the points* 0 *and* 1, *and verifies the condition C(-1), we have :*

$$D_q M_{n,q}^{\text{-}1,\text{-}1} f(x) = M_{n-1,q}^{0,0}\left(D_q f\left(\frac{\cdot}{q}\right)\right)(qx), x \in [0,1]. \tag{5.1}$$

Proof. The expressions for $(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})$ established in the proof of Proposition (2.1) hold if $\alpha = \beta = -1$ and $k = 1, \ldots, n-2$.

For the two other terms we have
$(f_{n,1}^{\text{-}1,\text{-}1} - f(0)) = -\left[(1-t)_q^{n-1}(f(t)-f(0))\right]_0^1 + \int_0^1 (1-qt)_q^{n-1} D_q f(t) d_q t$ and
$(f(1) - f_{n,n-1}^{\text{-}1,\text{-}1}) = -\left[t^{n-1}(f(t)-f(1))\right]_0^1 + \int_0^1 (qt)^{n-1} D_q f(t) d_q t$.

The first terms vanish since f is continuous at 0 and 1. □

Theorem 5.4

1. *If f is continuous on* $[0,1]$, *then* $\left\| M_{n,q}^{\text{-}1,\text{-}1} f - f \right\|_\infty \leq Cte\ \omega\left(f, \frac{1}{\sqrt{[n]_q}}\right)$, *(Theorem 3.1).*
2. *If* $\lim_{n\to\infty} q_n = 1$ *and if the function f is bounded on* $[0,1]$, *then*
   (a) $\lim_{n\to\infty} M_{n,q_n}^{\text{-}1,\text{-}1} f(x) = f(x)$ *if f is continuous at the point* $x \in [0,1]$, *(Theorem 3.5).*
   (b) $\lim_{n\to\infty} [n]_{q_n} \left(M_{n,q_n}^{\text{-}1,\text{-}1} f(x) - f(x)\right) = x(1-x)f''(x)$ *if the function f admits a second derivative at the point* $x \in [0,1]$, *(Theorem 3.6).*

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References

[1] G.E. Andrews, R. Askey and R. Roy, *Special functions,* Encyclopedia of Math. and its Appl., Cambridge Univ. Press, 1999.

[2] H. Berens and Y. Xu, On Bernstein-Durrmeyer Polynomials with Jacobi Weights, *Approximation Theory and Functional Analysis, C.K. Chui A.D* (1991), 25-46.

[3] M.M. Derriennic, Sur l'approximation de fonctions intégrables sur $[0,1]$ par des polynômes de Bernstein modifiés, *J. Approx. Theory* **31-4** (1981), 325-343.

[4] M.M. Derriennic, de la Vallée Poussin and Bernstein-type operators, *Approximation Theory, Proc. ID0MAT 95, M.W. Müller, M. Felten, D.H. Mache (eds), AK.Verlag Berlin* **86** (1995), 71-84.

[5] T.N.T. Goodman, Shape preserving representations, *Math. Methods in Computer Aided Geometric Design, T. Lyche, L.L. Schumaker(eds), Academic Press, New-York* (1989), 333-551.

[6] T.N.T. Goodman, H. Oruç and G.M. Phillips, Convexity and generalized Bernstein Polynomials, *Proc. of the Edinburg Math. Soc. Journal* **42** (1999), 179-190.

[7] V. Kac and P. Cheung, *Quantum Calculus,* Universitext, Spinger, 2002.

[8] S. Karlin, *Total Positivity,* Stanford Univ. Press, 1968.

[9] G.M. Phillips, Bernstein polynomials based on the q-integers, *Annals of Num. Math.* **4** (1997), 511-518.

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Overconvergence of Some Next-to-Interpolatory Polynomials

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Abstract. The next-to-interpolatory polynomial of an analytic function $f(z)$ is the unique solution of the following problem:

given $n+1$ distinct complex numbers $z_0, z_1, \ldots, z_n$ (*nodes*), given an integer $p \geq 1$, find a polynomial $P(f;z)$ of degree at most $p(n+1)-2$ with

$$P^{(i)}(f;z_k) = f^{(i)}(z_k),\ 0 \leq k \leq n,\ 0 \leq i \leq p-2, \qquad (a)$$

and that, at the same time, minimizes

$$\max_{0 \leq k \leq n} \left| P^{(p-1)}(f;z_k) - f^{(p-1)}(z_k) \right|. \qquad (b)$$

For $p = 1$ only condition (b) is imposed, condition (a) has to be omitted.

In this paper the solution will be given explicitly for the nodes given by the zeros of the polynomial

$$w(z) = (z+\alpha)^{n+1} - r^{n+1}(1+\alpha z)^{n+1},\ 0 \leq \alpha < 1,\ 0 < r \leq 1.$$

It will be proved that this solution converges geometrically and uniformly to $f(z)$ on compact subsets of its domain of analyticity.

Moreover, overconvergence for $n \to \infty$ of the difference of two next-to-interpolatory polynomials will be shown in the case that $f(z)$ has a singularity on the boundary of its domain of analyticity.

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1 Introduction

The concept of Hermite interpolation (see [3]) is well known and some facts will be given here without proof.

Given $n+1$ pairwise distinct points, $z_0, z_1, \ldots, z_n$ in the complex plane (the *nodes*) and an integer $p \geq 1$, the Hermite interpolation problem consists of finding a polynomial $P(z)$ of degree at most $p(n+1)-1$ that satisfies

$$P^{(i)}(z_k) = c_k^i,\ 0 \leq k \leq n,\ 0 \leq i \leq p-1, \tag{1.1}$$

where the complex *data* c_k^i are arbitrary.

This problem has a unique solution for any set of data, i.e. it is *regular*, and the so-called *fundamental polynomials* ($c_k^i = \delta_{i,k}$, Kronecker's delta) can be given explicitly.

In the case that the data are derived from a function that is $(p-1)$ times differentiable at the nodes, i.e.

$$c_k^i = f^{(i)}(z_k),\ 0 \leq k \leq n,\ 0 \leq i \leq p-1, \tag{1.2}$$

the unique solution of (1.1) is called the *interpolating polynomial of* f.

It is a matter of simple linear algebra that the restriction of the degree of $P(z)$ to be at most $p(n+1)-2$ leads to a problem that does not have a solution for all sets of data! This was addressed in [4] by modification of the interpolation problem in the following way.

Given data c_k^i, find a polynomial $P(z)$ of degree at most $p(n+1)-2$ that satisfies

$$P^{(i)}(f; z_k) = c_k^i,\ 0 \leq k \leq n,\ 0 \leq i \leq p-2, \tag{1.3}$$

(this condition has to be omitted for $p=1$) and that, at the same time, minimizes

$$\max_{0 \leq k \leq n} \left| P^{(p-1)}(f; z_k) - c_k^{p-1} \right|. \tag{1.4}$$

In [4], Theorem 3, the solution is proved to exist, to be unique and is exhibited explicitly as a weighted sum of the solutions of Hermite problems that omit one of the nodes in the conditions for the $(p-1)$th derivative:

$$P(z) = \sum_{k=0}^{n} g_k q_k(z),\ g_k = \frac{|w'(z_k)|^{-p}}{\sum_{j=0}^{n} |w'(z_j)|^{-p}}, \tag{1.5}$$

where

$$w(z) = \prod_{k=0}^{n} (z - z_k)$$

and $q_k(z)$ is the unique solution of

$$\begin{cases} q_k^{(i)}(z_j) = c_j^i, & 0 \leq j \leq n,\ 0 \leq i \leq p-2, \\ q_k^{(p-1)}(z_j) = c_j^{p-1}, & 0 \leq j \leq n,\ j \neq k. \end{cases} \tag{1.6}$$

In the next section the interpolation nodes will be taken at the zeros of the polynomial indicated in the abstract and the next-to-interpolatory polynomial will be given explicitly with the aid of Cauchy's integral formula. Moreover, it will be shown that this polynomial converges to the function $f(z)$ on its domain of analyticity for $n \to \infty$ in the case that $c_k^i = f^{(i)}(z_k)$.

In the study of different interpolation processes, the notion of *equiconvergence* or *overconvergence* appears, going back to [6]; it has been studied in different settings, see a.o. [1], [2], [7].

In [6] the author considered a function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

analytic on $|z| < \rho$, not on $|z| \leq \rho$, $\rho > 1$. If $P_n(z)$ is the Lagrange interpolating polynomial to $f(z)$ on the roots of unity (the zeros of $z^{n+1}-1$) and if

$$S_n(z) = \sum_{k=0}^{n} a_k z^k$$

are the partial sums of $f(z)$, Walsh showed that

$$\lim_{n\to\infty} (P_n(z) - S_n(z)) = 0 \text{ on } |z| < \rho^2,$$

locally uniformly and geometrically.

In [5] Sharma and Ziegler combined the two concepts next-to-interpolatory and overconvergence, using as nodes the Möbius transforms of the $(n+1)$th roots of unity in the case that $p = 1$ and they stated in a remark

> Motzkin and Sharma [4] have also defined next-to-interpolatory polynomials on sets with multiplicities.
> Cavaretta et al. [2] have proved an equiconvergence theorem on roots of unity when the multiplicity is 2.
> It would be interesting to obtain its analogue on the perturbed roots of unity.

The layout of the paper is now as follows: in section 2 the next-to-interpolatory polynomials will be given in integral form for $p \geq 1$, it will be stated that these polynomials converge for $n \to \infty$ to the function they were derived from on the domain of analyticity (locally uniformly and geometrically). Furthermore two interpolation processes will be compared and an over-convergence result for the difference will be stated. The results reduce to those in [5] on taking $p = 1$.

Section 3 contains the proofs of the results and the paper concludes with a short list of references.

2 New results

Let R, r, $\alpha \in \mathbb{R}$ be real numbers satisfying

$$R > 1,\ 0 \leq \alpha < 1,\ 0 < r \leq 1, \tag{2.1}$$

and let the function $f(z)$, analytic on $|z+\alpha| < R+\alpha$ but not on $|z+\alpha| \leq R+\alpha$, be given by

$$f(z) = \sum_{j=0}^{\infty} a_j (z+\alpha)^j,\ \limsup_{j\to\infty} |a_j|^{1/j} = \frac{1}{R}. \tag{2.2}$$

For an arbitrary integer p, with $p \geq 1$, and α, r as in (2.1), the interpolation nodes are given by the zeros of the polynomial of degree $n+1$, $n \geq 0$:

$$w(z) = w_{n,\alpha,r}(z) = \frac{(z+\alpha)^{n+1} - r^{n+1}(1+\alpha z)^{n+1}}{1 - r^{n+1}\alpha^{n+1}}. \tag{2.3}$$

In the sequel, the dependence on α, r and n will not be indicated explicitly for $w(z)$.

Theorem 2.1 *The next-to-interpolatory polynomial $P_n(f;z)$ of degree at most $p(n+1)-2$, satisfying*

$$P_n^{(i)}(f;z_k) = f^{(i)}(z_k),\ 0 \leq k \leq n,\ 0 \leq i \leq p-2 \tag{2.4}$$

and at the same time minimizing

$$\max_{0\leq k\leq n} \left| P_n^{(p-1)}(f;z_k) - f^{(p-1)}(z_k) \right|, \tag{2.5}$$

where the z_k $(0 \leq k \leq n)$ are the zeros of $w(z)$ from (2.3)*, are given explicitly by*

$$P_n(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{w(t)^p - w(z)^p}{w(t)^p}dt - \lambda_p \sum_{k=0}^{n} g_k \frac{w(z)^p}{z-z_k}, \tag{2.6}$$

with

$$\lambda_p = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{w(t)^p}dt \text{ and } g_k = \frac{|1-r\alpha\omega^k|^{p(n-1)}}{\sum_{\ell=0}^{n}|1-r\alpha\omega^\ell|^{p(n-1)}} \ (0 \leq k \leq n). \tag{2.7}$$

Here ω is a primitive $(n+1)$th root of unity and Γ is a closed, positively oriented smooth curve that has z and all the zeros z_k of $w(z)$ in its interior: $|t+\alpha| = \rho + \alpha$ for a certain ρ with $1 < \rho < R$.

Remark 2.2 *From* (2.6)*, we see that the next-to-interpolatory polynomial of order p is just the Hermite interpolation polynomial (cf. [3]), with a correction given using* (2.7)*. If now $f(z) = Q_m(z)$ is a polynomial of degree $m \leq p(n+1)-2$, then the 'full' Hermite interpolation problem (use in* (2.4) *$0 \leq i \leq p-1$) has as solution Q_m itself. Indeed: the interpolation problem has a unique solution of degree at most $p(n+1)-1$, so the 'candidate' Q_m is recovered (although the degree is lower).*

That formula (2.6) *leads to the same result follows from the fact that the integral representation for λ_p in* (2.7) *now has as integrand a rational function with numerator of degree at most $p(n+1)-2$ and denominator degree $p(n+1)$. Thus the integral, actually $-\mathrm{Res}[Q_m(t)/w(t)^p, t=\infty]$, is equal to zero.*

Remark 2.3 *The error made in* (2.5) *for the solution, turns out to be independent of k: for each k with $0 \leq k \leq n$ we have*

$$\left|P_n^{(p-1)}(f;z_k) - f^{(p-1)}(z_k)\right| = \max_{0\leq j\leq n}\left|P_n^{(p-1)}(f;z_j) - f^{(p-1)}(z_j)\right|.$$

For the proof, the reader is referred to section 3.

Just as in the case of Hermite interpolation, the interpolation process "recovers" $f(z)$ when $n \to \infty$:

Theorem 2.4 *Let p, α, r, $f(z)$ and $P_n(f;z)$ be as before, then*

$$\lim_{n\to\infty} P_n(f;z) = f(z) \text{ on } |z+\alpha| < R+\alpha, \tag{2.8}$$

geometrically and locally uniformly.

Introduce now a sort of 'shifted' coefficient g_k:

$$\varepsilon_k = g_k - \frac{1}{n+1},\ 0 \le k \le n. \tag{2.9}$$

This allows us to rewrite (2.6) as

$$P_n(f;z) = P_n^*(f;z) + R_n(f;z) \tag{2.10}$$

with

$$P_n^*(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{w(t)^p - w(z)^p - \frac{t-z}{n+1}w'(z)w(z)^{p-1}}{w(t)^p}dt \tag{2.11a}$$

$$R_n(f;z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{w(t)^p}dt \cdot \sum_{k=0}^{n} \varepsilon_k \frac{w(z)^p}{z-z_k}. \tag{2.11b}$$

Remark 2.5 *From the explicit form of g_k in* (2.7)*, it follows that $\varepsilon_k = 0$ for all k when $\alpha = 0$. In that case the interpolation nodes are $r\omega^k$ $(0 \le k \le n$, ω a primitive $(n+1)$th root of unity, and the "error term" $R_n(f;z)$ in* (2.10)*,* (2.11) *is identically zero.*

In the sequel we need an expression for $P_n^*(f;z)$ in (2.11) in terms of the coefficients of the power series for $f(z)$, this will be done in the following theorem, after the introduction of some abbreviations.

Put

$$w^*(z) = (1 - r^{n+1}\alpha^{n+1})w(z) = (z+\alpha)^{n+1} - (1+\alpha z)^{n+1}, \tag{2.12}$$

$$K_1(t,z) := (t+\alpha)^n w^*(t)^{p-1} - (z+\alpha)^n w^*(z)^{p-1}, \tag{2.13}$$

$$K_2(t,z) := (1+\alpha t)^n w^*(t)^{p-1} - (1+\alpha z)^n w^*(z)^{p-1}, \tag{2.14}$$

and for arbitrary m, N with $0 \le m \le N \le \infty$

$$S_m^N(t) = \sum_{j=m}^{N} \binom{p+j-1}{j} \left(r\frac{1+\alpha t}{t+\alpha}\right)^{j(n+1)}, \tag{2.15}$$

then we have

Theorem 2.6 *$P_n^*(f;z)$ can be rewritten as*

$$P_n^*(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{K_1(t,z)}{(t+\alpha)^{p(n+1)}}(t+\alpha)S_0^\infty(t)dt$$
$$- \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{K_2(t,z)}{(t+\alpha)^{p(n+1)}}r^{n+1}(1+\alpha t)S_0^\infty(t)dt \tag{2.16}$$

Remark 2.7 *Note that* $K_1(z,z) = K_2(z,z) = 0$*, showing that the integrand is analytic at* $t = z$*.*

In order to compare two different interpolation processes, in the vein of 'comparing data from a full function' with 'data from a part of the function', we introduce a *cut-off* parameter ℓ that is used to define a cutoff of (2.16)

Definition 2.8 *Let* $\ell \geq 1$ *be an arbitrary natural number, write*

$$S_n^{*,\ell}(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{K_1(t,z)}{(t+\alpha)^{p(n+1)}}(t+\alpha)S_0^{\ell-1}(t)dt - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{K_2(t,z)}{(t+\alpha)^{p(n+1)}}r^{n+1}(1+\alpha t)S_0^{\ell-2}(t)dt \quad (2.17)$$

(for $\ell = 1$ *the second integral has to be omitted).*

Introduce with $R_n(f;z)$ *from* (2.11b)

$$S_n^{\ell}(f;z) = S_n^{*,\ell}(f;z) + R_n(f;z). \quad (2.18)$$

Remark 2.9 *To keep the expose within moderate length, we just state the effect of the introduction of* ℓ*:*

working out the integral in (2.16) *explicitly, one sees that all coefficients* a_j *of* $f(z)$ *appear in* $P_n^*(f;z)$*,*

using the cut-off as in (2.17) *leads to the same range for the index* j *of the* a_j *that appear in the explicit form of both integrals; the use of* $\ell - 1$ *in the first and* $\ell - 2$ *in the second integral is fundamental here.*

Now we can formulate the overconvergence result:

Theorem 2.10 *Introduce the difference*

$$\Delta(f;z) := P_n(f;z) - S_n^{\ell}(f;z) = P_n^*(f;z) - S_n^{*,\ell}(f;z), \quad (2.19)$$

then the limit

$$\lim_{n\to\infty}\Delta(f;z) = 0, \quad (2.20)$$

holds geometrically and locally uniformly on

$$|z+\alpha| < (R+\alpha)\left(\frac{1}{r}\frac{R+\alpha}{1+\alpha R}\right)^{\ell/p}. \quad (2.21)$$

3 Proofs

First we prove two lemmas that will be used in the sequel.

Lemma 3.1 *Let k with $0 \le k \le n$ be fixed and define*

$$\pi_k(z) = \prod_{j=0, j\neq k}^{n} \frac{z - z_j}{z_k - z_j} = \frac{w(z)}{w'(z_k)(z - z_k)}. \tag{3.1}$$

Then:

$$\pi_k(z_m) = \delta_{k,m} = \begin{cases} 0, & k \neq m, \\ 1, & k = m, \end{cases} \tag{3.2}$$

and

$$\pi_k^{(i)}(z_k) = \frac{1}{i+1} \frac{w^{(i+1)}(z_k)}{w'(z_k)}, \ i \ge 1. \tag{3.3}$$

Proof. Follows by applying induction and l'Hospital's rule. □

Lemma 3.2 *Let $x \in \mathbb{R}$, $|x| < 1$, then for any integer $\ell \ge 1$:*

$$\sum_{j=0}^{\infty} \binom{p+j+\ell-1}{j+\ell} x^{j+\ell} = \frac{x^\ell}{1-x} \sum_{j=0}^{p-1} \binom{p+\ell-1}{j+\ell} \left(\frac{x}{1-x}\right)^j. \tag{3.4}$$

Proof. For real x with $|x| < 1$ and $\mathcal{D} = \frac{d}{dx}$ we have

$$\begin{aligned}
&\sum_{j=0}^{\infty} \binom{p+j+\ell-1}{j+\ell} x^{j+\ell} = \\
&\quad = \frac{1}{(p-1)!} \sum_{j=0}^{\infty} (p+j+\ell-1)(p+j+\ell-2)\cdots(j+\ell+1)x^{j+\ell} \\
&\quad = \frac{1}{(p-1)!} \mathcal{D}^{p-1} \sum_{j=0}^{\infty} x^{p+j+\ell-1} = \frac{1}{(p-1)!} \mathcal{D}^{p-1} \left(\frac{x^{p+\ell-1}}{1-x}\right) \\
&\quad = \frac{1}{(p-1)!} \sum_{j=0}^{p-1} \binom{p-1}{j} \mathcal{D}^{p-1-j}(x^{p+\ell-1}) \mathcal{D}^j \left(\frac{1}{1-x}\right) \\
&\quad = \frac{1}{(p-1)!} \sum_{j=0}^{p-1} \binom{p-1}{j} (p+\ell-1)(p+\ell-2)\cdots(j+\ell+1) \frac{j! x^{j+\ell}}{(1-x)^{j+1}} \\
&\quad = \frac{x^\ell}{1-x} \sum_{j=0}^{p-1} \binom{p+\ell-1}{j+\ell} \left(\frac{x}{1-x}\right)^j.
\end{aligned}$$

□

Now we turn to the explicit form of the next-to-interpolatory polynomial and prove Theorem 2.1:

Proof. According to [4], Theorem 3, the polynomial $P_n(f;z)$ is given by (1.5), (1.6) in the following form

$$P_n(f;z) = \sum_{k=0}^{n} g_k q_k(z) \tag{3.5}$$

with

$$\begin{cases} q_k^{(i)}(z_j) = f_j^{(i)}, & 0 \le j \le n,\ 0 \le i \le p-2, \\ q_k^{(p-1)}(z_j) = f_j^{(p-1)}, & 0 \le j \le n,\ j \ne k, \end{cases} \tag{3.6}$$

and

$$g_k = \frac{|w'(z_k)|^{-p}}{\sum_{j=0}^{n} |w'(z_j)|^{-p}}, \tag{3.7}$$

where $w(z)$ is defined in (2.3).

Note that we could use w^* from (2.12) instead of w in the form (3.7) as the factors $(1 - r^{n+1}\alpha^{n+1})$ cancel.

The construction of $q_k(z)$ is done in two steps. First we write

$$q_k(z) = \varphi(z) + w(z)^{p-1} u_k(z),\ \varphi \in \Pi_{p(n+1)-2},\ u_k \in \Pi_{n-1}, \tag{3.8}$$

with a fixed polynomial $\varphi(z)$ to be determined later.

Because

$$\mathcal{D}^i \left(w(z)^{p-1} u_k(z) \right) \big|_{z=z_j} = 0,\ 0 \le j \le n,\ 0 \le i \le p-2, \tag{3.9}$$

we thus have to construct $\varphi \in \Pi_{p(n+1)-2}$ satisfying

$$\varphi^{(i)}(z_j) = f^{(i)}(z_j),\ 0 \le j \le n,\ 0 \le i \le p-2. \tag{3.10}$$

As we have $(p-1)(n+1)$ conditions only, this leaves a total degree of freedom n.

In the second step we determine $u_k \in \Pi_{n-1}$ in such a way that

$$q_k^{(p-1)}(z_j) = f^{(p-1)}(z_j),\ 0 \le j \le n,\ j \ne k. \tag{3.11}$$

Thus

$$\begin{aligned} f^{(p-1)}(z_j) &= \varphi^{(p-1)}(z_j) + \mathcal{D}^{p-1} \left(w(z)^{p-1} u_k(z) \right) |_{z=z_j} \\ &= \varphi^{(p-1)}(z_j) + (p-1)! w'(z_j)^{p-1} u_k(z_j),\ 0 \le j \le n,\ j \ne k, \end{aligned} \tag{3.12}$$

which implies, using $w'(z_j) \neq 0$ since the zeros of $w(z)$ are simple, that

$$u_k(z_j) = \frac{f^{(p-1)}(z_j) - \varphi^{(p-1)}(z_j)}{(p-1)!w'(z_j)^{p-1}},\ 0 \leq j \leq n,\ j \neq k. \tag{3.13}$$

We see that for a *fixed* j $(0 \leq j \leq n)$, the quantities $\{u_k(z_j) : 0 \leq k \leq n,\ k \neq j\}$ do *not depend on* k. This will be exploited later on.

The explicit formula for $u_k(z)$ then follows using standard Lagrange interpolation on the zeros of $\pi_k(z)$ as introduced in (3.1), using the values from (3.13):

$$u_k(z) = \sum_{j=0, j\neq k}^{n} \frac{u_k(z_j)}{\pi_k'(z_j)} \frac{\pi_k(z)}{z - z_j}. \tag{3.14}$$

Rearranging this formula for fixed k with the aid of Lemma 3.1, we find

$$\begin{aligned} u_k(z) &= \sum_{j=0, j\neq k}^{n} \frac{u_k(z_j)}{w'(z_j)} \frac{w(z)(z_j - z_k)}{(z - z_j)(z - z_k)} \\ &= \sum_{j=0, j\neq k}^{n} \frac{u_k(z_j)}{w'(z_j)} \frac{w(z)}{z - z_j} - \frac{w(z)}{z - z_k} \sum_{j=0, j\neq k}^{n} \frac{u_k(z_j)}{w'(z_j)} \end{aligned} \tag{3.15}$$

Taking into account that the weights g_k sum to 1, this shows

$$\begin{aligned} P_n(f; z) &= \sum_{k=0}^{n} g_k \left(\varphi(z) + w(z)^{p-1} u_k(z)\right) \\ &= \varphi(z) + \sum_{k=0}^{n} \left[\sum_{j=0, j\neq k}^{n} g_k \lambda_{k,j} \frac{w(z)^p}{z - z_j} - \sum_{j=0, j\neq k}^{n} \lambda_{k,j} g_k \frac{w(z)^p}{z - z_k} \right] \\ &= \varphi(z) + \sum_{k=0}^{n} \left[\sum_{j=0}^{n} g_k \lambda_{k,j} \frac{w(z)^p}{z - z_j} - \sum_{j=0}^{n} \lambda_{k,j} g_k \frac{w(z)^p}{z - z_k} \right] \end{aligned} \tag{3.16}$$

with

$$\lambda_{k,j} = \begin{cases} \frac{u_k(z_j)}{w'(z_j)} = \frac{f^{(p-1)}(z_j) - \varphi^{(p-1)}(z_j)}{(p-1)!w'(z_j)^p} & \text{for } j \neq k, \\ \frac{f^{(p-1)}(z_k) - \varphi^{(p-1)}(z_k)}{(p-1)!w'(z_k)^p} & \text{for } j = k \end{cases} \tag{3.17}$$

This, for the moment rather arbitrary looking definition of $\lambda_{k,k}$, is allowed because the terms with $j = k$ in the inner sum of (3.16) cancel (this can also be seen from the first line of (3.15) for u_k).

Take r $(0 \leq r \leq n)$ fixed and let Γ_r be a small circle centered at z_r, traversed once in counterclockwise direction, then

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(t)-\varphi(t)}{w(t)^p}dt &= \frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(t)-\varphi(t)}{\pi_r(t)^p}\frac{dt}{w'(z_r)^p(t-z_r)^p}\\ &= \frac{1}{(p-1)!w'(z_r)^p}\mathcal{D}^{p-1}\left(\frac{f-\varphi}{\pi_r^p}\right)\Bigg|_{t=z_r}\\ &= \frac{1}{(p-1)!w'(z_r)^p}\sum_{j=0}^{p-1}\binom{p-1}{j}\left(f^{(j)}(z_r)-\varphi^{(j)}(z_r)\right)\mathcal{D}^{p-1-j}\left(\pi_r^{-p}\right)\Bigg|_{t=z_r}.\end{aligned} \tag{3.18}$$

Because of the way $\varphi(z)$ is constructed, we see that only the term with $j = p-1$ is different from zero and therefore

$$\frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(t)-\varphi(t)}{w(t)^p}dt = \frac{f^{(p-1)}(z_r)-\varphi^{(p-1)}(z_r)}{(p-1)!w'(z_r)^p} = \lambda_{k,r}. \tag{3.19}$$

This calculation shows the reason of the form for $\lambda_{k,k}$ introduced in (3.17).

Now apply the residue theorem to find for λ_p from formula (2.7)

$$\begin{aligned}\lambda_p = \sum_{r=0}^{n}\lambda_{k,r} &= \sum_{r=0}^{n}\frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(t)-\varphi(t)}{w(t)^p}dt\\ &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)-\varphi(t)}{w(t)^p}dt\\ &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{w(t)^p}dt - \frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(t)}{w(t)^p}dt\\ &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{w(t)^p}dt,\end{aligned} \tag{3.20}$$

since the integral with $\varphi(t)$ in the numerator vanishes as φ is a polynomial of degree at most $p(n+1)-2$ and the value of the integral is just $-\mathrm{Res}\left[\varphi(t)/w(t)^p; t=\infty\right] = 0$.

Moreover, we now see explicitly that the sum of the $\lambda_{k,r}$ over r does not depend on the value of k as a result of the adequate definition of $\lambda_{k,k}$ from (3.17).

We can now rewrite formula (3.16) as

$$P_n(f;z) = \varphi(z) + \sum_{j=0}^{n} \left[\sum_{k=0}^{n} g_k \lambda_{k,j} \right] \frac{w(z)^p}{z - z_j} - \sum_{k=0}^{n} \left[\sum_{j=0}^{n} \lambda_{k,j} \right] g_k \frac{w(z)^p}{z - z_k}. \tag{3.21}$$

Now (3.20) implies that the coefficient between brackets in the second sum is equal to λ_p and does not depend on k. Also the sum between brackets in the first sum can be written

$$\begin{aligned} \sum_{k=0}^{n} g_k \lambda_{k,j} &= \sum_{k=0, k\neq j} g_k \lambda_{k,j} + g_j \lambda_{j,j} \\ &= \left(\sum_{k=0, k\neq j} g_k \right) \frac{f^{(p-1)}(z_j) - \varphi^{(p-1)}(z_j)}{(p-1)! w'(z_j)^p} + g_j \lambda_{j,j} \\ &= (1 - g_j)\lambda_{j,j} + g_j \lambda_{j,j} \\ &= \lambda_{j,j}. \end{aligned} \tag{3.22}$$

With (3.21) and (3.22) the formula (3.16) for $P_n(f;z)$ finally turns into

$$P_n(f;z) = \varphi(z) + \sum_{j=0}^{n} \lambda_{j,j} \frac{w(z)^p}{z - z_j} - \lambda_p \sum_{k=0}^{n} g_k \frac{w(z)^p}{z - z_k}. \tag{3.23}$$

In view of Remark 2.2, formula (2.6) now follows if we prove

$$\mathcal{D}^i \left(\varphi(z) + \sum_{j=0}^{n} \lambda_{j,j} \frac{w(z)^p}{z - z_j} \right) \Bigg|_{z=z_m} = f^{(i)}(z_m), \ 0 \le m \le n, \ 0 \le i \le p-1,$$

as these are the conditions for the full Hermite interpolation polynomial.

This follows immediately from (3.10), (3.17) and (3.24) below:

$$\mathcal{D}^i \left(\sum_{j=0}^{n} \lambda_{j,j} \frac{w(z)^p}{z - z_j} \right) \Bigg|_{z=z_m} = \begin{cases} 0, & 0 \le i \le p-2, \\ \lambda_{m,m}(p-1)! w'(z_m)^p, & i = p-1, \end{cases} \tag{3.24}$$

Indeed, this gives for $0 \le m \le n$, $0 \le i \le p-2$

$$\mathcal{D}^i \left(\varphi(z) + \sum_{j=0}^{n} \lambda_{j,j} \frac{w(z)^p}{z - z_j} \right) \Bigg|_{z=z_m} = \varphi^{(i)}(z_m) = f^{(i)}(z_m),$$

and for $0 \le m \le n$, $i = p-1$

$$\mathcal{D}^{p-1}\left(\varphi(z) + \sum_{j=0}^{n} \lambda_{j,j} \frac{w(z)^p}{z - z_j}\right)\Bigg|_{z=z_m} = \\ = \varphi^{(p-1)}(z_m) + (p-1)!w'(z_m)^p \lambda_{k,m} = f^{(p-1)}(z_m).$$

Finally, the form of g_k in (2.7) follows from (1.5). The z_k satisfy

$$(z_k + \alpha)^{n+1} - r^{n+1}(1 + \alpha z_k)^{n+1} = 0,$$

and can be written as

$$z_k = \frac{r\omega^k - \alpha}{1 - r\alpha\omega^k},\ \omega \text{ a primitive } (n+1)\text{th root of unity.} \tag{3.25}$$

Insert

$$(w^*)'(z_k) = \prod_{\ell=0,\ell\neq k}^{n} (z_k - z_\ell) = \left(r\frac{1-\alpha^2}{1 - r\alpha\omega^k}\right)^n \prod_{\ell=0,\ell\neq k}^{n} \frac{\omega^k - \omega^\ell}{1 - r\alpha\omega^\ell}$$

into (1.5):

$$\begin{aligned} g_k &= \frac{|(w^*)'(z_k))|^{-p}}{\sum_{\ell=0}^{n} |(w^*)'(z_\ell)|^{-p}} \\ &= \frac{|1 - r\alpha\omega^k|^{pn} \prod_{\ell=0,\ell\neq k}^{n} \frac{|1-r\alpha\omega^\ell|^p}{|\omega^k - \omega^\ell|^p}}{\sum_{j=0}^{n} |1 - r\alpha\omega^j|^{pn} \prod_{\ell=0,\ell\neq j}^{n} \frac{|1-r\alpha\omega^\ell|^p}{|\omega^j - \omega^\ell|^p}} \\ &= \frac{|1 - r\alpha\omega^k|^{pn} \prod_{\ell=0,\ell\neq k}^{n} |1 - r\alpha\omega^\ell|^p \prod_{\ell=0,\ell\neq k}^{n} |\omega^k - \omega^\ell|^{-p}}{\sum_{j=0}^{n} |1 - r\alpha\omega^j|^{pn} \prod_{\ell=0,\ell\neq j}^{n} |1 - r\alpha\omega^\ell|^p \prod_{\ell=0,\ell\neq j}^{n} |\omega^j - \omega^\ell|^{-p}} \\ &= \frac{|1 - r\alpha\omega^k|^{pn} \prod_{\ell=0,\ell\neq k}^{n} |1 - r\alpha\omega^\ell|^p}{\sum_{j=0}^{n} |1 - r\alpha\omega^j|^{pn} \prod_{\ell=0,\ell\neq j}^{n} |1 - r\alpha\omega^\ell|^p} = \frac{|1 - r\alpha\omega^k|^{p(n-1)}}{\sum_{j=0}^{n} |1 - r\alpha\omega^j|^{p(n-1)}}, \end{aligned} \tag{3.26}$$

while the product over differences of powers of roots of unity does not depend on k

$$\prod_{\ell=0,\ell\neq k}^{n} |\omega^k - \omega^\ell|^{-p} = \prod_{\ell=0,\ell\neq k}^{n} |\omega^k|^{-p} |1 - \omega^{\ell-k}|^{-p} = \prod_{\ell=1}^{n} |1 - \omega^\ell|^{-p}.$$

The final form of (3.26) is just the value of g_k from (2.7). □

We now turn to the proof of Remark 2.3.

Proof. Introduce

$$\varepsilon_k(P_n) := \left| P_n^{(p-1)}(f; z_k) - f^{(p-1)}(z_k) \right|, \ 0 \le k \le n, \tag{3.27}$$

where $P_n(f; z)$ is the next-to-interpolatory polynomial from Theorem 2.1 and let

$$\varepsilon = \max_{0 \le k \le n} \varepsilon_k(P_n). \tag{3.28}$$

Assume that for a certain value ℓ with $0 \le \ell \le n$

$$\varepsilon_\ell(P_n) < \varepsilon.$$

Now define

$$Q_n(f; z) = P_n(f; z) - \delta \xi(z),$$

with $\xi(z)$ the unique polynomial of degree at most $p(n+1) - 2$ satisfying

$$\begin{cases} \xi^{(i)}(z_k) = 0, & 0 \le k \le n, \ 0 \le i \le p-2, \\ \xi^{(p-1)}(z_k) = P_n^{(p-1)}(f; z_k) - f^{(p-1)}(z_k), & 0 \le k \le n, \ k \ne \ell, \end{cases}$$

and $\delta \in (0, 1)$ satisfying

$$\delta |\xi^{(p-1)}(z_\ell)| < \varepsilon - \varepsilon_\ell(P_n).$$

Then

$$\begin{aligned} \varepsilon_k(Q_n) &= \left| Q_n^{(p-1)}(f; z_k) - f^{(p-1)}(z_k) \right| \\ &= (1 - \delta) \left| P_n^{(p-1)}(f; z_k) - f^{(p-1)}(z_k) \right| \\ &= (1 - \delta) \varepsilon_k(P_n) < \varepsilon \end{aligned}$$

for $0 \le k \le n, \ k \ne \ell$, while

$$\begin{aligned} \varepsilon_\ell(Q_n) &= \left| Q_n^{(p-1)}(f; z_\ell) - f^{(p-1)}(z_\ell) \right| \\ &\le \left| P_n^{(p-1)}(f; z_\ell) - f^{(p-1)}(z_\ell) \right| + \delta \left| \xi^{(p-1)}(z_\ell) \right| \\ &< \varepsilon_\ell(P_n) + (\varepsilon - \varepsilon_\ell(P_n)) = \varepsilon. \end{aligned}$$

Thus

$$\max_{0 \le k \le n} \varepsilon_k(Q_n) < \varepsilon,$$

in contradiction with the fact that the next-to-interpolatory polynomial P_n is the solution of the mini-max problem (2.4), (2.5). □

The proof of Theorem 2.4 runs as follows.

Proof. Let z, fixed, belong to $|z+\alpha| < R+\alpha$, then there exists a ρ with $1 < \rho < R$ such that $\Gamma = \{t \mid |t+\alpha| = \rho+\alpha\}$ encircles z.

Introducing δ_2, $0 < \delta_2 < 1$, by

$$\delta_2 = \frac{1}{\rho+\alpha}\left(\frac{\rho+1}{2}+\alpha\right), \tag{3.29}$$

we see that the fixed z belongs to D defined by

$$D := \{z \mid |z+\alpha| \leq \delta_2(\rho+\alpha)\}, \tag{3.30}$$

and, moreover, we have that Γ lies inside the boundary of D

$$\partial D = \{z \mid |z+\alpha| = \frac{\rho+1}{2}+\alpha\}. \tag{3.31}$$

Now (3.30) implies that

$$z \in D \Rightarrow |1+\alpha z| \leq \delta_3(1+\alpha\rho),\ 0 < \delta_3 < 1. \tag{3.32}$$

After these preliminaries, we use (2.6) and

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{w(t)^p}{w(t)^p}dt,$$

to find

$$f(z) - P_n(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{w(z)^p}{w(t)^p}dt + \lambda_p \sum_{k=0}^{n} g_k \frac{w(z)^p}{z-z_k}. \tag{3.33}$$

Equation (2.7) implies

$$|g_k| \leq 1,\ 0 \leq k \leq n. \tag{3.34}$$

Using

$$|t+\alpha| = \rho+\alpha \Rightarrow |1+\alpha t| \leq 1+\rho\alpha,$$

it easily follows that for t on Γ

$$\left|r\frac{1+\alpha t}{t+\alpha}\right| \leq \left|r\frac{1+\rho\alpha}{\rho+\alpha}\right| \leq \left|\frac{1+\rho\alpha}{\rho+\alpha}\right| =: \delta_1 < 1. \tag{3.35}$$

Therefore we have from

$$|w(t)| = \frac{|t+\alpha|^{n+1}}{1-r^{n+1}\alpha^{n+1}}\left|1-\left(r\frac{1+\alpha t}{t+\alpha}\right)^{n+1}\right|$$

the bounds

$$(1-\delta_1^{n+1})(\rho+\alpha)^{n+1} \le |w(t)| \le \frac{2}{1-\alpha r}(\rho+\alpha)^{n+1},\ t \in \Gamma. \tag{3.36}$$

Then (3.20) gives, together with the fact that $f(t)$ is bounded on $|t+\alpha| \le \rho+\alpha$, the inequality

$$|\lambda_p| \le \frac{M_1}{(\rho+\alpha)^{p(n+1)}},\ M_1 > 0. \tag{3.37}$$

Combining (3.30), (3.32), (3.35) and (3.36), it follows that for $t \in \Gamma$

$$\begin{aligned}\left|\frac{w(z)}{w(t)}\right| &\le \frac{\left|\frac{z+\alpha}{t+\alpha}\right|^{n+1} + r^{n+1}\left|\frac{1+\alpha z}{t+\alpha}\right|^{n+1}}{1-r^{n+1}\left(\frac{1+\rho\alpha}{\rho+\alpha}\right)^{n+1}} \\ &\le \frac{\left(\frac{\delta_2(\rho+\alpha)}{\rho+\alpha}\right)^{n+1} + \left(\frac{\delta_3(1+\rho\alpha)}{\rho+\alpha}\right)^{n+1}}{1-\delta_1^{n+1}} \\ &\le \frac{\delta_2^{n+1}+\delta_1^{n+1}\delta_3^{n+1}}{1-\delta_1^{n+1}} \\ &\le M_2\delta_4^{n+1},\ M_2 > 0,\ 0 < \delta_4 < 1.\end{aligned} \tag{3.38}$$

Again using that $f(t)$ is bounded, we can now estimate the integral in the difference (3.33):

$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\frac{w(z)^p}{w(t)^p}dt\right| \le M_3 \cdot M_2^p\delta_4^{p(n+1)} = M_4\delta_4^{p(n+1)},\ n \ge n_0. \tag{3.39}$$

Then we can find a $\delta_5,\ 0 < \delta_5 < 1$, such that the sum in (3.33) can be estimated by

$$\begin{aligned}\left|\sum_{k=0}^{n} g_k\frac{w(z)^p}{z-z_k}\right| &\le \sum_{k=0}^{n}\left\{[\delta_2(\rho+\alpha)]^{n+1} + [r\delta_3(1+\rho\alpha)]^{n+1}\right\}^{p-1}\left|\frac{w(z)}{z-z_k}\right| \\ &\le (n+1)M_5\delta_5^{p(n+1)}(\rho+\alpha)^{p(n+1)},\ M_5 > 0,\ 0 < \delta_5 < 1.\end{aligned} \tag{3.40}$$

Here we used $\delta_5 = \max(\delta_2, \delta_3)$, (3.30), (3.32), (3.34), (3.35) and the fact that $w(z)/(z-z_k)$ is an analytic function and the maximum of its absolute value over z with (3.30) is assumed on the boundary i.e. according to (3.31) and (3.36)

$$\begin{aligned}\left|\frac{w(z)}{z-z_k}\right| &\le \max_{z\in\partial D}\left|\frac{w(z)}{z-z_k}\right| \\ &\le \max_{z\in\partial D}|w(z)| / \min_{z\in\partial D}|z-z_k|,\end{aligned}$$

and

$$\max_{z\in\partial D}|w(z)| \le [\delta_2(\rho+\alpha)]^{n+1} + [r\delta_3(1+\rho\alpha)]^{n+1} \le 2\delta_5(\rho+\alpha)^{n+1},$$

$$\min_{z\in\partial D}|z-z_k| \ge \min_{z\in\partial D}||z+\alpha| - |z_k+\alpha|| \ge \frac{\rho+1}{2} + \alpha - 1 \ge \frac{\rho-1}{2}.$$

In this lower bound we used for the zero z_k the estimate

$$z_k+\alpha = re^{i\varphi}(1+\alpha z_k) \Rightarrow |z_k|^2 = \frac{r^2 - 2\alpha r\cos\varphi + \alpha^2}{1 - 2\alpha r\cos\varphi + \alpha^2 r^2} \le 1 \Rightarrow |z_k| \le 1.$$

The inequalities (3.39) and (3.40) imply (2.8). □

Next we have the proof of Theorem 2.6.

Proof. Start with (2.11) and put a star on each occurrence of the polynomial $w(\cdot)$ (see (2.12) for w^*):

$$P_n^*(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,\frac{w^*(t)^p - w^*(z)^p - \frac{t-z}{n+1}w^*(z)'w^*(z)^{p-1}}{w^*(t)^p}dt \quad (3.41)$$

and rewrite the numerator as follows

$$\begin{aligned}
&w^*(t)^p - w^*(z)^p - \frac{t-z}{n+1}w^*(z)'w^*(z)^{p-1} = \\
&\quad = w^*(t)^p - w^*(z)^p - (t-z)\left\{(z+\alpha)^n - \alpha r^{n+1}(1+\alpha z)^n\right\}w^*(z)^{p-1} \\
&\quad = w^*(t)^p - w^*(z)^p - (t-z)(z+\alpha)^n w^*(z)^{p-1} + \\
&\qquad\qquad + \alpha r^{n+1}(t-z)(1+\alpha z)^n w^*(z)^{p-1} \\
&\quad = w^*(t)^p - w^*(z)^p - \{(t+\alpha)-(z+\alpha)\}(z+\alpha)^n w^*(z)^{p-1} + \\
&\qquad\qquad + r^{n+1}\{(1+\alpha t)-(1+\alpha z)\}(1+\alpha z)^n w^*(z)^{p-1} \\
&\quad = w^*(t)^p - (t+\alpha)(z+\alpha)^n w^*(z)^{p-1} + r^{n+1}(1+\alpha t)(1+\alpha z)^n w^*(z)^{p-1}.
\end{aligned} \quad (3.42)$$

Since

$$w^*(t)^p = (t+\alpha)(t+\alpha)^n w^*(t)^{p-1} - r^{n+1}(1+\alpha t)(1+\alpha t)^n w^*(t)^{p-1},$$

(3.42) can be written as

$$\begin{aligned}
&(t+\alpha)\left\{(t+\alpha)^n w^*(t)^{p-1} - (z+\alpha)^n w^*(z)^{p-1}\right\} + \\
&\qquad - r^{n+1}(1+\alpha t)\left\{(1+\alpha t)^n w^*(t)^{p-1} - (1+\alpha z)^n w^*(z)^{p-1}\right\}.
\end{aligned} \quad (3.43)$$

The binomial series leads with

$$\left| r\frac{1+\alpha t}{t+\alpha} \right| \leq \frac{1+\alpha\rho}{\rho+\alpha} < 1$$

to the expansion

$$\begin{aligned} w(t)^{-p} &= (t+\alpha)^{-p(n+1)} \left\{ 1 - \left(r\frac{1+\alpha t}{t+\alpha} \right)^{n+1} \right\}^{-p} \\ &= (t+\alpha)^{-p(n+1)} \sum_{j=0}^{\infty} \binom{-p}{j} \left(r\frac{1+\alpha t}{t+\alpha} \right)^{j(n+1)} \qquad (3.44) \\ &= (t+\alpha)^{-p(n+1)} \sum_{j=0}^{\infty} \binom{p+j-1}{j} \left(r\frac{1+\alpha t}{t+\alpha} \right)^{j(n+1)}. \end{aligned}$$

Formulae (3.41), (3.43) and (3.44) imply (2.16). □

Finally we prove Theorem 2.10.

Proof. Using (2.16), (2.15), (2.17) and (2.18), the difference in (2.19) can be written as

$$\begin{aligned} \Delta(f;z) = &\frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{K_1(t,z)}{(t+\alpha)^{p(n+1)}} (t+\alpha) S_\ell^\infty(t) dt \\ &- \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{K_2(t,z)}{(t+\alpha)^{p(n+1)}} r^{n+1} (1+\alpha t) S_{\ell-1}^\infty(t) dt. \qquad (3.45) \end{aligned}$$

In view of Remark 2.7, the location of z is now arbitrary.

Introduce for integer $m \geq 1$ the series

$$T(m;t) := \sum_{j=0}^{\infty} \binom{p+j+m-1}{j+m} \left(r\frac{1+\alpha t}{t+\alpha} \right)^{(j+\ell)(n+1)} \qquad (3.46)$$

and shift the summations in (3.45) to find

$$\begin{aligned} \Delta(f;z) = &\frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{K_1(t,z)}{(t+\alpha)^{p(n+1)}} (t+\alpha) T(\ell;t) dt \\ &- \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{K_2(t,z)}{(t+\alpha)^{p(n+1)}} r^{n+1} (1+\alpha t) T(\ell-1;t) dt. \qquad (3.47) \end{aligned}$$

Because

$$\left| r\frac{1+\alpha t}{t+\alpha} \right| \leq \frac{1+\alpha\rho}{\rho+\alpha} < 1$$

for $t \in \Gamma$, we can apply Lemma 3.2 with

$$x := x(t) = \left(r \frac{1+\alpha t}{t+\alpha} \right)^{n+1}$$

and $m = \ell$, resp. $m = \ell - 1$ to write

$$\begin{aligned} \Delta(f;z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{K_1(t,z)}{(t+\alpha)^{p(n+1)}} (t+\alpha) U(\ell;t) dt \\ - \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{K_2(t,z)}{(t+\alpha)^{p(n+1)}} r^{n+1} (1+\alpha t) U(\ell-1;t) dt, \end{aligned} \tag{3.48}$$

where $U(m;t)$ is given by

$$U(m;t) := \sum_{j=0}^{p-1} \binom{p+m-1}{j+m} \left(\frac{x}{1-x} \right)^j \frac{x^m}{1-x}. \tag{3.49}$$

With

$$w^*(t) = \sum_{k=0}^{p-1} (-1)^{k(n+1)} \binom{p-1}{k} (t+\alpha)^{(p-k-1)(n+1)} (1+\alpha t)^{k(n+1)},$$

and the same for z, we can now write

$$\begin{aligned} (t+\alpha) K_1(t,z) = (t+\alpha) \sum_{k=0}^{p-1} (-1)^k r^{k(n+1)} \binom{p-1}{k} \times \\ \left\{ (t+\alpha)^{(p-k)(n+1)-1} (1+\alpha t)^{k(n+1)} - (z+\alpha)^{(p-k)(n+1)-1} (1+\alpha z)^{k(n+1)} \right\} \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} (1+\alpha t) r^{n+1} K_2(t,z) = r^{n+1} (1+\alpha t) \sum_{k=0}^{p-1} (-1)^k r^{k(n+1)} \binom{p-1}{k} \\ \times \left\{ (t+\alpha)^{(p-k-1)(n+1)} (1+\alpha t)^{(k+1)(n+1)-1} + \right. \\ \left. -(z+\alpha)^{(p-k-1)(n+1)} (1+\alpha z)^{(k+1)(n+1)-1} \right\}. \end{aligned} \tag{3.51}$$

Insert (3.50) and (3.51) in (3.48):

$$\Delta(f;z) = \sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}(t+\alpha)r^{k(n+1)}\frac{A(t,z,k)}{(t+\alpha)^{p(n+1)}}\left(\frac{x}{1-x}\right)^j\frac{x^\ell}{1-x}dt - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}(1+\alpha t)r^{(k+1)(n+1)}\frac{B(t,z,k)}{(t+\alpha)^{p(n+1)}}\left(\frac{x}{1-x}\right)^j\frac{x^{\ell-1}}{1-x}dt\right], \tag{3.52}$$

with the abbreviations

$$A(t,z,k) = (t+\alpha)^{(p-k)(n+1)-1}(1+\alpha t)^{k(n+1)} - (z+\alpha)^{(p-k)(n+1)-1}(1+\alpha z)^{k(n+1)}, \tag{3.53}$$

$$B(t,z,k) = (t+\alpha)^{(p-k-1)(n+1)}(1+\alpha t)^{(k+1)(n+1)-1} - (z+\alpha)^{(p-k-1)(n+1)}(1+\alpha z)^{(k+1)(n+1)-1}. \tag{3.54}$$

We are now only interested in the case where z belongs to $|z+\alpha| \geq R+\alpha$ (smaller values are covered by Theorem 2.4): then terms with $|z+\alpha|$ dominate those with $|t+\alpha| = \rho+\alpha < R+\alpha$. Therefore it is sufficient to show that we have for all values

$$0 \leq j,\, k \leq p-1, \tag{3.55}$$

the following limits for $n \to \infty$

$$(t+\alpha)^{-p(n+1)+1}r^{k(n+1)}(z+\alpha)^{(p-k)(n+1)-1}(1+\alpha z)^{k(n+1)} \times \left(\frac{x}{1-x}\right)^j\frac{x^\ell}{1-x} \to 0, \tag{3.56}$$

$$(t+\alpha)^{-p(n+1)}(1+\alpha t)r^{(k+1)(n+1)}(z+\alpha)^{(p-k-1)(n+1)}(1+\alpha z)^{(k+1)(n+1)-1} \times \left(\frac{x}{1-x}\right)^j\frac{x^{\ell-1}}{1-x} \to 0. \tag{3.57}$$

Indeed: the function $f(t)/(t-z)$ is bounded on Γ, the length of Γ is finite, the binomial coefficients are finite (non-zero) constants and there is a finite number of terms.

Taking the $(n+1)$th root, it is sufficient to have the following inequalities for j,k in the range (3.55):

$$r^k|t+\alpha|^{-p+\frac{1}{n+1}}|z+\alpha|^{p-k-\frac{1}{n+1}}|1+\alpha z|^k\frac{\left(r\left|\frac{1+\alpha t}{t+\alpha}\right|\right)^{j+\ell}}{\left|1-\left(r\frac{1+\alpha t}{t+\alpha}\right)^{n+1}\right|^{\frac{j+1}{n+1}}}<1, \quad (3.58)$$

$$r^{k+1}|t+\alpha|^{-p}|z+\alpha|^{p-k-1}|1+\alpha z|^{k+1}\frac{\left(r\left|\frac{1+\alpha t}{t+\alpha}\right|\right)^{j+\ell-1}}{\left|1-\left(r\frac{1+\alpha t}{t+\alpha}\right)^{n+1}\right|^{\frac{j+1}{n+1}}}<1, \quad (3.59)$$

As before, $|t+\alpha|=\rho+\alpha$ implies

$$\left|r\frac{1+\alpha t}{t+\alpha}\right|\leq\frac{1+\alpha\rho}{\rho+\alpha}<1 \quad (3.60)$$

and it turns out that $\Delta(f;z)\to 0$ for $n\to\infty$ follows from

$$\begin{cases} |z+\alpha|^{p-k}|1+\alpha z|^k<\dfrac{(\rho+\alpha)^p}{r^k}\left(r\dfrac{1+\alpha\rho}{\rho+\alpha}\right)^{-j-\ell}, \\ |z+\alpha|^{p-k-1}|1+\alpha z|^{k+1}<\dfrac{(\rho+\alpha)^p}{r^{k+1}}\left(r\dfrac{1+\alpha\rho}{\rho+\alpha}\right)^{-j-\ell+1}, \end{cases} \quad (3.61)$$

for $k,\ j=0,1,\ldots,p-1$, which can be written as

$$|z+\alpha|^p<\min_{0\leq j,k\leq p-1}\left\{(\rho+\alpha)^p\left(r\frac{1+\alpha\rho}{\rho+\alpha}\right)^{-j-\ell}\left|\frac{1}{r}\frac{z+\alpha}{1+\alpha z}\right|^k,\right.$$
$$\left.(\rho+\alpha)^p\left(r\frac{1+\alpha\rho}{\rho+\alpha}\right)^{-j-\ell-1}\left|\frac{1}{r}\frac{z+\alpha}{1+\alpha z}\right|^{k+1}\right\}. \quad (3.62)$$

Now $|z+\alpha|\geq R+\alpha>\rho+\alpha$ implies

$$\left|\frac{1}{r}\frac{1+\alpha z}{z+\alpha}\right|<\frac{1}{r}\frac{1+\alpha\rho}{\rho+\alpha},$$

which can be seen from

$$\left|\frac{1}{r}\frac{1+\alpha z}{z+\alpha}\right|/\frac{1}{r}\frac{1+\alpha\rho}{\rho+\alpha}=$$
$$=\left|\frac{1-\alpha^2}{|z+\alpha|}+\alpha\frac{z+\alpha}{|z+\alpha|}\right|/\left(\frac{1-\alpha^2}{\rho+\alpha}+\alpha\right)\leq$$
$$\leq\left(\frac{1-\alpha^2}{|z+\alpha|}+\alpha\right)/\left(\frac{1-\alpha^2}{\rho+\alpha}+\alpha\right)<1.$$

The conditions in (3.62) can therefore be written as

$$|z+\alpha|^p < \min_{0\le j,k\le p-1} \left\{ (\rho+\alpha)^p \left(\frac{1}{r}\frac{\rho+\alpha}{1+\alpha\rho}\right)^{k+j+\ell}, \right. \\ \left. (\rho+\alpha)^p \left(\frac{1}{r}\frac{\rho+\alpha}{1+\alpha\rho}\right)^{k+j+\ell+2} \right\}, \quad (3.63)$$

and this minimum is attained for $k+j+\ell\,(+2)$ minimal because the number $(\rho+\alpha)/[r(1+\alpha\rho)]$ is strictly larger than 1:

$$|z+\alpha| < (\rho+\alpha)\left(\frac{1}{r}\frac{\rho+\alpha}{1+\alpha\rho}\right)^{\ell/p}. \quad (3.64)$$

The bound (2.21) then follows from the fact that the right hand side of (3.64) is strictly increasing in ρ and the value of ρ was bounded by R. □

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References

[1] M.R. Akhlaghi, A. Jakimovski, A. Sharma, *Equiconvergence of some complex interpolatory polynomials,* Numerische Mathematik **57** (1990), 635–649.

[2] A.S. Cavaretta, A. Sharma, R.S. Varga, *Interpolation in the roots of unity: An extension of a theorem of J.L. Walsh,* Resultate der Mathematik **3** (1980), 155-191.

[3] G.G. Lorentz, K. Jetter, S.D. Riemenschneider, Birkhoff Interpolation, *Encycl. of Mathematics and its Applications* **19,** *G.-C. Rota, Reading, 1983.*

[4] T.S. Motzkin, A. Sharma, *Next-to-interpolatory approximation on sets with multiplicities,* Canadian J. of Mathematics **18** (1966), 1196–1211.

[5] A. Sharma, Z. Ziegler, *Hermite interpolation on some perturbed roots of unity,* Analysis **19** (1999), 1–12.

[6] J.L. Walsh, Interpolation and approximation by rational functions in the complex domain, *AMS Math. Soc. Colloquium Publ.* **XX,** *AMS, Providence 1969.*

[7] L. Yuan Ren, *Extension of a theorem of J.L. Walsh on the overconvergence,* Approximation Theory and its Applications **2** (1986), 19–32.

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On Multivariate Classical Orthogonal Polynomials

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Abstract. Classical orthogonal polynomials in one variable can be defined as the orthogonal polynomials associated to a moment functional satisfying a Pearson differential equation. We extend this concept to several variables defining classical multivariate orthogonal polynomials as those associated to a moment functional satisfying a matrix analogue of the Pearson differential equation. Furthermore, we generalize some characterizations of classical orthogonal polynomials to the multivariate case.

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1 Introduction

Hermite, Laguerre, Jacobi and Bessel polynomials constitute the so–called *classical orthogonal polynomials* in one variable, one of the most important topic in the general theory of orthogonal polynomials. The special differential properties of these four families of orthogonal polynomials characterize them. In the literature, several characterizations of classical orthogonal polynomials are known (see, for instance, [2], [13]). Next, we will enumerate some of the most important ones.

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If we denote by $\{P_n\}_{n\geq 0}$ a sequence of orthogonal polynomials in one variable and let u be the corresponding moment functional, then, the following statements are equivalent

C.1 $\{P_n\}_{n\geq 0}$ is a **classical family**, Hermite, Laguerre, Jacobi or Bessel.

C.2 **Pearson differential equation** There exist two polynomials ϕ and ψ with $\deg\phi \leq 2$, and $\deg\psi \leq 1$ such that u satisfies the distributional Pearson–type equation

$$D(\phi u) = \psi u,$$

where ϕu denotes the *left multiplication* of the functional u times the polynomial ϕ, defined by means of

$$\langle \phi u, p\rangle = \langle u, \phi p\rangle,$$

and Du denotes the *distributional derivative* of the functional u, defined as follows

$$\langle Du, p\rangle = -\langle u, p'\rangle.$$

C.3 **Structure relation** There exists a polynomial ϕ with $\deg\phi \leq 2$ such that

$$\phi(x)P_n'(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x),$$

where $a_n, b_n, c_n \in \mathbb{R}$, and $c_n \neq 0$.

C.4 **Orthogonality of the derivatives** $\{P_{n+1}'\}_{n\geq 0}$ is also an orthogonal polynomial system. In this case, there exists a polynomial ϕ with $\deg\phi \leq 2$ such that $\{P_{n+1}'\}_{n\geq 0}$ is orthogonal with respect to the linear functional $v = \phi u$.

C.5 **Differential equation** There exist two polynomials ϕ and ψ with $\deg\phi \leq 2$, and $\deg\psi \leq 1$ such that, for $n \geq 0$, P_n is solution of the differential equation

$$\phi(x)y'' + \psi(x)y' = \lambda_n y,$$

where $\lambda_n \in \mathbb{R}$.

The aim of this paper is to introduce the concept of multivariate classical orthogonal polynomials, as the natural extension of the definition in one variable. This will be achieved by means of a matrix Pearson–type partial differential equation satisfied by the multivariate linear functional. In addition, we will prove the analogous of the above characterizations for multidimensional classical orthogonal polynomials: the structure relation, the orthogonality of the gradients, and the second order partial differential

equation. These characterizations involve matrix polynomial coefficients, the usual gradient operator, and the divergence operator. Some similar results in two variables appears in [1, 4, 5, 7, 8, 11, 12].

The structure of the paper is as follows. Section 2 is devoted to introduce the basic definitions and tools for the rest of the paper. In Section 3, we expose the main result of the paper, the proof of the result is given in Section 4, and finally, in Section 5, we present some examples of multivariate classical orthogonal polynomials.

2 Preliminaries

Throughout this paper, we will denote by $\mathcal{P}$ the linear space of polynomials in d variables with real coefficients, and by $\mathcal{P}'$ its topological dual (see [14]). Analogously $\mathcal{P}_n$ will denote the set of real polynomials of total degree not greater than n, and we will denote by Π_n the space of homogeneous polynomials of degree n in d variables. It's well known that

$$\dim \mathcal{P}_n = \binom{n+d}{n} \qquad \text{and} \qquad \dim \Pi_n = \binom{n+d-1}{n} = r_n.$$

Let us denote by $\mathcal{M}_{h\times k}(\mathbb{R})$ the linear space of $(h \times k)$ real matrices, and by $\mathcal{M}_{h\times k}(\mathcal{P})$ the linear space of $(h \times k)$ matrices with polynomial elements. If $h = k$, we will denote $\mathcal{M}_{h\times h} \equiv \mathcal{M}_h$, and, in particular, I_h will represent the identity matrix of order h.

We will use the Kronecker product of matrices (see [6]) defined by means of

$$M \otimes N = \begin{pmatrix} m_{11}N & m_{12}N & \dots & m_{1k}N \\ m_{21}N & m_{22}N & \dots & m_{2k}N \\ \vdots & \vdots & & \vdots \\ m_{h1}N & m_{h2}N & \dots & m_{hk}N \end{pmatrix} \in \mathcal{M}_{hl\times km}(\mathcal{P})$$

where $M \in \mathcal{M}_{h\times k}(\mathcal{P})$ and $N \in \mathcal{M}_{l\times m}(\mathcal{P})$ are given matrices.

Now, we review some basic definitions and tools about multivariate orthogonal polynomials that we will need in the rest of the paper. For a complete description of this and other related subjects see, for instance, [3, 9, 10, 15].

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\mathrm{m} = (m_1, \dots, m_d) \in \mathbb{N}_0^d$ and $\mathrm{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ we write $\mathrm{x}^{\mathrm{m}} = x_1^{m_1} \cdots x_d^{m_d}$. The number $|\mathrm{m}| = m_1 + \cdots + m_d$ is called the *total degree* of x^{m}.

Let $\{\mu_{\mathrm{m}}\}_{\mathrm{m}\in\mathbb{N}_0^d}$ be a multi–sequence of real numbers and let us denote by u, the only linear functional in $\mathcal{P}'$ satisfying

$$\langle u, \mathrm{x}^{\mathrm{m}}\rangle = \mu_{\mathrm{m}}.$$

Then, u is called the *moment functional* determined by $\{\mu_{\mathrm{m}}\}_{\mathrm{m}\in\mathbb{N}_0^d}$, and the number μ_{m} is called the *moment of order* m. As usual, $\langle\ ,\ \rangle$ represent the duality bracket.

Let u be a moment functional, and let $M = (m_{i,j}(\mathrm{x}))_{i,j=1}^{h,k} \in \mathcal{M}_{h\times k}(\mathcal{P})$ be a polynomial matrix. The *action of u over M* (see [3]) is given by

$$\langle u, M\rangle = (\langle u, m_{i,j}(\mathrm{x})\rangle)_{i,j=1}^{h,k} \in \mathcal{M}_{h\times k}(\mathbb{R}),$$

and the *left product* of M times u as

$$\langle Mu, N\rangle = \langle u, M^t\ N\rangle, \qquad \forall N \in \mathcal{M}_{h\times l}(\mathcal{P}).$$

Let $(P_{\mathrm{m}}^n)_{|\mathrm{m}|=n}$ be a sequence of polynomials of total degree n. Then we can use the matrix notation introduced in [9] and [16], and denote by $\mathbb{P}_n$ the vector polynomial

$$\mathbb{P}_n = (P_{\mathrm{m}}^n)_{|\mathrm{m}|=n} = (P_{\mathrm{m}^{(1)}}^n, P_{\mathrm{m}^{(2)}}^n, \cdots, P_{\mathrm{m}^{(r_n)}}^n)^t \in \mathcal{M}_{r_n\times 1}(\mathcal{P}_n),$$

where $\mathrm{m}^{(1)}, \mathrm{m}^{(2)}, \cdots, \mathrm{m}^{(r_n)}$ are the elements in $\{\mathrm{m} \in \mathbb{N}_0^d : |\mathrm{m}| = n\}$ arranged according to the lexicographical order. Observe that $\mathbb{P}_0$ is a constant ($r_0 = 1$), and $\mathbb{P}_1$ is a column vector of multivariate polynomials of degree 1 of dimension $r_1 = d$. When $\{\mathbb{P}_m\}_{m=0}^n$ is a basis of $\mathcal{P}_n$ for each $n \geq 0$, then $\{\mathbb{P}_n\}_{n\geq 0}$ is called a *polynomial system* (PS).

Definition 2.1 *We will say that a PS $\{\mathbb{P}_n\}_{n\geq 0}$ is a* weak orthogonal polynomial system (WOPS) *with respect to u if*

$$\begin{aligned} &\langle u, \mathbb{P}_n\mathbb{P}_m^t\rangle = 0, \quad m \neq n, \\ &\langle u, \mathbb{P}_n\mathbb{P}_n^t\rangle = H_n, \end{aligned} \tag{2.1}$$

where $H_n \in \mathcal{M}_{r_n}(\mathbb{R})$ is a regular matrix.

If H_n is a diagonal matrix, we will say that the WOPS $\{\mathbb{P}_n\}_{n\geq 0}$ is an *orthogonal polynomial system (OPS)*. In the particular case where H_n is the identity matrix, we call $\{\mathbb{P}_n\}_{n\geq 0}$ an *orthonormal polynomial system*.

Moreover, a WOPS is called a *monomial* WOPS if every polynomial contains only one term of higher degree, that is,

$$P_{\mathrm{m}}(\mathrm{x}) = c_{\mathrm{m}}\mathrm{x}^{\mathrm{m}} + R(\mathrm{x}), \quad |\mathrm{m}| = n,$$

where $c_{\mathrm{m}} \neq 0$ is a real constant and $R(\mathrm{x}) \in \mathcal{P}_{n-1}$. If $c_{\mathrm{m}} = 1$, we will say that $\{\mathbb{P}_n\}_{n\geq 0}$ is a *monic* WOPS.

We will say that a moment functional u is *regular* if there exists a WOPS with respect to u.

Moreover, in this paper, we will use the *gradient operator* ∇, and the *divergence operator* div defined as usual. The natural extension of this operators for matrices is defined as follows. Let $M \in \mathcal{M}_{h\times k}(\mathcal{P})$ and $N_1, N_2, \ldots, N_d \in \mathcal{M}_{h\times k}(\mathcal{P})$ be polynomial matrices. We define

$$\nabla M = \begin{pmatrix} \partial_1 M \\ \partial_2 M \\ \vdots \\ \partial_d M \end{pmatrix} \in \mathcal{M}_{dh\times k}(\mathcal{P}), \quad \text{div} \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_d \end{pmatrix} = \sum_{i=1}^{p} \partial_i N_i \in \mathcal{M}_{h\times k}(\mathcal{P}),$$

where $\partial_i = \partial/\partial x_i$. Using the duality, we define the *distributional gradient operator* and the *distributional divergence operator* acting over linear functionals in the following way

$$\langle \nabla u, \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} \rangle = -\langle u, \text{div} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} \rangle = -\langle u, \sum_{i=1}^{p} \partial_i p_i \rangle,$$

$$\langle \text{div} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix}, p \rangle = -\langle \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix}, \nabla p \rangle = -\sum_{i=1}^{p} \langle u_i, \partial_i p \rangle.$$

3 The main result

Definition 3.1 *A regular moment functional u is said to be* classical *if it satisfies the matrix Pearson–type equation*

$$\text{div } (\Phi\ u) = \Psi^t\ u, \tag{3.1}$$

where

$$\Phi = (\phi_{ij})_{i,j=1}^{d} \in \mathcal{M}_d(\mathcal{P}_2), \qquad \Psi = (\psi_i)_{i=1}^{d} \in \mathcal{M}_{d\times 1}(\mathcal{P}_1),$$

are polynomial matrices such that Φ is symmetric, $\deg \phi_{ij} \leq 2$, $\deg \psi_i \leq 1$, *for $i, j = 1, \ldots, d$, and*

$$\det\langle u, \Phi\rangle \neq 0. \tag{3.2}$$

Using the Kronecker product, expression (3.1) can be written as follows

$$\operatorname{div}\left((\Phi \otimes I_h)\ u\right) = (\Psi^t \otimes I_h)\ u, \quad h \geq 1. \tag{3.3}$$

This expression means that, for all $M \in \mathcal{M}_{h\times k}(\mathcal{P})$, we have

$$\langle \operatorname{div}\left((\Phi \otimes I_h)\ u\right), M\rangle = \langle (\Psi^t \otimes I_h)\ u, M\rangle,$$

that is

$$-\langle u, (\Phi \otimes I_h)\nabla M\rangle = \langle u, (\Psi \otimes I_h)M\rangle.$$

A WOPS with respect to u is called a *classical WOPS*. From now on, we will denote by $\{\mathbb{P}_n\}_{n\geq 0}$ the classical monic WOPS with respect to u.

Observe that, for $n = 1$, we have $\nabla\mathbb{P}_1^t = I_d$, and therefore

$$-\langle u, \Phi\rangle = \langle \operatorname{div}\ (\Phi u), \mathbb{P}_1^t\rangle = \langle \Psi^t u, \mathbb{P}_1^t\rangle = \langle u, \Psi\mathbb{P}_1^t\rangle.$$

Now, for $n = 0$, we have $\mathbb{P}_0 = 1$, $\nabla\mathbb{P}_0 = 0$, and we get

$$0 = -\langle \Phi u, \nabla\mathbb{P}_0^t\rangle = \langle \operatorname{div}\ (\Phi u), \mathbb{P}_0^t\rangle = \langle \Psi^t\ u, \mathbb{P}_0^t\rangle = \langle u, \Psi\rangle,$$

that is, $\langle u, \psi_i\rangle = 0$ for $i = 1, \ldots, d$. Therefore using (3.2) and the regular character of u, we deduce that $\deg \psi_i = 1$ for all i. Finally, using again (3.2), we deduce that there exists a regular matrix $A_\Psi \in \mathcal{M}_d(\mathbb{R})$ such that

$$\Psi = A_\Psi \mathbb{P}_1. \tag{3.4}$$

Observe that expression (3.1) is equivalent to

$$\Phi\ \nabla u = \tilde{\Psi}\ u, \quad \tilde{\Psi} = \Psi - (\operatorname{div}\ \Phi)^t. \tag{3.5}$$

If we define the operator L acting over $\mathcal{P}$ by means of

$$L[p] \equiv \operatorname{div}\ (\Phi\nabla p) + \tilde{\Psi}^t\nabla p = \sum_{i,j=1}^{d} \phi_{ij}\partial_{ij}^2 p + \sum_{i=1}^{d} \psi_i \partial_i p,$$

then, L^*, the *formal Lagrange adjoint* of L is defined by

$$L^*[u] \equiv \operatorname{div}\ (\Phi\nabla u) - \operatorname{div}\ (\tilde{\Psi}u), \tag{3.6}$$

which satisfies

$$\langle L^*[u], p\rangle = \langle u, L[p]\rangle, \quad \forall p \in \mathcal{P}.$$

Obviously, if u is classical, then $L^*[u] = 0$.

The next theorem is devoted to characterize multivariate classical orthogonal polynomials. This is the main result of the paper.

Theorem 3.2 *Let u be a multivariate regular moment functional and let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic WOPS with respect to u. The following statements are equivalent*

(i) *u is a multivariate* **classical** *linear functional, that is, there exist polynomial matrices $\Phi \in \mathcal{M}_d(\mathcal{P}_2)$, $\Psi \in \mathcal{M}_{d\times 1}(\mathcal{P}_1)$, such that*

$$\operatorname{div}\,(\Phi\ u) = \Psi^t\ u,$$

and $\det\langle u, \Phi\rangle \neq 0$.

(ii) **Structure relation** *There exists a polynomial matrix $\Phi \in \mathcal{M}_d(\mathcal{P}_2)$, such that $\{\mathbb{P}_n\}_{n\geq 0}$ satisfies*

$$\Phi\ \nabla\mathbb{P}_n^t = (I_d \otimes \mathbb{P}_{n+1}^t)F_{n+1}^n + (I_d \otimes \mathbb{P}_n^t)F_n^n + (I_d \otimes \mathbb{P}_{n-1}^t)F_{n-1}^n, \quad (3.7)$$

for $n \geq 1$, where $F_m^n = (F_{m,1}^n, F_{m,2}^n, \ldots, F_{m,d}^n)^t$ is a block column matrix, with $F_{m,i}^n \in \mathcal{M}_{r_n\times r_m}(\mathbb{R})$, and F_0^1 regular.

(iii) **Orthogonality of the gradients** *There exists a polynomial matrix $\Phi \in \mathcal{M}_d(\mathcal{P}_2)$, such that $\{\nabla\mathbb{P}_n^t\}_{n\geq 1}$ satisfy the orthogonality relations*

$$\langle u, (\nabla\mathbb{P}_m^t)^t\Phi\nabla\mathbb{P}_n^t\rangle = K_n\delta_{n,m}, \quad n, m \geq 1,$$

where $K_n \in \mathcal{M}_{r_n}(\mathbb{R})$ is a symmetric matrix, and K_1 is regular.

(iv) **Matrix second order differential equation** *There exist polynomial matrices $\Phi \in \mathcal{M}_d(\mathcal{P}_2)$, $\tilde{\Psi} \in \mathcal{M}_{d\times 1}(\mathcal{P}_1)$, and there exist matrices $\Lambda_n \in \mathcal{M}_{r_n}(\mathbb{R})$ such that*

$$L[\mathbb{P}_n^t] \equiv \operatorname{div}\,(\Phi\nabla\mathbb{P}_n^t) + \tilde{\Psi}^t\nabla\mathbb{P}_n^t = \mathbb{P}_n^t\Lambda_n, \quad (3.8)$$

and $\Lambda_1 \in \mathcal{M}_d(\mathbb{R})$ is a regular matrix.

Remark 3.3 *Observe that in the four above statements the polynomial matrix Φ is always the same.*

Remark 3.4 *In the particular case when $\Lambda_n = \lambda_n\ I_{r_n}$, $n \geq 0$, all the orthogonal polynomials with total degree n satisfy the same second order partial differential equation. This is the case studied in [11] for bivariate orthogonal polynomials.*

4 Proof of the main result

First, we need a technical lemma involving the gradient and divergence operators, and the Kronecker product.

Lemma 4.1 *Let $M \in \mathcal{M}_{h\times k}(\mathcal{P})$ and $N \in \mathcal{M}_{k\times m}(\mathcal{P})$ be two polynomial matrices. Then, we have*

(i) $\nabla(MN) = (\nabla M)N + (I_d \otimes M)\nabla N \in \mathcal{M}_{dh\times m}(\mathcal{P})$.

(ii) $\operatorname{div}\,((I_d \otimes M)(\Phi \otimes I_k)\nabla N) =$
$= M \operatorname{div}\,((\Phi\otimes I_k)\nabla N)+(\nabla M^t)^t(\Phi\otimes I_k)\nabla N \in \mathcal{M}_{h\times m}(\mathcal{P})$.

Proof of Theorem 3.2

(i) ⇒ (ii) The expression $\Phi\nabla\mathbb{P}_n^t$ is a matrix of dimension $(d\times r_n)$, whose elements are polynomials of degree at most $n+1$. Then, there exists matrices $F_i^n \in \mathcal{M}_{dr_i\times r_n}(\mathbb{R})$, such that

$$\Phi\ \nabla\mathbb{P}_n^t = \sum_{i=0}^{n+1}(I_d \otimes \mathbb{P}_i^t)\ F_i^n,$$

where F_i^n can be obtained using the orthogonality as follows

$$\langle u, (I_d \otimes \mathbb{P}_m)(\Phi\ \nabla\mathbb{P}_n^t)\rangle = \langle u, (I_d \otimes \mathbb{P}_m)\sum_{i=0}^{n+1}(I_d \otimes \mathbb{P}_i^t)\ F_i^n\rangle = (I_d \otimes H_m)\ F_m^n.$$

On the other hand, using Lemma 4.1, and equation (3.3), we get

$$\begin{aligned}
&\langle u, (I_d \otimes \mathbb{P}_m)(\Phi\ \nabla\mathbb{P}_n^t)\rangle \\
&\qquad = \langle u, (\Phi \otimes I_{r_m})\nabla(\mathbb{P}_m\ \mathbb{P}_n^t)\rangle - \langle u, (\Phi \otimes I_{r_m})\nabla\mathbb{P}_m\ \mathbb{P}_n^t\rangle \\
&\qquad = -\langle \operatorname{div}\,((\Phi \otimes I_{r_m})u), \mathbb{P}_m\ \mathbb{P}_n^t\rangle - \langle u, (\Phi \otimes I_{r_m})\nabla\mathbb{P}_m\ \mathbb{P}_n^t\rangle \\
&\qquad = -\langle (\Psi^t \otimes I_{r_m})u, \mathbb{P}_m\ \mathbb{P}_n^t\rangle - \langle u, (\Phi \otimes I_{r_m})\nabla\mathbb{P}_m\ \mathbb{P}_n^t\rangle \\
&\qquad = -\langle u, [(\Psi \otimes I_{r_m})\mathbb{P}_m + (\Phi \otimes I_{r_m})\nabla\mathbb{P}_m]\mathbb{P}_n^t\rangle.
\end{aligned}$$

Since $(\Psi \otimes I_{r_m})\mathbb{P}_m + (\Phi \otimes I_{r_m})\nabla\mathbb{P}_m$ is a polynomial matrix of degree at most $m+1$, we deduce that $F_m^n = 0$, for $m < n-1$, and relation (3.7) holds. Moreover, using (3.4) in the above expression for $n = 1$ and $m = 0$, we deduce the regularity of F_0^1,

$$(I_d \otimes H_0)F_0^1 = -\langle u, (\Psi\mathbb{P}_0 + \Phi\nabla\mathbb{P}_0)\mathbb{P}_1^t\rangle = -\langle u, \Psi\mathbb{P}_1^t\rangle = -A_\Psi\ H_1.$$

(ii) ⇒ (iii) From (3.7), we can write

$$\begin{aligned}\langle u, (\nabla\mathbb{P}_m^t)^t\Phi\nabla\mathbb{P}_n^t\rangle &= \langle u, (\nabla\mathbb{P}_m^t)^t(I_d\otimes\mathbb{P}_{n+1}^t)\rangle F_{n+1}^n\\ &+\langle u, (\nabla\mathbb{P}_m^t)^t(I_d\otimes\mathbb{P}_n^t)\rangle F_n^n + \langle u, (\nabla\mathbb{P}_m^t)^t(I_d\otimes\mathbb{P}_{n-1}^t)\rangle F_{n-1}^n.\end{aligned}$$

and taking into account that

$$(\nabla\mathbb{P}_m^t)^t(I_d\otimes\mathbb{P}_j^t) = \left(\ (\partial_1\mathbb{P}_m)\mathbb{P}_j^t \ \middle|\ (\partial_2\mathbb{P}_m)\mathbb{P}_j^t \ \middle|\ \cdots \ \middle|\ (\partial_d\mathbb{P}_m)\mathbb{P}_j^t\ \right),$$

we obtain

$$\langle u, (\nabla\mathbb{P}_m^t)^t(I_d\otimes\mathbb{P}_j^t)\rangle = 0, \qquad m < j+1,$$

that is,

$$\langle u, (\nabla\mathbb{P}_m^t)^t\Phi\nabla\mathbb{P}_n^t\rangle = 0, \qquad m < n,$$

and the result follows by symmetry. Finally, the regularity of K_1 follows from the regularity of F_0^1 and $\nabla\mathbb{P}_1^t = I_d$.

(iii) ⇒ (i) Define $\Psi = -K_1H_1^{-1}\mathbb{P}_1$. Then, we want to prove that

$$\langle \text{div }(\Phi u), \mathbb{P}_n^t\rangle = \langle \Psi^t u, \mathbb{P}_n^t\rangle, \quad n \geq 0.$$

In fact, the above identity is trivial for $n = 0$. For $n \geq 2$, using that $\nabla\mathbb{P}_1^t = I_d$, we have

$$\langle \text{div }(\Phi u), \mathbb{P}_n^t\rangle = -\langle u, \Phi\nabla\mathbb{P}_n^t\rangle = -\langle u, (\nabla\mathbb{P}_1^t)^t\Phi\nabla\mathbb{P}_n^t\rangle = 0.$$

Using the explicit expression for Ψ, we get

$$\langle \Psi^t u, \mathbb{P}_n^t\rangle = \langle u, \Psi\mathbb{P}_n^t\rangle = 0.$$

For $n = 1$, we compute

$$\langle \text{div }(\Phi u), \mathbb{P}_1^t\rangle = -\langle u, \Phi\nabla\mathbb{P}_1^t\rangle = -\langle u, \Phi\rangle = -\langle u, (\nabla\mathbb{P}_1^t)^t\Phi\nabla\mathbb{P}_1^t\rangle = -K_1.$$

On the other hand, we have

$$\langle \Psi^t u, \mathbb{P}_1^t\rangle = \langle u, \Psi\mathbb{P}_n^t\rangle = -\langle u, K_1H_1^{-1}\mathbb{P}_1\mathbb{P}_1^t\rangle = -K_1.$$

(iii) ⇔ (iv) Using property (ii) of Lemma 4.1, we have

$$\langle u, \mathbb{P}_m\text{div }(\Phi\nabla\mathbb{P}_n^t)\rangle = \langle u, \text{div }((I_d\otimes\mathbb{P}_m)(\Phi\otimes I_1)\nabla\mathbb{P}_n^t)\rangle - \langle u, (\nabla\mathbb{P}_m^t)^t\Phi\nabla\mathbb{P}_n^t\rangle.$$

The first term in the right hand side can be computed by means of (3.5) in the following way,

$$\begin{aligned}&\langle u, \text{div }((I_d\otimes\mathbb{P}_m)(\Phi\otimes I_1)\nabla\mathbb{P}_n^t)\rangle = -\langle\nabla u, (\Phi\otimes I_{r_m})(I_d\otimes\mathbb{P}_m)\nabla\mathbb{P}_n^t\rangle\\ &= -\langle(\Phi\otimes I_{r_m})\nabla u, (I_d\otimes\mathbb{P}_m)\nabla\mathbb{P}_n^t\rangle = -\langle(\tilde\Psi\otimes I_{r_m})u, (I_d\otimes\mathbb{P}_m)\nabla\mathbb{P}_n^t\rangle\\ &= -\langle u, (\tilde\Psi^t\otimes I_{r_m})(I_d\otimes\mathbb{P}_m)\nabla\mathbb{P}_n^t\rangle = -\langle u, (\mathbb{P}_m\otimes I_1)(I_1\otimes\tilde\Psi^t)\nabla\mathbb{P}_n^t\rangle,\end{aligned}$$

where we have used a well known property of the Kronecker product (see [6, p. 252]): Let $M \in \mathcal{M}_{h\times k}$, $N \in \mathcal{M}_{l\times m}$ be given matrices, and let I_r denote the identity matrix of dimension r, then

$$(M \otimes I_l)(I_k \otimes N) = M \otimes N = (I_h \otimes N)(M \otimes I_m).$$

And therefore, we conclude

$$\langle u, \mathbb{P}_m L[\mathbb{P}_n^t]\rangle = -\langle u, (\nabla \mathbb{P}_m^t)^t \Phi \nabla \mathbb{P}_n^t\rangle. \tag{4.1}$$

The expression $L[\mathbb{P}_n^t]$ is a polynomial matrix of degree at most n, then, we can write

$$L[\mathbb{P}_n^t] = \text{div } (\Phi \nabla \mathbb{P}_n^t) + \tilde{\Psi}^t \nabla \mathbb{P}_n^t = \sum_{i=0}^{n} \mathbb{P}_i^t \Lambda_i^n,$$

where Λ_m^n can by obtained by orthogonality as follows

$$\langle u, \mathbb{P}_m L[\mathbb{P}_n^t]\rangle = \langle u, \mathbb{P}_m \sum_{i=0}^{n} \mathbb{P}_i^t \Lambda_i^n\rangle = \langle u, \mathbb{P}_m \mathbb{P}_m^t\rangle \Lambda_m^n = H_m \Lambda_m^n.$$

On the other hand, using (4.1), we have

$$H_m \Lambda_m^n = -\langle u, (\nabla \mathbb{P}_m^t)^t \Phi \nabla \mathbb{P}_n^t\rangle = 0, \quad n \neq m,$$

and (3.8) follows, defining $\Lambda_n = \Lambda_n^n$.

The above relation for $n = m = 1$ gives

$$H_1 \Lambda_1 = -\langle u, (\nabla \mathbb{P}_1^t)^t \Phi \nabla \mathbb{P}_1^t\rangle = -K_1,$$

and then, Λ_1 is regular.

The reciprocal can be obtained using again (4.1), because

$$\langle u, (\nabla \mathbb{P}_m^t)^t \Phi \nabla \mathbb{P}_n^t\rangle = \langle u, \mathbb{P}_m L[\mathbb{P}_n^t]\rangle = \langle u, \mathbb{P}_m \mathbb{P}_n^t\rangle \Lambda_n,$$

and the result follows from the orthogonality of the polynomials $\{\mathbb{P}_n\}_{n\geq 0}$. □

5 Examples

Example 5.1 *Multiple Hermite polynomials on* $\mathbb{R}^d$.

The multiple Hermite polynomials are defined by

$$H_{\mathrm{m}}(\mathrm{x}) = H_{m_1}(x_1)\cdots H_{m_d}(x_d),$$

where $\mathrm{x} = (x_1, \ldots, x_d)$ and $\mathrm{m} = (m_1, \ldots, m_d)$. These polynomials are orthogonal polynomials on the domain $\mathbb{R}^d$ with respect to the multiple Hermite weight function

$$\omega(\mathrm{x}) = e^{\|\mathrm{x}\|},$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

In this case, the matrices Φ and Ψ are given by

$$\Phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} -2x_1 \\ -2x_2 \\ \vdots \\ -2x_d \end{pmatrix},$$

and the matrix $\Lambda_n = -2n\, I_{r_n}$.

Example 5.2 *Multiple Laguerre polynomials on* $\mathbb{R}^d_+$.

The multiple Laguerre polynomials are defined by

$$L^{\alpha}_{\mathrm{m}}(\mathrm{x}) = L^{\alpha_1}_{m_1}(x_1)\cdots L^{\alpha_d}_{m_d}(x_d),$$

where $\mathrm{x} = (x_1, \ldots, x_d), \mathrm{m} = (m_1, \ldots, m_d)$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$, with $\alpha_i > -1$. These polynomials are orthogonal on the domain $\mathbb{R}^d_+$ with respect to the multiple Laguerre weight function

$$\omega_\alpha(\mathrm{x}) = \mathrm{x}^{\alpha}\, e^{-|\mathrm{x}|},$$

which is the product of the Laguerre weight functions in one variable.

In this case, the matrices Φ and Ψ are given by

$$\Phi = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_d \end{pmatrix}, \Psi = \begin{pmatrix} -x_1 + \alpha_1 + 1 \\ -x_2 + \alpha_2 + 1 \\ \vdots \\ -x_d + \alpha_d + 1 \end{pmatrix},$$

and the matrix $\Lambda_n = -n\, I_{r_n}$.

Example 5.3 *Multiple Jacobi polynomials on the d–cube.*

The multiple Jacobi polynomials are defined by

$$P_{\mathrm{m}}^{(\alpha,\beta)}(\mathrm{x}) = P_{m_1}^{(\alpha_1,\beta_1)}(x_1)\dots P_{m_d}^{(\alpha_d,\beta_d)}(x_d),$$

where $\mathrm{x} = (x_1,\dots,x_d), \mathrm{m} = (m_1,\dots,m_d), \alpha = (\alpha_1,\dots,\alpha_d)$ and $\beta = (\beta_1,\dots,\beta_d)$, with $\alpha_i > -1, \beta_i > -1$. These polynomials are orthogonal on the domain $[-1,1]^d$ of $\mathbb{R}^d$ with respect to the multiple Jacobi weight function

$$\omega_{(\alpha,\beta)}(\mathrm{x}) = \prod_{i=1}^{d}(1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}.$$

In this case, the matrices Φ and Ψ are given by

$$\Phi = \begin{pmatrix} 1-x_1^2 & 0 & \cdots & 0 \\ 0 & 1-x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-x_d^2 \end{pmatrix},$$

$$\Psi = \begin{pmatrix} -(\alpha_1+\beta_1+2)x_1 + \beta_1 - \alpha_1 \\ -(\alpha_2+\beta_2+2)x_2 + \beta_2 - \alpha_2 \\ \vdots \\ -(\alpha_d+\beta_d+2)x_d + \beta_d - \alpha_d \end{pmatrix}.$$

The matrix Λ_n is a diagonal matrix, whose entries in the main diagonal are given by

$$\lambda_{\mathrm{m}} = -\sum_{i=1}^{d} m_i(m_i+\alpha_i+\beta_i+1),$$

where m runs over the set of indexes $\mathrm{m} = (m_1,\dots,m_d)$ satisfying $|\mathrm{m}| = m_1 + \ldots + m_d = n$.

As we can see the elements in the diagonal of Λ_n are distinct, and therefore this example does not fit in the Krall and Sheffer classification of the bivariate classical orthogonal polynomials.

Example 5.4 *Orthogonal polynomials on the unit ball (see [3]).*

The classical orthogonal polynomials on the unit ball, $B_d = \{\mathrm{x} : \|\mathrm{x}\| \le 1\}$, are associated with the weight function

$$\omega_\mu(\mathrm{x}) = (1-\|\mathrm{x}\|)^{\mu-1/2}, \qquad \mu > -1/2.$$

In this case, the matrices Φ and Ψ are given by

$$\Phi = \begin{pmatrix} x_1^2-1 & x_1x_2 & \cdots & x_1x_d \\ x_1x_2 & x_2^2-1 & \cdots & x_2x_d \\ \vdots & \vdots & \ddots & \vdots \\ x_1x_d & x_2x_d & \cdots & x_d^2-1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} (2\mu+d)x_1 \\ (2\mu+d)x_2 \\ \vdots \\ (2\mu+d)x_d \end{pmatrix},$$

and the matrix $\Lambda_n = n(n+d+2\mu-1)\, I_{r_n}$.

Example 5.5 *Orthogonal polynomials on the simplex (see [3]).*

The weight function associated with the classical orthogonal polynomials on the d–simplex, $T_d = \{\mathrm{x} : x_1 \geq 0, \ldots, x_d \geq 0, 1-|\mathrm{x}| \geq 0\}$, is defined by

$$\omega_\alpha(\mathrm{x}) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}(1-|\mathrm{x}|)^{\alpha_{d+1}}, \qquad \alpha_i > -1$$

In this case, the matrices Φ and Ψ are given by

$$\Phi = \begin{pmatrix} x_1(x_1-1) & x_1x_2 & \cdots & x_1x_d \\ x_1x_2 & x_2(x_2-1) & \cdots & x_2x_d \\ \vdots & \vdots & \ddots & \vdots \\ x_1x_d & x_2x_d & \cdots & x_d(x_d-1) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} (|\alpha|+d)x_1 - (\alpha_1+1) \\ (|\alpha|+d)x_2 - (\alpha_2+1) \\ \vdots \\ (|\alpha|+d)x_d - (\alpha_d+1) \end{pmatrix},$$

and the matrix $\Lambda_n = n(n+|\alpha|+d)\, I_{r_n}$.

References

[1] H. Berens, H. Schmid and Y. Xu, *On two–dimensional definite orthogonal systems and a lower bound for the number of nodes of associated cubature formulas,* SIAM J. Math. Anal. **26** (1995), 468–487.

[2] T. S. Chihara, An introduction to Orthogonal Polynomials, *Gordon and Breach, 1978.*

[3] C. F. Dunkl and Y. Xu, Orthogonal Polynomials of several variables, *Encyclopedia of Mathematics and its Applications* **81,** *Cambridge University Press, 2001.*

[4] L. Fernández, T. E. Pérez and M. A. Piñar, *Classical Orthogonal Polynomials in two variables: a matrix approach*, to appear in Numerical Algorithm.

[5] L. Fernández, T. E. Pérez and M. A. Piñar, *Weak Classical Orthogonal Polynomials in two variables*, J. Comput. Appl. Math. **178** (2005), 191–203.

[6] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis. *Cambridge University Presss, Cambridge, 1991.*

[7] Y. J. Kim, K. H. Kwon and J. K. Lee, *Orthogonal polynomials in two variables and second–order partial differential equations,* J. Comput. Appl. Math. **82** (1997), 239–260.

[8] Y. J. Kim, K. H. Kwon and J. K. Lee, *Partial differential equations having orthogonal polynomial solutions,* J. Comput. Appl. Math. **99** (1998), 239–253.

[9] M. A. Kowalski, *The recursion formulas for orthogonal polynomials in n variables,* SIAM J. Math. Anal. **13** (1982), 309–315.

[10] M. A. Kowalski, *Orthogonality and recursion formulas for polynomials in n variables,* SIAM J. Math. Anal. **13** (1982), 316–323.

[11] H. L. Krall and I. M. Sheffer, *Orthogonal polynomials in two variables,* Ann. Mat. Pura Appl. Serie 4 **76** (1967), 325–376.

[12] L. L. Littlejohn, *Orthogonal polynomial solutions to ordinary and partial differential equations,* in Orthogonal Polynomials and their Applications. Proceedings Segovia (Spain), 1986, Lecture Notes in Mathematics, **1329, Springer, Berlin, (1988), 98–124.**

[13] F. Marcellán, A. Branquinho and J. Petronilho, *Classical orthogonal polynomials: A functional approach,* Acta Appl. Math. **34** (1994), 283–303.

[14] P. Maroni, *Une théorie algébrique des polynômes orthogonaux,* IMACS Ann. Comput. Appl. Math., **9** (1991), 95–130.

[15] P. K. Suetin, Orthogonal Polynomials in Two Variables, *Gordon and Breach, Amsterdam, 1999.*

[16] Y. Xu, *On multivariate orthogonal polynomials,* SIAM J. Math. Anal. **24** (1993), 783–794.

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Homogeneous Banach Spaces with respect to Jacobi Polynomials

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Abstract. Homogeneous Banach spaces determined by the Jacobi translation operator are introduced and studied. Based on this translation operator a Jacobi differential operator is analyzed. Approximation procedures in the homogeneous Banach spaces are presented.

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1 Introduction

Homogeneous Banach spaces, homogeneous Banach algebras and Segal algebras had found much interest in Fourier analysis. In particular, if one investigates functions on the unit circle $\mathcal{T}$ and if one deals with their Fourier expansions, in homogeneous Banach spaces all the classical approximation procedures, like Fejér- or de la Vallée-Poussin-approximation, can be applied. References for homogeneous Banach spaces on $\mathcal{T}$ are [5] or [6], for homogeneous Banach algebras and Segal algebras [15] and [12].

We extend the related investigation to Jacobi expansions. So we consider functions on the interval $S = [-1, 1]$ and reveal a lot of homogeneous Banach spaces, which are in fact Banach algebras, consisting of functions on $[-1, 1]$. For these homogeneous Banach spaces we give examples of appropriate approximation processes. The basic functions throughout are the

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Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$. They play the part that the trigonometric polynomials e^{int} play in Fourier series.

2 Jacobi polynomials and the continuous Jacobi translation

The Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$, $\alpha,\beta > -1$, are orthogonal with respect to $\pi^{(\alpha,\beta)}$, where supp $\pi^{(\alpha,\beta)} = [-1,1]$ and

$$d\pi^{(\alpha,\beta)}(x) \;=\; c_{\alpha,\beta}\,(1-x)^{\alpha}(1+x)^{\beta}\,dx$$

with

$$c_{\alpha,\beta} \;=\; 2^{-(\alpha+\beta+1)}\,\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

see [13].

We choose the normalization such that $R_n^{(\alpha,\beta)}(1) = 1$. By this normalization we have for $n \in \mathbb{N}_0$

$$h^{(\alpha,\beta)}(n) := \left(\int_{-1}^{1}\left(R_n^{(\alpha,\beta)}(x)\right)^2\,d\pi^{(\alpha,\beta)}(x)\right)^{-1}$$

$$= \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n(\alpha+1)_n}{(\alpha+\beta+1)\;n!\;(\beta+1)_n}.$$

We assume furtheron that $(\alpha,\beta) \in J$, where

$$J \;=\; \{(\alpha,\beta):\ \alpha \geq \beta > -1 \ \text{ and } \ (\beta \geq -\frac{1}{2} \ \text{ or } \alpha+\beta \geq 0)\} \qquad (2.1)$$

In that case, it is known that for any $x, y \in [-1,1]$ there exists a probability Borel measure $\mu_{x,y}^{(\alpha,\beta)} \in M([-1,1])$ such that

$$R_n^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(y) \;=\; \int_{-1}^{1} R_n^{(\alpha,\beta)}(z)\;d\mu_{x,y}^{(\alpha,\beta)}(z)$$

for every $n \in \mathbb{N}_0$, see [3]. These measures give rise to a generalized translation operator.

At first, let us agree on omitting the parameters (α,β) at all the notions in the context of Jacobi polynomials. So α,β are fixed throughout, fulfilling (2.1). Also we briefly write $S = [-1,1]$.

Let $L^p(S,\pi)$, $1 \le p < \infty$, be the Banach space endowed with the norm

$$\|f\|_p \;=\; \left(\int_{-1}^{1} |f(x)|^p \, d\pi(x)\right)^{1/p},$$

and let $C(S)$ be the Banach space with norm $\|f\|_\infty = \sup\limits_{x\in S}|f(x)|$. The translation operator induced by the measures $\mu_{x,y}$ is not the usual translation operator. Given an element $f \in L^1(S,\pi)$ and $y \in S$, define $T_y f$ by

$$T_y f(x) \;=\; \int_{-1}^{1} f(z)\, d\mu_{x,y}(z) \tag{2.2}$$

Now it is known that $T_y f \in L^p(S,\pi)$, if $f \in L^p(S,\pi)$, as well as $T_y f \in C(S)$ whenever $f \in C(S)$. Moreover $\|T_y f\|_p \le \|f\|_p$ and $\lim\limits_{y\to 1-} \|T_y f - f\|_p = 0$. An appropriate framework to prove these results is the theory of hypergroups, see e.g. [2] or [4]. In fact, due to Gasper's result on the existence of the $\mu_{x,y}$, it follows that S is endowed with a corresponding hypergroup structure, see [7]. An important property of the translation operator is its relation to the Jacobi transform, see (2.9) below. We denote by

$$\check{f}(n) \;=\; \int_{-1}^{1} f(x)\, R_n(x)\, d\pi(x), \tag{2.3}$$

if $f \in L^1(S,\pi)$, and by

$$\hat{d}(x) \;=\; \sum_{n=0}^{\infty} d(n)\, R_n(x)\, h(n), \tag{2.4}$$

if $d \in l^1(\mathbb{N}_0,h)$, wherein $d = (d(n))_{n\in\mathbb{N}_0} \in l^p(\mathbb{N}_0,h)$, if $\sum\limits_{n=0}^{\infty} |d(n)|^p\, h(n) < \infty$, $1 \le p < \infty$. Note that $\sup\limits_{x\in S}|R_n(x)| = R_n(1) = 1$ for all $n \in \mathbb{N}_0$.

An important fact for the Jacobi transform is the validity of the uniqueness theorem stating that $\check{f} \equiv 0$ implies $f = 0$ in $L^1(S,\pi)$, see [2] or [4].

Considering the two Hilbert spaces $L^2(S,\pi)$ and $l^2(\mathbb{N}_0,h)$ we have the following relations:

$$\int_{-1}^{1} |f(x)|^2 \, d\pi(x) \;=\; \sum_{n=0}^{\infty} |\check{f}(n)|^2 \, h(n) \tag{2.5}$$

for each $f \in L^2(S,\pi) \subset L^1(S,\pi)$, and

$$\sum_{n=0}^{\infty} |d(n)|^2\, h(n) \;=\; \int_{-1}^{1} |\hat{d}(x)|^2 \, d\pi(x) \tag{2.6}$$

for each $d \in l^1(\mathbb{N}_0, h) \subseteq l^2(\mathbb{N}_0, h)$.

Applying the generalized translation of (2.2) we can define a convolution. For $f, g \in L^1(S, \pi)$ the convolution

$$f * g\,(y) \;=\; \int_{-1}^{1} f(x)\, T_y g(x)\; d\pi(x), \quad y \in S \tag{2.7}$$

is an element of $L^1(S, \pi)$, such that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. So $L^1(S, \pi)$ is a Banach algebra with the convolution as multiplication. Similarly $L^1(S, \pi)$ acts on $L^p(S, \pi)$, $1 \leq p < \infty$, using definition (2.7), where $f \in L^1(S, \pi)$, $g \in L^p(S, \pi)$. Details may be found in [2] or [4]. An important fact is that for $f, g \in L^1(S, \pi)$

$$\int_{-1}^{1} f(x)\, T_y g(x)\; d\pi(x) \;=\; \int_{-1}^{1} T_y f(x)\, g(x)\; d\pi(x) \tag{2.8}$$

holds true. Now it is easy to show that

$$(T_y f)\check{}(n) \;=\; R_n(y)\, \check{f}(n) \tag{2.9}$$

for every $f \in L^1(S, \pi)$, $y \in S$.

If $f \in C(S) \subseteq L^1(S, \pi)$ the convolution of f with $g \in L^p(S, \pi)$ is nothing else than a $L^p(S, \pi)$-valued integral,

$$f * g \;=\; \int_{-1}^{1} f(x)\, T_x g\; d\pi(x),$$

since $T_x g(y) = T_y g(x)$.

We will also apply a dual hypergroup structure on $\mathbb{N}_0$ connected with the Jacobi polynomials $R_n(x)$. Indeed, the linearization coefficients $g(n, m; k)$ in the decomposition

$$R_n(x)\, R_m(x) \;=\; \sum_{k=|n-m|}^{n+m} g(n, m; k)\, R_k(x), \tag{2.10}$$

$n, m \in \mathbb{N}_0$, are all nonnegative, provided $(\alpha, \beta) \in J$.

This is a result of Gasper, too. Indeed, the range of parameters (α, β), for which $g(n, m; k) \geq 0$, is larger than J. For details, we refer to [7]. The nonnegativity of the $g(n, m; k)$ gives rise to a convolution on $\mathbb{N}_0$. The corresponding translation operator S_n acts on the space $l^\infty(\mathbb{N}_0)$ of all bounded sequences by

$$S_n\, d(m) \;=\; \sum_{k=|n-m|}^{n+m} g(n, m; k)\; d(k), \tag{2.11}$$

$d = (d(k))_{k\in\mathbb{N}_0} \in l^\infty(\mathbb{N}_0)$. It is easy to check that $\|S_n d\|_p \le \|d\|_p$ for all $d \in l^p(\mathbb{N}_0, h)$, $1 \le p \le \infty$.

3 Homogeneous Banach spaces

Fix some pair of parameters $(\alpha, \beta) \in J$.

Definition 3.1 *We call a linear subspace B of $L^1(S,\pi)$ a homogeneous Banach space on S with respect to (α,β), if it is endowed with a norm $\|\,.\,\|_B$ such that*

(B1) $R_n \in B$ *for all* $n \in \mathbb{N}_0$

(B2) B *is complete with respect to* $\|\,.\,\|_B$ *and* $\|\,.\,\|_1 \le \|\,.\,\|_B$

(B3) For every $f \in B$, $x \in S$ *we have* $T_x f \in B$ *and* $\|T_x f\|_B \le \|f\|_B$

(B4) For every $f \in B$ *the map* $x \mapsto T_x f$, $S \to B$ *is continuous.*

A homogeneous Banach space B is called character-invariant, if

(B5) For every $f \in B$, $n \in \mathbb{N}_0$ *we have* $R_n \cdot f \in B$ *and* $\|R_n \cdot f\|_B \le \|f\|_B$.

The main purpose of this chapter is to discover a lot of interesting examples of homogeneous Banach spaces B on S with respect to $(\alpha,\beta) \in J$. But we begin with showing that B is in fact an $L^1(S,\pi)$-module.

Proposition 3.2 *Let B be a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$. Then for each $g \in B$ and $f \in L^1(S,\pi)$, $f * g$ is an element of B and $\|f * g\|_B \le \|f\|_1 \, \|g\|_B$.*

Proof. Considering the B-valued integral we have for $g \in B$

$$f * g \;=\; \int_{-1}^{1} f(x)\, T_x g \; d\pi(x) \;\in B,$$

and $\|f * g\|_B \le \|f\|_1 \|g\|_B$, wherein f is a continuous function on S. If $f \in L^1(S,\pi)$ choose a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in C(S)$ such that $\|f - f_n\|_1 \to 0$ as $n \to \infty$. Now it is easily shown that $f * g \in B$ and $\|f * g\|_B \le \|f\|_1 \|g\|_B$. □

Corollary 3.3 *Every homogeneous Banach space B on S with respect to $(\alpha,\beta) \in J$ is a Banach algebra with convolution as multiplication.*

Proof. We have only to note that $B \subseteq L^1(S,\pi)$ and $\|\,.\,\|_1 \le \|\,.\,\|_B$. □

Remark 3.4 *A character-invariant homogeneous Banach space B on S is a Segal algebra, see [15].*

Proposition 3.5 *Let B be an arbitrary character-invariant homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$. Then for each $g \in B$ and $d \in l^1(\mathbb{N}_0, h)$, $\hat{d}g$ is an element of B and $\|\hat{d}\, g\|_B \leq \|d\|_1 \, \|g\|_B$.*

Proof. For each $n \in \mathbb{N}_0$ the partial sum $f_n = \sum_{k=0}^{n} d(k)h(k)R_k g$ is an element of B, and for $n > m$ we have

$$\|f_n - f_m\|_B \leq \left(\sum_{k=m+1}^{n} |d(k)| \, h(k) \right) \|g\|_B.$$

Now it is clear that f_n converges in B towards $\hat{d}g$ and $\|\hat{d}g\|_B \leq \|d\|_1 \, \|g\|_B$. □

We know already that $B = L^p(S, \pi)$, $1 \leq p < \infty$ and $B = C(S)$ are homogeneous Banach spaces on S with respect to $(\alpha, \beta) \in J$, . An important further example is the analog of the Wiener algebra.

3.1 Wiener algebra $W(S)$

Denote by $W(S) = \{f \in C(S) : \check{f} \in l^1(\mathbb{N}_0, h)\}$ and set $\|f\|_W := \|\check{f}\|_1$ for $f \in W(S)$.

Proposition 3.6 *$W(S)$ is a homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$. It is also character-invariant.*

Proof. Obviously $W(S)$ is a linear space, and $\|\,.\,\|_W$ is a norm by the uniqueness theorem. Since $\check{R}_m(n) = h(n)^{-1}\delta_{m,n}$ condition (B1) is fulfilled. $(W(S), \|\,.\,\|_W)$ is complete, since $f \mapsto \check{f}$, $W(S) \to l^1(\mathbb{N}_0, h)$ is an isometric isomorphism from $W(S)$ onto the Banach space $l^1(\mathbb{N}_0, h)$. Since $\|f\|_W = \|\check{f}\|_1 \geq \|\check{f}\|_2 = \|f\|_2 \geq \|f\|_1$, we see that (B2) holds true. For $f \in W(S)$, $x \in S$ we know $(T_x f)\check{}(n) = R_n(x)\check{f}(n)$, and hence $T_x f \in W(S)$ and $\|T_x f\|_W \leq \|f\|_W$. To show (B4) let $x_0 \in S$, $\epsilon > 0$. There exists $N \in \mathbb{N}$ and $g \in W(S)$ such that $\|g - f\|_W < \epsilon/4$, wherein $\check{g}(n) = \check{f}(n)$ for all $n \in \{0, ..., N\}$ and $\check{g}(n) = 0$ for all $n \geq N + 1$. Furthermore there is some $\delta > 0$ with $\|T_x g - T_{x_0} g\|_W < \epsilon/2$ for all $x \in S$ such that $|x - x_0| < \delta$. Thus we obtain $\|T_x f - T_{x_0} f\|_W < \epsilon$ for all $x \in S$ with $|x - x_0| < \delta$.

Finally we have for $f \in W(S)$ and $n \in \mathbb{N}_0$

$$(R_n \cdot f)\check{}\,(m) \;=\; S_n \check{f}(m) \qquad \text{for } m \in \mathbb{N}_0,$$

where S_n is the translation operator on $l^1(\mathbb{N}_0, h)$. We conclude that $R_n \cdot f \in W(S)$ and $\|R_n \cdot f\|_W \le \|f\|_W$. □

Proposition 3.7 *$W(S)$ is the smallest character-invariant homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$. That is: If B is a character-invariant homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$, then $W(S) \subseteq B$ and $\|\,.\,\|_B \le c\|\,.\,\|_W$, $c > 0$.*

Proof. For $f \in W(S)$ and any $n \in \mathbb{N}$ set $f_n = \sum_{k=0}^{n} \check{f}(k) R_k h(k)$. Then $f_n \in B$, and for $n > m$

$$\|f_n - f_m\|_B \;\le\; \sum_{k=m+1}^{n} |\check{f}(k)|\, \|R_k\|_B\, h(k).$$

Since B is character-invariant, we have

$$\|R_k\|_B \;=\; \|R_k R_0\|_B \;\le\; \|R_0\|_B$$

Putting $c = \|R_0\|_B$ we obtain that f_n converges in B to f, and $\|f\|_B \le c\|\check{f}\|_1 = c\|f\|_W$. □

3.2 Beurling space $W_*(S)$

Denote by

$$W_*(S) := \{f \in W(S) : \sum_{k=0}^{\infty} (\sup_{l \ge k} |\check{f}(l)|\,)\, h(k) < \infty\}.$$

$W_*(S)$ is a linear space. Setting $\tilde{f}(k) := \sup_{l \ge k} |\check{f}(l)|$ we define

$$\|f\|_{W_*} := \sum_{k=0}^{\infty} \tilde{f}(k)\, h(k).$$

The uniqueness theorem implies that $\|\,.\,\|_{W_*}$ is a norm on $W_*(S)$. The polynomials R_n are elements of $W(S)$, and clearly

$$\|f\|_{W_*} \;\ge\; \|f\|_W \;\ge\; \|f\|_1 \qquad \text{for all } f \in W_*(S).$$

Lemma 3.8 *$(W_*(S), \| \ \|_{W_*})$ is a Banach space.*

Proof. We have to show the completeness of $W_*(S)$. Let $(f_n)_{n\in\mathbb{N}_0}$ be a Cauchy sequence in $W_*(S)$. Then $(f_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $W(S)$. Put $f = \lim\limits_{n\to\infty} f_n \in W(S)$. Given $\epsilon > 0,\ M \in \mathbb{N},$ there exists $N_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{W_*} \ < \ \epsilon/2 \qquad \text{for all } n, m \geq N_0 \qquad\qquad (a)$$

and for $\delta := \frac{\epsilon}{2(M+1)} > 0,$ there is an $N_M \in \mathbb{N}_0$ such that

$$\delta \ \geq \ \|f - f_n\|_W \ \geq \ |\check{f}(l) - \check{f}_n(l)|\, h(l) \qquad \text{for all } n \geq N_M,\ \ l \in \mathbb{N}_0$$

In other words

$$\|\check{f}h - \check{f}_n h\|_\infty \ \leq \ \delta \qquad \text{for all } n \geq N_M \qquad\qquad (b)$$

We can assume that $N_M \geq N_0$. Write

$$\sum_{k=0}^{M}(\sup_{l\geq k} |\check{f}(l) \ - \ \check{f}_n(l)|\,)\, h(k)$$

$$\leq \ \sum_{k=0}^{M}(\sup_{l\geq k} |\check{f}(l) - \check{f}_{N_M}(l)|\,)\, h(k) \ + \ \sum_{k=0}^{\infty}(\sup_{l\geq k} |\check{f}_{N_M}(l) - \check{f}_n(l)|\,)\, h(k)$$

Applying (b) we get

$$\sum_{k=0}^{M}(\sup_{l\geq k} |\check{f}(l) - \check{f}_{N_M}(l)|\,)\, h(k) \ \leq \ \sum_{k=0}^{M} \sup_{l\geq k} |\check{f}(l) - \check{f}_{N_M}(l)|\, h(l)$$

$$\leq \ \sum_{k=0}^{M} \|\check{f}h - \check{f}_{N_M} h\|_\infty \ \leq \ (M+1)\,\delta \ = \ \frac{\epsilon}{2},$$

and by (a) it follows

$$\sum_{k=0}^{\infty}(\sup_{l\geq k} |\check{f}_{N_M}(l) - \check{f}_n(l)|\,)\, h(k) \ < \ \frac{\epsilon}{2} \qquad \text{for all } n \geq N_0.$$

So we have shown that for any $M \in \mathbb{N}$ and $n \geq N_0$

$$\sum_{k=0}^{M}(\sup_{l\geq k} |\check{f}(l) - \check{f}_n(l)|\,)\, h(k) \ < \ \epsilon,$$

which yields $f - f_n \in W_*(S)$ and $\|f - f_n\|_{W_*} \leq \epsilon$ for all $n \geq N_0$. Thus $f = (f - f_{N_0}) + f_{N_0} \in W_*(S)$ and $\|f - f_n\|_{W_*} \to 0$ as $n \to \infty$. □

In order to derive condition (B4) we show:

Lemma 3.9 *The linear subspace* $W_{*,c}(S) := \{f \in W_*(S) : \check{f}(k) \neq 0$ *at most for finitely many* $k \in \mathbb{N}_0\}$ *is dense in* $W_*(S)$.

Proof. For any $f \in W_*(S)$, $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \tilde{f}(k)h(k) < \frac{\epsilon}{2}$. Denote by $g \in W_*(S)$ that element for which $\check{g}(k) = \check{f}(k)$ for $k = 0, ..., 2N-1$ and $\check{g}(k) = 0$ for $k = 2N, 2N+1, ...$ Then $\|f-g\|_{W_*} = S_1 + S_2$, wherein

$$S_1 \;=\; \sum_{k=0}^{N} (\sup_{l \geq k} |\check{f}(l) - \check{g}(l)|\,)\, h(k)$$

and

$$S_2 \;=\; \sum_{k=N+1}^{\infty} (\sup_{l \geq k} |\check{f}(l) - \check{g}(l)|\,)\, h(k).$$

Since $|\check{f}(l) - \check{g}(l)| = 0$ for $l = 0, ..., 2N-1$ and $|\check{f}(l) - \check{g}(l)| = |\check{f}(l)|$ for $l \geq 2N$, we obtain

$$\sup_{l \geq k} |\check{f}(l) - \check{g}(l)| \;=\; \sup_{l \geq 2N} |\check{f}(l)| \;=\; \tilde{f}(2N)$$

for every $k = 0, ..., 2N-1$. In particular $S_1 = \sum_{k=0}^{N} \tilde{f}(2N)h(k)$.

Now the weights $h(n)$ are increasing (which can be checked directly), and also $\tilde{f}(2N) \leq ... \leq \tilde{f}(N+1) \leq \tilde{f}(N)$. Hence

$$S_1 \;\leq\; \sum_{k=0}^{N} \tilde{f}(k+N)\, h(k+N) \;=\; \sum_{k=N}^{2N} \tilde{f}(k)\, h(k) \;\leq\; \sum_{k=N}^{\infty} \tilde{f}(k)\, h(k) \;<\; \frac{\epsilon}{2}.$$

Considering S_2 we have

$$S_2 \;=\; \tilde{f}(2N)\, h(N+1) \;+ ... + \tilde{f}(2N)\, h(2N-1) \;+ \sum_{k=2N}^{\infty} \tilde{f}(k)\, h(k)$$

$$\leq\; \tilde{f}(N+1)\, h(N+1) \;+ ... + \tilde{f}(2N-1)\, h(2N-1) \;+ \sum_{k=2N}^{\infty} \tilde{f}(k)\, h(k)$$

$$= \sum_{k=N+1}^{\infty} \tilde{f}(k)\, h(k) \;<\; \frac{\epsilon}{2},$$

and the statement of the Lemma is proved. □

Proposition 3.10 *$W_*(S)$ is a homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$.*

Proof. (B1) and (B2) are already shown. Since

$$\widetilde{T_x f}(k) \;=\; \sup_{l \geq k} |R_l(x)\, \check{f}(l)| \;\leq\; \tilde{f}(k)$$

condition (B3) holds true.

If $f \in W_*(S)$, $x_0 \in S$ and $\epsilon > 0$, choose $g \in W_{*,c}(S)$ such that $\|f - g\|_{W_*} < \frac{\epsilon}{4}$. Let N be the smallest natural number such that $\check{g}(n) = 0$ for all $n \geq N + 1$. Since

$$\begin{aligned}(T_x g - T_{x_0} g)^{\sim}(k) \;&=\; \sup_{l \geq k} |R_l(x) - R_l(x_0)|\, |\check{g}(l)| \\ &\leq\; \tilde{g}(0) \sup_{l \leq N} |R_l(x) - R_l(x_0)|,\end{aligned}$$

there exists $\delta > 0$ such that

$$(T_x g - T_{x_0} g)^{\sim}(k)\, h(k) \;<\; \frac{\epsilon}{2(N+1)}$$

for $k = 0, ..., N$ and $x \in S$, $|x - x_0| < \delta$. Therefore $\|T_x g - T_{x_0} g\|_{W_*} < \frac{\epsilon}{2}$ for $x \in S$, $|x - x_0| < \delta$, and we obtain

$$\|T_x f - T_{x_0} f\|_{W_*} \;<\; 2\, \|f - g\|_{W_*} \;+\; \frac{\epsilon}{2} \;<\; \epsilon$$

for $x \in S$, $|x - x_0| < \delta$. □

Remark 3.11 *It is easy to find an $f \in W(S) \backslash W_*(S)$. Choose $f \in W(S)$ with $\check{f}(n) = 0$ whenever $n \neq 2^k$, $k \in \mathbb{N}_0$ and $\check{f}(2^k) = \frac{1}{2^k h(2^k)}$ for $k \in \mathbb{N}_0$. Since $\tilde{f}(2^{k-1} + 1) = ... = \tilde{f}(2^k) = \check{f}(2^k)$ for $k \in \mathbb{N}$, we have for the formal series*

$$\begin{aligned}\sum_{n=0}^{\infty} \tilde{f}(n)\, h(n) \;&\geq\; \sum_{k=1}^{\infty} \frac{1}{2^k h(2^k)}\, [h(2^{k-1} + 1) + ... + h(2^k)] \\ &\geq\; \sum_{k=1}^{\infty} \frac{1}{2}\, \frac{h(2^{k-1} + 1)}{h(2^k)}\end{aligned}$$

By the asymptotic behaviour of the Γ-function it follows that $h(n) \sim n^{2\alpha+1}$, and so $\frac{h(2^{k-1}+1)}{h(2^k)}$ does not converge to zero. Thus $f \notin W_*(S)$. By Proposition (3.7) it follows that $W_*(S)$ is not character-invariant. This is also a

consequence of the fact that $\|R_n\|_{W_*} = (\sum_{k=0}^{n} h(k))/h(n)$. *If* $W_*(S)$ *would be character-invariant, then* $\|R_n\|_{W_*} \leq \|R_0\|_{W_*} = 1$, *which contradicts* $(\sum_{k=0}^{n} h(k))/h(n) \to \infty$.

3.3 Lipschitz spaces $\mathbf{lip}_B(\lambda)$

Let B be a homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$, and choose $0 < \lambda < 1$. Denote by

$$\mathrm{lip}_B(\lambda) := \{f \in B : \lim_{x \to 1-} \frac{\|T_x f - f\|_B}{(1-x)^\lambda} = 0\}$$

and

$$\|f\|_L := \|f\|_B + \sup_{x \in [-1,1[} \frac{\|T_x f - f\|_B}{(1-x)^\lambda} \qquad \text{for } f \in \mathrm{lip}_B(\lambda).$$

We notify that

$$\mathrm{Lip}_B(\lambda) := \{f \in B : \sup_{x \in [-1,1[} \frac{\|T_x f - f\|_B}{(1-x)^\lambda} < \infty\}, \qquad 0 < \lambda \leq 1$$

is a Banach space with norm $\|\,.\,\|_L$, however in general not a homogeneous Banach space. It is easy to see that $\mathrm{lip}_B(\lambda)$ is a linear subspace of $\mathrm{Lip}_B(\lambda)$ and $\|f\|_L \geq \|f\|_B \geq \|f\|_1$ for every $f \in \mathrm{lip}_B(\lambda)$.

Proposition 3.12 *$lip_B(\lambda)$ is a homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$.*

Proof. We begin with proving the completeness of $\mathrm{lip}_B(\lambda)$ with respect to $\|\,.\,\|_L$. If $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathrm{lip}_B(\lambda)$ and $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in [-1,1[} \frac{\|T_x(f_n - f_m) - (f_n - f_m)\|_B}{(1-x)^\lambda} \leq \frac{\epsilon}{4}$$

for $n, m \geq N$. Since $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in B, there is an element $f \in B$ with $\lim_{n \to \infty} \|f - f_n\|_B = 0$. In particular, given $x \in [-1, 1[$, there is $N(x) \in \mathbb{N}$, $N(x) \geq N$, such that

$$\|f - f_n\|_B \leq \frac{\epsilon}{8} (1-x)^\lambda$$

for every $n \geq N(x)$. Considering an $x \in [-1,1[$, we get

$$\frac{\|T_x(f-f_n)-(f-f_n)\|_B}{(1-x)^\lambda}$$

$$\leq \frac{\|T_x(f-f_{N(x)})-(f-f_{N(x)})\|_B}{(1-x)^\lambda} + \frac{\|T_x(f_{N(x)}-f_n)-(f_{N(x)}-f_n)\|_B}{(1-x)^\lambda}$$

$$\leq \frac{2\|f_{N(x)}-f\|_B}{(1-x)^\lambda} + \sup_{y\in[-1,1[} \frac{\|T_y(f_{N(y)}-f_n)-(f_{N(y)}-f_n)\|_B}{(1-y)^\lambda}$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

for all $n \geq N$. This is valid for all $x \in [-1,1[$. If we can show $f \in \mathrm{lip}_B(\lambda)$, it follows $\lim_{n\to\infty} \|f-f_n\|_L = 0$. Since $f_N \in \mathrm{lip}_B(\lambda)$, there exists $\delta > 0$ such that

$$\frac{\|T_x f_N - f_N\|_B}{(1-x)^\lambda} \leq \frac{\epsilon}{2}$$

for all $x \in [-1,1[$ with $|x-1| < \delta$. Hence we get

$$\frac{\|T_x f - f\|_B}{(1-x)^\lambda} \leq \frac{\|T_x(f-f_N)-(f-f_N)\|_B}{(1-x)^\lambda} + \frac{\|T_x f_N - f_N\|_B}{(1-x)^\lambda} < \epsilon$$

for all $x \in [-1,1[$ with $|x-1| < \delta$. In particular, we have

$$\lim_{x\to 1-} \frac{\|T_x f - f\|_B}{(1-x)^\lambda} = 0,$$

and so condition (B2) is shown. For proving (B1) notice that

$$\frac{\|T_x R_n - R_n\|_B}{(1-x)^\lambda} = \frac{|R_n(x)-1|}{(1-x)^\lambda}\,\|R_n\|_B = |(1-x)^{1-\lambda}P(x)|\,\|R_n\|_B$$

$$\leq C\,(1-x)^\lambda,$$

wherein $P(x)$ is the polynomial given by $P(x) = \frac{R_n(x)-1}{1-x}$, $x \neq 1$. Since $0 < \lambda < 1$, we have $R_n \in \mathrm{lip}_B(\lambda)$.

If $f \in \mathrm{lip}_B(\lambda)$ and $x, y \in S$, then

$$\|T_x(T_y f) - T_y f\|_B \leq \|T_x f - f\|_B,$$

and hence $T_y f \in \mathrm{lip}_B(\lambda)$ and $\|T_y f\|_L \leq \|f\|_L$. It remains to derive (B4). If $f \in \mathrm{lip}_B(\lambda)$, $y_0 \in S$ and $\epsilon > 0$, then there is a $\delta > 0$ such that

$$\frac{\|T_x f - f\|_B}{(1-x)^\lambda} \leq \frac{\epsilon}{2} \qquad \text{for all } x \in [-1,1[\quad \text{with } |x-1| < \delta,$$

and an $\eta > 0$ such that

$$\|T_y f - T_{y_0} f\|_B \;<\; \frac{\epsilon\delta^\lambda}{2} \qquad \text{for all } y \in S \quad \text{with } |y - y_0| < \eta.$$

Therefore we obtain for any $y \in S$

$$\frac{\|T_x(T_y f - T_{y_0} f) \,-(T_y f - T_{y_0} f)\|_B}{(1-x)^\lambda}$$

$$\leq \frac{\|T_y(T_x f - f)\|_B \,+\|T_{y_0}(T_x f - f)\|_B}{(1-x)^\lambda} \;\leq\; \frac{2\,\|T_x f - f\|_B}{(1-x)^\lambda} \;<\; \epsilon \quad (a)$$

for all $x \in [-1,1[$ with $|x-1| < \delta$. If $|x-1| \geq \delta$ then $\frac{1}{(1-x)^\lambda} \leq \frac{1}{\delta^\lambda}$, and hence

$$\frac{\|T_x(T_y f - T_{y_0} f) \,-(T_y f - T_{y_0} f)\|_B}{(1-x)^\lambda}$$

$$\leq \frac{\|T_x(T_y f - T_{y_0} f)\|_B \,+\|T_y f - T_{y_0} f\|_B}{(1-x)^\lambda} \;\leq\; \frac{2\,\|T_y f - T_{y_0} f\|_B}{(1-x)^\lambda} \;<\; \epsilon \quad (b)$$

for all $y \in S$ with $|y - y_0| < \eta$.

The inequalities (a) and (b) give

$$\sup_{x\in[-1,1[} \frac{\|T_x(T_y f - T_{y_0} f) \,-(T_y f - T_{y_0} f)\|_B}{(1-x)^\lambda} \;<\; \epsilon$$

for all $y \in S$, $|y - y_0| < \eta$, and the continuity of $y \mapsto T_y f$, $S \to \mathrm{lip}_B(\lambda)$ follows. □

3.4 Dirichlet spaces $U_B(S)$

If $f \in L^1(S,\pi)$ denote by

$$D_n f(x) \;=\; \sum_{k=0}^{n} \check{f}(k)\; R_k(x)\; h(k), \qquad\qquad n \in \mathbb{N}_0,$$

the n-th Fourier-Jacobi partial sum. Let B be a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$. Set

$$U_B(S) := \{f \in B : \lim_{n\to\infty} \|D_n f - f\|_B = 0\},$$

$$\|f\|_U := \sup_{n\in\mathbb{N}_0} \|D_n f\|_B$$

Notice that $R_k \in B$ implies $D_n f \in B$ for any $f \in L^1(S,\pi)$. Furthermore, $\|\,.\,\|_U$ is a norm. Since $(D_n f)_{n\in\mathbb{N}_0}$ converges, we have $\sup\limits_{n\in\mathbb{N}_0} \|D_n f\|_B < \infty$. If $\|f\|_U = 0$, then $D_n f = 0$ for all $n \in \mathbb{N}_0$, and hence $\check{f}(k) = 0$ for all $k \in \mathbb{N}_0$. The uniqueness theorem gives $f = 0$ in $L^1(S,\pi)$. The other properties of a norm are easily shown.

Proposition 3.13 *Let B be a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$ satisfying $\sup\limits_{n\in\mathbb{N}_0} \|R_n\|_B < \infty$. $U_B(S)$ is a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$.*

Proof. Since $D_n R_k = R_k$ for $n \geq k$ we have $R_k \in U_B(S)$. By $\|f\|_1 \leq \|f\|_B \leq \|f - D_n f\|_B + \|D_n f\|_B \leq \|f - D_n f\|_B + \|f\|_U$ it follows $\|f\|_1 \leq \|f\|_U$. $U_B(S)$ is also complete. In fact, if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $U_B(S)$, $\epsilon > 0$, then there is $N_1 \in \mathbb{N}$ such that $\sup\limits_{n\in\mathbb{N}} \|D_n f_k - D_n f_l\|_B < \epsilon$ for all $k,l \geq N_1$. Because of the inequality $\|f\|_B \leq \|f\|_U$, the sequence $(f_n)_{n\in\mathbb{N}}$ converges in B. Let $f = \lim\limits_{n\to\infty} f_n$ be the limit in B, and $N_2 \in \mathbb{N}$ such that $\|f - f_n\|_B < \epsilon$ for every $n \geq N_2$. Hence

$$|\check{f}_k(m) - \check{f}(m)| \;\leq\; \|f_k - f\|_1 \;\leq\; \|f_k - f\|_B \;< \epsilon$$

for all $k \geq N_2$ and $m \in \mathbb{N}_0$. Since $\sup\limits_{n\in\mathbb{N}_0} \|R_n\|_B = C < \infty$ we can moreover find for each $n \in \mathbb{N}$ a $K_n \in \mathbb{N}$ such that

$$\|D_n f_k - D_n f\|_B \;\leq\; C \left(\sum_{m=0}^{n} |\check{f}_k(m) - \check{f}(m)|\; h(m) \right) \;<\; \epsilon$$

for all $k \geq K_n$. We can assume $K_n \geq N := \max\{N_1, N_2\}$, and obtain for all $k \geq N$ and $n \in \mathbb{N}_0$

$$\|D_n(f - f_k) - (f - f_k)\|_B \;\leq\; \|D_n(f - f_{K_n})\|_B \;+\; \|D_n f_{K_n} - D_n f_k\|_B$$
$$+ \|f - f_k\|_B \;<\; 3\epsilon.$$

Since $f_N \in U_B(S)$ there is some $n_0 \in \mathbb{N}$ such that

$$\|D_n f_N - f_N\|_B \;<\; \epsilon$$

for $n \geq n_0$, and we get finally

$$\|D_n f - f\|_B \;\leq\; \|D_n(f - f_N) - (f - f_N)\|_B \;+\; \|D_n f_N - f_N\|_B \;<\; 4\epsilon$$

for $n \geq n_0$. Therefore $f \in U_B(S)$. We have also $f = \lim_{k\to\infty} f_k$ in $U_B(S)$. In fact,

$$\|D_n f - D_n f_k\|_B \;\leq\; \|D_n(f - f_k) - (f - f_k)\|_B \;+\; \|f - f_k\|_B \;<\; 4\epsilon$$

for $k \geq N$. So far conditions (B1) and (B2) are proved. For showing (B3) notice, that

$$D_n T_x f(y) \;=\; \sum_{k=0}^{n} \check{f}(k)\, R_k(x)\, R_k(y)\, h(k) \;=\; T_x D_n f(y)$$

and hence

$$\|D_n T_x f - T_x f\|_B \;=\; \|T_x(D_n f - f)\|_B \;\leq\; \|D_n f - f\|_B,$$

and so $T_x f \in U_B(S)$, if $f \in U_B(S)$. We get also

$$\|T_x f\|_U \;=\; \sup_{n\in\mathbb{N}_0} \|D_n T_x f\|_B \;\leq\; \sup_{n\in\mathbb{N}_0} \|D_n f\|_B \;=\; \|f\|_U.$$

It remains to prove that $x \mapsto T_x f, \quad S \to U_B(S)$ is continuous. If $f \in U_B(S)$, $x_0 \in S$ and $\epsilon > 0$, there is $N \in \mathbb{N}_0$ with $\|D_n f - f\|_B < \frac{\epsilon}{3}$ for all $n \geq N$ and $\delta_1 > 0$ such that $\|T_x f - T_{x_0} f\|_B < \frac{\epsilon}{3}$ for all $x \in S$, $|x - x_0| < \delta_1$. Hence

$$\|D_n(T_x f) - D_n(T_{x_0} f)\|_B \;=\; \|T_x(D_n f) - T_{x_0}(D_n f)\|_B$$

$$\leq\; \|T_x(D_n f) - T_x f\|_B \;+\; \|T_x f - T_{x_0} f\|_B \;+\; \|T_{x_0} f - T_{x_0}(D_n f)\|_B \;<\; \epsilon$$

for all $n \geq N$ and $x \in S$, $|x - x_0| < \delta_1$. Obviously there exists also a $\delta_2 > 0$ such that

$$\|D_n(T_x f) - D_n(T_{x_0} f)\|_B \;\leq\; C \left(\sum_{m=0}^{n} |\check{(T_x f)}(m) - \check{(T_{x_0} f)}(m)|\, h(m) \right) \;<\; \epsilon$$

for all $n = 0, ..., N$ and $x \in S$, $|x - x_0| < \delta_2$. Setting $\delta = \min\{\delta_1, \delta_2\}$ we get

$$\|T_x f - T_{x_0} f\|_U \;<\; \epsilon \qquad \text{for all } x \in S,\ |x - x_0| < \delta.$$

□

A further example of a homogeneous Banach space on S will be given in the next section.

4 The Jacobi differential operator and a Sobolev space

The continuous Jacobi translation gives rise to a Jacobi differential operator.

Let $(\alpha, \beta) \in J$ be a fixed pair of parameters. By a direct calculation one can show that

$$s_n := R_n'(1) = \frac{n(n+\alpha+\beta+1)}{\alpha+\beta+2}, \tag{4.1}$$

for $n \in \mathbb{N}_0$, see [8].

Denote by $\mathcal{D}(A) := \text{lin } \{R_n : n \in \mathbb{N}_0\}$ the linear span of the R_n in $L^2(S, \pi)$ and define $A := \mathcal{D}(A) \to L^2(S, \pi)$ by

$$A(R_n) = (1 + s_n)\, R_n \tag{4.2}$$

for $n \in \mathbb{N}_0$. Notify that for $f = \sum_{k=0}^{n} c_k R_k \in \mathcal{D}(A)$ we have

$$A(f) = f + \sum_{k=0}^{n} c_k R_k'(1)\, R_k$$

$$= f + \sum_{k=0}^{n} c_k R_k \lim_{y \to 1-} \frac{1 - R_k(y)}{1-y} = f + \lim_{y \to 1-} \frac{f - T_y f}{1-y}. \tag{4.3}$$

Lemma 4.1 *$A : \mathcal{D}(A) \to L^2(S, \pi)$ is a symmetric positive definite operator with point spectrum $\sigma_p(A) = \{1 + s_n : n \in \mathbb{N}_0\}$ and corresponding eigenfunctions R_n, $n \in \mathbb{N}_0$.*

Proof. If $f = \sum_{k=0}^{n} c_k R_k$, $g = \sum_{l=0}^{n} d_l R_l \in \mathcal{D}(A)$, we obtain

$$\langle Af, g \rangle = \langle f, g \rangle + \sum_{k=0}^{n} \sum_{l=0}^{n} s_k c_k\, \overline{d}_l\, \langle R_k, R_l \rangle$$

$$= \langle f, g \rangle + \sum_{k=0}^{n} s_k c_k\, \overline{d}_k\, \frac{1}{h(k)} = \langle f, Ag \rangle,$$

because of $s_k \in \mathbb{R}$. Since the s_n are strictly increasing, it follows

$$\langle Af, f \rangle = \sum_{k=0}^{n} |c_k|^2 (1 + s_k)\, \frac{1}{h(k)} \geq \sum_{k=0}^{n} |c_k|^2\, \frac{1}{h(k)} = \|f\|_2^2.$$

Clearly, $1+s_n$ are eigenvalues of A with eigenfunctions R_n. If $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenfunction $f = \sum_{k=0}^{n} c_k R_k \in \mathcal{D}(A),\ \ f \neq 0,$ then

$$0 \ = \ (A - \lambda\,\mathrm{id})f \ = \ \sum_{k=0}^{n} (1 + s_k - \lambda)\ c_k R_k.$$

The R_k are linear independent, and so $1 + s_k - \lambda = 0$ for all $k = 0, ..., n$. Since $f \neq 0$, at least one c_k is different from zero, and then $\lambda = 1 + s_k$. □

Proposition 4.2 *The operator $A : \mathcal{D}(A) \to L^2(S,\pi)$ is essentially self-adjoint. Its closure $\overline{A} = A^{**} : \mathcal{D}(\overline{A}) \to L^2(S,\pi)$ fulfils*

(1) $\mathcal{D}(\overline{A}) \ = \ \{f \in L^2(S,\pi) : \ \sum_{k=0}^{\infty} (1+s_k)^2\, |\check{f}(k)|^2\, h(k) \ < \infty\}$

(2) $\sigma(\overline{A}) \ = \ \sigma_p(\overline{A}) \ = \ \sigma_p(A) \ = \ \{1 + s_k : k \in \mathbb{N}_0\}$ *with corresponding eigenfunctions R_k.*

Proof. By [14, Satz 21.4] the operator A is essentially self-adjoint. Notify that the linear span of the R_n is dense in $L^2(S,\pi)$. The statements (1) and (2) are part of [14, Satz 21.1]. □

Now we consider a slight modification of $\overline{A}$. Denote by

$$\mathcal{D}(B) := \ \{f \in L^2(S,\pi) : \sum_{k=0}^{\infty} s_k^2\, |\check{f}(k)|^2\, h(k) \ < \infty\}$$

and $B : \mathcal{D}(B) \to L^2(S,\pi), \quad Bf := \overline{A}f - f, \quad f \in \mathcal{D}(B)$. Obviously $\mathcal{D}(B) = \mathcal{D}(\overline{A})$. Moreover we can show:

Proposition 4.3 *The operator $B : \mathcal{D}(B) \to L^2(S,\pi)$ is self-adjoint, and explicitly given by*

$$Bf \ = \ \sum_{k=0}^{\infty} s_k \check{f}(k)\ R_k\, h(k), \qquad\qquad f \in \mathcal{D}(B) \tag{4.4}$$

Further $B = \overline{A - id}$ holds true.

Proof. By [14, Lemma 17.2] it follows $B^* = (\overline{A} - \mathrm{id})^* = \overline{A}^* - \mathrm{id} = \overline{A} - \mathrm{id} = B$, i.e. B is self-adjoint. If $f \in \mathcal{D}(B)$ and $k \in \mathbb{N}_0$, we get

$$\widehat{Bf}(k) \ = \ \langle (\overline{A} - \ \mathrm{id})f, R_k \rangle \ = \ \langle f, \overline{A} R_k \rangle \ - \ \langle f, R_k \rangle$$

$$= (1+s_k)\,\hat{f}(k) - \hat{f}(k) = s_k\hat{f}(k)$$

Therefore we have shown the second statement.

Concerning the third statement notice that $A-\mathrm{id}$ is closable, since $A-\mathrm{id} \subseteq \overline{A}-\mathrm{id}$. Moreover, $B^* = (\overline{A}-\mathrm{id})^* = \overline{A}^*-\mathrm{id} = A^*-\mathrm{id} = (A-\mathrm{id})^*$, and hence $B = B^{**} = (A-\mathrm{id})^{**} = \overline{A-\mathrm{id}}$. □

Lemma 4.4

$$\sup_{x\in S} |R_n'(x)| = R_n'(1) = s_n \qquad \textit{for each } n \in \mathbb{N}_0$$

Proof. Denote by $T_n(x) = R_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = \cos(nt)$, $x = \cos t$, the Chebyshev polynomials of the first kind, and by $U_n(x) = R_n^{(\frac{1}{2},\frac{1}{2})}(x) = \dfrac{\sin((n+1)t)}{(n+1)\sin t}$, $x = \cos t$, the Chebyshev polynomials of the second kind. It is known that in the representation

$$R_n(x) = \sum_{k=0}^{n} c_{n,k} T_k(x)$$

the connection coefficients $c_{n,k}$ are nonnegative (provided $(\alpha,\beta) \in J$), see [9] or [1]. Since

$$R_n'(x) = \sum_{k=1}^{n} c_{n,k}\, k^2\, U_{k-1}(x),$$

we obtain

$$\sup_{x\in S} |R_n'(x)| = \sum_{k=0}^{n} c_{n,k}\, k^2 = \sum_{k=0}^{n} c_{n,k}\, T_k'(1) = R_n'(1).$$

□

We notify that the idea to apply in Lemma (4.4) the nonnegative connection coefficients is due to R. Szwarc.

Lemma 4.5 *If $f \in \mathcal{D}(B)$ then*

$$\lim_{x\to 1-} \|\frac{f - T_x f}{1-x} - Bf\|_2 = 0$$

Proof. For each $x \in S,\ x < 1,$ we have

$$\|\frac{f - T_x f}{1-x} - Bf\|_2^2 = \|\sum_{k=0}^{\infty}\left(\frac{1-R_k(x)}{1-x} - s_k\right)\check{f}(k)\,R_k\,h(k)\|_2^2$$

$$= \sum_{k=0}^{\infty}\left|\frac{1-R_k(x)}{1-x} - s_k\right|^2 |\check{f}(k)|^2\,h(k)$$

$$= \sum_{k=0}^{\infty}|R_k'(\varrho_k(x)) - R_k'(1)|^2\,|\check{f}(k)|^2\,h(k),$$

wherein $x < \varrho_k(x) < 1$. Using Lemma (4.4) , we have

$$|R_k'(\varrho_k(x)) - R_k'(1)|\,|\check{f}(k)| \le 2s_k\,|\check{f}(k)|$$

Now we can apply Lebesgue's theorem of dominated convergence to obtain the assertion. □

Next we introduce the Jacobi differential operator.

Definition 4.6 *Let* $(\alpha,\beta) \in J$. *Put*

$$H_2^{(1)}(S) := \{f \in L^2(S,\pi) : \lim_{x\to 1-}\frac{f - T_x f}{1-x} \text{ exists in } L^2(S,\pi)\}$$

and

$$D : H_2^{(1)}(S) \longrightarrow L^2(S,\pi), \qquad Df = \lim_{x\to 1-}\frac{f - T_x f}{1-x}. \tag{4.5}$$

We call D *the* **Jacobi differential operator** *with respect to* $(\alpha,\beta) \in J$.

We show now that $H_2^{(1)}(S) = \mathcal{D}(B)$ and that D coincides with B given by the identity (4.4).

By Lemma (4.5) we have $\mathcal{D}(B) \subseteq H_2^{(1)}(S)$. If $f \in \mathcal{D}(A)$, we know from (4.3) that $(A-\mathrm{id})(R_n) = DR_n$. Hence $A-\mathrm{id} \subseteq D$, and therefore $D^* \subseteq (A-\mathrm{id})^* = B^*$, as we have already noted in the proof of Proposition (4.3). Since D is a symmetric ooperator, it follows $D \subseteq D^* \subseteq B^* = B$, due to Proposition (4.3) .

Thus we have proven that $H_2^{(1)}(S) = \mathcal{D}(B)$ and $B = D$. Collecting all facts shown above, we have

Theorem 4.7 *The Jacobi differential operator* $D : H_2^{(1)}(S) \to L^2(S,\pi)$ *with respect to* $(\alpha,\beta) \in J$ *fulfils*

(1) $H_2^{(1)}(S) = \mathcal{D}(B) = \mathcal{D}(\overline{A})$.

(2) $Df = \lim_{x\to 1-} \dfrac{f - T_x f}{1-x} = \sum_{k=0}^{\infty} s_k \check{f}(k)\, R_k\, h(k) = \overline{A}f - f$ *for all* $f \in H_2^{(1)}(S)$.

(3) D *is a self-adjoint operator.*

(4) $\sigma(D) = \sigma_p(D) = \{s_k : k \in \mathbb{N}_0\}$ *with corresponding eigenfunctions* R_k.

Now we present a homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$, which we call a Sobolev space induced by D. Put

$$\|f\|_{2,1} := \|f\|_2 + \|Df\|_2, \qquad f \in H_2^{(1)}(S),$$

as norm.

Proposition 4.8 *$H_2^{(1)}(S)$ is a homogeneous Banach space on S with respect to $(\alpha, \beta) \in J$.*

Proof. Since $DR_n = s_n R_n \in L^2(S, \pi)$, the condition (B1) is satisfied. Obviously $\|f\|_1 \le \|f\|_2 \le \|f\|_{2,1}$. $H_2^{(1)}(S)$ is complete. In fact, if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H_2^{(1)}(S)$, then f_n and Df_n converge in $L^2(S,\pi)$ towards $f \in L^2(S,\pi)$ and $g \in L^2(S,\pi)$, respectively. Since D is a closed operator, it follows that $Df = g$. Now it is routine to show that $(f_n)_{n\in\mathbb{N}}$ converges in $H_2^{(1)}(S)$ towards $f \in H_2^{(1)}(S)$, and (B2) is checked. If $f \in H_2^{(1)}(S)$, $x \in S$, it follows $(T_x Df)\check{}(n) = R_n(x)(Df)\check{}(n) = R_n(x) s_n \check{f}(n) = (DT_x f)\check{}(n)$ for every $n \in \mathbb{N}_0$. Hence $T_x Df = DT_x f$, and so $T_x f \in H_2^{(1)}(S)$ and $\|T_x f\|_{2,1} \le \|f\|_{2,1}$. It remains to prove (B4). If $f \in H_2^{(1)}(S)$, $x_0 \in S$, $\epsilon > 0$, there exists $\delta > 0$ such that $\|T_x f - T_{x_0} f\|_2 < \epsilon/2$ and $\|DT_x f - DT_{x_0} f\|_2 = \|T_x Df - T_{x_0} Df\|_2 < \epsilon/2$ for all $x \in S$ with $|x - x_0| < \delta$. Hence $\|T_x f - T_{x_0} f\|_{2,1} < \epsilon$ for $x \in S$, $|x - x_0| < \delta$. □

It is obvious how to introduce Sobolev spaces based on higher derivatives D^k. We omit this subject.

Remark 4.9 *There is a close relation between D and the Jacobi differential equation satisfied by $R_n(x)$, see [13, Theorem 4.2.1]. It was our intention to introduce a differential operator which is based only on the translation operator without appealing to the well-known Jacobi differential operator given in [13, Theorem 4.2.1].*

5 Approximation for homogeneous Banach spaces

In [10] and [11] there are investigated various approximation methods for Jacobi expansions on the Banach spaces $L^p(S,\pi)$, $1 \le p < \infty$, and $C(S)$. These results can be more generally applied to homogeneous Banach spaces B on S with respect to $(\alpha,\beta) \in J$. In fact, choosing weights $a_{n,k} \in \mathbb{C}$, $n \in \mathbb{N}_0$, $k = 0,...,n$, define a summation kernel $(A_n)_{n\in\mathbb{N}_0}$ by

$$A_n(x) \;=\; \sum_{k=0}^{n} a_{n,k}\, R_k(x)\, h(k) \tag{5.1}$$

for $x \in S$, $n \in \mathbb{N}_0$. For $f \in B$ consider the Jacobi expansion

$$A_n f(x) \;=\; \sum_{k=0}^{n} a_{n,k}\, \check{f}(k)\, R_k(x)\, h(k) \tag{5.2}$$

for $x \in S$, $n \in \mathbb{N}_0$.

Obviously, $A_n f \in B$ and using B-valued integration we obtain

$$A_n * f \;=\; \int_{-1}^{1} A_n(x)\, L_x f \; d\pi(x) \;=\; \sum_{k=0}^{n} \left(\int_{-1}^{1} R_k(x)\, L_x f \; d\pi(x) \right) a_{n,k}\, h(k)$$

$$= \sum_{k=0}^{n} a_{n,k}\, \check{f}(k)\, R_k(\,.\,)\, h(k) \;=\; A_n f. \tag{5.3}$$

The following theorem is a consequence of the Banach-Steinhaus theorem, see e.g. [11, Theorem 1].

Theorem 5.1 *Let B be a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$ and assume that the linear span of the R_n is dense in B. We have $\lim_{n\to\infty} \|A_n f - f\|_B = 0$ for all $f \in B$ if and only if the conditions (A1) and (A2) hold, wherein*

(A1) $\lim_{n\to\infty} a_{n,k} \;=\; 1$ *for every $k \in \mathbb{N}_0$*

(A2) $\|A_n f\|_B \;\le\; C\, \|f\|_B$ *for all $f \in B$, $n \in \mathbb{N}_0$,*

where C is a constant independent of f and n.

Applying (5.3) and Proposition (3.2) we have

Corollary 5.2 *Let B be a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$. If (A1) holds and $\|A_n\|_1 \le C$ for all $n \in \mathbb{N}_0$, then $\lim_{n\to\infty} \|A_n f - f\|_B = 0$ for all $f \in B$.*

An example of $a_{n,k}$ for which the sufficient conditions of Corollary (5.2) hold, is already given in [11]. There we presented a Fejér-like kernel:

$$A_{2n}(x) \;=\; \sum_{k=0}^{2n} a_{2n,k}\, R_k(x)\, h(k),$$

where

$$a_{2n,k} \;=\; \sum_{j=0}^{n} \sum_{i=|k-j|}^{min(n,k+j)} g(k,j;i)\, h(j) \,/\, \sum_{j=0}^{n} h(j),$$

see [11].

Another example is the de la Vallée-Poussin-type kernel. Putting

$$a_{n,k} \;=\; \frac{n!\,\Gamma(n+\alpha+\beta+2)}{(n-k)!\,\Gamma(n+k+\alpha+\beta+2)},$$

set

$$A_n(x) \;=\; \sum_{k=0}^{n} a_{n,k}\, R_k(x)\, h(k).$$

From equality (2.30) in [1] and (4.3.4) in [13] we can derive that

$$A_n(x) \;=\; \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+2)\,\Gamma(\alpha+1)2^{\alpha+\beta+1}} \left(\frac{1+x}{2}\right)^n.$$

In particular $\|A_n\|_1 = a_{n,0} = 1$. The asymptotic properties of the Gamma function also imply that $\lim\limits_{n\to\infty} a_{n,k} = 1$ for all $k \in \mathbb{N}_0$. Therefore Corollary (5.2) can be applied.

References

[1] R. Askey, Orthogonal polynomials and special functions. SIAM, Philadelphia, 1975.

[2] W. R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups. de Gruyter, Berlin, 1995.

[3] G. Gasper, Banach algebras for Jacobi series and positivity of a kernel. Ann. of Math. **95**, 261-280, 1972.

[4] R. I. Jewett, Spaces with an abstract convolution of measures. Adv. in Math. **18**, 1-101, 1975.

[5] Y. Katznelson, Introduction to Harmonic Analysis. Wiley, New York, 1968.

[6] R. Lasser, Introduction to Fourier Series. Marcel Dekker, New York, 1996.

[7] R. Lasser, Orthogonal polynomials and hypergroups. Rend. Mat. **3**, 185-209, 1983.

[8] R. Lasser, On the Lévy-Hinčin formula for commutative hypergroups. In: Probability Measures on Groups VII, 298-308, LNS **1064**, Springer, Berlin, 1984.

[9] R. Lasser and M. Rösler, A note on property (T) of orthogonal polynomials. Archiv für Math. **60**, 459-463, 1993.

[10] R. Lasser and J. Obermaier, On Fejér means with respect to orthogonal polynomials: A hypergroup theoretic approach. In: Progress in Approximation Theory, 551-565. Academic Press, Boston, 1991.

[11] R. Lasser and J. Obermaier, On the convergence of weighted Fourier expansions with respect to orthogonal polynomials. Acta Sci. Math. **61**, 345-355, 1995.

[12] H. Reiter, L^1-Algebras and Segal Algebras. LNS **231**, Springer, New York, 1971.

[13] O. Szegö, Orthogonal Polynomials. Amer. Math. Soc., New York, 1959.

[14] H. Triebel, Höhere Analysis. VEB, Berlin, 1972.

[15] H. C. Wang, Homogeneous Banach Algebras. Marcel Dekker, New York, 1977.

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Properties of Fejér Means of Fourier Series with respect to Unbounded Vilenkin Systems

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Abstract. It is a highly celebrated problem in dyadic harmonic analysis the pointwise convergence of the Fejér (or $(C,1)$) means of functions on unbounded Vilenkin groups. We give a résumé of the very recent developments concerning this matter. Besides, we prove that the maximal operator $\sup|\sigma_{M_n}|$ is of type $(L(\log^+ L)^{a+1}, L(\log^+ L)^a)$ for any $a \geq 0$ on unbounded Vilenkin groups.

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1 Introduction, some known results

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [28, 1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \dots\}, \mathbb{P} := \mathbb{N} \setminus \{0\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, with the group operation mod m_k addition. Since the group is discrete, then every subset is open. The normalized Haar measure

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on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ $(j \in \{0, 1, ..., m_k - 1\})$. Let

$$G_m := \overset{\infty}{\underset{k=0}{\times}} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m.

Furthermore, let $L^p(G_m)$ $(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$ $(f \in L^1)$.

Let a be a nonnegative real. We say that the function $f \in L^1(G_m)$ belongs to the logarithm space $L(\log^+ L)^a(G_m)$ if the integral

$$\|f\|_{L(\log^+ L)^a} := \int_{G_m} |f(x)| \left(\log^+(|f(x)|)\right)^a d\mu(x)$$

is finite. The positive logarithm $\log^+$ is defined as

$$\log^+(x) := \begin{cases} \log x & \text{if } x > \exp(1), \\ 1 & \text{otherwise.} \end{cases}$$

The definiton of $\log^+$ by $\log x \vee 1$ and not by $\log x \vee 0$ is a bit unusual. But just a little, and the two definition give the same space. For some technical reason later, in the paper it is more convenient the use $\log x \vee 1$.

Let X and Y be either $L(\log^+ L)^a(G_m)$ or $L^p(G_m)$ for some $1 \leq p \leq \infty$, and $a \geq 0$ with norms $\|.\|_X$ and $\|.\|_Y$. We say that operator T is of type (X, Y) if there exist an absolute constant $C > 0$ for which $\|Tf\|_Y \leq C\|f\|_X$ for all $f \in X$. If $X = Y = L^p(G_m)$ then we often say that T is of type (p, p) instead of type (L^p, L^p). T is of weak type (L^1, L^1) (or weak type $(1, 1)$) if there exist an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \leq C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps a function f to the maximal function f^* is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \leq \infty$ (see e.g. [3]).

Let $M_0 := 1, M_{n+1} := m_n M_n \, (n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\}, \ i \in \mathbb{N}),$$

where only a finite number of n_i 's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp(2\pi \imath \frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & , \text{ if } x_n \neq 0, \\ m_n & , \text{ if } x_n = 0 \end{cases}$ $(x \in G_m, \, n \in \mathbb{N})$. The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then , $\psi_{k \oplus n} = \psi_k \psi_n, \ \psi_n(x + y) = \psi_n(x)\psi_n(y), \ \psi_n(-x) = \bar{\psi}_n(x), \ |\psi_n| = 1 \ (k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect

to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y,x) = D_n(y-x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f,$$

$$K_n(y,x) = K_n(y-x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(y-x),$$

$$(n \in \mathbb{P}, y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f,\ S_0 f = D_0 = K_0 = 0,\ f \in L^1(G_m)).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x),$$

$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y-x) d\mu(x)$$

$$(n \in \mathbb{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0) \end{cases},$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Next we introduce some notation with respect to the theory of two-dimensional Vilenkin systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstood.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\hat{f}(n_1,n_2) := \int_{G_m\times G_{\tilde{m}}} f(x^1,x^2)\bar{\psi}_{n_1}(x^1)\bar{\psi}_{n_2}(x^2)d\mu(x^1,x^2),$$

$$S_{n_1,n_2}f(y^1,y^2) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\hat{f}(k_1,k_2)\psi_{k_1}(y^1)\psi_{k_2}(y^2),$$

$$D_{n_1,n_2}(y,x) = D_{n_1}(y^1-x^1)D_{n_2}(y^2-x^2)$$

$$:= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\psi_{k_1}(y^1)\psi_{k_2}(y^2)\bar{\psi}_{k_1}(x^1)\bar{\psi}_{k_2}(x^2),$$

$$\sigma_{n_1,n_2}f := \frac{1}{n_1n_2}\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}S_{k_1,k_2}f,$$

$$K_{n_1,n_2}(y,x) = K_{n_1,n_2}(y-x) := \frac{1}{n_1n_2}\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}D_{k_1,k_2}(y-x),$$

$$(y=(y^1,y^2),x=(x^1,x^2)\in G_m\times G_{\tilde{m}}).$$

It is also well-known that

$$\sigma_{n_1,n_2}f(y) = \int_{G_m\times G_{\tilde{m}}} f(x)K_{n_1,n_2}(y-x)d\mu(x),$$

$$S_{M_{n_1},\tilde{M}_{n_2}}f(x) = M_{n_1}\tilde{M}_{n_2}\int_{I_{n_1}(x^1)\times I_{n_2}(x^2)} f = (E^1_{n_1}\otimes E^2_{n_2})f(x).$$

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on one and two-dimensional unbounded Vilenkin groups.

Fine [5] proved every Walsh-Fourier series (in the Walsh case $m_j = 2$ for all $j\in\mathbb{N}$) is a.e. (C,α) summable for $\alpha>0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [17]. Schipp [23] gave a simpler proof for the case $\alpha=1$, i.e. $\sigma_n f\to f$ a.e. $(f\in L^1(G_m))$. He proved that σ^* is of weak type (L^1,L^1). That σ^* is bounded from H^1 to L^1 was discovered by Fujii [7].

The theorem of Schipp are generalized to the p-series fields ($m_j = p$ for all $j\in\mathbb{N}$) by Taibleson [26], and later to bounded Vilenkin systems by Pál and Simon [20].

Now, what about the Vilenkin groups with unbounded generating sequences? The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [21]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [21] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [21] that if $\frac{\log m_n}{M_n} \to \infty$, then there exists a function f continuous on G_m whose Fourier series are not $(C,1)$ summable on a set $S \subset G_m$ which is non-denumerable. On the other hand, Nurpeisov gave [19] a necessary and sufficient condition of the uniform convergence of the Fejér means $\sigma_{M_n} f$ of continuous functions on unbounded Vilenkin groups. Namely, define the uniform modulus of continuity as

$$\omega_n(f) := \sup_{h \in I_n(0), x \in G_m} |f(x+h) - f(x)|.$$

Nurpeisov proved [19]: A necessary and sufficient condition that the means $\sigma_{M_n} f$ of the Fourier series of the continuous function f converge uniformly to f on an unbounded Vilenkin group for all such an f is that

$$\omega_{n-1}(f) \log(m_n) = o(1).$$

Since the uniform modulus of continuity can be any nonincreasing real sequence which converges to zero (for the proof see [22, 6]), then as a consequence of this it is possible to give a sequence m increasing enough fast, and a function even in the Lipschitz class Lip(1), such that the M_nth Fejér means do not converge to the function uniformly.

So, it seems that it is impossible to give a (Hölder) function class such that the uniform convergence of the Fejér means would hold for all functions in this class if there is no condition on sequence m at all.

On the other hand, mean convergence of the full partial sums for $L^p, p > 1$, is known for the unbounded case. For the proof see [24]. This trivially implies the norm convergence $\sigma_n f \to f$ for all $f \in L^p$, where $1 < p < \infty$.

Concerning the a.e. convergence we can say a bit more. Namely, in 1999 the author [10] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to

the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups.

In 2001 Simon proved [25] the following theorem with respect to the Fejér means of L^1 functions. A sequence m is said to be strong quasi-bounded if

$$\frac{1}{M_{n+1}} \sum_{j=0}^{n-1} M_{j+1} < C \log m_n.$$

Then every bounded m is quasi-bounded, and there are also some unbounded ones. Let m be strong quasi-bounded. Then for all $f \in L^1(G_m)$

$$\sigma_{M_n} f(x) - f(x) = o(\max(\log m_0, \dots, \log m_{n-1})).$$

Later, in 2003, the author of this paper improved [12] this result, and gave a partial answer for L^1 case. He discussed this partial sequence of the sequence of the Fejér means. Namely, if $f \in L^1(G_m)$, then he proved (see [12]) that $\sigma_{M_n} f \to f$ almost everywhere, where m is any sequence. In my opinion, it is highly likely that the methods of the papers [10, 12] can be applied and improved in order to prove the a.e. relation $\sigma_n f \to f$ for all $f \in L \log^+ L$ and m. Anyway, it is not an easy task...

What can be said in the case of two-dimensional functions? This is "another story". For double trigonometric Fourier series Marcinkiewicz and Zygmund [16] proved that $\sigma_{m,n} f \to f$ a.e. as $m, n \to \infty$ provided the integral lattice points (m, n) remain in some positive cone, that is provided $\beta^{-1} \leq m/n \leq \beta$ for some fixed parameter $\beta \geq 1$. It is known that the classical Fejér means are dominated by decreasing functions whose integrals are bounded but this fails to hold for the one-dimensional Walsh-Fejér kernels. This growth difference is exacerbated in higher dimensions so that the trigonometric techniques are not powerful enough for the Walsh case.

In 1992 Móricz, Schipp and Wade [18] proved that $\sigma_{2^{n_1},2^{n_2}} f \to f$ a.e. for each $f \in L^1([0,1)^2)$, when $n_1, n_2 \to \infty$, $|n_1 - n_2| \leq \alpha$ for some fixed α. Later, Gát and Weisz proved (independently, in the same year) this for the whole sequence, that is, the theorem of Marcinkiewicz and Zygmund with respect to the Walsh-Paley system (see [9] and [30]). For the bounded Vilenkin case see the paper of Weisz [32], and the paper of Blahota and the author [2]. In the paper [2] they generalizes this theorem with respect to two-dimensional bounded Vilenkin-like systems.

If we do not provide a "cone restriction" for the indices in $\sigma_{n,k} f$ that is, we discuss the convergence of this two-dimensional Fejér means in the Pringsheim sense, then the situation changes. In 1992 Móricz, Schipp and

Wade [18] proved with respect to the Walsh-Paley system that $\sigma_{n,k}f \to f$ a.e. for each $f \in L\log^+ L([0,1)^2)$, when $\min\{n,k\} \to \infty$. Later, in 2002 Weisz generalized [31] this with respect to two-dimensional bounded Vilenkin systems.

In 2000 Gát proved [11] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved the existence of a function $f \in L^1([0,1)^2)$ such that $f \in L\log^+ L\delta(L)$, and $\sigma_{n,k}f$ does not converge to f a.e. as $\min\{n,k\} \to \infty$.

What can be said in the two-dimensional case with respect to unbounded Vilenkin systems? In 1997 Wade proved [29] the following. Let $\beta_{k,j} := \max\{m_0,\dots,m_{k-1},\tilde{m}_0,\dots,\tilde{m}_{j-1}\}$. The sequence m is called δ-quasi bounded, $0 \le \delta < 1$, if the sums

$$\sum_{j=0}^{n-1} m_j/(m_{j+1}\dots m_n)^\delta$$

are (uniformly) bounded. Let the generating sequences $m, \tilde{m}$ be δ-quasi bounded. Then for all $f \in L^1(G_m \times G_{\tilde{m}})$ we have

$$\sigma_{M_n,\tilde{M}_k} f(x) - f(x) = o(\beta_{n,k}\beta^{2r}_{n+r,k+r}),$$

as $n,k \to \infty$, provided that $|n-k| < \alpha$, where $\alpha, r \in \mathbb{N}$ are some constants for almost every $x \in G_m \times G_{\tilde{m}}$.

On the other hand, there was nothing concerning the pointwise convergence before the following manuscript of the author. In [13] he proved the following theorem. Let $f \in (L\log^+ L)(G_m \times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1},\tilde{M}_{n_2}} f \to f$ almost everywhere, where $\min\{n_1,n_2\} \to \infty$ provided that the distance of the indices is bounded, that is, $|n_1 - n_2| < \alpha$ for some fixed constant $\alpha > 0$. Here it is necessary to emphasize that in this paper $m, \tilde{m}$ can be any sequences.

It seems also to be interesting to discuss the almost everywhere convergence of Marcinkiewicz means $\frac{1}{n}\sum_{j=0}^{n-1} S_{j,j}f$ of integrable functions on two-dimensional unbounded Vilenkin groups. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. For the trigonometric, Walsh-Paley, and bounded Vilenkin case see the papers of Zhizhiasvili, Weisz and Gát [35, 33, 8]. With respect to the Walsh case see also the papers of Goginava [15, 14]. Some of the results summarized in this paper (including the proofs) can also be found in [34].

2 The theorem

At last, we prove a $(L(\log^+ L)^{a+1}, L(\log^+ L)^a)$ type inequality with respect to the one-dimensional Fejér means of integrable function on unbounded Vilenkin groups.
Define the maximal operator $\sigma^\dagger f := \sup_{n\in\mathbb{N}} |\sigma_{M_n} f|$, where f is an integrable function.

Theorem 2.1 *Let $f \in L(\log^+ L)^{a+1}$, and $a \geq 0$. Then we have*

$$\|\sigma^\dagger f\|_{L(\log^+ L)^a} \leq C_a(\|f\|_{L(\log^+ L)^{a+1}} + 1).$$

In order to prove Theorem 2.1 we need some assumptions with respect to the following operator: For an integrable function f we define:

$$H_1 f(y) := \sup_{A\in\mathbb{N}} \left| M_{A-1} \int_{I_A(y)\setminus I_{A+1}(y)} f(x) \frac{1}{1 - r_{A-1}(y-x)} d\mu(x) \right|.$$

A modified version of the lemma below can be found in the books of Torchinsky [27, p. 92], Weisz [34, p. 12] and also in the book (without proof) of Bennett and Sharpley [4, p. 254].

Lemma 2.2 *Let $f \in L(\log^+ L)^{a+1}$, and $a \geq 0$. Then we have*

$$\|H_1 f\|_{L(\log^+ L)^a} \leq C_a \|f\|_{L(\log^+ L)^{a+1}} + C_a.$$

Proof. In [12] we proved that the operator H_1 is of type (L^2, L^2), and of weak type (L^1, L^1). This was the base of the proof of the a.e. convergence relation $\sigma_{M_n} f \to f$.

The distribution function of a measurable function $f : G_m \to \mathbb{C}$ is given by:

$$\mu_f(\lambda) := \mu\{x \in G_m : |f(x)| > \lambda\} \quad (\lambda \geq 0).$$

The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) := \inf\{\lambda : \mu_f(\lambda) \leq t\} \quad (t \geq 0).$$

We use the convention that $\inf \emptyset = +\infty$. Thus, if $\mu_f(\lambda) > t$ for all $\lambda \geq 0$, then $f^*(t) = \infty$. Since μ_f is bounded by 1, and so $f^*(t) = 0$ for all $t \geq 1$, we can regard f^* as a function defined on the interval $[0, 1]$.

For the definition of f^* see e.g. the book of Bennett and Sharpley. [4, p. 36-39].

Since f and f^* are equimeasurable [4, p. 41], and the function $x \to \log^+ x$ is a nonnegative continuous increasing function, then we have

$$\int_{G_m} |f(x)|^p \left[\log^+ |f(x)|\right]^a d\mu(x) = \int_0^1 f^*(t)^p \left[\log^+(f^*(t))\right]^a dt$$

for all $1 \le p$, $0 \le a$. We will use the following inequalities

$$\begin{aligned}
&\int_0^1 f^*(t) \left(\log^+ \left(\frac{1}{t}\right)\right)^a dt \\
&\le \int_{\left\{f^*(t) > t^{-1/2}\right\}} f^*(t) \left(\log^+(f^*(t)^2)\right)^a dt \\
&+ \int_{\left\{f^*(t) \le t^{-1/2}\right\}} t^{-1/2} \left(\log^+(1/t)\right)^a dt \\
&\le C_a \int_0^1 f^*(t) \left(\log^+(f^*(t))\right)^a dt + \int_0^1 t^{-1/2} \left(\log^+ \left(\frac{1}{t}\right)\right)^a dt \\
&\le C_a \|f\|_{L(\log^+ L)^a} + C_a.
\end{aligned} \tag{2.1}$$

And also the following one. Since for any $f \in L(G_m)$ integrable function and $t > 0$ we have $f^*(t) \le \|f\|_1/t$, then assuming $\|f\|_1 > 0$ we have

$$\begin{aligned}
&\|f\|_{L(\log^+ L)^a} = \int_0^1 f^*(t) \left(\log^+(f^*(t))\right)^a dt \\
&\le \int_0^1 f^*(t) \left[\log^+ \left(\frac{\|f\|_1}{t}\right)\right]^a dt \\
&= \int_0^{\min(\|f\|_1/\exp(1),1)} f^*(t) \left[\log^+ \left(\frac{\|f\|_1}{t}\right)\right]^a dt \\
&+ \int_{\min(\|f\|_1/\exp(1),1)}^1 f^*(t) \left[\log^+ \left(\frac{\|f\|_1}{t}\right)\right]^a dt \\
&\le \int_0^1 f^*(t) \left[\log(\|f\|_1) + \log \left(\frac{1}{t}\right)\right]^a dt + \int_0^1 f^*(t) dt \\
&\le C_a \int_0^1 f^*(t) \left(\log^+ \left(\frac{1}{t}\right)\right)^a dt + C_a \|f\|_1 |\log(\|f\|_1)|^a + \|f\|_1.
\end{aligned} \tag{2.2}$$

Since the operator H_1 is of weak type $(1,1)$ and of type $(2,2)$ [12], then we have [4, p. 223]

$$(H_1 f)^*(t) \le C t^{-1} \int_0^t f^*(s) ds + C t^{-1/2} \int_t^1 s^{-1/2} f^*(s) ds. \tag{2.3}$$

Inequalities (2.2) and (2.3) imply:

$$\begin{aligned}
&\|H_1 f\|_{L(\log^+ L)^a} \le C_a \|H_1 f\|_1 \left(1 + |\log \|H_1 f\|_1|^a\right) \\
&+ C_a \int_0^1 (H_1 f)^*(t) (\log^+(1/t))^a dt \\
&=: s_1 + s_2.
\end{aligned}$$

Discuss the second addend by the help of (2.1):

$$\begin{aligned}
s_2 &\le C \int_0^1 \left(t^{-1} \int_0^t f^*(s) ds \right) (\log^+(1/t))^a dt \\
&+ C \int_0^1 t^{-1/2} \int_t^1 s^{-1/2} f^*(s)\, ds\, (\log^+(1/t))^a dt \\
&= C \int_0^1 f^*(s) \left(\int_s^1 \frac{1}{t} \left(\log^+(1/t)\right)^a dt \right) ds \\
&+ C \int_0^1 f^*(s) s^{-1/2} \int_0^s t^{-1/2} \left(\log^+(1/t)\right)^a dt ds \\
&\le C_a \int_0^1 f^*(s) \left(\log^+(1/s)\right)^{a+1} ds \\
&+ C_a \int_0^1 f^*(s) \left(\log^+(1/s)\right)^a ds \\
&\le C_a \int_0^1 f^*(s) \left(\log^+(1/s)\right)^{a+1} ds \\
&\le C_a (\|f\|_{L(\log^+ L)^{a+1}} + 1).
\end{aligned}$$

It is remained to give the same bound for addend s_1. Since the operator H_1 is of type (L^2, L^2) [12], and of weak type (L^1, L^1). In the book of Bennett and Sharpley [4, p. 243-249] one can find that an operator of this kind satisfies the following inequality

$$\|H_1 f\|_1 \le C \|f\|_{L \log^+ L} + C. \tag{2.4}$$

Apply this inequality. Then we get

$$\begin{aligned}
&s_1 = C_a \|H_1 f\|_1 \left(1 + |\log \|H_1 f\|_1|^a\right) \\
&\le C_a \left(\|f\|_{L \log^+ L} + \exp(1) \right) \left(1 + \left| \log C_a \left(\|f\|_{L \log^+ L} + \exp(1) \right) \right|^a \right) \\
&\le C_a \left(\|f\|_{L \log^+ L} + \exp(1) \right) \left(1 + \left| \log \left(\|f\|_{L \log^+ L} + \exp(1) \right) \right|^a \right) \\
&C_a \left(\|f\|_{L \log^+ L} + \exp(1) \right) \left(1 + \left(\log^+ \left(\|f\|_{L \log^+ L} \right) \right)^a \right).
\end{aligned}$$

Since the function $x \to x(\log^+ x)^a$ is convex on $[0, +\infty)$, then we have

$$s_1 \leq C_a + C_a \left(\|f\|_{L \log^+ L} \right) \left(\log^+ \left(\|f\|_{L \log^+ L} \right) \right)^a \leq C_a + C_a \|f\|_{L(\log^+ L)^{a+1}}.$$

This completes the proof of the Lemma. □

Proof of Theorem 2.1.

For any $1 \leq j \in \mathbb{N}$ define the operator H_j in the following way

$$\begin{aligned} &H_j f(y) \\ &:= \sup_{j \leq A \in \mathbb{N}} \left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x) \right. \\ &\left. \times \frac{1}{1 - r_{A-j}(y-x)} \, d\mu(x) \right|, \end{aligned}$$

where $y \in G_m$. Hereinafter, we prove for the operator H_j:

$$\|H_j f\|_{L(\log^+ L)^a} \leq C_a \frac{j}{2^j} \left(\|f\|_{L(\log^+ L)^{a+1}} + 1 \right). \tag{2.5}$$

for all $f \in L(\log^+ L)^{a+1}$. The proof applies Lemma 2.2 for a modified Vilenkin group. We apply a finite permutation for the coordinate groups of the Vilenkin group G_m such that for all $A \geq j, A \in \mathbb{N}$ the $A-j$th coordinate group and the $A-1$st coordinate group will be adjacent. Then we use the inequality in Lemma 2.2 for the modified group. Introduce the operators $H_{j,k}$ and $H_{j,k}^N$ as follows:

$$\begin{aligned} &H_j f(y) \\ &\leq \sup_{k=0,\dots,j-1} \sup_{\substack{j \leq A \in \mathbb{N} \\ A \equiv k \mod j}} \left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x) \right. \\ &\left. \times \frac{1}{1 - r_{A-j}(y-x)} \, d\mu(x) \right| \\ &=: \sup_{k=0,\dots,j-1} H_{j,k} f(y), \end{aligned}$$

and

$$\begin{aligned} &H_{j,k}^N f(y) \\ &:= \sup \Big\{ \Big| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x) \\ &\times \frac{1}{1 - r_{A-j}(y-x)} \, d\mu(x) \Big| : j \leq A \leq Nj + k, A \equiv k \mod j \Big\}. \end{aligned}$$

It is easy to have that $\|H_j f\|_{L(\log^+ L)^a} \leq \sum_{k=0}^{j-1} \|H_{j,k} f\|_{L(\log^+ L)^a}$. Since $H_{j,k}^N f$ is monotone increasing as N gets larger, then by the Beppo-Levi theorem we get that if we prove for the operators $H_{j,k}^N$ satisfy $\|H_{j,k}^N f\|_{L(\log^+ L)^a} \leq C_a(\|f\|_{L(\log^+ L)^{a+1}} + 1)/2^j$ (uniformly in N - it means that the constant C_a does not depend on N, j, k), then $H_{j,k}$ is also of this type. This would imply

$$\begin{aligned}
&\|H_j f\|_{L(\log^+ L)^a} \\
&\leq \sum_{k=0}^{j-1} \|H_{j,k} f\|_{L(\log^+ L)^a} \\
&\leq \sum_{k=0}^{j-1} \frac{C_a}{2^j} (\|f\|_{L(\log^+ L)^{a+1}} + 1) \\
&\leq \frac{C_a j}{2^j} \|f\|_{L(\log^+ L)^{a+1}}.
\end{aligned}$$

That is, the proof of inequality (2.5) would be complete.

Recall that the Vilenkin group G_m is the complete direct product of its coordinate groups Z_{m_l}, that is, $G_m = \overset{\infty}{\underset{l=0}{\times}} Z_{m_l}$. We define another Vilenkin group. Its coordinate groups will be the same, but with certain rearrangement. Let the function $\alpha : \mathbb{N} \to \mathbb{N}$ be defined in the following way. If $n \geq k + Nj$, or $n \not\equiv k, k-1 \mod j$, then

$$\alpha(n) := n,$$

and

$$\alpha(k + lj) := k + (l+1)j - 1, \quad \alpha(k + (l+1)j - 1) := k + lj,$$

for all $l < N, l \in \mathbb{N}$. Then define the Vilenkin group $G_m^{j,k}$ as:

$$G_m^{j,k} = \overset{\infty}{\underset{l=0}{\times}} Z_{m_{\alpha(l)}}.$$

We give a measure preserving bijection between the two Vilenkin groups. We denote it by β, or more precisely (if it is needed) by $\beta_{j,k}$. It will not cause any confusion. That is,

$$\beta = \beta_{j,k} : G_m \to G_m^{j,k},$$

and let the n^{th} coordinate of the sequence $\beta_{j,k}(x) \in G_m^{j,k}$ be $x_{\alpha(n)}$. That is,

$$(\beta(x))_n = x_{\alpha(n)} \quad (n \in \mathbb{N}).$$

Consequently, we have a finite permutation of the coordinates. This is very important for us, since when we discuss the operator H_1 on the Vilenkin group $G_m^{j,k}$, then we can apply the result given (H_1 satisfies (2.5)) for the operator $H_{j,k}^N$ on the Vilenkin group G_m.

Denote by $\tilde{m}$ the sequence for which $\tilde{m}_l = m_{\alpha(l)}$. Introduce the notations $\tilde{x} := \beta(x)\,(x \in G_m)$, $\tilde{r}_l := r_{\alpha(l)}\,(l \in \mathbb{N})$. Recall that $A \equiv k \mod j$. Then we have

$$\begin{aligned}
&1 - r_{A-j}(y-x) = 1 - \exp\left(2\pi\imath \frac{y_{A-j} - x_{A-j}}{m_{A-j}}\right)\\
&= 1 - \exp\left(2\pi\imath \frac{\tilde{y}_{A-1} - \tilde{x}_{A-1}}{m_{A-j}}\right)\\
&= 1 - \exp\left(2\pi\imath \frac{\tilde{y}_{A-1} - \tilde{x}_{A-1}}{\tilde{m}_{A-1}}\right)\\
&= 1 - \tilde{r}_{A-1}(\tilde{y} - \tilde{x}).
\end{aligned}$$

Moreover, denote by $\tilde{M}$ the sequence of the generalized powers with respect to the sequence $\tilde{m}$. This gives

$$\begin{aligned}
&\tilde{M}_{A-1}\\
&= \tilde{m}_0 \dots \tilde{m}_{A-2}\\
&= m_0 m_1 \dots m_{A-j-1} m_{A-j+1} \dots m_{A-1}\\
&= \frac{m_0 \dots m_{A-1}}{m_{A-j}}\\
&= M_{A-j} \frac{m_{A-j} m_{A-j+1} \dots m_{A-1}}{m_{A-j}}\\
&= M_{A-j} m_{A-j+1} \dots m_{A-1}.
\end{aligned}$$

This gives $M_{A-j} \le \tilde{M}_{A-1}/2^{j-1}$. The equalities above give

$$\begin{aligned}
&\left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x) \frac{1}{1 - r_{A-j}(y-x)}\, d\mu(x) \right|\\
&= \left| M_{A-j} \int_{\bigcup_{\tilde{x}_{A-1} \neq \tilde{y}_{A-1}} I_A(\tilde{y}_0,\dots,\tilde{y}_{A-2},\tilde{x}_{A-1})} \tilde{f}(\tilde{x}) \frac{1}{1 - \tilde{r}_{A-1}(\tilde{y} - \tilde{x})}\, d\mu(\tilde{x}) \right|\\
&\le \frac{1}{2^{j-1}} H_1 \tilde{f}(\tilde{y}),
\end{aligned}$$

where the function $\tilde{f}$ is defined on $G_m^{j,k}$ by $f(x) = \tilde{f}(\tilde{x})$ for all $x \in G_m$. The

definition of $H^N_{j,k}$ gives

$$H^N_{j,k}f(y) \le \frac{1}{2^{j-1}} H_1\tilde{f}(\tilde{y}).$$

By Lemma 2.2, and by the fact that $\log^+$ is monotone we have

$$\|H^N_{j,k}f\|_{L(\log^+ L)^a} \le \frac{1}{2^{j-1}}\|H_1 f\|_{L(\log^+ L)^a} \le C_a \frac{1}{2^j}\left(\|f\|_{L(\log^+ L)^{a+1}} + 1\right).$$

That is, we have proved inequality (2.5). In [12, Proof of Theorem 2.1] one can read

$$|\sigma_{M_A} f(y)| \le |f|^*(y) + \sum_{j=1}^{\infty} H_j f(y),$$

where f is any integrable function, and $f^* := \sup_n |E_n f|$. That is, for the maximal operator $\sigma^\dagger := \sup_A |\sigma_{M_A}|$ we have

$$\sigma^\dagger f \le |f|^* + \sum_{j=1}^{\infty} H_j f.$$

This inequality gives

$$\|\sigma^\dagger f\|_{L(\log^+ L)^a} \le C_a \||f|^*\|_{L(\log^+ L)^{a+1}} + C_a \|\sum_{j=1}^{\infty} H_j f\|_{L(\log^+ L)^a} =: I + II.$$

It is easy to have that $I \le C_a\|f\|_{L(\log^+ L)^{a+1}}$. To investigate II is a bit more difficult. Mainly, we use inequality (2.5). Since the function $x \to x(\log^+ x)^a$ is strictly monotone increasing and convex, then we have

$$\sum_{j=2^L}^{2^{L+1}-1} 2^{j/2-2^{L+2}} H_j f \left(\log^+\left(\sum_{j=2^L}^{2^{L+1}-1} 2^{j/2-2^{L+2}} H_j f)\right)\right)^a$$
$$\le \sum_{j=1}^{\infty} 2^{j/2-2^L} H_j f \left(\log^+ H_j f\right)^a.$$

Thus,

$$\left\| \sum_{j=2^L}^{2^{L+1}-1} 2^{j/2-2^{L+2}} H_j f \left(\log^+\left(\sum_{j=1}^{\infty} 2^{j/2-2^{L+2}} H_j f)\right)\right)^a \right\|_1$$
$$\le C_a \sum_{j=1}^{\infty} 2^{j/2-2^L} \|H_j f\|_{L(\log^+ L)^a}$$
$$\le C_a 2^{-2^L}(\|f\|_{L(\log^+ L)^{a+1}} + 1).$$

Consequently, we get

$$
\begin{aligned}
&C_a(\|f\|_{L(\log^+ L)^{a+1}}+1)\\
&\geq \left\| \sum_{j=2^L}^{2^{L+1}-1} 2^{j/2} H_j f \left(\log^+ \left(\sum_{j=2^L}^{2^{L+1}-1} 2^{j/2} H_j f) \right) \right)^a \right\|_1\\
&- \left\| \sum_{j=2^L}^{2^{L+1}-1} (2^{j/2}-2^L) H_j f \right\|_1\\
&\geq \| \sum_{j=2^L}^{2^{L+1}-1} 2^{j/2} H_j f\|_{L(\log^+ L)^{a+1}} - C\| \sum_{j=2^L}^{2^{L+1}-1} 2^{j/2} H_j f\|_1\\
&\geq \| \sum_{j=2^L}^{2^{L+1}-1} 2^{j/2} H_j f\|_{L(\log^+ L)^{a+1}} - C \sum_{j=2^L}^{2^{L+1}-1} 2^{-j/2}(\|f\|_{L(\log^+ L)^1}+1)\\
&\geq C2^{2^{L-1}}\| \sum_{j=2^L}^{2^{L+1}-1} H_j f\|_{L(\log^+ L)^{a+1}} - C\|f\|_{L(\log^+ L)^1} - C.
\end{aligned}
$$

In other words,

$$
2^{2^{L-1}}\| \sum_{j=2^L}^{2^{L+1}-1} H_j f\|_{L(\log^+ L)^{a+1}} \leq C_a(\|f\|_{L(\log^+ L)^{a+1}}+1).
$$

By the application of this inequality we have for II

$$
\begin{aligned}
II &= \| \sum_{L=1}^{\infty} \sum_{j=2^L}^{2^{L+1}-1} H_j f\|_{L(\log^+ L)^a} \leq C_a\| \sum_{j=2^0}^{2^1-1} H_j f\|_{L(\log^+ L)^a}\\
&+ C_a^2\| \sum_{j=2^1}^{2^2-1} H_j f\|_{L(\log^+ L)^a} + C_a^3\| \sum_{j=2^2}^{2^3-1} H_j f\|_{L(\log^+ L)^a} + \dots\\
&\leq C_a(\|f\|_{L(\log^+ L)^{a+1}}+1),
\end{aligned}
$$

since the sum $\sum_{L=1}^{\infty} C_a^L 2^{-2^{L-1}}$ is a convergent one. This means the proof of Theorem 2.1 is complete. □

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References

[1] G.H. Agaev, N.Ja. Vilenkin, G.M. Dzhafarli, and A.I. Rubinstein, *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups*, Izd.("ELM"), Baku, 1981 (Russian).

[2] I. Blahota and G. Gát, *Pointwise convergence of double Vilenkin-Fejér means*, Stud. Sci. Math. Hungar. **36** (2000), 49–63.

[3] D.L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probability **1** (1973), 19–42.

[4] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Inc., Orlando, Florida, 1988.

[5] N.J. Fine, *Cesàro summability of Walsh-Fourier series*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 558–591.

[6] S. Fridli, *On the modulus of continuity with respect to functions defined on Vilenkin groups*, Acta Mat. Hungar. **45** (1985), no. 3-4, 393–396.

[7] N. Fujii, *A maximal inequality for H^1 functions on the generalized Walsh-Paley group*, Proc. Amer. Math. Soc. **77** (1979), 111–116.

[8] G. Gát, *Convergence of Marcinkiewicz means of integrable functions with respect to two-dimensional Vilenkin systems*, Georgian Math. Journal, to appear.

[9] G. Gát, *Pointwise convergence of the Cesàro means of double Walsh series*, Ann. Univ. Sci. Budap. Rolando Eoetvoes, Sect. Comput. **16** (1996), 173–184.

[10] G. Gát, *Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups*, Journal of Approximation Theory **101** (1999), no. 1, 1–36.

[11] G. Gát, *On the divergence of the $(C,1)$ means of double Walsh-Fourier series*, Proc. Am. Math. Soc. **128** (2000), no. 6, 1711–1720.

[12] G. Gát, *Cesàro means of integrable functions with respect to unbounded Vilenkin systems*, Journal of Approximation Theory **124** (2003), no. 1, 25–43.

[13] G. Gát, *On the pointwise convergence of Cesàro means of two-variable functions with respect to unbounded Vilenkin systems*, Journal of Approximation Theory **128** (2004), no. 1, 69–99.

[14] U. Goginava, *Almost everywhere summability of multiple Fourier series*, Math. Anal. and Appl. **287** (2003), no. 1, 90–100.

[15] U. Goginava, *Pointwise convergence of the Marcinkiewicz means of double Walsh series*, Bull. Georgian Acad. Sci. **161** (2000), no. 3, 382-384.

[16] J. A. Zygmund, J. Marcinkiewicz, *On the summability of double Fourier series*, Fund. Math. **32** (1939), 122–132.

[17] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann Soc. Polon. Math. **16** (1937), 85–96.

[18] F. Móricz, F. Schipp, and W.R. Wade, *Cesàro summability of double Walsh-Fourier series*, Trans Amer. Math. Soc. **329** (1992), 131–140.

[19] R.Ž. Nurpeisov, *On the uniform* $(C, 1)$ *summability of Fourier series and integrability of series with respect to multiplicative orthonormal systems*, Analysis Math. **15** (1989), 127–143.

[20] J. Pál and P. Simon, *On a generalization of the concept of derivative*, Acta Math. Acad. Sci. Hungar. **29** (1977), 155–164.

[21] J. Price, *Certain groups of orthonormal step functions*, Canadian J. Math. **9** (1957), 413–425.

[22] A.I. Rubinshtein, *On the modulus of continuity of functions defined on 0-dimesnional groups*, Mat. Zametki **23** (1978), 379–388.

[23] F. Schipp, *Über gewiessen Maximaloperatoren*, Annales Univ. Sci. Budapestiensis , Sectio Math. **18** (1975), 189–195.

[24] F. Schipp, *On* L^p*-norm convergence of series with respect to product systems*, Analysis Math. **2** (1976), 49–63.

[25] P. Simon, *Cesàro means of Vilenkin-Fourier series*, Publ. Math. **59** (2001), no. 1-2, 203-219.

[26] M.H. Taibleson, *Fourier Series on the Ring of Integers in a p-series Field*, Bull. Amer. Math. Soc. **73** (1967), 623–629.

[27] A. Torchinsky *Real-variable Methods in Harmonic Analysis*, Dover Publications, Inc. New York, 2004.

[28] N.Ja. Vilenkin, *On a class of complete orthonormal systems*, Izv. Akad. Nauk. SSSR, Ser. Math. **11** (1947), 363–400 (Russian).

[29] W.R. Wade, *Growth of Cesàro means of double Vilenkin-Fourier series of unbounded type*, Analysis of divergence. Control and management of divergent processes. Proceedings of the 7th international workshop in analysis and its applications, IWAA, Orono, ME, USA June 1-6, 1997. Boston, MA: Birkhauser. Applied and Numerical Harmonic Analysis (1999), 41–50.

[30] F. Weisz, *Cesàro summability of two-dimensional Walsh-Fourier series*, Trans. Amer. Math. Soc. **348** (1996), 2169–2181.

[31] F. Weisz, *Summability results of Walsh- and Vilenkin-Fourier series*, Functions, Series, Operators – Alexits Memorial Conference, Budapest (Hungary) 1999 (2002), 443–464.

[32] F. Weisz, *Hardy spaces and Cesàro means of two-dimensional Fourier series*, Bolyai Soc. Math. Studies, **5**, Approximation Theory and Function Series, Budapest (Hungary), 1995, Budapest, (1996), 353-367.

[33] F. Weisz, *Convergence of double Walsh-Fourier series and Hardy spaces*, Appr. Theory Appl. **17** (2001), 32-44.

[34] F. Weisz, *Summability of multi-dimensional Fourier series and Hardy spaces. Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London. 2002.

[35] L.V. Zhizhiasvili, *Generalization of a theorem of Marcinkiewicz*, Izv. Akad. nauk USSR Ser Mat. **32** (1968), 1112–1122 (Russian).

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The Rate of Convergence by Certain New Meyer-König and Zeller Operators of Finite Type

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Abstract. We introduce a new sequence $(L_n)_{n\in\mathbf{N}}$ of Meyer-König and Zeller operators of finite type, and we investigate its approximation properties. By deriving a sharp estimate of the second moment of the L_n-operators, we prove both a theorem on the rate of convergence by the first modulus of continuity and a Voronovskaja-type formula. Furthermore, we study the rate of convergence for a Durrmeyer version of the L_n-operators.

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1 Introduction

Starting from the identity

$$(1-x)^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{k}x^k=1,\qquad x\in[0,1[, \tag{1.1}$$

W. Meyer-König and K. Zeller [8] have defined a sequence of linear positive operators, associating to each continuous real-valued function defined on $[0,1]$, a so-called "Bernstein power series." In the slight modification by E. W. Cheney and A. Sharma [4], the Meyer-König and Zeller

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operators are given by

$$M_n f(1) = f(1), \tag{1.2}$$

$$M_n f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \qquad x \in [0,1[, \tag{1.3}$$

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}, \qquad k = 0, 1, 2, \ldots, \tag{1.4}$$

where $n \in \mathbf{N}$, the set of positive integers, and $f \in C[0,1]$, the space of continuous real-valued functions defined on $[0,1]$. It is well-known that the sequence $(M_n f)_{n\in\mathbf{N}}$ converges uniformly to f on $[0,1]$ whenever $f \in C[0,1]$. For approximation properties of the sequence $(M_n)_{n\in\mathbf{N}}$ the reader is referred to the paper by A. Lupaş and M. W. Müller [6].

If the series in (1.3) is truncated after the first $n+1$ terms, then we obtain another sequence of linear positive operators, associating to each function $f \in C[0,1]$ the sequence $(\underline{M}_n f)_{n\in\mathbf{N}}$, of polynomials defined by

$$\underline{M}_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n+k}\right) m_{n,k}(x). \tag{1.5}$$

Unlike the Meyer-König and Zeller operators, the sequence $(\underline{M}_n f)_{n\in\mathbf{N}}$ does not converge uniformly to f on $[0,1]$ whenever $f \in C[0,1]$. A simple explanation would be that, unlike the series (1.3), only arguments of f lying between 0 and $\frac{1}{2}$ are taken into consideration in the sum (1.5). So a natural question arising here is how to complete the sum (1.5), in order to obtain a sequence of operators with good approximation properties? A possible answer to this question is suggested by the following combinatorial identity:

$$\sum_{k=0}^{n} m_{n,k}(x) + \sum_{k=0}^{n} m_{n,k}(1-x) = 1, \qquad x \in \mathbf{R}. \tag{1.6}$$

The reader is referred to [10] for a simple probabilistic proof of (1.6).

Starting from (1.6), we introduce the sequence $(L_n)_{n\in\mathbf{N}}$, of linear positive operators defined by

$$\begin{aligned} L_n f(x) \;=\; & \sum_{k=0}^{n} f\left(\frac{k}{n+k}\right) m_{n,k}(x) \\ & + \sum_{k=0}^{n} f\left(\frac{n}{n+k}\right) m_{n,k}(1-x), \qquad x \in [0,1]. \end{aligned} \tag{1.7}$$

Recently, U. Abel, V. Gupta, and M. Ivan [1] have investigated a Durrmeyer counterpart of the M_n-operators. With a slight modification in the notation, their operators are defined by

$$\widetilde{M}_n f(x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t) f(t)dt, \qquad x \in [0,1],$$
$$b_{n,k}(t) = (n+1)\binom{n+k+1}{k} t^k (1-t)^n, \qquad k = 0, 1, 2, \ldots.$$

Inspired by the $\widetilde{M}_n$-operators, we also consider a Durrmeyer version of the L_n-operators, defined by

$$\begin{aligned} \widetilde{L}_n f(x) &= \sum_{k=0}^{n} m_{n,k}(x) \int_0^1 b_{n,k}(t) f(t)dt \\ &\quad + \sum_{k=0}^{n} m_{n,k}(1-x) \int_0^1 b_{n,k}(1-t) f(t)dt. \end{aligned} \tag{1.8}$$

The main purpose of the present paper is to investigate the approximation properties of the (seemingly new) sequences $(L_n)_{n\in\mathbf{N}}$ and $(\widetilde{L}_n)_{n\in\mathbf{N}}$. In Section 2 the relationship between the L_n-operators and the classical M_n-operators is established. It enables us to transfer at once the qualitative properties of the M_n-operators to the L_n-operators. Quantitative properties of the L_n-operators, including an estimate on the rate of convergence by the first modulus of continuity and a Voronovskaja-type formula, are obtained in Section 3. Similar results, but involving the $\widetilde{L}_n$-operators, are proved in Section 4.

2 Auxiliary results and qualitative properties of the L_n- and $\widetilde{L}_n$-operators

In order to derive the main qualitative properties of the L_n-operators we will use the following combinatorial identity: for all positive integers n and N such that $N \geq n+1$ and for all $x \in \mathbf{R}$ it holds that

$$\sum_{k=0}^{N-n-1} m_{n,k}(x) = \sum_{k=n+1}^{N} p_{N,k}(1-x), \tag{2.1}$$

where $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$. This identity can be easily proved by induction on N.

Lemma 2.1. *For every nonnegative integer r, every $n \in \mathbf{N}$ such that $n \geq r$, and every $x \in \mathbf{R}$ the following equalities hold true:*

$$\sum_{k=0}^{n-r} m_{n,k}(x) + \sum_{k=0}^{n} m_{n-r,k}(1-x) = 1, \tag{2.2}$$

$$\sum_{k=0}^{n-r} \frac{k}{n+k}\, m_{n,k}(x) + \sum_{k=0}^{n} \frac{n-r}{n-r+k}\, m_{n-r,k}(1-x) = x, \tag{2.3}$$

$$\sum_{k=0}^{n-r} \frac{n}{n+k}\, m_{n,k}(x) + \sum_{k=0}^{n} \frac{k}{n-r+k}\, m_{n-r,k}(1-x) = 1-x. \tag{2.4}$$

Proof. Taking into account (2.1) and the fact that $p_{N-k,k}(1-x) = p_{N,k}(x)$, we deduce that

$$\begin{aligned}
&\sum_{k=0}^{n-r} m_{n,k}(x) + \sum_{k=0}^{n} m_{n-r,k}(1-x) \\
&\quad = \sum_{k=n+1}^{2n+1-r} p_{2n+1-r,k}(1-x) + \sum_{k=n-r+1}^{2n+1-r} p_{2n+1-r,k}(x) \\
&\quad = \sum_{k=0}^{n-r} p_{2n+1-r-k,k}(1-x) + \sum_{k=n-r+1}^{2n+1-r} p_{2n+1-r,k}(x) \\
&\quad = \sum_{k=0}^{n-r} p_{2n+1-r,k}(x) + \sum_{k=n-r+1}^{2n+1-r} p_{2n+1-r,k}(x) \\
&\quad = \sum_{k=0}^{2n+1-r} p_{2n+1-r,k}(x) = 1.
\end{aligned}$$

Likewise, using (2.2) we have

$$\begin{aligned}
&\sum_{k=0}^{n-r} \frac{k}{n+k} m_{n,k}(x) + \sum_{k=0}^{n} \frac{n-r}{n-r+k} m_{n-r,k}(1-x) \\
&= x \sum_{k=0}^{n-r-1} \binom{n+k}{k} x^k (1-x)^{n+1} + x \sum_{k=0}^{n} \binom{n-r-1+k}{k} (1-x)^k x^{n-r} \\
&= x \left[\sum_{k=0}^{n-r-1} m_{n,k}(x) + \sum_{k=0}^{n} m_{n-r-1,k}(1-x) \right] = x.
\end{aligned}$$

The equality (2.4) follows from (2.2) and (2.3). □

For $r = 0$, the equalities (2.2) and (2.3) ensure that

$$L_n e_0 = e_0 \qquad \text{and} \qquad L_n e_1 = e_1 \qquad \text{for all } n \in \mathbf{N}, \tag{2.5}$$

where e_k $(k = 0, 1, 2, \ldots)$ denotes, as usual, the monomial $e_k(t) = t^k$.

Next we establish the relationship between the L_n-operators and the classical M_n-operators. Given an arbitrary function $f : [0,1] \to \mathbf{R}$, let $\underline{f} : [0,1] \to \mathbf{R}$ and $\overline{f} : [0,1] \to \mathbf{R}$ be the functions defined by

$$\underline{f}(x) = \begin{cases} f(x) & \text{if} \quad 0 \le x \le \frac{1}{2} \\ 0 & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

and

$$\overline{f}(x) = \begin{cases} f(1-x) & \text{if} \quad 0 \le x \le \frac{1}{2} \\ 0 & \text{if} \quad \frac{1}{2} < x \le 1, \end{cases}$$

respectively. Then we have

$$\underline{f}(x) + \overline{f}(1-x) = \begin{cases} f(x) & \text{if} \quad x \ne \frac{1}{2} \\ 2f\left(\frac{1}{2}\right) & \text{if} \quad x = \frac{1}{2}. \end{cases} \tag{2.6}$$

Lemma 2.2. *For every function $f : [0,1] \to \mathbf{R}$, every $n \in \mathbf{N}$, and every $x \in [0,1]$ it holds that*

$$L_n f(x) = M_n \underline{f}(x) + M_n \overline{f}(1-x). \tag{2.7}$$

Proof. Straightforward. □

By means of (2.5) and (2.7) the qualitative properties of the classical M_n-operators can be immediately transferred to the L_n-operators.

Theorem 2.3. *The following assertions are true:*

1° *For every function $f \in C[0,1]$, the sequence $(L_n f)_{n \in \mathbf{N}}$ converges uniformly to f on $[0,1]$.*

2° *If $f \in C[0,1]$ is nondecreasing, then $L_n f$ is also nondecreasing for all $n \in \mathbf{N}$.*

2° *If $f \in C[0,1]$ is convex, then $L_n f$ is also convex and it holds that $L_n f \ge L_{n+1} f \ge f$ for all $n \in \mathbf{N}$.*

Proof. We will prove only 1° and the inequality $L_n f \ge L_{n+1} f$ in 3°. The other assertions can be proved analogously.

1° Let $f \in C[0,1]$ and let $g := f - f\left(\frac{1}{2}\right)$. Then $g\left(\frac{1}{2}\right) = 0$, hence $\underline{g} \in C[0,1]$ and $\overline{g} \in C[0,1]$. By the well-known approximation properties of the

M_n-operators, it follows that $M_n\underline{g}(x) \longrightarrow \underline{g}(x)$ and $M_n\overline{g}(1-x) \longrightarrow \overline{g}(1-x)$ uniformly on $[0,1]$. Using now (2.7) and (2.6) we deduce that

$$L_n g(x) = M_n\underline{g}(x) + M_n\overline{g}(1-x) \longrightarrow \underline{g}(x) + \overline{g}(1-x) = g(x),$$

uniformly on $[0,1]$. But $L_n g(x) = L_n f(x) - f\left(\frac{1}{2}\right)$, according to (2.5). Therefore $L_n f(x) - f\left(\frac{1}{2}\right) \longrightarrow f(x) - f\left(\frac{1}{2}\right)$ uniformly on $[0,1]$. This proves assertion 1°.

3° Let $y = \alpha x + \beta$ be a support line to the graph of f at the point $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$, and let $g \in C[0,1]$ be the function defined by $g(x) = f(x) - \alpha x - \beta$. Then g is convex, $g \geq 0$, and $g\left(\frac{1}{2}\right) = 0$. Consequently, both $\underline{g}$ and $\overline{g}$ are convex. By [6, Theorem 3.3] it follows that

$$M_n\underline{g}(x) \geq M_{n+1}\underline{g}(x) \qquad \text{and} \qquad M_n\overline{g}(1-x) \geq M_{n+1}\overline{g}(1-x)$$

for all $x \in [0,1]$. Taking into account (2.7) and (2.6) we deduce that

$$L_n g(x) \geq L_{n+1} g(x) \qquad \text{for all } x \in [0,1].$$

But, according to (2.5), we have

$$L_n g(x) = L_n f(x) - \alpha x - \beta \qquad \text{and} \qquad L_{n+1} g(x) = L_{n+1} f(x) - \alpha x - \beta.$$

Consequently, it holds that $L_n f \geq L_{n+1} f$. □

Theorem 2.4. *If $f : [0,1] \to \mathbf{R}$ is a nondecreasing function, then $\widetilde{L}_n f$ is also nondecreasing for all $n \in \mathbf{N}$.*

Proof. Taking into account that

$$m'_{n,0}(x) = -(n+1)m_{n-1,0}(x)$$

and that

$$m'_{n,k}(x) = \frac{(n+k)(n+k-1)}{n}\, m_{n-1,k-1}(x) - \frac{(n+k+1)(n+k)}{n}\, m_{n-1,k}(x)$$

for $k \geq 1$, it is easily seen that

$$\begin{aligned}
&(\widetilde{L}_n f)'(x) \\
&\quad = \sum_{k=0}^{n-1} \frac{(n+k+1)(n+k)}{n}\, m_{n-1,k}(x) \int_0^1 [b_{n,k+1}(t) - b_{n,k}(t)] f(t)\,dt \\
&\qquad - \sum_{k=0}^{n-1} \frac{(n+k+1)(n+k)}{n}\, m_{n-1,k}(1-x) \times \\
&\qquad\qquad \times \int_0^1 [b_{n,k+1}(1-t) - b_{n,k}(1-t)] f(t)\,dt.
\end{aligned}$$

It remains to show that for all $k \in \{0, 1, \ldots, n-1\}$, the following inequalities hold true:

$$\int_0^1 [b_{n,k+1}(t) - b_{n,k}(t)] f(t)dt \geq 0, \tag{2.8}$$

$$\int_0^1 [b_{n,k+1}(1-t) - b_{n,k}(1-t)] f(t)dt \leq 0. \tag{2.9}$$

It suffices to consider only the case when f is differentiable on $[0,1]$. Since

$$b_{n,k+1}(t) - b_{n,k}(t) = -(n+1)(n+k+1)! \left(\frac{t^{k+1}}{(k+1)!} \cdot \frac{(1-t)^{n+1}}{(n+1)!} \right)',$$

integration by parts yields

$$\int_0^1 [b_{n,k+1}(t) - b_{n,k}(t)] f(t)dt$$
$$= (n+1)(n+k+1)! \int_0^1 \frac{t^{k+1}}{(k+1)!} \cdot \frac{(1-t)^{n+1}}{(n+1)!} f'(t)dt.$$

This proves the validity of (2.8). The inequality (2.9) can be proved analogously. □

Remark 2.5. Unfortunately, the $\widetilde{L}_n$-operators do not preserve convexity. For instance,

$$\widetilde{L}_2 e_1(x) = \frac{1}{4} + \frac{9}{20}\, x + \frac{3}{20}\, x^2 - \frac{1}{10}\, x^3$$

has an inflexion point at $x = \frac{1}{2}$.

3 The rate of convergence by the L_n-operators

It should be noted that **optimal** quantitative properties of the L_n-operators (e.g. degree of approximation) cannot be deduced directly from the corresponding properties of the M_n-operators, via (2.7). Such quantitative properties are established in the present section.

As we will see later, in estimating the degree of approximation by both the L_n-operators and the $\widetilde{L}_n$-operators, the key role is played by the following sum:

$$S_n(x,m,r) := \sum_{k=0}^{n-r} \frac{1}{n+m+r+k}\, m_{n,k}(x) + \sum_{k=0}^{n} \frac{1}{n+m+k}\, m_{n-r,k}(1-x),$$

where m and r are nonnegative integers, $n \in \mathbf{N}$ satisfies $n \geq r$, and $x \in [0,1]$. Likewise, throughout the rest of the paper we will use the following notation:

$$\phi(x) := \max(x,\, 1-x) = \begin{cases} 1-x & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ x & \text{if} \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

Lemma 3.1. *Let m and r be non-negative integers and let $n \in \mathbf{N}$ satisfying $n > r$ and $m(n-r) \geq r^2$. Then for each $x \in [0,1]$ it holds that*

$$\begin{aligned} \frac{\phi(x)}{n+m+r} \quad &\leq \quad S_n(x,m,r) \qquad (3.1) \\ &\leq \frac{2n-r-1}{(n-r)(2n+m-1)}\,\phi(x) + \frac{\binom{2n-r}{n} x^{n-r}(1-x)^n}{2n+m}. \end{aligned}$$

Proof. Taking into account (2.3), for every $x \in [0,1]$ we have

$$\begin{aligned} &S_n(x,m,r) - \frac{x}{n+m+r} \\ &\quad = \frac{(1-x)^{n+1}}{n+m+r} + \sum_{k=1}^{n-r} \left[\frac{\binom{n+k}{k}}{n+m+r+k} - \frac{\binom{n+k-1}{k-1}}{n+m+r} \right] x^k (1-x)^{n+1} \\ &\qquad + \sum_{k=0}^{n} \left[\frac{\binom{n-r+k}{k}}{n+m+k} - \frac{\binom{n-r+k-1}{k}}{n+m+r} \right] (1-x)^k x^{n-r+1}. \end{aligned}$$

But

$$\frac{\binom{n+k}{k}}{n+m+r+k} - \frac{\binom{n+k-1}{k-1}}{n+m+r} \geq 0 \qquad \text{for all } k = 1, \ldots, n-r$$

and

$$\frac{\binom{n-r+k}{k}}{n+m+k} - \frac{\binom{n-r+k-1}{k}}{n+m+r} \geq 0 \qquad \text{for all } k = 0, 1, \ldots, n.$$

It follows that

$$\frac{x}{n+m+r} \leq S_n(x,m,r) \qquad \text{for all } x \in [0,1]. \qquad (3.2)$$

Taking now into account (2.4), for every $x \in [0,1]$ we have

$$\begin{aligned} &S_n(x,m,r) - \frac{1-x}{n+m+r} \\ &\quad = \frac{x^{n-r+1}}{n+m} + \sum_{k=0}^{n-r} \left[\frac{\binom{n+k}{k}}{n+m+r+k} - \frac{\binom{n+k-1}{k}}{n+m+r} \right] x^k (1-x)^{n+1} \\ &\qquad + \sum_{k=1}^{n} \left[\frac{\binom{n-r+k}{k}}{n+m+k} - \frac{\binom{n-r+k-1}{k-1}}{n+m+r} \right] (1-x)^k x^{n-r+1}. \end{aligned}$$

But

$$\frac{\binom{n+k}{k}}{n+m+r+k}-\frac{\binom{n+k-1}{k}}{n+m+r}\geq 0 \qquad \text{for all } k=0,1,\ldots,n-r$$

and

$$\frac{\binom{n-r+k}{k}}{n+m+k}-\frac{\binom{n-r+k-1}{k-1}}{n+m+r}\geq 0 \qquad \text{for all } k=1,\ldots,n.$$

It follows that

$$\frac{1-x}{n+m+r}\leq S_n(x,m,r) \qquad \text{for all } x\in[0,1]. \tag{3.3}$$

From (3.2) and (3.3) we conclude that the lower estimate in (3.1) holds true.

In order to prove the upper estimate in (3.1), assume that $0\leq x\leq \frac{1}{2}$ (the case $\frac{1}{2}\leq x\leq 1$ can be treated analogously). We have

$$\begin{aligned} S_n(x,m,r) &= \frac{1-x}{n}\sum_{k=0}^{n-r-1}\frac{n+k}{n+m+r+k}\,m_{n-1,k}(x) \\ &\quad +\frac{x}{n-r}\sum_{k=0}^{n-1}\frac{n-r+k}{n+m+k}\,m_{n-r-1,k}(1-x) \\ &\quad +\frac{\binom{2n-r}{n}x^{n-r}(1-x)^n}{2n+m}. \end{aligned}$$

Since

$$\begin{aligned} &\frac{n+k}{n+m+r+k}\leq\frac{2n-r-1}{2n+m-1} \qquad \text{for all } k=0,1,\ldots,n-r-1, \\ &\frac{n-r+k}{n+m+k}\leq\frac{2n-r-1}{2n+m-1} \qquad \text{for all } k=0,1,\ldots,n-1, \end{aligned}$$

we get

$$\begin{aligned} S_n(x,m,r) &\leq (1-x)\,\frac{2n-r-1}{n(2n+m-1)}\sum_{k=0}^{n-r-1}m_{n-1,k}(x) \\ &\quad +x\,\frac{2n-r-1}{(n-r)(2n+m-1)}\sum_{k=0}^{n-1}m_{n-r-1,k}(1-x) \\ &\quad +\frac{\binom{2n-r}{n}x^{n-r}(1-x)^n}{2n+m}. \end{aligned}$$

Using (2.2) and taking into account that $x \leq 1-x = \phi(x)$, we conclude that

$$\begin{aligned} &S_n(x,m,r) \\ &\leq \frac{2n-r-1}{(n-r)(2n+m-1)}\,\phi(x)\left[\sum_{k=0}^{n-r-1} m_{n-1,k}(x) + \sum_{k=0}^{n-1} m_{n-r-1,k}(1-x)\right] \\ &\quad + \frac{\binom{2n-r}{n}x^{n-r}(1-x)^n}{2n+m} \\ &= \frac{2n-r-1}{(n-r)(2n+m-1)}\,\phi(x) + \frac{\binom{2n-r}{n}x^{n-r}(1-x)^n}{2n+m}. \end{aligned}$$

This completes the proof of (3.1). □

Proposition 3.2. *For every $n \in \mathbf{N}$, $n \geq 2$ and every $x \in [0,1]$ it holds that*

$$L_n e_2(x) = x^2 + x(1-x)S_{n-1}(x,2,0), \tag{3.4}$$

$$L_n e_2(x) = x^2 + x(1-x)\phi(x)r_n^{[2]}(x), \tag{3.5}$$

where

$$\frac{1}{n+1} \leq r_n^{[2]}(x) \leq \frac{1}{n}\left(1 + \frac{1}{\sqrt{2n-1}}\right). \tag{3.6}$$

Proof. Elementary calculations show that

$$\begin{aligned} L_n e_2(x) &= x^2\left[\sum_{k=0}^{n-2} m_{n,k}(x) + \sum_{k=0}^{n} m_{n-2,k}(1-x)\right] \\ &+ x(1-x)\left[\sum_{k=0}^{n-1}\frac{1}{n+k+1}m_{n-1,k}(x) + \sum_{k=0}^{n-1}\frac{1}{n+k+1}m_{n-1,k}(1-x)\right]. \end{aligned}$$

Taking into account (2.2), it follows that (3.4) holds true. Next we prove that

$$\frac{\phi(x)}{n+1} \leq S_{n-1}(x,2,0) \leq \frac{\phi(x)}{n}\left(1 + \frac{1}{\sqrt{2n-1}}\right). \tag{3.7}$$

Indeed, according to Lemma 3.1, we have

$$\frac{\phi(x)}{n+1} \leq S_{n-1}(x,2,0) \leq \frac{2n-3}{(n-1)(2n-1)}\,\phi(x) + \frac{\binom{2n-2}{n-1}x^{n-1}(1-x)^{n-1}}{2n},$$

hence the lower estimate in (3.7) holds true. In order to prove the upper estimate, note that $\frac{2n-3}{(n-1)(2n-1)} < \frac{1}{n}$ and that

$$\begin{aligned}\frac{\binom{2n-2}{n-1}x^{n-1}(1-x)^{n-1}}{2n} &= x(1-x)\cdot\frac{1}{2n}\cdot\frac{(2n-2)!}{(n-1)!^2}\cdot[x(1-x)]^{n-2}\\ &\leq \frac{1}{2}\,\phi(x)\cdot\frac{1}{2n}\cdot\frac{(2n-2)!}{(n-1)!^2}\cdot\frac{1}{2^{2n-4}}\\ &= \frac{\phi(x)}{n}\cdot\frac{(2n-2)!}{2^{2n-2}(n-1)!^2}\leq\frac{\phi(x)}{n}\cdot\frac{1}{\sqrt{2n-1}}.\end{aligned}$$

Consequently, it holds that

$$\frac{2n-3}{(n-1)(2n-1)}\,\phi(x)+\frac{\binom{2n-2}{n-1}x^{n-1}(1-x)^{n-1}}{2n}\leq\frac{\phi(x)}{n}+\frac{\phi(x)}{n\sqrt{2n-1}}.$$

This proves the upper estimate in (3.7). The relations (3.4) and (3.7) ensure now that (3.5) and (3.6) hold true. □

From (3.5) and (3.6) it follows that

$$L_n e_2(x)-x^2\leq\frac{4x(1-x)\phi(x)}{3n}$$

for all $x\in[0,1]$ and all $n\in\mathbf{N}$, $n\geq 5$. This estimate, the relation (2.5), and a standard argument (see, for instance, [2, Theorem 5.1.2]) enable us to state the main result of this section, giving the local order of approximation in terms of the first modulus of continuity.

Theorem 3.3. *For every $n\in\mathbf{N}$, $n\geq 5$, every $f\in C[0,1]$, and all $x\in[0,1]$ it holds that*

$$|L_n f(x)-f(x)|\leq\left(1+\frac{2}{\sqrt{3}}\right)\omega\left(f;\sqrt{\frac{x(1-x)\phi(x)}{n}}\right).$$

Moreover, if f is differentiable on $[0,1]$ and f' is bounded on $[0,1]$, then it holds that

$$|L_n f(x)-f(x)|\leq\frac{4+2\sqrt{3}}{3}\sqrt{\frac{x(1-x)\phi(x)}{n}}\,\omega\left(f';\sqrt{\frac{x(1-x)\phi(x)}{n}}\right).$$

We end this section with a Voronovskaja-type formula for the L_n-operators. Given $x\in[0,1]$, let ψ_x be the function defined by $\psi_x(t)=t-x$.

Theorem 3.4. *For every $x \in [0,1]$ and every function $f \in C[0,1]$ which is twice differentiable at x it holds that*

$$\lim_{n\to\infty} n[L_n f(x) - f(x)] = \frac{x(1-x)\phi(x)}{2}\, f''(x). \tag{3.8}$$

Proof. For $x \in \{0,1\}$, the equality (3.8) is obvious because $L_n f(0) = f(0)$ and $L_n f(1) = f(1)$ for all $n \in \mathbf{N}$. Suppose next that $x \in]0,1[$. From (3.5) and (3.6) we deduce that

$$L_n e_2(x) = x^2 + \frac{x(1-x)\phi(x)}{n} + o\left(n^{-1}\right). \tag{3.9}$$

Consider the sum

$$\sigma_n(x) := \frac{1}{1-x} \sum_{k=0}^{\infty} \left(k - \frac{nx}{1-x}\right)^4 m_{n,k}(x).$$

It satisfies (see [6, p. 21])

$$0 \le \sigma_n(x) \le \frac{Cn^2}{(1-x)^5} \qquad \text{for all } n \in \mathbf{N},$$

where C is an absolute constant.

We have

$$L_n\psi_x^4(x) = \sum_{k=0}^{n} \left(\frac{k}{n+k} - x\right)^4 m_{n,k}(x) + \sum_{k=0}^{n} \left(\frac{n}{n+k} - x\right)^4 m_{n,k}(1-x).$$

Since

$$\left(\frac{k}{n+k} - x\right)^4 = \frac{(1-x)^4}{(n+k)^4} \left(k - \frac{nx}{1-x}\right)^4 \le \frac{(1-x)^4}{n^4} \left(k - \frac{nx}{1-x}\right)^4,$$
$$\left(\frac{n}{n+k} - x\right)^4 = \frac{x^4}{(n+k)^4} \left(k - \frac{n(1-x)}{x}\right)^4 \le \frac{x^4}{n^4} \left(k - \frac{n(1-x)}{x}\right)^4,$$

we deduce that

$$\begin{aligned} L_n\psi_x^4(x) &\le \frac{(1-x)^4}{n^4} \sum_{k=0}^{n} \left(k - \frac{nx}{1-x}\right)^4 m_{n,k}(x) \\ &\qquad + \frac{x^4}{n^4} \sum_{k=0}^{n} \left(k - \frac{n(1-x)}{x}\right)^4 m_{n,k}(1-x) \\ &\le \frac{(1-x)^5}{n^4}\sigma_n(x) + \frac{x^5}{n^4}\sigma_n(1-x) \le \frac{2C}{n^2}. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} nL_n\psi_x^4(x) = 0. \tag{3.10}$$

From (2.5), (3.9), and (3.10), by virtue of a standard result (for instance [7, Theorem 3] or [9, Theorem 1]) we conclude that (3.8) holds true. □

Remark 3.5. An alternative proof of Theorem 3.4 can be obtained by using the Voronovskaja-type formula for the classical M_n-operators. Indeed, for $x \in [0,1]$, $x \neq \frac{1}{2}$, equality (3.8) follows directly from [6, Eq. (2.4)], via (2.7). Suppose now that $x = \frac{1}{2}$. Let

$$p(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2$$

be the second degree Taylor polynomial of f around $\frac{1}{2}$ and consider the function $g := f - p$. We have

$$L_n f\left(\frac{1}{2}\right) = M_n\underline{g}\left(\frac{1}{2}\right) + M_n\overline{g}\left(\frac{1}{2}\right) + L_n p\left(\frac{1}{2}\right). \tag{3.11}$$

Since f is twice differentiable at $\frac{1}{2}$, it follows that both $\underline{g}$ and $\overline{g}$ are twice differentaible at $\frac{1}{2}$. Moreover, $\underline{g}''\left(\frac{1}{2}\right) = \overline{g}''\left(\frac{1}{2}\right) = 0$. Using again the estimation (2.4) from [6], we deduce that

$$M_n\underline{g}\left(\frac{1}{2}\right) = o\left(n^{-1}\right) \qquad \text{and} \qquad M_n\overline{g}\left(\frac{1}{2}\right) = o\left(n^{-1}\right). \tag{3.12}$$

On the other hand, by (2.5), (3.5), and (3.6) it follows that

$$L_n p(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left[L_n e_2(x) - x + \frac{1}{4}\right],$$

hence

$$L_n p\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) + \frac{1}{16}f''\left(\frac{1}{2}\right)\left(n^{-1} + o\left(n^{-1}\right)\right). \tag{3.13}$$

From (3.11), (3.12), and (3.13) we conclude that

$$\lim_{n\to\infty} n\left[L_n f\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right)\right] = \frac{1}{16}f''\left(\frac{1}{2}\right).$$

Therefore, (3.8) is proved also for $x = \frac{1}{2}$.

However, the estimation of the fourth central moment given by (3.10) will be used in the proof of the Voronovskaja-type formula for the $\widetilde{L}_n$-operators (Theorem 4.4).

4 The rate of convergence by the $\widetilde{L}_n$-operators

Using the Beta integrals it is easily seen that

$$\int_0^1 b_{n,k}(t)e_m(t)dt = \frac{(n+k+1)!}{(n+k+m+1)!}\cdot\frac{(k+m)!}{k!}, \tag{4.1}$$

$$\int_0^1 b_{n,k}(1-t)e_m(t)dt = \frac{(n+k+1)!}{(n+k+m+1)!}\cdot\frac{(n+m)!}{n!}, \tag{4.2}$$

for every $n \in \mathbf{N}$ and all nonnegative integers k and m.

Proposition 4.1. *The following equalities hold true:*

$$\widetilde{L}_n e_0 = e_0 \qquad \text{for all } n \in \mathbf{N}, \tag{4.3}$$

$$\widetilde{L}_n e_1(x) = x + S_n(x,2,0) - 2xS_n(x,2,1) \qquad \text{for all } n \geq 1, \tag{4.4}$$

$$\widetilde{L}_n e_2(x) = x^2 + 4xS_n(x,2,1) - 6x^2 S_n(x,2,2) + \widetilde{r}_n^{[2]}(x) \tag{4.5}$$

for all $n \geq 2$, where

$$|\widetilde{r}_n^{[2]}(x)| \leq \frac{2(6x^2+6x+1)}{(n+2)(n+3)}. \tag{4.6}$$

Proof. The equality (4.3) is an immediate consequence of (4.1), (4.2), and (2.2). Taking into account (4.1) and (4.2), for every $n \geq 1$ we have

$$\begin{aligned}\widetilde{L}_n e_1(x) = x &\left[\sum_{k=0}^{n-1} m_{n,k}(x) + \sum_{k=0}^{n} m_{n-1,k}(1-x)\right] \\ &-2x\left[\sum_{k=0}^{n-1}\frac{1}{n+3+k}\,m_{n,k}(x) + \sum_{k=0}^{n}\frac{1}{n+2+k}\,m_{n-1,k}(1-x)\right] \\ &+\sum_{k=0}^{n}\frac{1}{n+2+k}\,m_{n,k}(x) + \sum_{k=0}^{n}\frac{1}{n+2+k}\,m_{n,k}(1-x).\end{aligned}$$

By virtue of (2.2), we conclude that (4.4) holds true.

On the other hand, elementary calculations show that

$$\begin{aligned}\widetilde{L}_n e_2(x) &= x^2\left[\sum_{k=0}^{n-2} m_{n,k}(x)+\sum_{k=0}^{n} m_{n-2,k}(1-x)\right]\\ &\quad -6x^2\left[\sum_{k=0}^{n-2}\frac{1}{n+4+k}\,m_{n,k}(x)+\sum_{k=0}^{n}\frac{1}{n+2+k}\,m_{n-2,k}(1-x)\right]\\ &\quad +4x\left[\sum_{k=0}^{n-1}\frac{1}{n+3+k}\,m_{n,k}(x)+\sum_{k=0}^{n}\frac{1}{n+2+k}\,m_{n-1,k}(1-x)\right]\\ &\quad +\widetilde{r}_n^{[2]}(x),\end{aligned}$$

where

$$\begin{aligned}&\widetilde{r}_n^{[2]}(x)\\ &\quad = 12x^2\left[\sum_{k=0}^{n-2}\frac{m_{n,k}(x)}{(n+k+4)(n+k+5)}+\sum_{k=0}^{n}\frac{m_{n-2,k}(1-x)}{(n+k+2)(n+k+3)}\right]\\ &\qquad -12x\left[\sum_{k=0}^{n-1}\frac{m_{n,k}(x)}{(n+k+3)(n+k+4)}+\sum_{k=0}^{n}\frac{m_{n-1,k}(1-x)}{(n+k+2)(n+k+3)}\right]\\ &\qquad +2\left[\sum_{k=0}^{n}\frac{m_{n,k}(x)}{(n+k+2)(n+k+3)}+\sum_{k=0}^{n}\frac{m_{n,k}(1-x)}{(n+k+2)(n+k+3)}\right].\end{aligned}$$

Taking into account (2.2) it follows that (4.6) holds true and that

$$\begin{aligned}|\widetilde{r}_n^{[2]}(x)| &\le \frac{12x^2}{(n+2)(n+3)}\left[\sum_{k=0}^{n-2} m_{n,k}(x)+\sum_{k=0}^{n} m_{n-2,k}(1-x)\right]\\ &\qquad +\frac{12x}{(n+2)(n+3)}\left[\sum_{k=0}^{n-1} m_{n,k}(x)+\sum_{k=0}^{n} m_{n-1,k}(1-x)\right]\\ &\qquad +\frac{2}{(n+2)(n+3)}\left[\sum_{k=0}^{n} m_{n,k}(x)+\sum_{k=0}^{n} m_{n,k}(1-x)\right]\\ &= \frac{2(6x^2+6x+1)}{(n+2)(n+3)}.\end{aligned}$$

This proves the estimate (4.6). □

Proposition 4.2. *For every $n \geq 4$ and every $x \in [0,1]$ it holds that*

$$|\widetilde{L}_n e_1(x) - x| \leq \frac{14\phi(x)}{3n}, \tag{4.7}$$

$$\widetilde{L}_n \psi_x^2(x) \leq \frac{38x\phi(x) + 4}{n-1}. \tag{4.8}$$

Proof. We claim that

$$\frac{\phi(x)}{n+2} \leq S_n(x,2,0) \leq \frac{\phi(x)}{n+1}\left(1 + \frac{1}{\sqrt{2n+1}}\right), \tag{4.9}$$

$$\frac{\phi(x)}{n+3} \leq S_n(x,2,1) \leq \frac{\phi(x)}{n}\left(1 + \frac{2}{\sqrt{2n+1}}\right), \tag{4.10}$$

$$\frac{\phi(x)}{n+4} \leq S_n(x,2,2) \leq \frac{\phi(x)}{n-1}\left(1 + \frac{8}{\sqrt{2n+1}}\right), \tag{4.11}$$

for every $n \geq 4$ and every $x \in [0,1]$. Indeed, replacing n by $n+1$ in (3.7), we see that (4.9) holds true. On the other hand, according to Lemma 3.1 we have

$$\frac{\phi(x)}{n+3} \leq S_n(x,2,1) \leq \frac{2}{2n+1}\,\phi(x) + \frac{\binom{2n-1}{n} x^{n-1}(1-x)^n}{2(n+1)}.$$

By proceeding as in the proof of Proposition 3.2, it is not difficult to show that

$$\frac{\binom{2n-1}{n} x^{n-1}(1-x)^n}{2(n+1)} \leq \frac{2\phi(x)}{(n+1)\sqrt{2n+1}} < \frac{2\phi(x)}{n\sqrt{2n+1}}.$$

Since $\frac{2}{2n+1} < \frac{1}{n}$, it follows that (4.10) holds true, too.

Finally, by virtue of Lemma 3.1, we have

$$\frac{\phi(x)}{n+4} \leq S_n(x,2,2) \leq \frac{2n-3}{(n-2)(2n+1)}\,\phi(x) + \frac{\binom{2n-2}{n} x^{n-2}(1-x)^n}{2(n+1)}.$$

Since $\frac{2n-3}{(n-2)(2n+1)} < \frac{1}{n-1}$ and

$$\frac{\binom{2n-2}{n} x^{n-2}(1-x)^n}{2(n+1)} \leq \frac{8\phi(x)}{(n+1)\sqrt{2n+1}} < \frac{8\phi(x)}{(n-1)\sqrt{2n+1}},$$

the estimate (4.11) is also proved.

From (4.4), (4.9), and (4.10) we deduce that

$$\begin{aligned}|\widetilde{L}_n e_1(x)-x| &\leq S_n(x,2,0)+2xS_n(x,2,1)\\ &\leq \frac{\phi(x)}{n+1}\left(1+\frac{1}{\sqrt{2n+1}}\right)+\frac{2x\phi(x)}{n}\left(1+\frac{2}{\sqrt{2n+1}}\right)\\ &\leq \frac{\phi(x)}{n}\left(1+2x+\frac{1+4x}{\sqrt{2n+1}}\right)\leq\frac{14\phi(x)}{3n}\end{aligned}$$

for every $n\geq 4$ and every $x\in[0,1]$. This proves the estimate (4.7).

Likewise, from (4.5), (4.10), and (4.11) we get

$$\begin{aligned}|\widetilde{L}_n e_2(x)-x^2| &\leq 4xS_n(x,2,1)+6x^2S_n(x,2,2)+|\widetilde{r}_n^{[2]}(x)|\\ &\leq \frac{4x\phi(x)}{n}\left(1+\frac{2}{\sqrt{2n+1}}\right)+\frac{6x^2\phi(x)}{n-1}\left(1+\frac{8}{\sqrt{2n+1}}\right)\\ &\quad+\frac{26}{(n+2)(n+3)}\\ &< \frac{2x\phi(x)}{n-1}\left(2+3x+\frac{4+24x}{\sqrt{2n+1}}\right)+\frac{26}{7(n+2)}\\ &< \frac{86x\phi(x)}{3(n-1)}+\frac{4}{n-1}\end{aligned}$$

for every $n\geq 4$ and every $x\in[0,1]$. Consequently,

$$\begin{aligned}\widetilde{L}_n\psi_x^2(x) &= \widetilde{L}_n e_2(x)-2x\widetilde{L}_n e_1(x)+x^2\\ &\leq |\widetilde{L}_n e_2(x)-x^2|+2x|\widetilde{L}_n e_1(x)-x|\\ &\leq \frac{38x\phi(x)+4}{n-1},\end{aligned}$$

completing the proof. □

Combining the estimates (4.7), (4.8), and a standard argument (for instance [2, Theorem 5.1.2]) we obtain the main result of this section.

Theorem 4.3. *For every $n\in\mathbf{N}$, $n\geq 4$, every $f\in C[0,1]$, and all $x\in[0,1]$ it holds that*

$$|\widetilde{L}_n f(x)-f(x)|\leq 2\omega\left(f;\sqrt{\frac{38x\phi(x)+4}{n-1}}\right).$$

Moreover, if f is differentiable on $[0,1]$ and f' is bounded on $[0,1]$, then it holds that

$$|\widetilde{L}_n f(x) - f(x)| \leq \frac{14\phi(x)}{3n}\,|f'(x)| + 2\sqrt{\frac{38x\phi(x)+4}{n-1}}\,\omega\left(f';\sqrt{\frac{38x\phi(x)+4}{n-1}}\right).$$

Our final result is a Voronovskaja-type formula for the $\widetilde{L}_n$-operators.

Theorem 4.4. *For every $x \in [0,1]$ and every function $f \in C[0,1]$ which is twice differentiable at x it holds that*

$$\lim_{n\to\infty} n[\widetilde{L}_n f(x) - f(x)] = (1-2x)\phi(x)f'(x) + x(1-x)\phi(x)f''(x). \quad (4.12)$$

Proof. From (4.4), (4.9), and (4.10) we get

$$\widetilde{L}_n e_1(x) = x + \frac{(1-2x)\phi(x)}{n} + o\left(n^{-1}\right). \quad (4.13)$$

Likewise, from (4.5), (4.10), and (4.11) we get

$$\widetilde{L}_n e_2(x) = x^2 + \frac{2x(2-3x)\phi(x)}{n} + o\left(n^{-1}\right). \quad (4.14)$$

We claim also that

$$\lim_{n\to\infty} n\widetilde{L}_n \psi_x^4(x) = 0. \quad (4.15)$$

To see this, let

$$V_{n,k} := \int_0^1 b_{n,k}(t)\left(t - \frac{k}{n+k}\right)^4 dt,$$

where $n \in \mathbf{N}$, $k = 0, 1, \ldots, n$. Using (4.1) we get

$$V_{n,k} = \frac{Q(n,k)}{(n+k)^4(n+k+2)(n+k+3)(n+k+4)(n+k+5)},$$

where Q is a polynomial of degree 6. Letting $V_n := \max\limits_{0\leq k\leq n} V_{n,k}$, it follows that $V_n = o\left(n^{-1}\right)$. Since

$$(t-x)^4 \leq 8\left(t - \frac{k}{n+k}\right)^4 + 8\left(x - \frac{k}{n+k}\right)^4,$$
$$(t-x)^4 \leq 8\left(t - \frac{n}{n+k}\right)^4 + 8\left(x - \frac{n}{n+k}\right)^4,$$

it is immediately seen that

$$\sum_{k=0}^{n} m_{n,k}(x) \int_0^1 b_{n,k}(t)\psi_x^4(t)dt \leq 8V_n \sum_{k=0}^{n} m_{n,k}(x) + 8\sum_{k=0}^{n} \left(x - \frac{k}{n+k}\right)^4 m_{n,k}(x)$$

and

$$\sum_{k=0}^{n} m_{n,k}(1-x) \int_0^1 b_{n,k}(1-t)\psi_x^4(t)dt \leq 8V_n \sum_{k=0}^{n} m_{n,k}(1-x) + 8\sum_{k=0}^{n} \left(x - \frac{n}{n+k}\right)^4 m_{n,k}(1-x).$$

Therefore

$$\widetilde{L}_n\psi_x^4(x) \leq 8V_n + 8L_n\psi_x^4(x).$$

This inequality and (3.10) ensure the validity of (4.15). From (4.3), (4.13), (4.14), and (4.15), by virtue of [7, Theorem 3] or [9, Theorem 1], we deduce that (4.12) holds true. □

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References

[1] U. Abel, V. Gupta, and M. Ivan, On the rate of convergence of a Durrmeyer variant of the Meyer-König and Zeller operators, *Arch. Inequal. Appl.* **1** (2003), 1–9.

[2] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and its Applications,* De Gruyter, Berlin–New York, 1994.

[3] W. Chen, On the integral type Meyer-König and Zeller operators, *Approx. Theory Appl.* **2** (1986), no. 3, 7–18.

[4] E. W. Cheney and A. Sharma, Bernstein power series, *Canad. J. Math.* **16** (1964), 241–253.

[5] A. Lupaş, The approximation by means of some linear positive operators, *"Approximation Theory, Proceedings, IDoMat95"* (M. W. Müller, M. Felten, and D. H. Mache, eds.), Mathematical Research, vol. 86, 201–229, Akademie Verlag, Berlin, 1995.

[6] A. Lupaş and M. W. Müller, Approximation properties of the M_n-operators, *Aequationes Math.* **5** (1970), 19–37.

[7] R. G. Mamedov, Asymptotic approximation of differentiable functions by linear positive operators, *Dokl. Akad. Nauk SSSR* **128** (1959), 471–474 (in Russian).

[8] W. Meyer-König and K. Zeller, Bernsteinsche Potenzreihen, *Studia Math.* **19** (1960), 89–94.

[9] P. C. Sikkema, On some linear positive operators, *Indag. Math.* **32** (1970), 327–337.

[10] D. Zeilberger, On an identity of Daubechies, *Amer. Math. Monthly* **100** (1993), 487.

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Approximation Numbers of Inclusions Between Schatten Classes

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Abstract. Complementing classical and recent results on s-numbers of identity operators between symmetric sequence spaces, we prove asymptotically optimal formulas for the approximation numbers and Gelfand numbers of identities $\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n$, where $\mathcal{S}_E^n$ and $\mathcal{S}_F^n$ denote the n-th sections of the Schatten ideals associated to symmetric Banach sequence spaces E and F, respectively.

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1 Introduction

The calculation of s-numbers of finite-dimensional identity maps is crucial for applications in approximation theory. Authors of some classical papers within this direction mainly considered identity maps $\mathrm{id} : \ell_p^n \hookrightarrow \ell_q^n$, e.g., Pietsch in [22] for $q < p$ and Gluskin in [10, 11] for $p < q$, respectively. In [12] as well as in [4], analogues of some of Gluskin's results for identity maps $\mathrm{id} : \mathcal{S}_p^n \hookrightarrow \mathcal{S}_q^n$ between finite-dimensional Schatten classes were given. However, the case $q < p$ has not been treated, and an investigation of the techniques used shows that they are not sufficient to prove asymptotically optimal estimates for this case. A first attempt in this case using absolutely summing norms of identity maps can be found in [9].

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In this article, we continue these studies. More generally, we give asymptotically optimal estimates for the approximation numbers and Gelfand numbers of identity maps $\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n$, where E and F are symmetric Banach sequence spaces such that F is continuously embedded in E (we will simply write $F \hookrightarrow E$ in the following) with some additional properties. As in our recent article [14] where we have considered identity maps between symmetric Banach sequence spaces (for related material see also [6, 7, 13, 19]), our main tool will be interpolation theory. Finally, we also give some lower estimates in the case $E \hookrightarrow F$, continuing the work of [12, 4] described above. In an appendix, we consider certain vector-valued inclusions.

2 Preliminaries

For this is a supplement to our article [14], we keep this section as short as possible and simply refer the reader to [14] for all other notions and notations used. Nevertheless, we will recall some of the most crucial notions, notations and results needed.

For two real sequences (a_n) and (b_n) we mean by $a_n \prec b_n$ that there exists $C > 0$ such that $a_n \le C\, b_n$ for all $n \in \mathbb{N}$, and by $a_n \succ b_n$ that $b_n \prec a_n$. If $a_n \prec b_n$ and $a_n \succ b_n$ simultaneously, then we write $a_n \asymp b_n$. For two functions defined on general domains, the notation is defined accordingly.

For a Banach sequence space E (that is, a Banach lattice modeled on the set of natural numbers) we denote by E^n the linear span of the first n standard unit vectors $e_1, \dots, e_n$ endowed with the norm induced by E. A Banach sequence space E is called symmetric whenever the norm is invariant under permutations of the indices and choices of signs, and the standard unit vectors e_n have norm equal to 1. It is said to be maximal if the unit ball B_E is closed in the topology of pointwise convergence of the space of all real sequences. The Köthe dual $E^\times$ is the set of all $\lambda = (\lambda_n) \in \ell_\infty$ such that for any $\mu = (\mu_n) \in E$, the sequence $(\lambda_n \mu_n) \in \ell_1$. Endowed with the norm $\|\lambda\|_{E^\times} := \sup_{\|\mu\|_E \le 1} \|(\lambda_n \mu_n)\|_{\ell_1}$, this forms again a Banach sequence space which is symmetric whenever E is. Note that $(E^n)' = (E^\times)^n$, and $E' = E^\times$ if and only if E is maximal; here, for a Banach space X, the topological dual space is denoted by X'. The fundamental sequence of a symmetric Banach sequence space E is defined by $\lambda_E(n) := \|\sum_{i=1}^n e_i\|_E$. Whenever necessary, λ_E will also denote any monotone extension of this discrete function to the interval $[1, \infty)$. We will frequently use the fact (see,

e.g., [17, 3.a.6]) that for all n,

$$\lambda_E(n)\,\lambda_{E^\times}(n) = n. \tag{2.1}$$

For the definition of Lorentz sequence spaces $d(w,p)$ and Orlicz sequence spaces ℓ_φ we refer to [17] (see also [14] for the concrete notation).

By $\mathcal{S}^n_E$, we denote the linear space $\mathcal{L}(\mathbb{C}^n,\mathbb{C}^n)$ equipped with the norm $\|T\|_{\mathcal{S}^n_E} := \|(s_k(T))\|_E$, where $(s_k(T))$ denotes the (finite) sequence of singular numbers of the operator T. As usual, we set $\mathcal{S}^n_p := \mathcal{S}^n_{\ell_p}$ and $\mathcal{S}^n_{p,q} := \mathcal{S}^n_{\ell_{p,q}}$ for $1 \le p \le \infty$ and $1 < p < \infty$, $1 \le q \le \infty$, respectively.

For a linear and continuous operator $T \in \mathcal{L}(X,Y)$ (between Banach spaces X and Y) recall the definition of the k-th approximation number

$$a_k(T) := \inf\{\|T-R\|;\, R \in \mathcal{L}(X,Y),\, \operatorname{rank} R < k\},$$

the k-th Gelfand number

$$c_k(T) := \inf\{\|T_{|G}\|;\, G \subset X,\, \operatorname{codim} G < k\},$$

and the k-th Weyl number

$$x_k(T) := \sup\{a_k(ST);\, S \in \mathcal{L}(\ell_2,X),\, \|S\| \le 1\}.$$

We will frequently use the following easy facts: let $s \in \{a,c,x\}$ and $T \in \mathcal{L}(X,Y)$. Then

(i) $\|T\| \ge s_1(T) \ge s_2(T) \ge s_3(T) \ge \ldots \ge 0$;

(ii) $s_k(STR) \le \|S\|\, s_k(T)\, \|R\|$ for any suitable combination of R, S and T;

(iii) $x_k(T) \le c_k(T) \le a_k(T)$ (hence equality for X a Hilbert space);

(iv) $a_k(T') = a_k(T)$ whenever T is a compact operator between Banach spaces and T' its dual operator.

For this and more information on s-numbers of operators we refer to [16, 24].

Given a quasi-Banach couple $\overline{X} = (X_0, X_1)$, a quasi-Banach space X satisfying $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$ is called an intermediate space with respect to $\overline{X}$. If in addition for any linear and bounded operator $T : X_0 + X_1 \to X_0 + X_1$ with $T : X_0 \to X_0$ and $T : X_1 \to X_1$ one also has $T : X \to X$ with

$$\|T : X \to X\| \le C \max(\|T : X_0 \to X_0\|, \|T : X_1 \to X_1\|), \quad C > 0,$$

then X is called a C-exact interpolation space with respect to $\overline{X}$. Where the exact value of C does not matter, we simply omit 'C-exact'. For more information concerning interpolation theory we refer to [2].

3 Basic tools

Let (X_0, X_1) be a quasi-Banach couple and X be an intermediate space with respect to the couple (X_0, X_1). We define for $s, t > 0$ the positive function

$$\psi(s,t) := \psi_X^{\overline{X}}(s,t) := \sup\{\|x\|_X;\, x \in X_0 \cap X_1,\, \|x\|_{X_0} \le s,\, \|x\|_{X_1} \le t\}.$$

This function is monotone in each coordinate and homogeneous of degree one, that is, $\psi(\alpha t, \alpha s) = \alpha\psi(s,t)$ for all $\alpha > 0$ and $s, t > 0$, and satisfies

$$\|x\|_X \le \psi(\|x\|_{X_0}, \|x\|_{X_1}), \qquad x \in X_0 \cap X_1. \tag{3.1}$$

In the following, when we say that (X_0, X_1) is an n-dimensional Banach couple we always mean that each of the involved spaces is $\mathbb{K}^n$ equipped with some norm. Here, $\mathbb{K}$ denotes as usual either $\mathbb{R}$ or $\mathbb{C}$.

The following proposition is crucial for our lower estimates and was stated in [14, 3.1].

Proposition 3.1. *Let (X_0, X_1) be an n-dimensional quasi-Banach couple and X intermediate. Then for all $1 \le k \le n$,*

$$c_k(\mathrm{id} : X \hookrightarrow X_1) \ge \psi_X^{\overline{X}}(1/c_k(\mathrm{id} : X_0 \hookrightarrow X_1), 1)^{-1}. \tag{3.2}$$

The following result is due to Defant and Mastyło [5, 2.3] and stated in this form in [14, 3.2].

Lemma 3.2. *Let E_0, E_1, E be symmetric Banach sequence spaces such that E is a C-exact interpolation space with respect to the couple (E_0, E_1). Then for all $n \in \mathbb{N}$ and $1 \le l \le n$,*

$$\lambda_E(l) \le \psi_{E^n}^{(E_0^n, E_1^n)}(\lambda_{E_0}(l), \lambda_{E_1}(l)) \le \psi_E^{\overline{E}}(\lambda_{E_0}(l), \lambda_{E_1}(l)) \le C\,\lambda_E(l). \tag{3.3}$$

The following lemma is a refinement of [14, 3.3] and can be easily deduced therefrom.

Lemma 3.3. *Let E, F, G be symmetric Banach sequence spaces such that E is a C-exact interpolation space with respect to the couple (G, F) and F a D-exact interpolation space with respect to the couple (G, ℓ_1). Then*

$$\frac{\lambda_F(n)}{\lambda_E(n)} \le \|\mathrm{id} : E^n \hookrightarrow F^n\| \le C\,D\,\frac{\lambda_F(n)}{\lambda_E(n)}. \tag{3.4}$$

We omit the proof of the following lemma which is obvious.

Lemma 3.4. *Let (E_0, E_1) be a couple of symmetric Banach sequence spaces and E an intermediate symmetric Banach sequence space. Then*

$$\psi_{\mathcal{S}_E^n}^{(\mathcal{S}_{E_0}^n, \mathcal{S}_{E_1}^n)}(s,t) = \psi_{E^n}^{(E_0^n, E_1^n)}(s,t), \quad s, t > 0. \tag{3.5}$$

4 Inclusions between Schatten classes

As announced in the introduction, we focus on the case $F \hookrightarrow E$. For simplicity, we will write $X \hookrightarrow Y$ instead of $\mathrm{id} : X \hookrightarrow Y$ in the following. We can immediately exclude large indices k from our considerations:

Proposition 4.1. *Let E and F symmetric Banach sequence spaces such that $F \hookrightarrow E$. Then*

$$a_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \asymp c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \asymp 1, \qquad n^2 - n + 1 \le k \le n^2.$$

Proof. By the monotonicity of the Gelfand numbers the lower estimate follows from the assumption:

$$c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \ge c_{n^2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) = \|F^n \hookrightarrow E^n\|^{-1}.$$

The upper estimate follows immediately from the fact that the natural injection $\ell_2^n \hookrightarrow \mathcal{S}_G^n$, $x \mapsto x \otimes e_1$ is an isometry for any symmetric Banach sequence space G. □

For smaller indices k, we have the following general upper estimate:

Proposition 4.2. *Let E, F, G be symmetric Banach sequence spaces such that either $E = \ell_\infty$, or E a C-exact interpolation space with respect to the couple (G, F) and F a D-exact interpolation space with respect to the couple (G, ℓ_1). Then for $1 \le k \le n^2 - n + 1$,*

$$c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \le a_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \le 2CD\, \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\lambda_E\left(\frac{n^2-k+1}{n}\right)}. \tag{4.1}$$

Proof. Choose an integer $h \in \{2, \dots, n\}$ such that $n^2 - nh + 1 \le k < n^2 - n(h-1) + 1$. Identify $\mathcal{L}(\ell_2^n)$ with the set of all $n \times n$-matrices. Let M_0 be the linear subspace of all matrices with zero entries in rows $1, \dots, h$ and P_0 the projection onto M_0. Then $\operatorname{rank} P_0 = n^2 - nh < k$. Now let $M_1 := M_0^\perp$ (the linear subspace of matrices with zero entries in rows $h+1, \dots, n$) and $P_1 := \mathrm{id} - P_0$ be the projection onto M_1. Observe that any matrix in M_1 has at most h nonzero singular values. Since $s_k(P_1 A) = c_k(P_1 A) \le c_k(A) = s_k(A)$ for all $A \in \mathcal{L}(\ell_2^n)$, we have $\|P_1 : \mathcal{S}_E^n \to \mathcal{S}_E^n\| \le 1$. Hence, for $A \in \mathcal{L}(\ell_2^n)$ and $\sigma = (\sigma_1, \dots, \sigma_l)$ the nonzero singular values of $P_1 A$, $l \le h$,

$$\|P_1 A | \mathcal{S}_F^n\| = \|\sigma\|_F \le \|E^h \hookrightarrow F^h\|\, \|\sigma\|_E \le \|E^h \hookrightarrow F^h\|\, \|A | \mathcal{S}_E^n\|.$$

Thus,

$$a_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \le \|P_1 : \mathcal{S}_E^n \to \mathcal{S}_F^n\| \le \|E^h \hookrightarrow F^h\|.$$

Clearly, $\|\ell_\infty^h \hookrightarrow F^h\| = \lambda_F(h)$ for any symmetric Banach sequence space F, which settles the case $E = \ell_\infty$. For the second case, the claim now follows from (3.4). □

In the commutative case, one of the main starting points for interpolation of Gelfand numbers in [14] was the equality

$$c_k(\ell_\infty^n \hookrightarrow F^n) = \lambda_F(n-k+1)$$

for any symmetric Banach sequence space F, based on a result of Milman in [20]. Unfortunately, we do not know of any analogue of this formula in the noncommutative case, i.e., for $c_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_F^n)$. Instead, we use a lower estimate for $c_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_2^n)$ to start things going.

Lemma 4.3. *For $1 \le k \le n^2 - n + 1$,*

$$a_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_2^n) \asymp c_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_2^n) \asymp \sqrt{\frac{n^2-k+1}{n}}. \tag{4.2}$$

Proof. By [4, p. 252] we have

$$c_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_2^n) \succ \frac{\sqrt{n^2-k+1}}{\ell(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n)} \asymp \sqrt{\frac{n^2-k+1}{n}},$$

which gives the lower estimate. Here, $\ell(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n}$ denotes the so-called ℓ-norm of the operator concerned (see, e.g., [25, p. 329]). The upper one holds true by (4.1). □

Extrapolation now allows us to extend this result to range spaces $\mathcal{S}_F^n$ with F a symmetric Banach sequence space somewhat between ℓ_1 and ℓ_2, not too close to ℓ_2, and with fundamental function satisfying the upcoming regularity condition (4.3). This condition is fulfilled, e.g., whenever $\lambda_F(n) \asymp n^{1/p}$, $1 \le p < 2$. In particular, ℓ_p, $1 \le p < 2$ as well as $\ell_{p,r}$, $1 < p < 2$ and $1 \le r \le \infty$, satisfy this condition.

Proposition 4.4. *Let F be a symmetric Banach sequence space such that*

$$\sum_{h=t}^{\infty} \frac{1}{\lambda_F(h)^2} \le C \frac{t}{\lambda_F(t)^2} \tag{4.3}$$

with some constant $C > 0$ for all t. Then for $1 \le k \le n^2 - n + 1$

$$a_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_F^n) \asymp c_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_F^n) \asymp \lambda_F\left(\frac{n^2-k+1}{n}\right). \tag{4.4}$$

Proof. The upper estimate is already shown in (4.1) for any symmetric Banach sequence space F. For the lower estimate, let M be a subspace of $\mathcal{S}_\infty^n$ with $\mathrm{codim} M < k$. Set $u = \frac{n^2-k+1}{n}$. By (4.2), $c_k(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_2^n) \geq c\sqrt{u}$ for some $c > 0$, hence, there exists $A \in M$ with $\|A|\mathcal{S}_\infty^n\| = 1$ and $\|A|\mathcal{S}_2^n\| \geq c\sqrt{u}$. Let $s = (s_1, \ldots, s_n)$ be the sequence of singular numbers of A. To verify the lower estimate it is enough to show that $\|s\|_F \geq d\lambda_F(u)$ for some constant $d > 0$ not depending on u and n. So assume that $\|s\|_F < d\lambda_F(u)$. Since $\|s\|_F \geq \lambda_F(h)s_h$ for $h = 1, \ldots, n$ we conclude that

$$s_h < d\frac{\lambda_F(u)}{\lambda_F(h)}.$$

Hence for any $t > 0$

$$c^2 u \leq \sum_{h=1}^{n} s_h^2 < \sum_{h=1}^{t} 1 + d^2\lambda_F(u)^2 \sum_{h=t}^{n} \frac{1}{\lambda_F(h)^2} \leq t + d^2 Ct\frac{\lambda_F(u)^2}{\lambda_F(t)^2}.$$

Choosing $t = [c^2u/2]$ and d small enough (not depending on u!) gives a contradiction. □

Proposition 4.5. *Let F be a symmetric Banach sequence space satisfying* (4.3) *or $F = \ell_2$, and E be a symmetric Banach sequence space which is an interpolation space with respect to the couple (F, ℓ_∞). Then for $1 \leq k \leq n^2 - n + 1$*

$$a_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \asymp c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \asymp \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\lambda_E\left(\frac{n^2-k+1}{n}\right)}. \tag{4.5}$$

Proof. By (4.2) and (4.4), respectively, (3.2) and (3.5)

$$c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \succ \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\psi_{E^n}^{(F^n,\ell_\infty^n)}((\lambda_F((n^2-k+1)/n), 1)}.$$

Now let the interpolation space E be C-exact, and choose $l \geq 1$ such that $l \leq (n^2 - k + 1)/n < l + 1$. Writing ψ instead of $\psi_{E^n}^{(F^n,\ell_\infty^n)}$, it then follows by the monotonicity of ψ and (3.3) that

$$\begin{aligned}\psi((\lambda_F((n^2-k+1)/n), 1) &\leq \psi(\lambda_F(l+1), 1) = \psi(\lambda_F(l+1), \lambda_{\ell_\infty}(l+1)) \\ &\leq C\,\lambda_E(l+1) \leq 2\,C\,\lambda_E((n^2-k+1)/n),\end{aligned}$$

which gives the lower estimate. The upper one follows from (4.1) when taking $G = \ell_\infty$. □

We immediately get the following result which we believe to be true also for c_k instead of a_k – an affirmative answer to this would be essential for getting far more general results, as it was pointed out above.

Corollary 4.6. *Let F be a symmetric Banach sequence space. Then for $1 \le k \le n^2 - n + 1$*

$$a_k(\mathcal{S}^n_\infty \hookrightarrow \mathcal{S}^n_F) \asymp \lambda_F\left(\frac{n^2-k+1}{n}\right).$$

Proof. This follows by the duality property of the approximation numbers from the case $F = \ell_1$ in the proposition above, using the fact that by a result of Mitiagin [21] any symmetric Banach sequence space is an interpolation space with respect to the couple (ℓ_1, ℓ_∞) (see also [16, 1.b.10]). □

Example 4.7. *Let $1 \le p \le 2$ and $p < q \le \infty$. Then for $1 \le k \le n^2 - n + 1$ the following asymptotic formulas hold (also valid for a_k instead of c_k):*

(i) Let E be a p-convex symmetric Banach sequence space. Then

$$c_k(\mathcal{S}^n_E \hookrightarrow \mathcal{S}^n_p) \asymp \frac{\left(\frac{n^2-k+1}{n}\right)^{1/p}}{\lambda_E\left(\frac{n^2-k+1}{n}\right)}.$$

(ii)

$$c_k(\mathcal{S}^n_q \hookrightarrow \mathcal{S}^n_p) \asymp \left(\frac{n^2-k+1}{n}\right)^{1/p-1/q}.$$

(iii) Let $1 < p < 2$ and $1 \le r, s \le \infty$. Then

$$c_k(\mathcal{S}^n_{q,s} \hookrightarrow \mathcal{S}^n_{p,r}) \asymp \left(\frac{n^2-k+1}{n}\right)^{1/p-1/q}.$$

(iv) For $0 < \alpha < 1 - p/2$ and $\beta \in \mathbb{R}$ let $w(t) := t^{-\alpha}\log^\beta(1+t)$, $t \ge 1$. Then

$$c_k(\mathcal{S}^n_{d(w,q)} \hookrightarrow \mathcal{S}^n_{d(w,p)}) \asymp \left(\left(\frac{n^2-k+1}{n}\right) w((n^2-k+1)/n)\right)^{1/p-1/q}.$$

Proof. Any p-convex symmetric Banach sequence space E (see, e.g., [18] for details on this notion) is an interpolation space with respect to the couple (ℓ_p, ℓ_∞) as was shown in [8] (for the case $p = 2$ see also [7, 4.3]). It can be easily calculated directly that ℓ_q for $q \geq p$ is p-convex. Moreover, $\ell_{q,s}$ is an interpolation space with respect to the couple $(\ell_{p,r}, \ell_\infty)$ (see, e.g., [2]). Hence, (i)–(iii) directly follow from (4.5). To see (iv), first note that

$$\lambda_{d(w,p)}(h) = (\textstyle\sum_{j=1}^{h} w(j))^{1/p} \asymp h^{\frac{1-\alpha}{p}} \log^{\frac{\beta}{p}}(1+h).$$

Hence,

$$\begin{aligned}\sum_{h=t}^{\infty} \frac{1}{\lambda_{d(w,p)}(h)^2} &\asymp \sum_{h=t}^{\infty} h^{\frac{2(\alpha-1)}{p}} \log^{\frac{-2\beta}{p}}(1+h) \\ &\asymp t^{\frac{2(\alpha-1)}{p}+1} \log^{\frac{-2\beta}{p}}(1+t) \asymp \frac{t}{\lambda_{d(w,p)}(t)^2}.\end{aligned}$$

Consequently, $d(w,p)$ satisfies (4.3). Now the claim follows as in the proof of (4.5), using the facts that $\psi_{d(w,q)}^{(d(w,p),\ell_\infty)}(s,1) \leq s^{p/q}$ and

$$\|d^h(w,q) \hookrightarrow d^h(w,p)\| \leq (\textstyle\sum_{i=1}^{h} w(i))^{1/p-1/q}$$

(see, e.g., the proof of [14, 4.4 (v)]). □

Clearly, similarly as in (iv) and [14], one may construct examples of inclusions between certain Orlicz sequence spaces – we leave the details to the interested reader. We believe that in the above examples the restriction on p can be dropped, and that (iv) holds for arbitrary Lorentz sequences w. In fact, for approximation numbers instead of Gelfand numbers, the restrictions $p \leq 2$ and $p < 2$ in (ii) and (iii), respectively, are superfluous:

Corollary 4.8. *(i) Let $1 \leq p \leq q \leq \infty$. Then for $1 \leq k \leq n^2 - n + 1$*

$$a_k(\mathcal{S}_q^n \hookrightarrow \mathcal{S}_p^n) \asymp \left(\frac{n^2-k+1}{n}\right)^{1/p-1/q}.$$

(ii) Let $1 < p < q < \infty$ and $1 \leq r, s \leq \infty$. Then for $1 \leq k \leq n^2 - n + 1$

$$a_k(\mathcal{S}_{q,s}^n \hookrightarrow \mathcal{S}_{p,r}^n) \asymp \left(\frac{n^2-k+1}{n}\right)^{1/p-1/q}.$$

Proof. This follows by the duality property of the approximation numbers from the examples (ii) and (iii) above, respectively. □

We continue with considering inclusions starting from $\mathcal{S}_2^n$; these have already been investigated in [9].

Corollary 4.9. *Let F be a symmetric Banach sequence space which is an interpolation space with respect to the couple (ℓ_2, ℓ_1). Then for all $1 \le k \le n^2 - n + 1$*

$$a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) \asymp c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) \asymp \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\sqrt{\frac{n^2-k+1}{n}}},$$

and for $n \le k \le n^2$

$$x_k(\mathcal{S}_F^n \hookrightarrow \mathcal{S}_2^n) \asymp \frac{\sqrt{\frac{k}{n}}}{\lambda_F\left(\frac{k}{n}\right)}.$$

Proof. By duality, $F^\times$ is an interpolation space with respect to (ℓ_2, ℓ_∞), hence by (4.5) together with (2.1)

$$\begin{aligned} c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) = a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) &= a_k(\mathcal{S}_{F^\times}^n \hookrightarrow \mathcal{S}_2^n) \\ &\ge c_k(\mathcal{S}_{F^\times}^n \hookrightarrow \mathcal{S}_2^n) \succ \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\sqrt{\frac{n^2-k+1}{n}}}. \end{aligned}$$

The upper estimate follows from (4.1) when taking $E = G = \ell_2$. The asymptotic formula for the Weyl numbers now follows from the fact (cf. [3, 2.3.2]) that for any $S \in \mathcal{L}(X, \ell_2^m)$ with $\dim X = \operatorname{rank} S = m$ and for all $1 \le k \le m$, $x_k(S) c_{m-k+1}(S^{-1}) = 1$. □

Examples of symmetric Banach sequence spaces which are interpolation spaces with respect to the couple (ℓ_2, ℓ_1) are the following: ℓ_p for $1 \le p \le 2$, $\ell_{p,q}$ for $1 < p < 2$ and $1 \le q \le \infty$, and any 2-concave symmetric Banach sequence space (this follows by duality from what was mentioned above for p-convex spaces). By [15], Orlicz sequence spaces as in the following example are 2-concave.

Example 4.10. *Let φ be an Orlicz function such that the function $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function in a neighborhood of zero. Then for $1 \le k \le n^2 - n + 1$*

$$c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\varphi^n) \asymp \left(\varphi^{-1}(n/(n^2-k+1))\sqrt{(n^2-k+1)/n)}\right)^{-1},$$

and for $n \le k \le n^2$

$$x_k(\mathcal{S}_\varphi^n \hookrightarrow \mathcal{S}_2^n) \asymp \varphi^{-1}(n/k)\,\sqrt{k/n}. \tag{4.6}$$

These estimates are sharper than those obtained with the approach by summing identity maps in [9]. Indeed, the techniques used there would give, e.g.,

$$x_k(\mathcal{S}_\varphi^n \hookrightarrow \mathcal{S}_2^n) \prec \sup_{r>0} \frac{\varphi^{-1}(n/r^2)}{\varphi^{-1}(1/r^2)}\,\varphi^{-1}(1/k)\,\sqrt{k/n}. \tag{4.7}$$

Taking $r = \sqrt{k}$, it is clear that the right hand-side in (4.7) is larger or equal than $\varphi^{-1}(n/k)\,\sqrt{k/n}$. In particular, if φ is submultiplicative, then the right hand-side in (4.7) is equivalent to

$$\frac{\varphi^{-1}(1/k)}{\varphi^{-1}(1/n)}\,\sqrt{k/n},$$

which is not equivalent to the right hand-side in (4.6) as long as φ is not also supermultiplicative.

Proposition 4.11. *Let E and F be symmetric Banach sequence spaces such that F is an interpolation space with respect to (ℓ_2, ℓ_1) and E an interpolation space with respect to (ℓ_2, F). Then for $1 \le k \le n^2 - n + 1$*

$$a_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \asymp c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \asymp \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\lambda_E\left(\frac{n^2-k+1}{n}\right)}.$$

Proof. By Corollary 4.9,

$$c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) = a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) \asymp \frac{\lambda_F\left(\frac{n^2-k+1}{n}\right)}{\sqrt{\frac{n^2-k+1}{n}}}.$$

Hence, by (3.2), (3.5) and the homogenity of $\psi_{E^n}^{(\ell_2^n, F^n)}$,

$$\begin{aligned} c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) &\succ \psi_{E^n}^{(\ell_2^n, F^n)}\left(\sqrt{(n^2-k+1)/n}/\lambda_F((n^2-k+1)/n), 1\right)^{-1} \\ &= \frac{\lambda_F((n^2-k+1)/n)}{\psi_{E^n}^{(\ell_2^n, F^n)}(\sqrt{(n^2-k+1)/n}, \lambda_F((n^2-k+1)/n))}. \end{aligned}$$

Now proceed as before to obtain the lower estimate. The upper one follows from (4.1) for $G = \ell_2$. □

We close this section with some lower estimates in the case $\ell_2 \hookrightarrow E \hookrightarrow F$ and F close to ℓ_q, $2 < q \le \infty$.

Proposition 4.12. *Let E be a symmetric Banach sequence space which is an interpolation space with respect to (ℓ_∞, ℓ_2). Then for $1 \le k \le n^2$*

$$c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_\infty^n) \succ \max\left(\lambda_E(n)^{-1}, \lambda_E\left(\frac{n^2}{n^2-k+1}\right)^{-1}\right).$$

Proof. In [4, p. 251] it was shown that

$$c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \max\left(\sqrt{\frac{n^2-k+1}{n^2}}, \frac{1}{\sqrt{n}}\right).$$

Now proceed as in the proof for $c_k(E^n \hookrightarrow \ell_\infty^n)$ given in [14, 5.2]. □

Proposition 4.13. *For $2 < q < \infty$ let F be a symmetric Banach sequence space such that $\lambda_F(n) \asymp n^{1/q}$. Then for any interpolation space E with respect to the couple (ℓ_2, F),*

$$c_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \succ \begin{cases} \left(\frac{n^2}{n^2-k+1}\right)^{\frac{1}{q-2}} \lambda_E\left(\left(\frac{n^2}{n^2-k+1}\right)^{\frac{q}{q-2}}\right)^{-1} & 1 \le k \le n_q \\ n^{1/q}\lambda_E(n)^{-1} & n_q \le k \le n \end{cases}$$

where $n_q = n^2 - n^{2/q+1} + 1$.

Proof. By what was said in the previous proof,

$$c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) \ge c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \succ \sqrt{(n^2-k+1)/n^2}.$$

Also, since $\lambda_F(n) \succ n^{1/q}$, we have $F \hookrightarrow \ell_{q,\infty}$, hence

$$c_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_F^n) \succ c_n(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_{q,\infty}^n) = \|\ell_{q,\infty}^n \hookrightarrow \ell_2^n\|^{-1} \asymp n^{1/q-1/2}.$$

Now proceed as in the proof for $c_k(E^n \hookrightarrow F^n)$ given in [14, 6.2]. □

Example 4.14. *Let $2 < p_0 < q_0 < \infty$ and $1 \le p_1, q_1 \le \infty$. Then*

$$c_k(\mathcal{S}_{p_0,p_1}^n \hookrightarrow \mathcal{S}_{q_0,q_1}^n) \succ \begin{cases} \left(\frac{n^2}{n^2-k+1}\right)^{\frac{p_0-q_0}{p_0(q_0-2)}} & 1 \le k \le n_{q_0} \\ n^{1/q_0-1/p_0} & n_{q_0} \le k \le n \end{cases}$$

where $n_{q_0} = n^2 - n^{2/q_0+1} + 1$.

5 Appendix: inclusions of vector-valued spaces

In this appendix, we illustrate how our techniques can be used for vector-valued inclusions. However, since this is not the main concern of this paper, we keep it as short as possible, our only aim to prove a certain vector-valued version of the classical Pietsch result. As usual, for a symmetric Banach sequence space E and a Banach space X we set

$$E^n(X) := \{(x_1, \dots, x_n);\, x_i \in X,\, 1 \le i \le n\},$$

equipped with the norm

$$\|(x_1, \dots, x_n)|E^n(X)\| := \|(\|x_1\|_X, \dots, \|x_n\|_X)\|_{E^n}.$$

Again, for large k's everything is clear:

Proposition 5.1. *Let E, F be symmetric Banach sequence spaces such that $F \hookrightarrow E$. Then for any m-dimensional normed space X and all k such that $nm - m + 1 \le k \le nm$,*

$$a_k(E^n(X) \hookrightarrow F^n(X)) \asymp c_k(E^n(X) \hookrightarrow F^n(X)) \asymp 1.$$

The same asymptotic estimates hold for the inclusion $H^m(E^n) \hookrightarrow H^m(F^n)$, where H is an arbitrary symmetric Banach sequence space.

Proof. Since $c_{nm}(E^n(X) \hookrightarrow F^n(X)) = \|F^n \hookrightarrow E^n\|^{-1} \asymp 1$, the lower estimate is clear for all other k's. On the other hand,

$$a_{nm-m+1}(E^n(X) \hookrightarrow F^n(X)) \le \|X \hookrightarrow X\| = 1,$$

which gives the upper estimate. The proof for the second inclusion stated above is similar. □

Proposition 5.2. *Let E, F, G be symmetric Banach sequence spaces such that E is a C-exact interpolation space with respect to the couple (G, F) and F a D-exact interpolation space with respect to the couple (G, ℓ_1). Then, for $1 \le k \le nm - m + 1$ and any m-dimensional Banach space X,*

$$a_k(E^n(X) \hookrightarrow F^n(X)) \prec 2CD\, \frac{\lambda_F\left(\frac{nm-k+1}{m}\right)}{\lambda_E\left(\frac{nm-k+1}{m}\right)}. \tag{5.1}$$

The same estimate holds for the inclusion $H^m(E^n) \hookrightarrow H^m(F^n)$, where H is an arbitrary symmetric Banach sequence space.

Proof. The proof is similar to the one of (4.1), but we state it for the convenience of the reader. Choose an integer $h \in \{2, \dots, n\}$ such that $nm - mh + 1 \le k < nm - m(h-1) + 1$. Identify $E^n(X)$ and $F^n(X)$ with the set of all $n \times m$-matrices. Let M_0 be the linear subspace of all matrices with zero entries in rows $1, \dots, h$ and P_0 the projection onto M_0. Then $\operatorname{rank} P_0 = nm - mh < k$. Now let $M_1 := M_0^{\perp}$ (the linear subspace of matrices with zero entries in rows $h+1, \dots, n$) and $P_1 := \mathrm{id} - P_0$ be the projection onto M_1. Clearly, $\|P_1 : E^n(X) \to E^n(X)\| \le 1$. Hence, for $A \in \mathbb{K}^{n \times m}$,

$$\begin{aligned}\|P_1 A | F^n(X)\| &= \|P_1 A | F^h(X)\| \le \|E^h \hookrightarrow F^h\| \, \|P_1 A | E^h(X)\| \\ &= \|E^h \hookrightarrow F^h\| \, \|P_1 A | E^n(X)\| \le \|E^h \hookrightarrow F^h\| \, \|A | E^n(X)\|.\end{aligned}$$

Thus,

$$a_k(E^n(X) \hookrightarrow F^n(X)) \le \|P_1 : E^n(X) \hookrightarrow F^n(X)\| \le \|E^h \hookrightarrow F^h\|.$$

The claim now follows from (3.4). The proof of the estimate for the inclusion $H^m(E^n) \hookrightarrow H^m(F^n)$ goes along the same lines, swapping rows and columns. □

Proposition 5.3. *Let $1 \le p < q \le \infty$ and H be a symmetric Banach sequence space such that $\ell_p \hookrightarrow H$ with $\|H^m \hookrightarrow \ell_p^m\| \asymp m^{1/p}/\lambda_H(m)$. Then for $1 \le k \le nm - m + 1$,*

$$\begin{aligned}a_k(\ell_q^n(H^m) \hookrightarrow \ell_p^n(H^m)) &\asymp c_k(\ell_q^n(H^m) \hookrightarrow \ell_p^n(H^m)) \\ &\asymp \left(\frac{nm - k + 1}{m}\right)^{1/p - 1/q}. \end{aligned} \tag{5.2}$$

The same asymptotic estimates hold for the inclusion $H^m(\ell_q^n) \hookrightarrow H^m(\ell_p^n)$.

Proof. By Pietsch [22] we know that $c_k(\ell_\infty^{nm} \hookrightarrow \ell_p^{nm}) = (nm - k + 1)^{1/p}$. Hence, by factorization and the assumption,

$$c_k(\ell_\infty^n(H^m) \hookrightarrow \ell_p^n(H^m)) \succ \left(\frac{nm - k + 1}{m}\right)^{1/p}$$

and

$$c_k(H^m(\ell_\infty^n) \hookrightarrow H^m(\ell_p^n)) \succ \left(\frac{nm - k + 1}{m}\right)^{1/p}.$$

Now let $\theta := p/q$. Then it holds $[\ell_\infty^n(H^m), \ell_p^n(H^m)]_\theta = \ell_q^n(H^m)$ and $[H^m(\ell_\infty^n), H^m(\ell_p^n)]_\theta = H^m(\ell_q^n)$ (see, e.g., [2]). Hence,

$$\psi_{\ell_q^n(H^m)}^{(\ell_\infty^n(H^m),\ell_p^n(H^m))}(s,t) = \psi_{H^m(\ell_q^n)}^{(H^m(\ell_\infty^n),H^m(\ell_p^n))}(s,t) = s^{1-\theta}t^\theta.$$

Thus, (3.4) and the above give the lower estimates. The upper ones are clear by the above proposition, taking $X = H^m$, $E = \ell_q$, $F = \ell_p$ and $G = \ell_\infty$. □

Corollary 5.4. *Let $1 \le p < q \le \infty$ and H be a symmetric Banach sequence space such that $\lambda_H(m) \asymp m^{1/r}$ for some $1 \le r \le \infty$. Then for all $1 \le k \le nm - m + 1$*

$$a_k(\ell_q^n(H^m) \hookrightarrow \ell_p^n(H^m)) \asymp \left(\frac{nm-k+1}{m}\right)^{1/p-1/q}.$$

The same asymptotic estimate holds for the inclusion $H^m(\ell_q^n) \hookrightarrow H^m(\ell_p^n)$.

Proof. Without loss of generality let H be maximal. If $r = \infty$, then $H = \ell_\infty$ (with equivalent norms), and the claim follows from the proposition above for $E = \ell_q$. The case $r = 1$ then is clear by the duality property of approximation numbers. Now let $1 < r < \infty$. If $p < r$, then $\ell_p \hookrightarrow H$ and $\|H^m \hookrightarrow \ell_p^m\| \asymp m^{1/p}/\lambda_H(m)$, since $\ell_{r,1} \hookrightarrow H \hookrightarrow \ell_{r,\infty}$. Hence, the proposition above can be applied. If $p \ge r$, then $q' < p' \le r'$. By (2.1) we have $\lambda_{H^\times}(m) \asymp m^{1/q'}$. Thus, the previous case gives

$$a_k(\ell_{p'}^n((H^\times)^m) \hookrightarrow \ell_{q'}^n((H^\times)^m)) \asymp \left(\frac{nm-k+1}{m}\right)^{1/p-1/q}.$$

The claim then follows by the duality property of the approximation numbers. The inclusion $H^m(\ell_q^n) \hookrightarrow H^m(\ell_p^n)$ can be treated similarly. □

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References

[1] J. Arazy, Some remarks on interpolation theorems and the boundness of the triangular projection in unitary matrix spaces, *Integral Equations Oper. Theory* **1** (1978), 453–495.

[2] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, Berlin, 1978.

[3] B. Carl and A. Defant, Tensor products and Grothendieck type inequalities of operators in L_p-spaces, *Trans. Amer. Math. Soc.* **331** (1992), 55–76.

[4] B. Carl and A. Defant, Asymptotic estimates for approximation quantities of tensor product identities, *J. Approx. Theory* **88** (1997), 228–256.

[5] A. Defant and M. Mastyło, Composition of $(E, 2)$-summing operators, *Studia Math.* **159** (2003), 51–65.

[6] A. Defant, M. Mastyło and C. Michels, Summing inclusion maps between symmetric sequence spaces, a survey, *Recent progress in functional analysis (Valencia, 2000),* 43–60, North-Holland Math. Stud., 189, North-Holland, Amsterdam, 2001.

[7] A. Defant, M. Mastyło and C. Michels, Summing inclusion maps between symmetric sequence spaces, *Trans. Amer. Math. Soc.* **354** (2002), 4473–4492.

[8] A. Defant, M. Mastyło and C. Michels, Eigenvalue estimates for operators on symmetric Banach sequence spaces, *Proc. Amer. Math. Soc.* **132** (2004), 513–521.

[9] A. Defant, M. Mastyło and C. Michels, Summing norms of identities between unitary ideals, preprint, 21p.

[10] E. D. Gluskin, On some finite-dimensional problems of width theory, *Physis—Riv. Internaz. Storia Sci.* **23** (1981), 5–10 (in Russian).

[11] E. D. Gluskin, Norms of random matrices and diameters of finite-dimensional sets, *Mat. Sb. (N.S.)* **120(162)** (1983), 180–189 (in Russian).

[12] Y. Gordon, H. König and C. Schütt, Geometric and probabilistic estimates for entropy and approximation numbers of operators, *J. Approx. Theory* **49** (1987), 219–239.

[13] A. Hinrichs, Approximation numbers of identity operators between symmetric Banach sequence spaces, *J. Approx. Theory* **118** (2002), 305–315.

[14] A. Hinrichs and C. Michels, Gelfand numbers of identity operators between symmetric sequence spaces, *Positivity*, to appear.

[15] I. A. Komarchev, 2-absolutely summable operators in certain Banach spaces, *Math. Notes* **25** (1979), 306–312.

[16] H. König, *Eigenvalue distributions of compact operators*, Birkhäuser, Basel, 1986.

[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: Sequence Spaces*, Springer-Verlag, Berlin, 1977.

[18] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function Spaces*, Springer-Verlag, Berlin, 1979.

[19] C. Michels, On absolutely summing operators on $C(K)$ with values in Banach lattices, *Function Spaces. Proceedings of the Sixth Conference (Wrocław, 2001)*, 190–197, World Scientific, 2003.

[20] V. D. Milman, Operators of class C_0 and C_0^*, *Teor. Funkcij Funkcional. Anal. i Prilozhen.* **10** (1970), 15–26 (in Russian).

[21] B. S. Mitiagin, An interpolation theorem for modular spaces, *Mat. Sb.* **66** (1965), 473–482 (in Russian).

[22] A. Pietsch, s-numbers of operators in Banach spaces, *Studia Math.* **51** (1974), 201–223.

[23] A. Pietsch, *Operator Ideals*, North-Holland, 1980.

[24] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, 1987.

[25] N. Tomczak-Jaegermann, *Banach–Mazur distances and finite-dimensional operator ideals*, Longman Scientific & Technical, 1989.

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Persistence of Eigenvalues and Multiplicity in the Neumann Problem for the Laplace Operator on Nonsmooth Domains

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Abstract. We consider the Neumann eigenvalue problem for the Laplace operator on a variable nonsmooth domain. We extend a result of Lupo and Micheletti concerning the structure of the set of domain deformations which leave the multiplicity of an eigenvalue unchanged, and we study the set of deformations which leave a certain eigenvalue unchanged.

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1 Introduction

In this paper we are mainly interested into two problems. The first has been considered by Lupo and Micheletti [9], [10], [11], Teytel [12], and by [7], and consists in analyzing the set of domain deformations of a reference domain for which a given eigenvalue of an elliptic partial differential operator preserves its multiplicity. The second problem consists in analyzing the set of domain deformations for which a given eigenvalue remains unchanged. In other words, the second problem consists in analyzing a weaker version of the isospectrality problem, which is concerned with the analysis

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of the domains for which the entire spectrum is the same. We refer to [7] for historical remarks and references to related contributions.

The papers of Lupo and Micheletti [9], [10], [11], and of Teytel [12] concern the first problem for various elliptic operators on regular domains and for regular domain deformations. In particular, Dirichlet boundary conditions have been considered in Lupo and Micheletti [9], [10], [11], Teytel [12], and Robin boundary conditions have been considered in Lupo and Micheletti [11]. In [7] the results of Lupo and Micheletti for the first problem have been extended to the case of the Dirichlet eigenvalue problem for the Laplace operator on a nonsmooth domain, and for a class of Lipschitz continuous deformations. In this paper we plan to consider the same approach of [7] to analyze the Neumann eigenvalue problem on a nonsmooth domain, and for a class of Lipschitz continuous deformations.

We first recast the boundary value problem on a deformed domain into a problem on a fixed reference domain by means of changing the variables with the function representing the deformation of the reference domain. Then we recast our problem in a Hilbert space with a scalar product depending on the deformation. At this point we are able to exploit the abstract results of [7] for Hilbert spaces with variable scalar products, some of which generalize the work of Lupo and Micheletti, to draw our conclusions. Namely, we show that the set of deformations for which a given eigenvalue preserves its multiplicity, and the set of deformations for which a given eigenvalue remains unchanged are contained in submanifolds of certain codimensions (cf. Theorems 3.1, 4.1.) Then we also provide some information of the codimension of such manifolds. In particular, for the problem of the persistence of the multiplicity our conditions are of the same sort of those of [7] for the Dirichlet problem which in turn are equivalent to those of Lupo and Micheletti for the Dirichlet problem in case the reference domain and deformations are regular (cf. Theorems 3.7, 4.2.)

This paper is organized as follows. Section 2 is a section of preliminaries. In section 3 we discuss the problem of persistence of the multiplicity. In section 4 we discuss the problem of persistence of an eigenvalue.

2 Preliminaries

We first introduce some technical preliminaries and notation. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be real Banach spaces. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Banach space of linear and continuous maps of $\mathcal{X}$ to $\mathcal{Y}$ endowed with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by $\mathcal{B}_s(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$

the space of the bilinear symmetric and continuous maps of $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$ endowed with the norm of the uniform convergence on the cross product of the unit sphere of $\mathcal{X}$ and of the unit sphere of $\mathcal{Y}$. We say that the space $\mathcal{X}$ is continuously imbedded in the space $\mathcal{Y}$ provided that $\mathcal{X}$ is a linear subspace of $\mathcal{Y}$, and that the inclusion map is continuous. We denote by $\mathbb{Z}$ the set of integer numbers, and by $\mathbb{N}$ the set of natural numbers including 0. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. We denote by $M_m(\mathbb{R})$ the set of $m \times m$ matrices with real entries, and by $S_m(\mathbb{R})$ the set of the symmetric elements of $M_m(\mathbb{R})$. If $A \in M_m(\mathbb{R})$, we denote by A^t the transpose matrix of A. If A is invertible, we set $A^{-t} \equiv \left(A^{-1}\right)^t$. All elements of $\mathbb{R}^n$ are thought as row vectors.

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathrm{cl}\Omega$ the closure of Ω and by $\partial\Omega$ the boundary of Ω. Throughout this paper, we shall consider only case $n \geq 2$. We denote by $L^2(\Omega)$ the space of square summable real valued measurable functions defined on Ω, and by $W^{1,2}(\Omega)$ the Sobolev space of distributions in Ω which have weak derivatives up to the first order in $L^2(\Omega)$, endowed with the norm defined by

$$\|u\|_{W^{1,2}(\Omega)} \equiv \left\{ \|u\|^2_{L^2(\Omega)} + \sum_{l=1}^{n} \|u_{x_l}\|^2_{L^2(\Omega)} \right\}^{1/2} \qquad \forall u \in W^{1,2}(\Omega)\,. \tag{2.1}$$

Now, we are interested in open connected subsets Ω of $\mathbb{R}^n$ of finite measure $|\Omega|$ such that

$$W^{1,2}(\Omega) \text{ is compactly imbedded in } L^2(\Omega)\,. \tag{2.2}$$

As is well known, if (2.2) holds, then the Poincaré-Wirtinger inequality holds in Ω (cf. *e.g.*, Evans [2, Proof of Thm. 1, p. 275].) Then we deform Ω by a Lipschitz continuous homeomorphism of the class $\mathcal{A}_\Omega$ which we now introduce. We denote by $\mathrm{Lip}(\Omega)$ the set of Lipschitz continuous functions of Ω to $\mathbb{R}$, and we set

$$\mathcal{A}_\Omega \equiv \Bigg\{ \phi \in (\mathrm{Lip}(\Omega))^n : \tag{2.3}$$
$$l_\Omega[\phi] \equiv \inf \left\{ \frac{|\phi(x) - \phi(y)|}{|x - y|} : x, y \in \Omega, x \neq y \right\} > 0 \Bigg\}\,.$$

We note that

$$l_\Omega[\phi] \leq |\det D\phi(x)|^{1/n}\,, \tag{2.4}$$

for almost all $x \in \Omega$ (cf. *e.g.*, [8, Lem. 4.22].) Now it can be verified that if Ω satisfies (2.2) and if $\phi \in \mathcal{A}_\Omega$, then $\phi(\Omega)$ also satisfies (2.2) (cf. [3, §2].) Accordingly, the Neumann eigenvalue problem

$$\int_{\phi(\Omega)} DvDw^t\,dy = \lambda \int_{\phi(\Omega)} vw\,dy \qquad \forall w \in W^{1,2}(\phi(\Omega)) \tag{2.5}$$

in the unknowns $v \in W^{1,2}(\phi(\Omega))$ (the Neumann eigenfunctions), $\lambda \in \mathbb{R}$ (the Neumann eigenvalues) has a sequence of eigenvalues

$$0 = \lambda_0[\phi] < \lambda_1[\phi] \le \lambda_2[\phi] \le \dots\,,$$

which we write as many times as their multiplicity. In this paper we are interested in the dependence of $\lambda_j[\phi]$ on ϕ for all $j \in \mathbb{N}\setminus\{0\}$. Thus we need to introduce a topology in $\mathcal{A}_\Omega$. As usual, we introduce the seminorm

$$|f|_1 \equiv \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : x,y \in \Omega,\ x \ne y\right\} \qquad \forall f \in \mathrm{Lip}(\Omega)\,,$$

on $\mathrm{Lip}(\Omega)$. It is easily seen that $\mathcal{A}_\Omega$ is open in $(\mathrm{Lip}(\Omega))^n$ (cf. [8, Prop. 4.29], [5, Thm. 3.11].) As is well known, $(\mathrm{Lip}(\Omega), |\cdot|_1)$ is a complete seminormed space. However, we prefer to deal with a normed space, rather than with a seminormed space. Then we will state our results for an arbitrary Banach space $\mathcal{X}_\Omega$, continuously imbedded in $(\mathrm{Lip}(\Omega), |\cdot|_1)$. Alternatively, one could also endow $\mathrm{Lip}(\Omega)$ with a norm which renders $\mathrm{Lip}(\Omega)$ a Banach space continuously imbedded in $(\mathrm{Lip}(\Omega), |\cdot|_1)$, and take $\mathcal{X}_\Omega$ equal to such a Banach space. We now wish to analyze the set of ϕ's in $\mathcal{A}_\Omega \cap \mathcal{X}_\Omega^n$ for which the multiplicity of a certain eigenvalue $\lambda_j[\phi]$ remains constant in ϕ, and the set of ϕ's for which a certain eigenvalue $\lambda_j[\phi]$ remains constant in ϕ.

Here one of the main difficulties is that the domain $\phi(\Omega)$, on which the Sobolev space $W^{1,2}(\phi(\Omega))$ is defined, changes as ϕ is perturbed. In order to overcome this difficulty, we plan to change the variables in problem (2.5) by means of ϕ, and to obtain a problem in $W^{1,2}(\Omega)$. Actually , we will obtain a problem in $W^{1,2}(\Omega)/\mathbb{R}$, as we need to get rid of the constants which generate the eigenspace corresponding to the trivial eigenvalue $\lambda_0[\phi] = 0$. See Theorem 2.1.

To do so, we need to introduce some notation, under the assumption that Ω is a nonempty open connected subset of $\mathbb{R}^n$, and that $\phi \in \mathcal{A}_\Omega$. We denote by $\mathcal{I}$ the canonical imbedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$. We denote by $\mathcal{J}_\phi$ the operator of $L^2(\Omega)$ to the strong dual $\big(W^{1,2}(\Omega)\big)'$ which takes $u \in L^2(\Omega)$ to the the functional $\mathcal{J}_\phi[u] \in \big(W^{1,2}(\Omega)\big)'$ defined by

$$\mathcal{J}_\phi[u][w] \equiv \int_\Omega uw|\det D\phi|\,dx \qquad \forall w \in W^{1,2}(\Omega)\,. \tag{2.6}$$

We denote by Q_ϕ the bilinear form of $\left(W^{1,2}(\Omega)\right)^2$ to $\mathbb{R}$ defined by

$$Q_\phi[u_1,u_2] \equiv \int_{\phi(\Omega)} D\left(u_1\circ\phi^{(-1)}\right) D\left(u_2\circ\phi^{(-1)}\right)^t dy\,, \tag{2.7}$$

for all $u_1, u_2 \in W^{1,2}(\Omega)$. We set

$$W^{1,2,0}_\phi(\Omega) \equiv \left\{u\in W^{1,2}(\Omega):\ \int_\Omega u|\det D\phi|\,dx = 0\right\}.$$

Then we simply write $W^{1,2,0}(\Omega)$ in case ϕ is the identity. We also find convenient to denote by $w^{1,2,0}(\Omega)$ the space $W^{1,2,0}(\Omega)$ endowed with the energy scalar product

$$< u_1, u_2 > \equiv \int_\Omega Du_1 Du_2^t\,dx \qquad \forall u_1,u_2\in W^{1,2,0}(\Omega)\,.$$

We denote by π_ϕ the map of $W^{1,2}(\Omega)$ to $W^{1,2,0}_\phi(\Omega)$ defined by

$$\pi_\phi[u] = u - \frac{\int_\Omega u|\det D\phi|\,dx}{\int_\Omega |\det D\phi|\,dx}\,,$$

for all $u\in W^{1,2}(\Omega)$. We denote by $\pi^\sharp_\phi$ the map of $W^{1,2}(\Omega)/\mathbb{R}$ onto $W^{1,2,0}_\phi(\Omega)$ defined by equality $\pi_\phi = \pi^\sharp_\phi\circ p$, where p is the canonical projection of $W^{1,2}(\Omega)$ onto the quotient $W^{1,2}(\Omega)/\mathbb{R}$. We denote by $Q^\sharp_\phi$ the bilinear form

$$Q^\sharp_\phi[p[u_1],p[u_2]] \equiv Q_\phi[u_1,u_2] \qquad \forall u_1,u_2\in W^{1,2}(\Omega),$$

which is clearly a scalar product on the quotient $W^{1,2}(\Omega)/\mathbb{R}$. We denote by $w^{1,2}_\phi(\Omega)/\mathbb{R}$ the quotient $W^{1,2}(\Omega)/\mathbb{R}$ endowed with $Q^\sharp_\phi$. Then we write simply $w^{1,2}(\Omega)/\mathbb{R}$ in case ϕ is the identity. Thus $w^{1,2}(\Omega)/\mathbb{R}$ is endowed with its usual energy scalar product. We denote by $\Delta_{\phi,N}$ the operator of $W^{1,2,0}_\phi(\Omega)$ to

$$\mathbf{F}(\Omega) \equiv \left\{F\in\left(W^{1,2}(\Omega)\right)':\ F[1]=0\right\},$$

which takes $u\in W^{1,2,0}_\phi(\Omega)$ to the element $\Delta_{\phi,N}[u]$ of $\mathbf{F}(\Omega)$ defined by equality

$$\Delta_{\phi,N}[u][w] = -Q_\phi[u,w] \qquad \forall w\in W^{1,2}(\Omega). \tag{2.8}$$

If ϕ is the identity, we write Δ_N instead of $\Delta_{\phi,N}$.

Then we have the following Theorem which states that the eigenvalues $\lambda_j[\phi]$ coincide with the reciprocals of the eigenvalues of a suitable compact selfadjoint operator $T_{\phi,N}$ in $w^{1,2}_\phi(\Omega)/\mathbb{R}$.

Theorem 2.1 *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ of finite measure such that (2.2) holds. Let $\phi \in \mathcal{A}_\Omega$. Then the following statements hold.*

(i) The operator $\Delta_{\phi,N}$ of $W^{1,2,0}_\phi(\Omega)$ onto $\mathbf{F}(\Omega)$ is a linear homeomorphism, and the operator $T_{\phi,N} \equiv -\left(\pi^\sharp_\phi\right)^{(-1)} \circ \Delta^{(-1)}_{\phi,N} \circ \mathcal{J}_\phi \circ \mathcal{I} \circ \pi^\sharp_\phi$ is compact and selfadjoint in $w^{1,2}_\phi(\Omega)/\mathbb{R}$.

(ii) The pair (λ, v) of the set $\mathbb{R} \times \left(w^{1,2,0}(\phi(\Omega)) \setminus \{0\}\right)$ satisfies equation (2.5) if and only if $\lambda > 0$ and the pair $\left(\mu \equiv \lambda^{-1}, \breve{u} \equiv p[v \circ \phi]\right)$ of the set $\mathbb{R} \times \left(\left(w^{1,2}_\phi(\Omega)/\mathbb{R}\right) \setminus \{0\}\right)$ satisfies equation

$$\mu \breve{u} = T_{\phi,N} \breve{u}\,. \tag{2.9}$$

(iii) Equation (2.9) has a decreasing sequence $\{\mu_j[\phi]\}_{j\in\mathbb{N}\setminus\{0\}}$ of eigenvalues in $]0, +\infty[$, and $\mu_j[\phi] = \lambda_j^{-1}[\phi]$ for all $j \in \mathbb{N} \setminus \{0\}$, where the numbers $\{\lambda_j[\phi]\}_{j\in\mathbb{N}\setminus\{0\}}$ are all the nonzero eigenvalues of (2.5) counted with their multiplicity. Each eigenvalue of (2.5), or of (2.9) has finite multiplicity.

A proof of Theorem 2.1 can be obtained by a standard argument (see [3, §2].)

3 On the set of deformations which preserve the multiplicity

By Theorem 2.1, we have reduced our original problem (2.5) to problem (2.9). We now consider problem (2.9) in the Hilbert space $w^{1,2}(\Omega)/\mathbb{R}$ endowed with the family of ϕ-dependent scalar products $Q^\sharp_\phi$. Since $T_{\phi,N}$ is selfadjoint in $w^{1,2}_\phi(\Omega)/\mathbb{R}$ but not in $w^{1,2}(\Omega)/\mathbb{R}$, we resort to the abstract scheme of [7, §3] which generalizes to variable scalar products a result of Lupo and Micheletti for a fixed scalar product, and we prove the following.

Theorem 3.1 *Let Ω be a nonempty connected open subset of $\mathbb{R}^n$ of finite measure such that (2.2) holds. Let $\mathcal{X}_\Omega$ be a Banach space continuously*

imbedded in $\mathrm{Lip}(\Omega)$. *Let* F *be a finite nonempty subset of* $\mathbb{N}\setminus\{0\}$. *Let*

$$\Theta_{\Omega,N}[F]\equiv\Big\{\phi\in\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega:\ \lambda_l[\phi]\notin\{\lambda_j[\phi]:\,j\in F\}\,\forall l\in\mathbb{N}\setminus(F\cup\{0\}),\ \lambda_j[\phi]\text{ have a common value }\lambda_F[\phi]\ \forall j\in F\Big\}.$$

Let $\tilde\phi\in\Theta_{\Omega,N}[F]$. *Let* $\tilde v_1,\dots,\tilde v_{|F|}$ *be an orthonormal basis in* $w^{1,2,0}(\tilde\phi(\Omega))$ *of the eigenspace of equation (2.5) in* $\tilde\phi(\Omega)$ *corresponding to* $\lambda_F[\tilde\phi]$. *Let*

$$\Gamma_{hk}[\psi]\equiv\int_{\tilde\phi(\Omega)}\Big(D\tilde v_hD\tilde v_k^t-\lambda_F[\tilde\phi]\tilde v_h\tilde v_k\Big)\,\mathrm{div}\,\Big(\psi\circ\tilde\phi^{(-1)}\Big)\,dy \tag{3.1}$$
$$-\int_{\tilde\phi(\Omega)}D\tilde v_h\left[D\Big(\psi\circ\tilde\phi^{(-1)}\Big)+D\Big(\psi\circ\tilde\phi^{(-1)}\Big)^t\right]D\tilde v_k^t\,dy,$$

for all $\psi\in\mathcal{X}^n_\Omega$, $h,k=1,\dots,|F|$. *Let* θ *be the dimension of the vector subspace*

$$\Big\{(\Gamma_{hk}[\psi]-\delta_{hk}\Gamma_{11}[\psi])_{h,k=1,\dots,|F|}:\,\psi\in\mathcal{X}^n_\Omega\Big\} \tag{3.2}$$

of

$$S^0_{|F|}\equiv\big\{A\equiv(a_{li})_{l,i=1,\dots,|F|}\in S_{|F|}(\mathbb{R}):\,a_{11}=0\big\}.$$

Then $0\le\theta\le\frac{|F|(|F|+1)}{2}-1$, *and we can choose an open neighborhood* $\tilde V$ *of* $\tilde\phi$ *in* $\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega$ *such that* $\Theta_{\Omega,N}[F]\cap\tilde V$ *is contained in a real analytic Banach submanifold of* $\mathcal{X}^n_\Omega$ *of codimension* θ. *Furthermore, if* $\theta=\frac{|F|(|F|+1)}{2}-1$, *then we can choose* $\tilde V$ *so that* $\Theta_{\Omega,N}[F]\cap\tilde V$ *is a real analytic Banach submanifold of* $\mathcal{X}^n_\Omega$ *of codimension* $\frac{|F|(|F|+1)}{2}-1$, *whose tangent space at* $\tilde\phi$ *equals the space*

$$\Big\{\psi\in\mathcal{X}^n_\Omega:\,(\Gamma_{hk}[\psi]-\delta_{hk}\Gamma_{11}[\psi])_{h,k=1,\dots,|F|}=0\Big\}. \tag{3.3}$$

Proof. By standard calculus in Banach space, one can prove that the map $\phi\mapsto(Q^\sharp_\phi,T_\phi)$ is real analytic from $\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega$ to the set

$$\mathcal{O}_\Omega \tag{3.4}$$
$$\equiv\Big\{(Q,T)\in\mathcal{Q}\Big(\big(w^{1,2}(\Omega)/\mathbb{R}\big)^2,\mathbb{R}\Big)\times\mathcal{K}\big(w^{1,2}(\Omega)/\mathbb{R},w^{1,2}(\Omega)/\mathbb{R}\big):\ T\in\mathcal{K}_s\big(\big(W^{1,2}(\Omega)/\mathbb{R},Q\big),\big(W^{1,2}(\Omega)/\mathbb{R},Q\big)\big)\Big\},$$

where $\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2,\mathbb{R}\right)$ denotes the open set of scalar products on $w^{1,2}(\Omega)/\mathbb{R}$ which are topologically equivalent to the canonical scalar product of $w^{1,2}(\Omega)/\mathbb{R}$, and where $\mathcal{K}\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right)$ denotes the Banach space of linear and compact operators in $w^{1,2}(\Omega)/\mathbb{R}$, and where the symbol $\mathcal{K}_s\left(\left(W^{1,2}(\Omega)/\mathbb{R},Q\right),\left(W^{1,2}(\Omega)/\mathbb{R},Q\right)\right)$ denotes the set of linear compact selfadjoint operators in $\left(W^{1,2}(\Omega)/\mathbb{R},Q\right)$ (cf. [6, §2].) We think of $\mathcal{O}_\Omega$ as a subset of the space $\mathcal{B}_s\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2,\mathbb{R}\right)\times\mathcal{K}\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right)$. (Actually, it can be proved that $\mathcal{O}_\Omega$ is a real analytic Banach manifold modelled on $\mathcal{B}_s\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2,\mathbb{R}\right)\times\mathcal{K}_s\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right)$ (cf. [7, §2].)) If T is compact and selfadjoint in $\left(W^{1,2}(\Omega)/\mathbb{R},Q\right)$ for a $Q\in\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2,\mathbb{R}\right)$ then the spectrum $\sigma[T]$ of T is a finite or countable subset of $\mathbb{R}$ and each element of $\sigma[T]$ different from zero is an eigenvalue of finite multiplicity. We need to enumerate such eigenvalues of T in a convenient way. To do so, we denote by $j^+[T]$ the (possibly infinite) number of elements of $\sigma[T]\cap]0,+\infty[$, each counted with its multiplicity, and we denote by $j^-[T]$ the (possibly infinite) number of elements of $\sigma[T]\cap]-\infty,0[$, each counted with its multiplicity. We also set $J^+[T]\equiv\{j\in\mathbb{Z}:1\le j\le j^+[T]\}$ and $J^-[T]\equiv\{j\in\mathbb{Z}:-j^-[T]\le j\le -1\}$. Then there exists a uniquely determined function $j\mapsto\mu_j[T]$ of $J[T]\equiv J^-[T]\cup J^+[T]$ to $\mathbb{R}\setminus\{0\}$ such that $j\mapsto\mu_j[T]$ is decreasing on $J^-[T]$ and on $J^+[T]$, and such that

$$\sigma[T]\setminus\{0\}=\{\mu_j[T]:j\in J[T]\},$$

and such that each eigenvalue $\mu_j[T]$ is repeated as many times as its multiplicity. Then we note that $\Theta_{\Omega,N}[F]$ coincides with the set

$$\left\{\phi\in\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega:(Q^\sharp_\phi,T_{\phi,N})\in\Theta[F]\right\},$$

where

$$\Theta[F]\equiv\{(Q,T)\in\mathcal{O}_\Omega:F\subseteq J[T],\ \mu_j[T]\text{ have a common value }\mu_F[T]\text{ for all }j\in F,\ \mu_l[T]\neq\mu_F[T]\text{ for all }l\in J[T]\setminus F\}. \tag{3.5}$$

Now let $\tilde\phi\in\Theta_{\Omega,N}[F]$. We set $\breve u_h=p[\tilde v_h\circ\tilde\phi]$ for $h=1,\dots,|F|$. By [6, §2] we have

$$\Gamma_{hk}[\psi]=-\lambda_F[\tilde\phi]Q^\sharp_{\tilde\phi}\left[d_{|\phi=\tilde\phi}T_{\phi,N}[\breve u_h][\psi],\breve u_k\right]\qquad\forall\psi\in\mathcal{X}^n_\Omega. \tag{3.6}$$

Thus we can invoke an abstract result of [7, §3] concerning families of compact selfadjoint operators in Hilbert space with variable scalar product. Here the family of operators is $\{T_{\phi,N}\}_{\phi\in\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega}$ and the variable scalar

product is $Q^{\sharp}_{\phi}$. By using such a result and by equality (3.6), we conclude that there exists a neighborhood $\tilde{V}$ of $\tilde{\phi}$ in $\mathcal{A}_{\Omega} \cap \mathcal{X}^n_{\Omega}$ such that $\Theta_{\Omega,N}[F] \cap \tilde{V}$ is contained in a Banach submanifold of $\mathcal{X}^n_{\Omega}$ of codimension θ, where θ is the dimension of the vector space in (3.2), and that if $\theta = \frac{|F|(|F|+1)}{2} - 1$, then the second part of the statement holds. □

Our goal is now to give more information on the codimension θ. To do so, we follow the strategy of [7, §5] and we prove Proposition 3.5 and Theorem 3.7 below. We first need some notation and some preliminary technical results. We set

$$\mathcal{D}_{\Omega} \equiv \left\{ \eta \in \mathbb{R}^{\Omega} : \text{there exists } \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ such that } \varphi_{|\Omega} = \eta \right\} ,$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of C^{∞} functions of $\mathbb{R}^n$ to $\mathbb{R}$ with compact support. If u is a function of Ω to $\mathbb{R}$, we denote by u_{Ω} the function of $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$u_{\Omega}(x) \equiv u(x) \quad \text{if } x \in \Omega, \qquad u_{\Omega}(x) \equiv 0 \quad \text{if } x \in \mathbb{R}^n \setminus \Omega \, .$$

We note that the symbol u_{Ω} should not be confused with the symbol $u_{|\Omega}$, which denotes the restriction of u to Ω.

As is well known, the space $L^{\infty}(\Omega)$ is canonically isometric to the strong dual of $L^1(\Omega)$. Accordingly, if $\{f_l\}_{l \in \mathbb{N}}$ is a sequence in $L^{\infty}(\Omega)$, we shall say that $\{f_l\}_{l \in \mathbb{N}}$ has a weak* limit f in $L^{\infty}(\Omega)$, if

$$\lim_{l \to \infty} \int_{\Omega} f_l g \, dx = \int_{\Omega} f g \, dx \qquad \forall g \in L^1(\Omega) \, .$$

As is well known, all bounded sequences in $L^{\infty}(\Omega)$ have weakly* convergent subsequences. Similarly, we shall speak about weakly* convergent subsequences of $(L^{\infty}(\Omega))^n$. Then we have the following technical Lemma of [7, §5].

Lemma 3.2 *Let Ω be a connected open subset of $\mathbb{R}^n$. Let $m \in \mathbb{N}$. Let $\tilde{\phi} \in \mathcal{A}_{\Omega}$. Let $\mathcal{X}_{\Omega}$ be a linear subspace of* $\mathrm{Lip}(\Omega)$ *containing $\mathcal{D}_{\Omega}$. Let Ξ be a linear map of* $(\mathrm{Lip}(\Omega))^n$ *to $\mathbb{R}^m$. Assume that* $\lim_{l \to \infty} \Xi[\psi_l] = \Xi[\psi]$ *in $\mathbb{R}^m$ whenever $\{\psi_l\}_{l \in \mathbb{N}}$ is a sequence in* $(\mathrm{Lip}(\Omega))^n$, *$\psi \in (\mathrm{Lip}(\Omega))^n$, and*

$$\lim_{l \to \infty} D\left(\psi_l \circ \tilde{\phi}^{(-1)}\right) = D\left(\psi \circ \tilde{\phi}^{(-1)}\right) \qquad \text{weakly}^* \text{ in } \left(L^{\infty}(\tilde{\phi}(\Omega))\right)^{n^2} . \tag{3.7}$$

Then the following equality holds

$$\{\Xi[\psi] : \psi \in \mathcal{X}_\Omega^n\} = \left\{\Xi[\xi \circ \tilde{\phi}] : \xi \in \left(\mathcal{D}_{\tilde{\phi}(\Omega)}\right)^n\right\} . \tag{3.8}$$

Then we have the following.

Lemma 3.3 *Let the same assumptions of Theorem 3.1 hold. Let $\mathcal{X}_\Omega$ be a linear subspace of* $\mathrm{Lip}(\Omega)$ *containing $\mathcal{D}_\Omega$. Then the space in (3.2) equals the space*

$$\left\{\left(\Gamma_{hk}[\xi \circ \tilde{\phi}] - \delta_{hk}\Gamma_{11}[\xi \circ \tilde{\phi}]\right)_{h,k=1,\dots,|F|} : \xi \in \left(\mathcal{D}_{\tilde{\phi}(\Omega)}\right)^n\right\} . \tag{3.9}$$

Proof. By the membership of $\tilde{v}_h$ in $W^{1,2}(\tilde{\phi}(\Omega))$, and by the Hölder inequality, and by the definition of Γ_{hk}, and by Lemma 3.2, we conclude that the space in (3.2) equals the space in (3.9). □

Since the functions $\xi \circ \tilde{\phi}$ for $\xi \in \left(\mathcal{D}_{\tilde{\phi}(\Omega)}\right)^n$ are bounded, we can invoke the following technical fact of [6, §2].

Proposition 3.4 *Let the same assumptions of Theorem 3.1 hold. Let $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ for all $h = 1, \dots, |F|$. Then*

$$\Gamma_{hk}[\psi] = \int_{\tilde{\phi}(\Omega)} \mathrm{div}\left\{\left[D\tilde{v}_h D\tilde{v}_k^t - \lambda_F[\tilde{\phi}]\tilde{v}_h\tilde{v}_k\right]\left(\psi \circ \tilde{\phi}^{(-1)}\right)\right\} dy\,,$$

for all $h, k = 1, \dots, |F|$, $\psi \in (\mathrm{Lip}(\Omega) \cap L^\infty(\Omega))^n$.

Then we have the following Proposition which gives an alternative description of the space in (3.2) under somewhat more restrictive assumptions.

Proposition 3.5 *Let the same assumptions of Theorem 3.1 hold. Let $\mathcal{X}_\Omega$ contain $\mathcal{D}_\Omega$. Let $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ for all $h = 1, \dots, |F|$. Let*

$$M_{hk} \equiv D\tilde{v}_h D\tilde{v}_k^t - \lambda_F[\tilde{\phi}]\tilde{v}_h\tilde{v}_k\,, \tag{3.10}$$

for all $h, k = 1, \dots, |F|$. Then the space in (3.2) equals the space

$$\left\{\left(\int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[(M_{hk} - \delta_{hk}M_{11})\,\xi\right]\,dy\right)_{h,k=1,\dots,|F|} : \xi \in \left(\mathcal{D}_{\tilde{\phi}(\Omega)}\right)^n\right\} . \tag{3.11}$$

Thus our problem is now to understand what is the dimension of the space in (3.11). To do so, we set

$$\overset{o}{W}{}^{1,1}(\Omega) \equiv \left\{ u \in W^{1,1}(\Omega) : u_\Omega \in W^{1,1}(\mathbb{R}^n) \right\} .$$

Clearly,

$$W_0^{1,1}(\Omega) \subseteq \overset{o}{W}{}^{1,1}(\Omega) ,$$

and equality holds for example if Ω is of class C^1. Then we have the following variant of a known technical statement (see also [7, §5].)

Lemma 3.6 *Let Ω be an open subset of $\mathbb{R}^n$ such that $\partial\Omega$ has zero n-dimensional Lebesgue measure. Let $r \in \mathbb{N} \setminus \{0\}$. Let $f_1, \dots, f_r \in W^{1,1}(\Omega)$. Let q denote the canonical projection of $W^{1,1}(\Omega)$ onto the quotient space $W^{1,1}(\Omega)/\overset{o}{W}{}^{1,1}(\Omega)$. The dimension of the space generated by $\{q(f_l)\}_{l=1,\dots,r}$ in $W^{1,1}(\Omega)/\overset{o}{W}{}^{1,1}(\Omega)$ equals the dimension of the space*

$$V \equiv \left\{ \left(\int_\Omega \operatorname{div}(\xi f_1)\, dx, \dots, \int_\Omega \operatorname{div}(\xi f_r)\, dx \right) : \xi \in (\mathcal{D}_\Omega)^n \right\} .$$

Now, it can be easily verified that if $\phi \in \mathcal{A}_\Omega$, then ϕ can be extended uniquely to a Lipschitz continuous homeomorphism of $\mathrm{cl}\Omega$ onto $\mathrm{cl}\phi(\Omega)$, which we still denote by ϕ, and

$$\phi(\partial\Omega) = \partial\phi(\Omega) , \tag{3.12}$$

(cf. *e.g.*, [7, §5].) Then we have the following, which generalizes to the Laplace operator with Neumann boundary conditions on a nonsmooth domain a result for the Dirichlet problem on a regular domain of Lupo and Micheletti [9, Rem. 8, p. 116].

Theorem 3.7 *Let the same assumptions of Theorem 3.1 hold. Let $\partial\Omega$ have zero n-dimensional Lebesgue measure. Let $\mathcal{X}_\Omega$ contain $\mathcal{D}_\Omega$. Let $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ for all $h = 1, \dots, |F|$. Let q denote the canonical projection of $W^{1,1}(\tilde{\phi}(\Omega))$ onto the quotient space $W^{1,1}(\tilde{\phi}(\Omega))/\overset{o}{W}{}^{1,1}(\tilde{\phi}(\Omega))$. Then θ equals the dimension of the subspace of $W^{1,1}(\tilde{\phi}(\Omega))/\overset{o}{W}{}^{1,1}(\tilde{\phi}(\Omega))$ generated by*

$$\{ q\left[M_{hk} - \delta_{hk} M_{11} \right] : h, k = 1, \dots, |F|, (h,k) \neq (1,1) \} , \tag{3.13}$$

where M_{hk} has been defined in (3.10).

Proof. Since the measure of $\partial\Omega$ is zero, equality (3.12) and the Lipschitz continuity of $\tilde{\phi}$ imply that the measure of $\tilde{\phi}(\partial\Omega)$ is zero. Since $D\tilde{v}_h D\tilde{v}_k^t \in W^{1,1}(\tilde{\phi}(\Omega))$, then Lemma 3.6 implies that the dimension of the space in (3.11) equals the dimension of the subspace generated by the set in (3.13). Then we conclude by Proposition 3.5. □

Remark 3.8 *We note that if $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$, then by standard elliptic regularity theory, we have $\tilde{v}_r \in W^{2,2}(\tilde{\phi}(\Omega))$ for $r = 1, \ldots, |F|$ (cf. e.g., Troianiello [13, Thm. 3.17 (ii), p. 180].) We point out however, that the membership of $\tilde{v}_r$ in $W^{2,2}(\tilde{\phi}(\Omega))$ may hold also under weaker regularity assumptions on $\tilde{\phi}(\Omega)$.*

Furthermore, we note that if we assume that Ω is of class $C^{1,1}$, and that $\tilde{\phi} \in \mathcal{A}_\Omega$ has continuous partial derivatives in Ω satisfying a Lipschitz condition in Ω, then $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$ (cf. e.g., [4, Lem. 2.4].)

Remark 3.9 *Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,1}$. Let $\mathcal{X}_\Omega$ be a linear subspace of $\mathrm{Lip}(\Omega)$ containing $\mathcal{D}_\Omega$. Let F be a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let $\tilde{\phi} \in \Theta_{\Omega,N}[F]$ be of class $C^{1,1}$ in $\mathrm{cl}\Omega$. Then by standard elliptic regularity theory, we have $\tilde{v}_r \in W^{2,2}(\tilde{\phi}(\Omega))$ and $D\tilde{v}_r$ has a trace on the boundary for $r = 1, \ldots, |F|$. Then by applying the Divergence Theorem, one can easily see that the dimension of the space generated by the set in (3.11) equals that of the subspace of $L^1(\partial\tilde{\phi}(\Omega))$ generated by the set*

$$\{M_{hk} - \delta_{hk} M_{11} : h, k = 1, \ldots, |F|, (h,k) \neq (1,1)\} .$$

Since $\partial\Omega$ has zero n-dimensional Lebesgue measure, such dimension coincides with that of the space in (3.13). Thus our Theorems 3.1, and 3.7 can be considered as counterparts for the Neumann problem on nonsmooth domains of the corresponding result proved by Lupo and Micheletti [9, Rem. 8, p. 116] for the Dirichlet problem in case Ω is of class C^3, and ϕ is of class C^3.

Example 3.10 *Let $\Omega = (0,\pi) \times (0,\pi)$ be the square of side π in the Euclidean plane, $\tilde{\phi}$ the identity map in Ω. Then it is well known that $\lambda_1[\Omega] = 1$ is an eigenvalue of multiplicity 2 and that the corresponding eigenspace is spanned by $\tilde{v}_1(y_1, y_2) = \frac{\sqrt{2}}{\pi} \cos y_1$ and $\tilde{v}_2(y_1, y_2) = \frac{\sqrt{2}}{\pi} \cos y_2$ (cf. e.g., Courant and Hilbert [1, Ch. 5, §5,4].) Then by arguing as in the previous Remark, the dimension θ of the space generated by the set in (3.11) is easily seen to be equal to 2. Thus in this case condition $\theta = \frac{m(m+1)}{2} - 1$ is fulfilled.*

4 On the set of deformations which preserve a certain eigenvalue

We now turn to study the structure of the set of ϕ's which preserve a certain eigenvalue. Then we obtain the following.

Theorem 4.1 *Let Ω be a nonempty connected open subset of $\mathbb{R}^n$ of finite measure such that (2.2) holds. Let $\mathcal{X}_\Omega$ be a Banach space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Let F be a finite nonempty subset of $\mathbb{N}\setminus\{0\}$. Let $\tilde{\lambda}>0$. Let*

$$\Theta_{\Omega,N}[F,\tilde{\lambda}]\equiv\Big\{\phi\in\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega:\ \lambda_j[\phi]=\tilde{\lambda}\ \forall j\in F,\ \lambda_l[\phi]\neq\tilde{\lambda}\ \forall l\in\mathbb{N}\setminus(F\cup\{0\})\Big\}.$$

Let $\tilde{\phi}\in\Theta_{\Omega,N}[F,\tilde{\lambda}]$. Let $\tilde{v}_1,\dots,\tilde{v}_{|F|}$ be an orthonormal basis in $w^{1,2,0}(\tilde{\phi}(\Omega))$ of the eigenspace of equation (2.5) corresponding to $\tilde{\lambda}$. Let ϑ be the dimension of the vector subspace

$$\Big\{(\Gamma_{hk}[\psi])_{h,k=1,\dots,|F|}:\ \psi\in\mathcal{X}^n_\Omega\Big\} \tag{4.1}$$

of $S_{|F|}(\mathbb{R})$, where Γ_{hk} has been introduced in (3.1). Then $1\leq\vartheta\leq\frac{|F|(|F|+1)}{2}$, and we can choose an open neighborhood $\tilde{V}$ of $\tilde{\phi}$ in $\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega$ such that $\Theta_{\Omega,N}[F,\tilde{\lambda}]\cap\tilde{V}$ is contained in a real analytic Banach submanifold of $\mathcal{X}^n_\Omega$ of codimension ϑ. Furthermore, if $\vartheta=\frac{|F|(|F|+1)}{2}$, then we can choose $\tilde{V}$ so that $\Theta_{\Omega,N}[F,\tilde{\lambda}]\cap\tilde{V}$ is a real analytic Banach submanifold of $\mathcal{X}^n_\Omega$ of codimension $\frac{|F|(|F|+1)}{2}$, whose tangent space at $\tilde{\phi}$ equals the space

$$\Big\{\psi\in\mathcal{X}^n_\Omega:\ (\Gamma_{hk}[\psi])_{h,k=1,\dots,|F|}=0\Big\}. \tag{4.2}$$

Proof. We proceed as in the proof of Theorem 3.1. We note that $\Theta_{\Omega,N}[F,\tilde{\lambda}]$ coincides with the set

$$\Big\{\phi\in\mathcal{A}_\Omega\cap\mathcal{X}^n_\Omega:\ (Q_\phi,T_\phi)\in\Theta[F,\tilde{\lambda}^{-1}]\Big\},$$

where

$$\Theta[F,\tilde{\lambda}^{-1}]\equiv\Big\{(Q,T)\in\mathcal{O}_\Omega:\ \mu_j[T]=\tilde{\lambda}^{-1}\ \forall j\in F,\ \mu_l[T]\neq\tilde{\lambda}^{-1}\text{ for all }l\in J[T]\setminus F\Big\}, \tag{4.3}$$

(see (3.4).) Then by equality (3.6), we conclude by exploiting the abstract result of [7, §3]. Finally, we note that by setting $\psi = \tilde{\phi}$ in (3.1), we obtain $\Gamma_{hk}[\tilde{\phi}] = -2\delta_{hk}$. Accordingly $\vartheta \geq 1$. □

By arguing so as to prove Theorem 3.7, we can prove the following.

Theorem 4.2 *Let the same assumptions of Theorem 4.1 hold. Let $\partial\Omega$ have zero n-dimensional Lebesgue measure. Let $\mathcal{X}_\Omega$ contain $\mathcal{D}_\Omega$. Let $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ for all $h = 1, \dots, |F|$. Let q denote the canonical projection of $W^{1,1}(\tilde{\phi}(\Omega))$ onto the quotient space $W^{1,1}(\tilde{\phi}(\Omega))/\overset{o}{W}{}^{1,1}(\tilde{\phi}(\Omega))$. Then ϑ equals the dimension of the subspace of $W^{1,1}(\tilde{\phi}(\Omega))/\overset{o}{W}{}^{1,1}(\tilde{\phi}(\Omega))$ generated by*

$$\left\{ q\left[D\tilde{v}_h D\tilde{v}_k^t - \lambda_F[\tilde{\phi}]\tilde{v}_h\tilde{v}_k \right] : h,k = 1,\dots,|F| \right\}. \tag{4.4}$$

Remark 4.3 *As in Remark 3.9, we observe that if Ω is a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,1}$, and if $\mathcal{X}_\Omega$ is a linear subspace of $\mathrm{Lip}(\Omega)$ containing $\mathcal{D}_\Omega$, and if $\tilde{\phi} \in \Theta_{\Omega,N}[F,\tilde{\lambda}]$ is of class $C^{1,1}$ in $\mathrm{cl}\Omega$, then ϑ equals the dimension of the subspace of $L^1(\partial\tilde{\phi}(\Omega))$ generated by the set*

$$\left\{ D\tilde{v}_h D\tilde{v}_k^t - \lambda_F[\tilde{\phi}]\tilde{v}_h\tilde{v}_k : h,k = 1,\dots,|F| \right\}.$$

Example 4.4 *If Ω is as in Example 3.10, then by arguing as in the previous Remark it follows that $\vartheta = 3$. Thus, in this case condition $\vartheta = \frac{m(m+1)}{2}$ is fulfilled.*

References

[1] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol. I, *Interscience, New York, 1953.*

[2] L.C. Evans, Partial Differential Equations, *American Mathematical Society, Providence, RI, 1998.*

[3] P.D. Lamberti, M. Lanza de Cristoforis, *A global Lipschitz continuity result for a domain dependent Neumann eigenvalue problem for the Laplace operator*, to appear (2005).

[4] P.D. Lamberti, M. Lanza de Cristoforis, *An analyticity result for the dependence of multiple eigenvalues and eigenspaces of the Laplace operator upon perturbation of the domain*, Glasgow Math. J. **44** (2002), 29–43.

[5] P.D. Lamberti, M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator*, Journal of Nonlinear and Convex Analysis **5** (2004), 19–42.

[6] P.D. Lamberti, M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Neumann problem for the Laplace operator*, submitted (2003).

[7] P.D. Lamberti, M. Lanza de Cristoforis, *Persistence of eigenvalues and multiplicity in the Dirichlet problem for the Laplace operator on nonsmooth domains*, submitted (2003).

[8] M. Lanza de Cristoforis, *Properties and pathologies of the composition and inversion operators in Schauder spaces*, Rend. Accad. Naz. delle Scienze detta dei XL, Memorie di Matematica **15** (1991), 93–109.

[9] D. Lupo, A.M. Micheletti, *On multiple eigenvalues of selfadjoint compact operators*, Journal of Mathematical Analysis and Applications **172** (1993), 106–116.

[10] D. Lupo, A.M. Micheletti, *A remark on the structure of the set of perturbations which keep fixed the multiplicity of two eigenvalues*, Revista de Matemáticas Aplicadas **16** (1995), 47–56.

[11] D. Lupo, A.M. Micheletti, *On the persistence of the multiplicity of eigenvalues for some variational elliptic operator depending on the domain*, Journal of Mathematical Analysis and Applications **193** (1995), 990–1002.

[12] M. Teytel, *How rare are multiple eigenvalues?*, Comm. Pure Appl. Math. **52** (1999) 917–934.

[13] G.M. Troianiello, Elliptic differential equations and obstacle problems, *Plenum Press, New York and London, 1987.*

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On Weighted (0,2)-Interpolation

Margit Lénárd

Abstract. The weighted (0,2)-interpolation is studied in a unified way without any additional constraint with respect to the existence, uniqueness and representation (explicit formulae). We give sufficient conditions on the nodes and the weight function, for the problem to be regular. Examples for regular weighted (0,2)-interpolation are given on the zeros of the classical orthogonal polynomials.

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1 Introduction

P. Turán and his associates [13] have started the study of the (0,2)-interpolation in order to get an approximate solution to the differential equation

$$y'' + f \cdot y = 0.$$

In 1961 J. Balázs [2] introduced a generalization of this problem, the *weighted (0,2)-interpolation problem*: For $n \geqslant 1$ let $\{x_k\}_{k=1}^n$ be a set of distinct points (the *nodes*) in the possibly infinite open or closed interval (a, b) and let $w \in C^2(a, b)$ be a function (the *weight function*). Find a polynomial R_n of minimal degree satisfying the conditions

$$R_n(x_k) = y_k, \quad (wR_n)''(x_k) = y_k'' \qquad (k = 1, \dots, n), \tag{1.1}$$

where y_k, y_k'' are arbitrary given real numbers.

This paper is in final form and no version of it will be submitted for publication elsewhere.

If for any choice of y_k and y_k'' $(k = 1, \dots, n)$ there exists a unique polynomial R_n of degree less than $2n$ satisfying the conditions (1.1), the weighted (0,2)-interpolation problem is called *regular* on the given nodes with respect to the weight function w.

The question is how to choose the nodal points and the weight function, so that the problem is regular and in the regular case find simple explicit form of R_n.

J. Balázs [2] proved that on the interval $[-1, 1]$ if the weight function is $w(x) = (1 - x^2)^{(\alpha+1)/2}$, then the problem is not regular on the zeros of the ultraspherical polynomial $P_n^{(\alpha)}$ $(\alpha > -1)$ (there does not exist a polynomial of degree $\leqslant 2n - 1$ satisfying the requirements (1.1)). He proved, that if n is even, then under the condition

$$R_n(0) = \sum_{k=1}^{n} y_k l_k^2(0) \tag{1.2}$$

the problem is regular, but for odd n the uniqueness fails (here l_k represent the fundamental polynomials of Lagrange interpolation on the nodal points x_i). He gave the explicit form of this polynomial and proved a convergence theorem.

Several authors investigated the weighted (0,2)-interpolation with the additional Balázs-type condition (1.2) on the roots of the classical orthogonal polynomials (Joó [5], Joó and Szili [6], Prasad [9], [10], [11], [12], Szili [15], [16]). In special cases Bajpai [1], Eneduanya [4], and Balázs [3] substituted the additional condition (1.2) with interpolatory type conditions. Recently the author [7], [8] studied the weighted (0,2)-interpolation with additional interpolatory conditions in a unified way with respect to the existence, uniqueness and representation (explicit formulae). For more results on (0,2) interpolation we refer to the survey paper of Szili [17].

In this paper we study the weighted (0,2)-interpolation problem without any additional condition. We give sufficient conditions on the nodes and the weight function, for the problem to be regular and in these cases we construct the explicit formulae for the fundamental polynomials. We give examples for regular weighted (0,2)-interpolation on the zeros of the classical orthogonal polynomials.

2 Preliminaries

In what follows, let n be a fixed positive integer and let $[a, b]$ be a finite or infinite interval. Let

$$x_0, x_1, \dots, x_n \in [a, b] \tag{2.1}$$

be distinct points, the nodal points of the interpolation. Let $p_{n-1}(x)$ be a polynomial of degree $n-1$, for which

$$p_{n-1}(x_i) = 0 \qquad (i = 1, \dots, n-1), \tag{2.2}$$

and let

$$l_j(x) = \frac{p_{n-1}(x)}{p'_{n-1}(x_j)(x - x_j)} \qquad (j = 1, \dots, n-1), \tag{2.3}$$

and hence

$$l_j(x_i) = \delta_{i,j} \qquad (i, j = 1, \dots, n-1), \tag{2.4}$$

We recall the general explicit formulae of the fundamental polynomials of first and second kind for the weighted (0,2)-interpolation ([7]):

Lemma 2.1 *Let r and q be polynomials for which $q|r$, and let $w \in C^2(a, b)$ be a weight function. If on the nodal points* (2.1) *the polynomials r and q fulfil*

$$r(x_i) \neq 0, \quad q(x_i) \neq 0 \qquad (i = 1, \dots, n-1), \tag{2.5}$$

and for the weight function

$$w(x_i) \neq 0, \quad (qwp_{n-1})''(x_i) = 0 \qquad (i = 1, \dots, n-1), \tag{2.6}$$

then for $k = 1, \dots, n-1$ the polynomials

$$\begin{aligned} A_k(x) = {} & \frac{r(x)}{r(x_k)}\, l_k^2(x) + \frac{q(x)p_{n-1}(x)}{r(x_k)p'_{n-1}(x_k)} \\ & \times \Big\{ c_k + \int_{x_0}^{x} \Big[\frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k} \cdot \frac{r(t)}{q(t)} + a_k l_k(t) + b_k p_{n-1}(t) \Big] dt \Big\} \end{aligned} \tag{2.7}$$

satisfy the weighted $(0, 2)$-interpolational conditions

$$A_k(x_i) = \delta_{i,k}, \qquad (wA_k)''(x_i) = 0 \qquad (i = 1, \dots, n-1), \tag{2.8}$$

where

$$a_k = -\frac{(rw)''(x_k)}{2(qw)(x_k)} - 2l'_k(x_k)\Big(\frac{r}{q}\Big)'(x_k), \tag{2.9}$$

and b_k, c_k are arbitrary constants.

Furthermore, for $k = 1, \dots, n-1$ the polynomials

$$B_k(x) = \frac{q(x)p_{n-1}(x)}{2q(x_k)w(x_k)p'_{n-1}(x_k)}\Big\{\tilde{a}_k + \int_{x_0}^{x}\Big[l_k(t) + \tilde{b}_k p_{n-1}(t)\Big]dt\Big\} \tag{2.10}$$

satisfy the weighted (0,2)-interpolational conditions

$$B_k(x_i) = 0, \qquad (wB_k)''(x_i) = \delta_{i,k} \qquad (i = 1, \dots, n-1), \tag{2.11}$$

where $\tilde{a}_k$ and $\tilde{b}_k$ are arbitrary constants.

3 Results

Theorem 3.1 *Let $\{x_i\}_{i=0}^{n-1}$ $(n \geqslant 2)$ be a set of distinct nodes in $[a, b]$, let $w \in C^2(a, b)$ be a weight function and $p_{n-1}(x) = c(x - x_1)\dots(x - x_{n-1})$. If*

$$w(x_i) \neq 0, \qquad \big((x - x_0)wp_{n-1}\big)''(x_i) = 0 \qquad (i = 1, \dots, n-1), \tag{3.1}$$

then the weighted $(0, 2)$-interpolation is regular on the nodes $\{x_i\}_{i=0}^{n-1}$ with respect to the weight function w if and only if

$$(wp_{n-1})'(x_0) \neq 0. \tag{3.2}$$

Proof. We apply Lemma 2.1 with $r(x) = (x - x_0)^2$ and $q(x) = (x - x_0)$. In order to get the minimal degree $2n - 1$ for A_k, let $b_k = 0$, and we have for $k = 1, \dots, n-1$

$$\begin{aligned} A_k(x) =& \frac{(x - x_0)^2}{(x_k - x_0)^2}\, l_k^2(x) + \frac{(x - x_0)p_{n-1}(x)}{(x_k - x_0)^2\, p'_{n-1}(x_k)} \\ &\times \Big\{c_k + \int_{x_0}^{x} \frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k}(t - x_0)\, dt + a_k \int_{x_0}^{x} l_k(t)\, dt\Big\}, \end{aligned} \tag{3.3}$$

where

$$a_k = -\frac{\big((x - x_0)^2\, w\big)''(x_k)}{2(x_k - x_0)w(x_k)} - 2l'_k(x_k). \tag{3.4}$$

It is clear, that $A_k(x_i) = \delta_{k,i}$ for $i = 0, 1, \dots, n-1$, and by Lemma 2.1 also $(wA_k)''(x_i) = 0$ for $i = 1, \dots, n-1$. From the condition $(wA_k)''(x_0) = 0$ we have the constant

$$c_k = -\frac{w(x_0)\, l_k(x_0)}{(wp_{n-1})'(x_0)}\big[p'_{n-1}(x_k)l_k(x_0) + p_{n-1}(x_0)a_k\big]. \tag{3.5}$$

In a similar way we obtain

$$B_k(x) = \frac{(x-x_0)p_{n-1}(x)}{2(x_k-x_0)w(x_k)p'_{n-1}(x_k)}\Big\{\tilde{a}_k + \int_{x_0}^{x} l_k(t)dt\Big\} \tag{3.6}$$

for $k = 1, \dots, n-1$, where

$$\tilde{a}_k = -\frac{(wp_{n-1})(x_0)l_k(x_0)}{(wp_{n-1})'(x_0)} \tag{3.7}$$

is determined from the condition $(wB_k)''(x_0) = 0$.

The polynomial A_0, which fulfils the conditions

$$A_0(x_i) = \delta_{0,i}\,, \qquad (wA_0)''(x_i) = 0 \qquad (i = 0, 1, \dots, n-1),$$

will be determined in the form

$$A_0(x) = \frac{p_{n-1}^2(x)}{p_{n-1}^2(x_0)} + (x-x_0)p_{n-1}(x)g_0(x), \tag{3.8}$$

where g_0 is a polynomial of degree at most $n-1$. From the conditions $(wA_0)''(x_i) = 0$ $(i = 1, \dots, n-1)$, we get

$$g_0'(x_i) = -\frac{1}{p_{n-1}^2(x_0)} \cdot \frac{p'_{n-1}(x_i)}{x_i - x_0},$$

which inspires us to define the polynomial g_0' as it follows

$$g_0'(x) = -\frac{1}{p_{n-1}^3(x_0)} \cdot \frac{p'_{n-1}(x)p_{n-1}(x_0) - p'_{n-1}(x_0)p_{n-1}(x)}{x - x_0}.$$

Thus

$$g_0(x) = a_0 - \frac{1}{p_{n-1}^3(x_0)} \int_{x_0}^{x} \frac{p'_{n-1}(t)p_{n-1}(x_0) - p'_{n-1}(x_0)p_{n-1}(t)}{t - x_0}dt, \tag{3.9}$$

where

$$a_0 = \frac{-(wp_{n-1}^2)''(x_0) + 2w(x_0)(p''_{n-1}p_{n-1} - p'^2_{n-1})(x_0)}{2(wp_{n-1})'(x_0)p_{n-1}^2(x_0)} \tag{3.10}$$

is calculated from the condition $(wA_0)''(x_0) = 0$.

It is easy to verify that the polynomial

$$B_0(x) = \frac{(x - x_0)p_{n-1}(x)}{2(wp_{n-1})'(x_0)} \tag{3.11}$$

fulfils the conditions

$$B_0(x_i) = 0\,, \qquad (wB_0)''(x_i) = \delta_{0,i} \qquad (i = 0, 1, \ldots, n-1).$$

As the polynomials A_k and B_k $(k = 0, 1, \ldots, n-1)$, defined by (3.3)-(3.11) are the basis polynomials of the interpolational problem, the polynomial

$$R_n(x) = \sum_{k=0}^{n-1} y_k \; A_k(x) + \sum_{k=0}^{n-1} y_k'' \; B_k(x) \tag{3.12}$$

is of degree at most $2n-1$ and fulfils the conditions of the weighted (0,2)-interpolation.

For the proof of the uniqueness we study the homogeneous problem: Find a polynomial $\bar{R}_n$ of degree at most $2n-1$ such that $\bar{R}_n(x_i) = 0$ and $(w\bar{R}_n)''(x_i) = 0$ $(i = 0, 1, \ldots, n-1)$. From these conditions it is obvious, that

$$\bar{R}_n(x) = (x - x_0)p_{n-1}(x)g_{n-1}(x),$$

where g_{n-1} is a polynomial of degree at most $n-1$. As for $i = 1, \ldots, n-1$

$$(w\bar{R}_n)''(x_i) = 2(x_i - x_0)w(x_i)p_{n-1}'(x_i)g_{n-1}'(x_i) = 0,$$

and $x_i \neq x_0$, $w(x_i) \neq 0$, $p_{n-1}'(x_i) \neq 0$, therefore with a constant C we get $g_{n-1}(x) \equiv C$. Finally, from the condition

$$(w\bar{R}_n)''(x_0) = 2C\,(wp_{n-1})'(x_0) = 0,$$

on using (3.2) we obtain $C = 0$, that is $\bar{R}_n(x) \equiv 0$, which completes the proof. □

Theorem 3.2 *Let $\{x_i\}_{i=0}^{n-1}$ $(n \geqslant 2)$ be a set of distinct nodes in $[a, b]$, let $w \in C^2(a, b)$ be a weight function and $p_{n-1}(x) = c(x - x_1)\ldots(x - x_{n-1})$. If*

$$w(x_i) \neq 0, \qquad \big(wp_{n-1}\big)''(x_i) = 0 \qquad (i = 1, \ldots, n-1), \tag{3.13}$$

then the weighted $(0, 2)$-interpolation is regular on the nodes $\{x_i\}_{i=0}^{n-1}$ with respect to the weight function w if and only if

$$\Big(wp_{n-1} \int_{x_0}^{x} p_{n-1}(t)dt\Big)''(x_0) \neq 0. \tag{3.14}$$

Proof. We apply Lemma 2.1 with $r(x) = (x - x_0)^3$ and $q(x) = 1$. From $A_k(x_0) = 0$ we get $c_k = 0$ and hence for $k = 1, \dots, n-1$

$$\begin{aligned} A_k(x) =& \frac{(x-x_0)^3}{(x_k-x_0)^3}\, l_k^2(x) + \frac{p_{n-1}(x)}{(x_k-x_0)^3 p'_{n-1}(x_k)} \Big\{ a_k \int_{x_0}^{x} l_k(t)\, dt \\ &+ b_k \int_{x_0}^{x} p_{n-1}(t)\, dt + \int_{x_0}^{x} \frac{l'_k(x_k) l_k(t) - l'_k(t)}{t - x_k} (t-x_0)^3\, dt \Big\}, \end{aligned} \tag{3.15}$$

where

$$a_k = -\frac{\big((x-x_0)^3 w\big)''(x_k)}{2w(x_k)} - 6 l'_k(x_k)(x_k - x_0)^2. \tag{3.16}$$

From the condition $(wA_k)''(x_0) = 0$ we have the constant

$$b_k = -a_k \cdot \frac{\Big(w p_{n-1} \int_{x_0}^{x} l_k(t)dt\Big)''(x_0)}{\Big(w p_{n-1} \int_{x_0}^{x} p_{n-1}(t)dt\Big)''(x_0)}. \tag{3.17}$$

In a similar way we obtain

$$B_k(x) = \frac{p_{n-1}(x)}{2w(x_k) p'_{n-1}(x_k)} \Big\{ \int_{x_0}^{x} l_k(t)dt + \tilde{b}_k \int_{x_0}^{x} p_{n-1}(t)dt \Big\} \tag{3.18}$$

for $k = 1, \dots, n-1$, where

$$\tilde{b}_k = -\frac{\Big(w p_{n-1} \int_{x_0}^{x} l_k(t)dt\Big)''(x_0)}{\Big(w p_{n-1} \int_{x_0}^{x} p_{n-1}(t)dt\Big)''(x_0)} \tag{3.19}$$

is determined from the condition $(wB_k)''(x_0) = 0$.

In a similar way to the construction of the polynomials A_0 and B_0 in the proof of Theorem 3.1 we find

$$A_0(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_0)} \left\{ 1 - \frac{(w p_{n-1})''(x_0) \int_{x_0}^{x} p_{n-1}(t)dt}{\Big(w p_{n-1} \int_{x_0}^{x} p_{n-1}(t)dt\Big)''(x_0)} \right\}, \tag{3.20}$$

and

$$B_0(x) = \frac{p_{n-1}(x) \int\limits_{x_0}^{x} p_{n-1}(t)dt}{\Big(wp_{n-1} \int\limits_{x_0}^{x} p_{n-1}(t)dt\Big)''(x_0)}. \tag{3.21}$$

As the polynomials A_k and B_k $(k = 0, 1, \ldots, n-1)$ defined by (3.15)-(3.21) are the basis polynomials of the interpolational problem, the polynomial

$$R_n(x) = \sum_{k=0}^{n-1} y_k \; A_k(x) + \sum_{k=0}^{n-1} y_k'' \; B_k(x) \tag{3.22}$$

is of degree at most $2n-1$ and fulfils the conditions of the weighted (0,2)-interpolation. The uniqueness can be proved in a similar way as in Theorem 3.1, which completes the proof.

□

Theorem 3.3 *Let $\{x_i\}_{i=0}^n$ $(n \geqslant 2)$ be a set of distinct nodes in $[a, b]$, let $w \in C^2(a, b)$ be a weight function and $p_{n-1}(x) = c(x - x_1) \ldots (x - x_{n-1})$. If*

$$w(x_i) \neq 0, \quad \big((x-x_0)(x-x_n)wp_{n-1}\big)''(x_i) = 0 \quad (i = 1, \ldots, n-1), \tag{3.23}$$

then the weighted $(0, 2)$-interpolation is regular on the nodes $\{x_i\}_{i=0}^n$ with respect to the weight function w if and only if

$$\begin{aligned} &((x-x_n)wp_{n-1})'(x_0)((x-x_0)wp_{n-1})'(x_n) \int\limits_{x_0}^{x_n} p_{n-1}(t)dt \\ &\quad + (x-x_n)wp_{n-1})'(x_0)((x-x_0)wp_{n-1}^2)(x_n) \\ &\quad - (x-x_0)wp_{n-1})'(x_n)((x-x_n)wp_{n-1}^2)(x_0) \neq 0. \end{aligned} \tag{3.24}$$

Proof. Now let $r(x) = (x-x_0)^2(x-x_n)^2$ and $q(x) = (x-x_0)(x-x_n)$, then by Lemma 2.1 we have for $k = 1, \ldots, n-1$

$$\begin{aligned} A_k(x) = &\frac{(x-x_0)(x-x_n)}{(x_k-x_0)^2(x_k-x_n)^2} \Bigg[(x-x_0)(x-x_n)l_k^2(x) + \frac{p_{n-1}(x)}{p_{n-1}'(x_k)} \\ &\times \Big\{c_k + \int\limits_{x_0}^{x} \frac{l_k'(x_k)l_k(t) - l_k'(t)}{t-x_k}(t-x_0)(t-x_n)dt \\ &\quad + a_k \int\limits_{x_0}^{x} l_k(t)dt + b_k \int\limits_{x_0}^{x} p_{n-1}(t)dt\Big\}\Bigg], \end{aligned} \tag{3.25}$$

where

$$a_k = -\frac{\big((x-x_0)^2(x-x_n)^2 w\big)''(x_k)}{2(x_k-x_0)(x_k-x_n)w(x_k)} - 2l_k'(x_k)(2x_k-x_0-x_n). \tag{3.26}$$

It is clear, that $A_k(x_i) = \delta_{k,i}$ for $i = 0, 1, \ldots, n$, and by Lemma 2.1 also $(wA_k)''(x_i) = 0$ for $i = 1, \ldots, n-1$. The constants b_k and c_k can be determined from the conditions $(wA_k)''(x_0) = 0$ and $(wA_k)''(x_n) = 0$, that is from the equivalent linear system

$$\begin{aligned} \big((x-x_n)wp_{n-1}\big)'(x_0)\, c_k + \big((x-x_n)wp_{n-1}^2\big)(x_0)\, b_k &= A_1 \\ \big((x-x_0)wp_{n-1}\big)'(x_n)\, c_k + \Big[\big((x-x_0)wp_{n-1}\big)'(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt & \\ + \big((x-x_0)wp_{n-1}^2\big)(x_n)\Big] b_k &= A_2 \end{aligned} \tag{3.27}$$

where

$$A_1 = (x_n-x_0)w(x_0)p_{n-1}(x_0)l_k(x_0)\Big[\frac{x_n-x_0}{x_k-x_0} + a_k\Big], \tag{3.28}$$

$$\begin{aligned} A_2 = &-\frac{(x_n-x_0)^2}{x_n-x_k} w(x_n)p_{n-1}(x_n)l_k(x_n) - \big((x-x_0)wp_{n-1}\big)'(x_n) \\ &\times \int_{x_0}^{x_n} \Big[\frac{l_k'(x_k)l_k(t) - l_k'(t)}{t-x_k}(t-x_0)(t-x_n) + a_k l_k(t)\Big]dt \\ &-(x_n-x_0)w(x_n)p_{n-1}(x_n)l_k(x_n)a_k. \end{aligned} \tag{3.29}$$

The linear system (3.27) has unique solution for b_k and c_k if and only if its determinant is not 0, that is, the condition (3.24) is fulfilled.

On using (2.10) with $q(x) = (x-x_0)(x-x_n)$ and the condition (3.24) we have for $k = 1, \ldots, n-1$

$$\begin{aligned} B_k(x) = &\frac{(x-x_0)(x-x_n)p_{n-1}(x)}{2(x_k-x_0)(x_k-x_n)w(x_k)p_{n-1}'(x_k)} \\ &\times \Big\{\tilde{a}_k + \int_{x_0}^{x} l_k(t)dt + \tilde{b}_k \int_{x_0}^{x} p_{n-1}(t)dt\Big\}, \end{aligned} \tag{3.30}$$

where $\tilde{a}_k$ and $\tilde{b}_k$ are the unique solution of the linear system

$$\begin{aligned}\big((x-x_n)wp_{n-1}\big)'(x_0)\,\tilde{a}_k + \big((x-x_n)wp_{n-1}^2\big)(x_0)\,\tilde{b}_k &= B_1\\ \big((x-x_0)wp_{n-1}\big)'(x_n)\tilde{a}_k + \Big[\big((x-x_0)wp_{n-1}\big)'(x_n)\int_{x_0}^{x_n} p_{n-1}(t)dt &\\ +\big((x-x_0)wp_{n-1}^2\big)(x_n)\Big]\,\tilde{b}_k &= B_2\end{aligned} \tag{3.31}$$

where

$$B_1 = (x_n - x_0)w(x_0)p_{n-1}(x_0)l_k(x_0), \tag{3.32}$$

$$B_2 = -\Big((x-x_0)wp_{n-1}\int_{x_0}^{x} l_k(t)dt\Big)'(x_n). \tag{3.33}$$

Following the steps of the proof of Theorem 3.1, under the condition (3.24) we obtain the polynomial

$$A_0(x) = \frac{(x-x_n)^2\,p_{n-1}^2(x)}{(x_0-x_n)^2\,p_{n-1}^2(x_0)} + (x-x_0)(x-x_n)p_{n-1}(x)g_0(x) \tag{3.34}$$

with

$$\begin{aligned}g_0(x) =& d_0 + c_0\int_{x_0}^{x} p_{n-1}(t)dt + \frac{1}{p_{n-1}^3(x_0)(x_0-x_n)^2}\\ &\int_{x_0}^{x}\frac{p_{n-1}'(t)p_{n-1}(x_0) - p_{n-1}'(x_0)p_{n-1}(t)}{t-x_0}(t-x_n)dt,\end{aligned} \tag{3.35}$$

where the parameters c_0 and d_0 are the unique solution of the linear system

$$\begin{aligned}\big((x-x_n)wp_{n-1}\big)'(x_0)d_0 + \big((x-x_n)wp_{n-1}^2\big)(x_0)c_0 &= C_1\\ \big((x-x_0)wp_{n-1}\big)'(x_n)d_0 + \Big[\big((x-x_0)wp_{n-1}\big)'(x_n)\int_{x_0}^{x_n} p_{n-1}(t)dt &\\ +\big((x-x_0)wp_{n-1}^2\big)(x_n)\Big]c_0 &= C_2\end{aligned} \tag{3.36}$$

where

$$C_1 = \frac{p_{n-1}''(x_0)p_{n-1}(x_0) - p_{n-1}'^2(x_0)}{p_{n-1}^2(x_0)}w(x_0) - \frac{\big((x-x_n)^2wp_{n-1}^2\big)''(x_0)}{2(x_0-x_n)^2\,p_{n-1}^2(x_0)}, \tag{3.37}$$

$$C_2 = -\frac{w(x_n)p_{n-1}^2(x_n)}{(x_0-x_n)^2 p_{n-1}^2(x_0)} + \frac{\big((x-x_0)wp_{n-1}\big)'(x_n)}{(x_0-x_n)^2\, p_{n-1}^3(x_0)} \int_{x_0}^{x_n} \frac{p'_{n-1}(t)p_{n-1}(x_0) - p'_{n-1}(x_0)p_{n-1}(t)}{t-x_0}(t-x_n)dt. \tag{3.38}$$

In a similar way, just by interchanging the roles of x_0 and x_n, we obtain the polynomial A_n of degree at most $2n+1$, which fulfils the conditions

$$A_n(x_i) = \delta_{n,i}\,, \qquad (wA_n)''(x_i) = 0 \qquad (i = 0, 1, \ldots, n).$$

Finally, the polynomial B_0, which is of degree at most $2n+1$ and fulfils the conditions

$$B_0(x_i) = 0\,, \qquad (wB_0)''(x_i) = \delta_{0,i} \qquad (i = 0, 1, \ldots, n),$$

can be written in the form

$$B_0(x) = (x-x_0)(x-x_n)p_{n-1}(x)\bar{c}_0 \int_{x_0}^{x} p_{n-1}(t)dt + \bar{d}_0, \tag{3.39}$$

where the constants $\bar{c}_0$ and $\bar{d}_0$ are defined by the linear system

$$\begin{aligned} \big((x-x_n)wp_{n-1}\big)'(x_0)\bar{d}_0 + \big((x-x_n)wp_{n-1}^2\big)(x_0)\bar{c}_0 &= \frac{1}{2} \\ \big((x-x_0)wp_{n-1}\big)'(x_n)\bar{d}_0 + \Big[\big((x-x_0)wp_{n-1}\big)'(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt & \\ + \big((x-x_0)wp_{n-1}^2\big)(x_n)\Big]\bar{c}_0 &= 0 \end{aligned} \tag{3.40}$$

The linear system (3.40) has unique solution for $\bar{c}_0$ and $\bar{d}_0$ if and only if the condition (3.24) is fulfilled. In a similar way we get the polynomial B_n, for which

$$B_n(x_i) = 0\,, \qquad (wB_n)''(x_i) = \delta_{n,i} \qquad (i = 0, 1, \ldots, n).$$

As the polynomials A_k and B_k $(k = 0, 1, \ldots, n)$ defined above are the basis polynomials of the interpolational problem, the polynomial

$$R_n(x) = \sum_{k=0}^{n} y_k\ A_k(x) + \sum_{k=0}^{n} y_k''\ B_k(x) \tag{3.41}$$

is of degree at most $2n+1$ and fulfils the conditions of the weighted (0,2)-interpolation. The uniqueness can be proved in a similar way as in Theorem 3.1, which completes the proof. □

4 Examples on the zeros of the classical orthogonal polynomials

Lemma 4.1 *The classical orthogonal polynomials (Jacobi-, Laguerre-, and Hermite polynomials) satisfy the homogeneous differential equation*

$$(wy)'' + f \cdot (wy) = 0 \tag{4.1}$$

with an appropriate weight function w and function f.

Proof. We refer to [14] ((4.24.1), (5.1.2), (5.5.2)):

For the Jacobi polynomials $y = P_n^{(\alpha,\beta)}(x) \quad (\alpha, \beta > -1)$

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}},$$

and

$$f(x) = \frac{1}{4}\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\frac{1-\beta^2}{(1+x)^2} + \frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-x^2)}.$$

For the Laguerre polynomials $y = L_n^{(\alpha)}(x) \quad (\alpha > -1)$

$$w(x) = e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}},$$

and

$$f(x) = \frac{2n+\alpha+1}{2x} + \frac{1-\alpha^2}{4x^2} - \frac{1}{4}.$$

For the Hermite polynomials $y = H_n(x)$

$$w(x) = e^{-\frac{x^2}{2}}, \qquad f(x) = 2n + 1 - x^2. \qquad \square$$

4.1 On the zeros of Jacobi polynomials

Let $x_0 = 1$, $p_{n-1}(x) = P_{n-1}^{(\alpha,\beta)}(x)$ for $\alpha, \beta > -1$, and the weight function $w(x) = (1-x)^{\frac{\alpha-1}{2}}(1+x)^{\frac{\beta+1}{2}}$.

It is clear that $w(x_i) \neq 0$ for $i = 1, \ldots, n-1$, and by Lemma 4.1

$$((x-x_0)wp_{n-1})''(x_i) = 0 \qquad (i = 1, \ldots, n-1).$$

From

$$(wp_{n-1})'(x) = (1-x)^{\frac{\alpha-3}{2}}(1+x)^{\frac{\beta-1}{2}}\Big\{\frac{1}{2}\big[(1-\alpha)(1+x) + (1+\beta)(1-x)\big]P_{n-1}^{(\alpha,\beta)}(x) + (1-x^2)P_{n-1}^{(\alpha,\beta)'}(x)\Big\}$$

it can be verified that the condition $(wp_{n-1})'(1) \neq 0$ is not fulfilled for $\alpha \neq 1$ or 3.

For $\alpha = 1$

$$\begin{aligned}(wp_{n-1})'(1) &= 2^{\frac{\beta-1}{2}}\big[\frac{\beta+1}{2}P_{n-1}^{(1,\beta)}(1) + 2P_{n-1}^{(1,\beta)'}(1)\big] \\ &= 2^{\frac{\beta-1}{2}}\big[\frac{\beta+1}{2}P_{n-1}^{(1,\beta)}(1) + (n+\beta+1)P_{n-2}^{(2,\beta+1)}(1)\big] \\ &= 2^{\frac{\beta-3}{2}}n^2(n+\beta) \neq 0,\end{aligned}$$

and for $\alpha = 3$

$$(wp_{n-1})'(1) = -2^{\frac{\beta+1}{2}}P_{n-1}^{(3,\beta)}(1) \neq 0.$$

Hence, applying Theorem 3.1, we have proved

Corollary 4.2 *The weighted* $(0,2)$*-interpolation problem is regular on the zeros of*

$$P_n^{(-1,\beta)}(x) = \frac{n+\beta}{2n}(x-1)P_{n-1}^{(1,\beta)}(x) \quad (\beta > -1)$$

with respect to the weight function

$$w(x) = (1+x)^{\frac{\beta+1}{2}},$$

and on the zeros of $(x-1)P_{n-1}^{(3,\beta)}(x)$ *with respect to* $w(x) = (1-x)(1+x)^{\frac{\beta+1}{2}}$ *(*$\beta > -1$*).*

Remark. On using $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ (c.f. [14] (4.1.3)) we obtain, that the weighted (0,2)-interpolation is regular on the zeros of

$$(x+1)P_n^{(\alpha,1)}(x) \qquad \text{and} \qquad (x+1)P_n^{(\alpha,3)}(x)$$

with respect to the weight functions

$$w(x) = (1-x)^{\frac{\alpha+1}{2}} \qquad \text{and} \qquad w(x) = (1+x)(1-x)^{\frac{\alpha+1}{2}},$$

respectively, for $\alpha > -1$.

If $x_0 = -1$, $x_n = 1$, $\alpha = \beta = 0$ and $p_{n-1}(x) = P_{n-1}^{(1,1)}(x)$, then the nodes are the zeros of $(1-x^2)P_n'(x)$, where P_n is the Legendre polynomial of degree n. For the weight function $w(x) = 1$, applying Theorem 3.3 and Lemma 4.1 we obtain the classical result of P. Turán [13]:

Corollary 4.3 *For odd* n *the* $(0,2)$*-interpolation is regular on the zeros of* $(1-x^2)P_n'(x)$.

4.2 On the zeros of Laguerre polynomials

Let $x_0 = 0$, $p_{n-1}(x) = L_{n-1}^{(\alpha)}(x)$ for $\alpha > -1$, and let $w(x) = e^{-\frac{x}{2}} x^{\frac{\alpha-1}{2}}$ be the weight function.

It is clear that $w(x_i) \neq 0$ for $i = 1, \ldots, n-1$, and by Lemma 4.1

$$((x - x_0) w p_{n-1})''(x_i) = 0 \qquad (i = 1, \ldots, n-1).$$

From

$$(w p_{n-1})'(x) = e^{-\frac{x}{2}} x^{\frac{\alpha-3}{2}} \left\{ \frac{\alpha - 1 - x}{2} L_{n-1}^{(\alpha)}(x) + x L_{n-1}^{(\alpha)'}(x) \right\}$$

it can be verified that the condition $(w p_{n-1})'(0) \neq 0$ is not fulfilled for $\alpha \neq 1$ or 3.

For $\alpha = 1$

$$(w p_{n-1})'(0) = -\frac{1}{2} L_{n-1}^{(1)}(0) + L_{n-1}^{(1)'}(0) = -\frac{n^2}{2} \neq 0,$$

and for $\alpha = 3$

$$(w p_{n-1})'(0) = L_{n-1}^{(3)}(0) \neq 0.$$

Hence, applying Theorem 3.1, we have proved

Corollary 4.4 *The weighted $(0, 2)$-interpolation problem is regular on the zeros of*

$$L_n^{(-1)}(x) = -\frac{1}{n} x L_{n-1}^{(1)}(x)$$

with respect to the weight function

$$w(x) = e^{-\frac{x}{2}},$$

and on the zeros of $x L_{n-1}^{(3)}(x)$ with respect to $w(x) = x e^{-\frac{x}{2}}$.

4.3 On the zeros of Hermite polynomials

Applying Theorem 3.2 and Lemma 4.1 one can prove

Corollary 4.5 *Let $\{x_i\}_{i=0}^n$ be the nodes, where $x_1, \ldots, x_n$ are the zeros of the Hermite polynomial $H_n(x)$. If $H_n'(x_0) = 0$, then the weighted $(0, 2)$-interpolation is regular, with respect to the weight function $w(x) = e^{-x^2/2}$. For even n, if $x_0 = 0$ the same problem is not regular.*

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References

[1] Bajpai, P., Weighted (0,2) interpolation on the extended Tchebycheff nodes of second kind, *Acta Math. Hung.*, **63** (1994), no. 2, 167–181.

[2] Balázs, J., Weighted (0,2)-interpolation on the zeros of the ultaspherical polynomials (in Hungarian: Súlyozott $(0, 2)$-interpoláció ultraszférikus polinom gyökein), *MTA III.oszt. Közl.*, **11** (1961), 305–338.

[3] Balázs, J., Modified weighted (0,2)interpolation, *Approximation Theory. In memory of A. K. Varma.* New York, NY: Marcel Dekker. Pure Appl. Math., 212, Govil N. K. (ed.) et al., (1998), 61-85.

[4] Eneduanya, S. A. N., The weighted (0,2) interpolation I, *Demonstracio Mathematica*, **18** (1985), no. 1, 9–13.

[5] Joó, I., On weighted $(0, 2)$ interpolation, *Annales Univ. Sci. Budapest., Sect. Math.*, **38** (1995), 185–222.

[6] Joó, I. and Szili, L., On weighted $(0, 2)$-interpolation on the roots of Jacobi polynomials, *Acta Math. Hung.*, **66** (1995), no. 1-2, 25–50.

[7] Lénárd, M., Weighted (0,2)-interpolation with interpolatory boundary conditions, *Annales Univ. Sci. Budapest., Sect. Comput.*, **24** (2004), 253–273.

[8] Lénárd, M., Weighted (0,2)-interpolation with additional interpolatory condition, Proceedings of the International Conference on Mathematics and its Applications, ICMA04, April 5-7, 2004, Kuwait, 329–342.

[9] Prasad, J., Some interpolatory polynomials on Hermite abscissas, *Math. Japonica*, **12** (1967), 73–80.

[10] Prasad, J., Balázs-type interpolation on Laguerre abscissas, *Math. Japonica*, **13** (1967), 47–53.

[11] Prasad, J., On the weighted $(0, 2)$-interpolation, *SIAM J. Numer. Anal.*, **7** (1970), 428–446.

[12] Prasad, J.,On the representation of functions by interpolatory polynomials, *Mathematica (Cluj)*, **15** (1973), 289–305.

[13] Surányi, J. and Turán, P., Notes on Interpolation I, On some interpolatorical properties of the ultraspherical polynomials, *Acta Math. Acad. Sci. Hung.*, **6** (1955), 66–79.

[14] Szegő, G., *Orthogonal polynomials*, Amer. Math. Soc. Coll. Publ., New York, 1959. (Fourth Edition in 1975)

[15] Szili, L., Weighted $(0, 2)$-interpolation on the roots of Hermite polynomials, *Annales Univ. Sci. Budapest., Sect. Math.*, **27** (1985), 153–166.

[16] Szili, L., Weighted $(0, 2)$-interpolation on the roots of the classical orthogonal polynomials, *Bull. Allahabad Math. Soc.*, **8–9** (1993–94), 111–120.

[17] Szili, L., A survey on (0,2) interpolation, *Annales Univ. Sci. Budapest., Sect. Comput.*, **16** (1996), 377–390.

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On the Pointwise Strong Approximation by Orthogonal Expansions

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Abstract. There is introduced a modificated local modulus of continuity as a measure of pointwise strong summability of expansion with respect to polynomial-like orthonormal system. The approximation versions and generalizations of the some special cases of the known results of K. Tandori [10],[11] , G. Freud [4] and of B.P. Osilenker [9] are obtained.

Key words: Strong approximation by orthogonal expansion, rate of pointwise strong summability.

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1 Introduction

Let ρ be a real–valued, non-negative function integrable in the Lebesgue sense in $[a,b]$ such that $\rho(t)=0$ almost nowhere.

Let $L^p_\rho = L^p_\rho[a,b]$ $(1 \le p \le \infty)$ be the class of all real–valued functions integrable in the Lebesgue sense with p–th power in $[a,b]$ with the weight ρ or bounded almost everywhere in $[a,b]$,respectively. Let us define the norm of $f \in L^p_\rho$ as

$$\|f\|_{L^p_\rho} = \|f\|_{L^p_\rho[a,b]} = \begin{cases} \left(\int_a^b \mid f(t) \mid^p \rho(t)\, dt\right)^{1/p} & \text{when} \quad 1 \le p < \infty, \\ ess\sup\{\mid f(t) \mid : t \in [a,b]\} & \text{when} \quad p = \infty. \end{cases}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Consider the orthonormal system (ϕ_ν) and the expansion of $f \in L_\rho^p$

$$Sf(x) = \sum_{\nu=0}^{\infty} a_\nu(f)\,\phi_\nu(x) \qquad \text{with} \quad x \in [a,b],$$

where $a_\nu(f)$ is the Fourier coefficient of f with respect to the system (ϕ_ν). Denote by

$$S_k f(x) = \sum_{\nu=0}^{k} a_\nu(f)\,\phi_\nu(x) = \int_a^b f(t) K_k(t,x)\,\rho(t)\,dt$$

the partial sums of Sf, where $K_k(\cdot,\cdot)$ is the kernel of integral, and let

$$H_n^q f(x) := \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |S_k f(x) - f(x)|^q \right\}^{\frac{1}{q}}, \qquad (q>0)$$

or

$$G_{M_n}^q f(x) := \left\{ \frac{1}{M_n} \sum_{k=0}^{n} C_k^{2-q} |S_k f(x) - f(x)|^q \right\}^{\frac{1}{q}}, \qquad (q>0)$$

where C_k and M_n will be denote later.

The estimate of the quantity $H_n^q f$ in connection with the orthogonal expansion was began by K. Tandori [10] but the quantity $G_{M_n}^q f$ was considered by B.P. Osilenker [9].

We will consider constant-preserving polynomial-like orthonormal system (ϕ_ν), i.e. such that $\phi_0 = const$, and the kernel has the following structure

$$K_k(t,x) = \sum_{\nu=0}^{k} \phi_\nu(x)\,\phi_\nu(t) = \sum_{\mu=1}^{m} F_\mu(t,x) \sum_{i,j=-s}^{s} \gamma_{i,j,\mu}^{(k)} \phi_{k+i}(x)\,\phi_{k+j}(t)$$

with $t,x \in [a,b]$, where s and m are natural numbers, independent of k, and the constants $\left|\gamma_{i,j,\mu}^{(k)}\right|$ have a common bound, independent of k, while the measurable functions F_μ satisfy the condition $F_\mu(t,x) = O\left(|t-x|^{-1}\right)$ for every $t,x \in [a,b]$, (here ϕ_{k+i} with eventually negative index should be considered to be identically equal zero) (see [1, p.177]).

The pointwise characteristic

$$\overline{w}_{x,p}f(\delta) := \sup_{0<h\leq\delta}\left\{\frac{1}{h}\int_{x-h}^{x+h}|f(t)-f(x)|^p\,\rho(t)\,dt\right\}^{1/p},$$

of $f\in L^p_\rho$ $(1\leq p<\infty)$, constructed on the base of definition of Lebesgue points $(L^p_\rho-points)$, was firstly(with $p=1$ and $\rho\equiv 1$) used, as a measure of approximation, by S.Aljančič, R.Bojanic and M.Tomić in [2]. Here for the estimate of quantities H^q_nf and $G^q_{M_n}f$ we will use the following modified characteristic of such type

$$w_{x,p}f(\delta) := \left\{\frac{1}{\delta}\int_{x-\delta}^{x+\delta}\chi_{[a,b]}(t)\,|f(t)-f(x)|^p\,\rho(t)\,dt\right\}^{1/p},$$

where $\chi_{[a,b]}$ is the characteristic function of interval $[a,b]$. We will also use the classical integral modulus of continuity in the following form (for $p>1$)

$$w_pf(\delta)=w_{L^p_\rho[a,b]}f(\delta) := \sup_{|h|\leq\delta}\|f(\cdot+h)-f(\cdot)\|_{L^p_\rho}.$$

It is easily seen that under the additional assumption that $|\rho(t)|\leq C$ almost everywhere in $[a,b]$, by the Minkowski inequality,

$$\|w_{\cdot,p}f(\delta)\|_{L^{p'}_\rho}\leq(2C)^{1/p}\,w_{p'}f(\delta),\quad \text{for } f\in L^{p'}_\rho \ \ (\ p'\geq p>1 \text{ and } p<\infty).$$

By C we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same of each occurrence.

2 Statement of the results

In this section we will present the rate of pointwise strong summability of orthogonal expansion of Lebesgue integrable functions.

At the begin for compare we formulate without proof three theorems, from [7], which are the approximation versions of results of K. Tandori type [10], [11] and G. Freud type [4] (see also [3]) and contain the estimate of quantity H^q_nf.

Theorem 1 ([7]) *Let $\phi_\nu\in L^r_\rho$ $(2\leq r\leq\infty)$ create an constant-preserving orthonormal polynomial-like system (ϕ_ν) satisfying the following conditions*

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}\|\phi_k(\cdot)\|^{2q}_{r,\rho}\right\}^{1/2q}\leq C \qquad \begin{array}{c}\text{for every positive integer } n \text{ with}\\ 2\leq 2q\leq r<\infty \ \text{ or } \ 2\leq 2q<r=\infty,\end{array}$$

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\phi_k(x)\right|^{2q}\right\}^{1/2q}\leq C(x) \qquad \text{for every positive integer } n$$
$$\left\{\frac{1}{\delta}\int_{x-\delta}^{x+\delta}\frac{1}{n+1}\sum_{k=0}^{n}\left|\phi_k(t)\right|^{2q}\rho(t)\,dt\right\}^{1/2q}\leq C(x) \qquad \begin{array}{l}\text{with } \delta=\frac{b-a}{n+1} \text{ and } 2\leq 2q<\infty,\\ \text{at the considered point } x\in(a,b).\end{array}$$

If $f\in L_\rho^p$ *with* $1<p\leq 2$ *such that* $\frac{1}{2q}+\frac{1}{p}=1$, *then for any* $q\prime\in(0,q]$,

$$H_n^{q'}f(x)\leq C(x)\left\{\frac{1}{(n+1)^{p-1}}\sum_{k=0}^{n}\frac{1}{(k+1)^{2-p}}\left(w_{x,p}f(\frac{b-a}{k+1})\right)^p\right\}^{1/p},$$

for all $x\in(a,b)$ *and every positive integer* n.

Theorem 2 ([7]) *Let* $f\in L_\rho^p$ $(1<p\leq 2)$ *and let an constant-preserving orthonormal polynomial-like system* (ϕ_ν) *satisfying the following conditions*

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}\|\phi_k(\cdot)\|_{r,\rho}^q\right\}^{1/q}\leq C \qquad \begin{array}{c}\text{for every positive integer } n \text{ with}\\ 2\leq q\leq r<\infty \quad or \quad 2\leq q<r=\infty,\end{array}$$

$$\begin{array}{c}|\phi_n(x)|\leq C(x)\\ \left\{\frac{1}{\delta}\int_{x-\delta}^{x+\delta}\left(\frac{1}{n+1}\sum_{k=0}^{n}|\phi_k(t)|\right)^q\rho(t)\,dt\right\}^{1/q}\leq C(x)\end{array} \qquad \begin{array}{c}\text{for every positive integer } n\\ \text{with } \delta=\frac{b-a}{n+1} \text{ and } 2\leq q<\infty\\ \text{such that } \frac{1}{q}+\frac{1}{p}=1, \text{ at}\\ \text{considered point } x\in(a,b).\end{array}$$

If $0<q'\leq q$, *then*

$$H_n^{q'}f(x)\leq C(x)\left\{\frac{1}{(n+1)^{p-1}}\sum_{k=0}^{n}\frac{1}{(k+1)^{2-p}}\left(w_{x,p}f(\frac{b-a}{k+1})\right)^p\right\}^{1/p},$$

for all $x\in(a,b)$ *and every positive integer* n.

Theorem 3 ([7]) *Let* $\phi_\nu\in L_\rho^\infty$ *create an constant-preserving orthonormal polynomial-like system* (ϕ_ν) *satisfying the following condition*

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\phi_k(t)\right|^{2q}\right\}^{1/2q}\leq C, \qquad \begin{array}{c}\text{for almost all } t\in(a,b) \text{ and every}\\ \text{positive integer } n \text{ with } 2\leq 2q<\infty.\end{array}$$

If $f\in L_\rho^p$ *with* $1<p\leq 2$ *such that* $\frac{1}{2q}+\frac{1}{p}=1$, *then for any* $q\prime\in(0,q]$,

$$H_n^{q'}f(x)\leq C\left\{\frac{1}{(n+1)^{p-1}}\sum_{k=0}^{n}\frac{1}{(k+1)^{2-p}}\left(w_{x,p}f(\frac{b-a}{k+1})\right)^p\right\}^{1/p},$$

holds, for all $x \in (a,b)$ *and sufficiently great positive integer* n *being dependent on* x, *or for all* $x \in [c,d] \subset (a,b)$ *and sufficiently great positive integer* n *being independent of* x, *or for all* $x \in (a,b)$ *and every positive integer* n *with* $C = C(x)$.

Remark 1 *Theorems 1 and 2 (for p=2) are the approximation versions of the results of K.Tandori[10] and G.Freud[4] type, and by analysis of the proof of Theorem 2 we note that slight modification of it gives the approximation version of the another result of K.Tandori[11] type.*

From the last theorem was also in [7] derived the corollary.

Corollary 1 *Let* $f \in L_\rho^{p'}[a,b]$ $(p' \geq p > 1)$ *, with bounded weight* ρ, *and let an constant-preserving orthonormal polynomial-like system* (ϕ_n) *satisfying the following condition*

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |\phi_k(t)|^{2q} \right\}^{\frac{1}{2q}} \leq C \qquad \begin{array}{l} for \quad almost\ all\ t \in [a,b], \\ q \in [1,+\infty)\ ,\ n = 1,2,3... \end{array} .$$

If $0 < q' \leq q$ *and* $\frac{1}{2q} + \frac{1}{p} = 1$ *,* $(p \leq 2)$, *then*

$$\left\| H_n^{q'} f(\cdot) \right\|_{L_\rho^{p'}[c,d]} \leq C \left\{ \frac{1}{(n+1)^{p-1}} \sum_{k=0}^{n} \frac{1}{(k+1)^{2-p}} \left(w_{L_\rho^{p'}[c,d]} f(\frac{b-a}{k+1}) \right)^p \right\}^{1/p}$$

holds, for all positive sufficiently great integer n *, where* $[c,d] \subset (a,b)$.

Next we present the approximation versions of the result of B.P. Osilenker [9, pp. 179-182] contain the estimate of quantity $G_{M_n}^q f$.

Theorem 4 *For every* $\nu = 0,1,2,...$ *let* $\phi_\nu \in L_\rho^r$ *with* $2 < r \leq \infty$ *and with* ρ *bounded a.e. on* $[a,b]$ *, create an constant-preserving orthonormal polynomial-like system* (ϕ_ν) *satisfying the following conditions*

$$\|\phi_\nu\|_{L_\rho^r} \leq C_\nu \quad with\ an\ almost\ decreasing\ sequence\ (C_\nu),$$

$$|\phi_\nu(x)| \leq C(x) \quad at\ considered\ point\ x \in (a,b).$$

If $f \in L_\rho^p$ *with* $p \in (r',2]$ *, where* $r' = \frac{r}{r-1}$ *, such that* $\frac{r'}{p} + \frac{2-r'}{q} = 1$, *then*

$$G_{M_n}^q f(x) \leq C(x) \left\{ \frac{1}{B_n^{p-1}} \sum_{k=0}^{B_n} \frac{1}{(k+1)^{2-p}} \left(w_{x,p} f(\frac{b-a}{k+1}) \right)^p \right\}^{1/p},$$

for all $x \in (a,b)$ *and every positive integer* n, *where* $M_n := A_n^{r'/(r'+p')}$ *with* $p\prime = \frac{p}{p-1}$ *and* $B_n := \left[A_n^{r'/(2-r'+qr')}\right]$ *while* $A_n := \sum_{k=0}^{n} C_k^{2-q} \left(\sum_{j=0}^{k} C_j\right)^q$.

Theorem 5 *Under the assumptions of Theorem 4 we have*

$$G_{M_n}^q f(x) \leq C(x) \left\{ \sum_{k=0}^{N_n} 2^{(1-p)k} \left(w_{x,p} f(2^k (b-a) A_n^{-r'/(2-r'+qr')}) \right)^p \right\}^{1/p},$$

for all $x \in (a,b)$ *and every positive integer* n, *where* $N_n := \left[1 + \log_2 A_n^{r'/(2-r'+qr')}\right]$.

A slight modification of our assumptions leads us to the following theorem presented without proof

Theorem 6 *Let* $\phi_\nu \in L_\rho^\infty$ *with* ρ *bounded a.e. on* $[a,b]$, *create an constant-preserving orthonormal polynomial-like system* (ϕ_ν) *satisfying, for every* $\nu = 0,1,2,3..$, *the following condition*

$$\|\phi_\nu\|_{L_\rho^\infty} \leq C_\nu \quad \text{with an almost decreasing sequence } (C_\nu).$$

If $f \in L_\rho^p$ *with* $p \in (1,2]$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$G_{M_n}^q f(x) \leq C \left\{ \frac{1}{B_n^{p-1}} \sum_{k=0}^{B_n} \frac{1}{(k+1)^{2-p}} \left(w_{x,p} f(\frac{b-a}{k+1}) \right)^p \right\}^{1/p}$$

and

$$G_{M_n}^q f(x) \leq C \left\{ \sum_{k=0}^{N_n} 2^{(1-p)k} \left(w_{x,p} f(2^k (b-a) A_n^{-1/(1+q)}) \right)^p \right\}^{1/p},$$

holds, for all $x \in (a,b)$ *and sufficiently great positive integer* n *being dependent on* x, *or for all* $x \in [c,d] \subset (a,b)$ *and sufficiently great positive integer* n *being independent of* x, *or for all* $x \in (a,b)$ *and every positive integer* n *with* $C = C(x)$, *where* $M_n = A_n^{1/(1+p')}$ *with* $p\prime = \frac{p}{p-1}$, $B_n = \left[A_n^{1/(1+q)}\right]$, $N_n = \left[1 + \log_2 A_n^{1/(1+q)}\right]$ *while* $A_n := \sum_{k=0}^{n} C_k^{2-q} \left(\sum_{j=0}^{k} C_j\right)^q$.

From the last theorem we conclude that

Corollary 2 *Under the assumptions of Theorem 6 we have*

$$\left\|G_{M_n}^q f(\cdot)\right\|_{L_\rho^p[c,d]} \leq C\left\{\frac{1}{B_n^{p-1}}\sum_{k=0}^{B_n}\frac{1}{(k+1)^{2-p}}\left(w_{L_\rho^p[c,d]}f(\frac{b-a}{k+1})\right)^p\right\}^{1/p}$$

and

$$\left\|G_{M_n}^q f(\cdot)\right\|_{L_\rho^p[c,d]} \leq C\left\{\sum_{k=0}^{N_n}2^{(1-p)k}\left(w_{L_\rho^p[c,d]}f(2^k(b-a)A_n^{-1/(1+q)})\right)^p\right\}^{1/p},$$

for all positive sufficiently great integer n, where $[c,d]\subset(a,b)$ and where $M_n=A_n^{1/(1+p')}$ with $p\prime=\frac{p}{p-1}$, $B_n=\left[A_n^{1/(1+q)}\right]$, $N_n=\left[1+\log_2 A_n^{1/(1+q)}\right]$ while $A_n:=\sum_{k=0}^n C_k^{2-q}\left(\sum_{j=0}^k C_j\right)^q$.

3 Auxiliary results

In the proofs of our theorems we will use the following Lemmas.

The first one, had used in [7] for the proofs of the Theorems 1-3, is a slight generalization of some version of inequality of F. Riesz [5, Theorem 6.3.1] from [6]

Lemma 1 ([6]) *Let $\phi_\nu\in L_\rho^r[c,d]$ for every $\nu=0,1,2,3...$ with $2\leq r\leq\infty$ and $[c,d]\subset[a,b]$, and let*

$$\sum_{k=0}^n|\phi_k(t)|^q\leq C^q(n+1)\qquad \text{for almost all}\quad t\in[c,d]\quad \text{when } 2\leq q\leq\ r=\infty$$

or

$$\sum_{k=0}^n\|\phi_k\|_{L_\rho^r[c,d]}^q\leq C^q(n+1)(b-a)^{q/r}\quad \text{when}\ \ 2\leq q\leq r<\infty$$

with C independent of n. If $f\in L_\rho^p[c,d]$ with p such that $\frac{1}{p}+\frac{1}{q}=1$ wanishing outside $[c,d]$, then

$$\left(\sum_{\nu=0}^\infty|a_\nu(f)|^q\right)^{1/q}\leq C^{\frac{2-p}{p}}\|f\|_{L_\rho^p[c,d]}\ .$$

Proof. Since the generalization based on introduction of the weight function ρ, thus the proof is exactly the same as the proof in [6]. ■

The second one which we use for the proof of the Theorems 4,5, is the another generalization of the inequality of F. Riesz [5, Theorem 6.3.1] from [8, Theorem 1](cf. [9, p. 166]).

Lemma 2 ([8]) *Let* $\phi_\nu \in L^r_\rho[c,d]$ *with* $2 < r \leq \infty$ *and let*

$$\|\phi_\nu\|_{L^r_\rho[c,d]} \leq C_\nu \quad \textit{for every} \quad \nu = 0,1,2,3...$$

If $f \in L^p_\rho[c,d]$ *with* $p \in (r',2]$ *, where* $r' = \frac{r}{r-1}$ *, such that* $\frac{r'}{p} + \frac{2-r'}{q} = 1$ *wanishing outside* $[c,d]$*, then*

$$\left(\sum_{\nu=0}^{\infty} C_\nu^{2-q} \left|a_\nu(f)\right|^q\right)^{1/q} \leq \|f\|_{L^p_\rho[c,d]} \quad .$$

The next one contains the function $w_{x,p}f$ and based on the easy calculation.

Lemma 3 *If* $f \in L^p_\rho$ $(1 < p \leq 2)$ *and* $\delta \in (0,C)$*, then*

$$\delta^{1-p}\left(w_{x,p}f(\delta)\right)^p \leq C\sum_{k=1}^{[C/\delta]} k^{p-2}\left(w_{x,p}f(\frac{C}{k})\right)^p$$

for all $x \in (a,b)$*.*

Proof. Applying monotonicity of the function $\delta\left(w_{x,p}f(\delta)\right)^p$ we obtain

$$\begin{aligned}\sum_{k=1}^{[C/\delta]} k^{p-2}\left(w_{x,p}f(\frac{C}{k})\right)^p &\geq \frac{1}{C}\delta\left(w_{x,p}f(\delta)\right)^p\sum_{k=1}^{[C/\delta]} k^{p-1} \geq \\ &\geq \frac{1}{C}\delta\left(w_{x,p}f(\delta)\right)^p\int_1^{[C/\delta]+1}(t-1)^{p-1}\,dt \geq \\ &\geq C\delta^{1-p}\left(w_{x,p}f(\delta)\right)^p,\end{aligned}$$

and our result follows. ■

4 Proofs of the results

We will proof the Theorems 4 and 5 only.

4.1 Proof of Theorem 4

Let x be a fixed point of (a,b) and let $\delta_n = \delta_n(x)$ be an arbitrary sequence of positive numbers which definition will be done below. Denoting the non-empty sets I_n and I_n' by

$$I_n = [a,b] \cap (x-\delta_n, x+\delta_n) \quad , \quad I_n' = [a,b] - I_n$$

we can write, for $0 \le k \le n$,

$$\begin{aligned} S_k f(x) - f(x) &= \int_a^b \{f(t)-f(x)\} K_k(t,x)\,\rho(t)\,dt = \\ &= \left(\int_{I_n} + \int_{I_n'}\right) \{f(t)-f(x)\} K_k(t,x)\,\rho(t)\,dt = \\ &= R_{1,k}(x) \quad + \quad R_{2,k}(x), \end{aligned}$$

and consequently, by the Minkowski inequality,

$$G_{M_n}^q f(x) \le \sum_{i=1}^{2} \left\{ \frac{1}{M_n} \sum_{k=0}^{n} C_k^{2-q} |R_{i,k}(x)|^q \right\}^{1/q} =: S_1(x) + S_2(x).$$

For the first term we can observe that

$$\begin{aligned} |R_{1,k}(x)| &\le C \int_{I_n} |f(t)-f(x)|\,|K_\nu(t,x)|\,\rho(t)\,dt \\ &\le C(x) \sum_{\nu=0}^{k} \left\{ \int_{I_n} |\phi_\nu(t)|^r \rho(t)\,dt \right\}^{1/r} \left\{ \int_{I_n} |f(t)-f(x)|^{r'} \rho(t)\,dt \right\}^{1/r'} \\ &\le C(x) \sum_{\nu=0}^{k} \|\phi_\nu\|_{r,\rho} \left\{ \left(\int_{I_n} |f(t)-f(x)|^p \rho(t)\,dt \right)^{r'/p} \left(\int_{I_n} \rho(t)\,dt \right)^{1-r'/p} \right\}^{1/r'}, \end{aligned}$$

whence by our assumption, we obtain

$$\begin{aligned} |R_{1,k}(x)| &\le C(x)\,\delta_n^{1/r'-1/p} \left(\int_{I_n} |f(t)-f(x)|^p \rho(t)\,dt \right)^{1/p} \sum_{\nu=0}^{k} C_\nu \\ &= C(x)\,\delta_n^{1/r'} \left\{ \delta_n^{-1} \int_{x-\delta_n}^{x+\delta_n} \chi_{[a,b]}(x)\,|f(t)-f(x)|^p \rho(t)\,dt \right\}^{1/p} \left(\sum_{\nu=0}^{k} C_\nu \right) \\ &= C(x)\,\delta_n^{1/r'} w_{x,p} f(\delta_n) \left(\sum_{\nu=0}^{k} C_\nu \right). \end{aligned}$$

but, by Lemma 3 with $C = b - a$,

$$w_{x,p}f(\delta_n) \leq C \left\{ \delta_n^{p-1} \sum_{k=1}^{[(b-a)/\delta_n]} k^{p-2} \left(w_{x,p}f(\frac{b-a}{k}) \right)^p \right\}^{1/p}$$

and thus

$$\begin{aligned} S_1(x) &\leq C(x)\, \delta_n^{1/r'} w_{x,p}f(\delta_n) \left\{ \frac{1}{M_n} \sum_{k=0}^{n} C_k^{2-q} \left(\sum_{\nu=0}^{k} C_\nu \right)^q \right\}^{1/q} \\ &\leq C(x)\, \delta_n^{1/r'} A_n^{1/q} M_n^{-1/q} \left\{ \delta_n^{p-1} \sum_{k=1}^{[(b-a)/\delta_n]} k^{p-2} \left(w_{x,p}f(\frac{b-a}{k}) \right)^p \right\}^{1/p}. \end{aligned}$$

Consequently, we estimate the term $S_2(x)$ .We can easily seen that

$$\begin{aligned} |R_{2,k}(x)| &= \left| \int_{I_n'} \{f(t) - f(x)\} \sum_{\mu=1}^{m} F_\mu(t,x) \sum_{i,j=-s}^{s} \gamma_{i,j,\mu}^{(k)} \phi_{k+i}(x)\, \phi_{k+j}(t)\, \rho(t)\, dt \right| \\ &\leq \sum_{\mu=1}^{m} \sum_{i,j=-s}^{s} \left| \gamma_{i,j,\mu}^{(k)} \right| |\phi_{k+i}(x)| \left| \int_{I_n'} \{f(t) - f(x)\} F_\mu(t,x)\, \phi_{k+j}(t)\, \rho(t)\, dt \right| \\ &\leq C(x) \sum_{\mu=1}^{m} \sum_{j=-s}^{s} \left| \int_{I_n'} \{f(t) - f(x)\} F_\mu(t,x)\, \phi_{k+j}(t)\, \rho(t)\, dt \right| \end{aligned}$$

and putting

$$g_\mu(t,x) = \begin{cases} \{f(t) - f(x)\} F_\mu(t,x) & \text{for } t \in I_n', \\ 0 & \text{otherwise.} \end{cases}$$

we obtain

$$\begin{aligned} |R_{2,k}(x)| &\leq C(x) \sum_{\mu=1}^{m} \sum_{j=-s}^{s} \left| \int_a^b g_\mu(t,x)\, \phi_{k+j}(t)\, \rho(t)\, dt \right| \\ &= C(x) \sum_{\mu=1}^{m} \sum_{j=-s}^{s} |a_{k+j}(g_\mu(\cdot,x))|. \end{aligned}$$

Hence, by the Minkowski inequality and almost monotonicity of (C_k),

Lemma 2 gives

$$\begin{aligned} S_2(x) &\leq C(x) M_n^{-1/q} \sum_{\mu=1}^{m} \sum_{j=-s}^{s} \left\{ \sum_{k=0}^{n} C_k^{2-q} \left| a_{k+j} \left(g_\mu (\cdot, x) \right) \right|^q \right\}^{1/q} \\ &\leq C(x) M_n^{-1/q} \sum_{\mu=1}^{m} \sum_{j=-s}^{s} \left\{ \sum_{k=0}^{n} C_{k+j}^{2-q} \left| a_{k+j} \left(g_\mu (\cdot, x) \right) \right|^q \right\}^{1/q} \\ &\leq C(x) M_n^{-1/q} \sum_{\mu=1}^{m} \left\{ \int_a^b |g_\mu (t, x)|^p \rho(t) \, dt \right\}^{1/p} \\ &\leq C(x) M_n^{-1/q} \left\{ \int_{I_n'} \left| \frac{f(t) - f(x)}{t - x} \right|^p \rho(t) \, dt \right\}^{1/p}. \end{aligned}$$

Let us observe that instead of the integral $\int_{I_n'}$, introducing the function $\chi_{[a,b]}$, we can consider the sum of integrals $\int_a^{x-\delta_n} \chi_{[a,b]} + \int_{x+\delta_n}^b \chi_{[a,b]}$, but since the estimates of these integrals are the same we will write the estimate of the first one only. Denoting by

$$\Phi_{x,p}(h) := \int_0^h \chi_{[a,b]}(x - v) \left| f(x - v) - f(x) \right|^p \rho(x - v) \, dv$$

and using the partial integration, we obtain

$$\begin{aligned} \int_a^{x-\delta_n} \chi_{[a,b]}(t) \left| \frac{f(t) - f(x)}{t - x} \right|^p \rho(t) \, dt &= \\ \int_{\delta_n}^{x-a} \chi_{[a,b]}(x - v) \left| \frac{f(x - v) - f(x)}{v} \right|^p \rho(x - v) \, dv &= \\ \int_{\delta_n}^{x-a} v^{-p} d\left(\Phi_{x,p}(v) \right) &= \\ (x - a)^{-p} \Phi_{x,p}(x - a) + p \int_{\delta_n}^{x-a} v^{-p-1} \Phi_{x,p}(v) \, dv &= \\ = \quad P_1 \quad + \quad P_2. \end{aligned}$$

By the monotonicity of the functions $u (w_{x,p} f(u))^p$, and by the obvious inequality

$$\Phi_{x,p}(h) \leq h \left(w_{x,p} f(h) \right)^p,$$

the term P_2 does not exceed

$$\begin{aligned} & \frac{1}{(b-a)^p}\int_{\frac{b-a}{x-a}}^{\frac{b-a}{\delta_n}} u^{p-1}\Phi_{x,p}\left(\frac{b-a}{u}\right)du \\ \leq\ & \frac{1}{(b-a)^{p-1}}\int_1^{\frac{b-a}{\delta_n}} u^{p-2}\left(w_{x,p}f(\frac{b-a}{u})\right)^p du \\ \leq\ & \frac{C}{(b-a)^{p-1}}\sum_{k=1}^{\left[\frac{b-a}{\delta_n}\right]} k^{p-2}\left(w_{x,p}f(\frac{b-a}{k})\right)^p. \end{aligned}$$

The same estimate we can obtain for the term P_1 by the inequality for $\Phi_{x,p}$ and Lemma 3.

The integral $\int_{x+\delta_n}^b$ can be estimated in the same way, the only difference be in the definition of the function $\Phi_{x,p}$. Instead of this one we have to take the following function

$$\Phi'_{x,p}(h) := \int_0^h \chi_{[a,b]}(x+v)\,|f(x+v)-f(x)|^p\,\rho(x+v)\,dv.$$

Therefore

$$S_2(x) \leq C(x)\,M_n^{-1/q}\delta_n^{1/p-1}\left\{\delta_n^{p-1}\sum_{k=0}^{[(b-a)/\delta_n]}(k+1)^{p-2}\left(w_{x,p}f(\frac{b-a}{k+1})\right)^p\right\}^{1/p}.$$

Collecting the above estimates we have

$$\begin{aligned} G^q_{M_n}f(x) \ \leq\ & C(x)\,M_n^{-1/q}\left(\delta_n^{1/r'}A_n^{1/q}+\delta_n^{1/p-1}\right)\cdot \\ & \cdot\left\{\delta_n^{p-1}\sum_{k=1}^{[(b-a)/\delta_n]}k^{p-2}\left(w_{x,p}f(\frac{b-a}{k})\right)^p\right\}^{1/p}. \end{aligned}$$

Finally, let $\delta_n := (b-a)\,A_n^{-r'/(2-r'+qr')}$, then

$$\begin{aligned} \delta_n^{1/r'}A_n^{1/q} &= (b-a)^{1/r'}\,A_n^{(-q+2+qr'-r')/(2-r'+qr')q} = \\ &= (b-a)^{1/r'}\,A_n^{r'/(r'+p')q} = (b-a)^{1/r'}\,M_n^{1/q} \end{aligned}$$

and

$$\delta_n^{1/p-1} = (b-a)^{1/p-1}\,A_n^{r'/(r'+p')q} = (b-a)^{1/p-1}\,M_n^{1/q},$$

because $M_n := A_n^{r'/(r'+p')}$, and our thesis follows.■

4.2 Proof of Theorem 5

Under the notation of the above proof

$$G_{M_n}^q f(x) \le S_1(x) + S_2(x)$$

and by the same arguments as the above

$$S_1(x) \le C(x)\,\delta_n^{1/r'} A_n^{1/q} M_n^{-1/q} w_{x,p} f(\delta_n).$$

In order to get the estimate of the expression $S_2(x)$ we take in to account the integral $\int_a^{x-\delta_n}$ only. Writing $N_n = \left[1 + \log_2 A_n^{r'/(2-r'+qr')}\right]$ we see that

$$\begin{aligned}
&\int_a^{x-\delta_n} \chi_{[a,b]}(t) \left|\frac{f(t)-f(x)}{t-x}\right|^p \rho(t)\,dt \\
\le\ & \sum_{k=1}^{N_n} \int_{x-2^k\delta_n}^{x-2^{k-1}\delta_n} \chi_{[a,b]}(t) \left|\frac{f(t)-f(x)}{t-x}\right|^p \rho(t)\,dt \\
\le\ & \sum_{k=1}^{N_n} \left(2^k\delta_n\right)^{-p} \int_{x-2^k\delta_n}^{x-2^{k-1}\delta_n} \chi_{[a,b]}(t) \left|f(t)-f(x)\right|^p \rho(t)\,dt \\
\le\ & (\delta_n)^{1-p} \sum_{k=1}^{N_n} 2^{-(p-1)k} \left(w_{x,p} f(2^k\delta_n)\right)^p,
\end{aligned}$$

whence

$$S_2(x) \le C(x)\, M_n^{-1/q} \left\{(\delta_n)^{1-p} \sum_{k=1}^{N_n} 2^{(1-p)k} \left(w_{x,p} f(2^k\delta_n)\right)^p\right\}^{1/p}.$$

Thus

$$\begin{aligned}
G_{M_n}^q f(x) \le\ & C(x)\,\delta_n^{1/r'} A_n^{1/q} M_n^{-1/q} w_{x,p} f(\delta_n) + \\
& + C(x) M_n^{-1/q} \delta_n^{1/p-1} \left\{\sum_{k=1}^{N_n} 2^{(1-p)k} \left(w_{x,p} f(2^k\delta_n)\right)^p\right\}^{1/p}.
\end{aligned}$$

Putting as above $\delta_n = (b-a)\, A_n^{-r'/(2-r'+qr')}$ we have

$$G_{M_n}^q f(x) \le C(x) \left\{\sum_{k=0}^{N_n} 2^{(1-p)k} \left(w_{x,p} f(2^k\delta_n)\right)^p\right\}^{1/p}.$$

and our proof is completed.■

References

[1] G. Alexits, Convergence problems of orthogonal series, Budapest 1961.

[2] S. Aljančič, R. Bojanic and M. Tomić, On the degree of convergence of Féjer-Lebesgue sums, L'Enseignement Mathematique, Geneve, Tome XV (1969) 21-28 .

[3] G. Freud, Orthogonal polynomials, Budapest 1971.

[4] G. Freud, Űber die starke (C,1)-Summierbarkeit von orthogonalen Polynomreihen, Acta Math. Acad. Sci. Hungaricae 3 (1952) 83-88.

[5] S. Kaczmarz and H. Steinhaus, Teorija ortogonalnych riadov, Moskva 1958,

[6] W. Łenski, Remark on the inequality of F. Riesz, Proceedings of The Władysław Orlicz Centenary Conference and Function Spaces VII, Poznań 2003.(in press).

[7] W. Łenski and R. Kranz, On the rate of pointwise strong summability of orthogonal expansions, Approximation Theory and its Applications, (submitted for publication).

[8] J. Marcinkiewicz and A. Zygmund, Some theorems on orthogonal system, Fund. Math. 28 (1937) pp.309-335

[9] B.P. Osilenker, Fourier series in orthogonal polynomials, Singapore-New Jersey-Hong Kong 2001.

[10] K. Tandori, Űber die Cesàrosche Summierbarkeit der Orthogonalen Polynomreihen I, Acta. Math. Acad. Sci. Hungaricae 3 (1952) 73-82.

[11] K. Tandori, Űber die Cesàrosche Summierbarkeit der Orthogonalen Polynomreihen II, Acta Math. Acad. Sci. Hungaricae 5 (1954) 236-253.

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Some C_0-Semigroups Related to Polynomial Operators

Detlef Mache and Ioan Rasa

Abstract. We consider the C_0-semigroups $(T(t))_{t\geq 0}$ generated by some differential operators and approximated by polynomial operators on $C[0,1]$. The action of $T(t)$ on monomials is completely described. As application the moments of some diffusions are computed.

Mathematical Subject Classification (2000). 47D06, 41A36

Keywords and phrases. One-parameter semigroups, positive approximation processes

1 Introduction

Consider the differential operator

$$W = x(1-x)\frac{d^2}{dx^2} + (a+1-(a+b+2)x)\frac{d}{dx},\ 0 < x < 1. \tag{1.1}$$

With a suitable domain, W is the infinitesimal generator of a positive contraction semigroup $(T(t))_{t\geq 0}$ on $C[0,1]$ which can be approximated by iterates of Durrmeyer-type operators.

Several properties of the semigroup are investigated in [1] and [6].

In this paper we study the action of $T(t)$ on the monomials $e(x) := x^k,\ k = 0, 1, \ldots$. In order to compute $T(t)e_k$ we give recurrence relations and explicit expressions. The eigenstructure of $T(t)$ is also investigated. The proofs are based on the fact that $T(t)e_k$ is the solution of a Cauchy problem associated with W.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Similar results for the semigroup generated by $\frac{x(1-x)}{2}\frac{d^2}{dx^2}$ were obtained in [5], [4], [3], [2] with a different method. In these papers $T(t)$ is approximated by iterates of Bernstein operators and the properties of these operators are extensively used.

In the final section $T(t)e_k$ is interpreted as the k-th moment of the diffusion Y_t generated by W.

2 The semigroup generated by W

Let $a, b \geq 0$ and

$$D(W) := \{u \in C^1[0,1] \cap C^2(0,1) : \lim_{x \to 0,1} x(1-x)u''(x) = 0\}.$$

Then $(W, D(W))$ is the infinitesimal generator of a positive contraction semigroup $T(t))_{t \geq 0}$ on $C[0,1]$ (with supremum norm and usual order). Moreover, $T(t)e_0 = e_0$ and $T(t)$ can be approximated by iterates of suitable positive linear operators. Details concerning these results can be found in [1].

Let $\gamma_j := j(j+a+b+1),\ j = 0, 1, \ldots.$

We need the following result from [6].

Theorem 2.1 *Let $k \geq 1$ be given. Then*

$$T(t)e_k = \sum_{i=0}^{k} c_{ik}(t)e_i, \tag{2.1}$$

where

$$c_{ik}(t) = \sum_{j=i}^{k} \alpha_{ijk} exp(-\gamma_j t) \tag{2.2}$$

with

$$\alpha_{ijk} \in \mathbb{R},\ 0 \leq i \leq j \leq k.$$

For $y \in \mathbb{R}$ and $l \geq 1$ let $(y)_0 := 1,\ (y)_l := y(y-1)\ldots(y-l+1)$.

In what follows we give recurrence relations and explicit expressions for the numbers α_{ijk}.

Theorem 2.2 *The following relations hold true:*

$$\alpha_{kkk} = 1, \tag{2.3}$$

$$\sum_{j=i}^{k} \alpha_{ijk} = 0,\ i = 0, \ldots, k-1, \tag{2.4}$$

$$\alpha_{ijk} = \frac{(i+1)(i+a+1)}{(i-j)(i+j+a+b+1)} \alpha_{i+1,jk}, \tag{2.5}$$

$i = 0, \ldots, k-1\ ;\ j = i+1, \ldots, k.$

Moreover, for $0 \le i \le j \le k$,

$$\alpha_{ijk} = (-1)^{i+j} \binom{k}{i} \binom{k-i}{j-i} (k+a)_{k-i} \frac{2j+a+b+1}{(k+j+a+b+1)_{k-i+1}}. \tag{2.6}$$

Proof. The function

$$u(t,x) := T(t)e_k(x) = \sum_{i=0}^{k} \sum_{j=i}^{k} \alpha_{ijk} x^i exp(-\gamma_j t) \tag{2.7}$$

is a solution of the Cauchy problem

$$u_t(t,x) = Wu(t,x), \tag{2.8}$$

$$u(0,x) = x^k. \tag{2.9}$$

From (2.7) and (2.9) we get

$$\sum_{i=0}^{k} \sum_{j=i}^{k} \alpha_{ijk} x^i = x^k,$$

which entails (2.3) and (2.4).

Now (2.7) and (2.8) yield

$$\begin{aligned} &\sum_{i=0}^{k} x^i \sum_{j=i}^{k} \alpha_{ijk} \gamma_j exp(-\gamma_j t) = \\ &\sum_{i=0}^{k} (\gamma_i x^i - i(i+a)x^{i-1}) \sum_{j=i}^{k} \alpha_{ijk} exp(-\gamma_j t). \end{aligned} \tag{2.10}$$

This implies

$$\begin{aligned} &\sum_{i=0}^{k-1} x^i \sum_{j=i+1}^{k} (i-j)(i+j+a+b+1)\alpha_{ijk} exp(-\gamma_j t) = \\ &\sum_{i=0}^{k-1} x^i \sum_{j=i+1}^{k} (i+1)(i+a+1)\alpha_{i+1,jk} exp(-\gamma_j t). \end{aligned} \tag{2.11}$$

Clearly (2.5) is a consequence of (2.11).

We shall prove (2.6) by induction on i. For $i = k$, (2.6) follows from (2.3). Suppose that (2.6) is true for a given $i \in \{1, \dots, k\}$ and all $j \in \{i, \dots, k\}$. Then (2.6) for $i-1$ and $j \in \{i, \dots, k\}$ is a consequence of (2.5) and the induction hypothesis. Finally, (2.6) for $i-1$ and $j = i-1$ reads as follows:

$$\alpha_{i-1,i-1,k} = \binom{k}{i-1}(k+a)_{k-i+1}\frac{2i+a+b-1}{(k+i+a+b)_{k-i+2}}. \tag{2.12}$$

To prove it we need the following identity which can be established by using elementary algebra:

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{x-p+2l}{(x+l)(x+l-1)\dots(x+l-p)} = 0. \tag{2.13}$$

Setting $p := k-i+1,\ x := k+i+a+b,$, from (2.13) we derive

$$\sum_{j=i-1}^{k}(-1)^{i+j-1}\binom{k}{i-1}\binom{k-i+1}{j-i+1}(k+a)_{k-i+1}\frac{2j+a+b+1}{(k+j+a+b+1)_{k-i+2}} = 0. \tag{2.14}$$

We already have seen that (2.6) is true for $i-1$ and $j \in \{i, \dots, k\}$; thus (2.14) yields

$$\binom{k}{i-1}(k+a)_{k-i+1}\frac{2i+a+b-1}{(k+i+a+b)_{k-i+2}} + \sum_{j=i}^{k}\alpha_{i-1,jk} = 0. \tag{2.15}$$

On the other hand, from (2.4) we deduce

$$\sum_{j=i-1}^{k}\alpha_{i-1,jk} = 0. \tag{2.16}$$

Now (2.12) is a consequence of (2.15) and (2.16); so the proof is finished. □

For $0 \le i \le k$ and $m \ge 0$ define the numbers U_{ikm} by

$$c_{ik}(t) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}U_{ikm}t^m. \tag{2.17}$$

Using (2.2) we get

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}U_{ikm}t^m=\sum_{j=i}^{k}\alpha_{ijk}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\gamma_j^m t^m,$$

which implies

$$U_{ikm}=\sum_{j=i}^{k}\alpha_{ijk}\gamma_j^m. \tag{2.18}$$

Theorem 2.3 *The following relations hold true:*

$$U_{kk0}=1, \tag{2.19}$$

$$U_{ik0}=0,\ i=0,\dots,k-1, \tag{2.20}$$

$$U_{kkm}=\gamma_k U_{kk,m-1},\ m\geq 1, \tag{2.21}$$

$$U_{ikm}=\gamma_i U_{ik,m-1}-(i+1)(i+a+1)U_{i+1,k,m-1}, \tag{2.22}$$

$m\geq 1;\ i=0,\dots,k-1.$

Proof. (2.19) and (2.20) follow from (2.3) and (2.4). Now return to (2.10), consider the derivative of order $m-1$ with respect to t, then put $t=0$; we get

$$\sum_{i=0}^{k}x^i U_{ikm}=\sum_{i=0}^{k}(\gamma_i x^i-i(i+a)x^{i-1})U_{ik,m-1}.$$

This yields

$$\sum_{i=0}^{k}x^i U_{ikm}=\sum_{i=0}^{k}x^i\gamma_i U_{ik,m-1}-\sum_{i=0}^{k-1}x^i(i+1)(i+a+1)U_{i+1,k,m-1},$$

which entails (2.21) and (2.22).

Let $p_j(x)=x^j+\dots$ be the monic Jacobi polynomials orthogonal with weight $x^a(1-x)^b$ on the interval $[0,1]$. Then

$$Wp_j=-\gamma_j p_j,\ j\geq 0. \tag{2.23}$$

It follows that

$$T(t)p_j=p_j exp(-\gamma_j t),\ j\geq 0. \tag{2.24}$$

□

Theorem 2.4 *For each $k \geq 0$,*

$$e_k = \sum_{j=0}^{k} \binom{k}{j} \frac{2j+a+b+1}{(k+j+a+b+1)_{k-j+1}} (k+a)_{k-j} p_j, \tag{2.25}$$

and

$$p_k = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \frac{(k+a)_{k-i}}{(2k+a+b)_{k-i}} e_i. \tag{2.26}$$

Proof. Set

$$e_k = \sum_{j=0}^{k} A_{jk} p_j, \ A_{jk} \in \mathbb{R}. \tag{2.27}$$

Then we have

$$T(t)e_k = \sum_{j=0}^{k} A_{jk} p_j exp(-\gamma_j t). \tag{2.28}$$

From (2.1) and (2.2) it follows that

$$T(t)e_k = \sum_{j=0}^{k} (\sum_{i=0}^{j} \alpha_{ijk} e_i) exp(-\gamma_j t). \tag{2.29}$$

Now (2.28) and (2.29) yield

$$A_{jk} p_j = \sum_{i=0}^{j} \alpha_{ijk} e_i, \ 0 \leq j \leq k. \tag{2.30}$$

Since p_j is a monic polynomial we get

$$A_{jk} = \alpha_{jjk}$$

and (2.6) shows that

$$A_{jk} = \binom{k}{j} \frac{2j+a+b+1}{(k+j+a+b+1)_{k-j+1}} (k+a)_{k-j}.$$

Combined with (2.27), this proves (2.25). For $j = k$, (2.30) gives

$$A_{kk} p_k = \sum_{i=0}^{k} \alpha_{ikk} e_i.$$

Since $A_{kk} = \alpha_{kkk} = 1$ and

$$\alpha_{ikk} = (-1)^{k+i} \binom{k}{i} \frac{(k+a)_{k-i}}{(2k+a+b)_{k-i}},$$

we get (2.26) and the proof is finished. □

3 The semigroup generated by $\frac{x(1-x)}{2}\frac{d^2}{dx^2}$

The differential operator $\frac{x(1-x)}{2}\frac{d^2}{dx^2}$ with domain

$$\{u \in C[0,1] \cap C^2(0,1) : \lim_{x\to 0,1} x(1-x)u''(x) = 0\}$$

is the infinitesimal generator of a positive contraction semigroup $(S(t))_{t\geq 0}$, firstly investigated in [4].

We present here some results which can be proved by the method of the preceding section. In fact, they were proved (see [5], [4], [3], [2]) using the approximation of $S(t)$ by iterates of Bernstein operators.

Let $\mu_j := \frac{1}{2}j(j-1)$, $j \geq 1$. We have $S(t)e_0 = e_0$ and, for $k \geq 1$,

$$T(t)e_k = \sum_{i=1}^{k} b_{ik}(t)e_i, \tag{3.1}$$

$$b_{ik}(t) = \sum_{j=i}^{k} \beta_{ijk} exp(-\mu_j t), \tag{3.2}$$

$$\beta_{kkk} = 1, \tag{3.3}$$

$$\sum_{j=i}^{k} \beta_{ijk} = 0,\ i = 1, \ldots, k-1, \tag{3.4}$$

$$\beta_{ijk} = \frac{i(i+1)}{(i-j)(i+j-1)}\beta_{i+1,jk}, \tag{3.5}$$

$i = 1, \ldots, k-1;\ j = i+1, \ldots, k$.

For $1 \leq i \leq k$ and $m \geq 0$ define the numbers V_{ikm} by

$$b_{ik}(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} V_{ikm} t^m.$$

Then

$$V_{kk0} = 1, \tag{3.6}$$

$$V_{ik0} = 0,\ i = 1, \ldots, k-1, \tag{3.7}$$

$$V_{kkm} = \mu_k V_{kk,m-1},\ m \geq 1, \tag{3.8}$$

$$V_{ikm} = \mu_i V_{ik,m-1} - \mu_{i+1} V_{i+1,k,m-1}, \tag{3.9}$$

$m \geq 1;\ i = 1, \ldots, k-1$.

Let $q_j(x) = x^j + \dots$ be the monic eigenpolynomials of $S(t)$, with eigenvalues $exp(-\mu_j t)$. Then $q_0 = 1$ and for $k \geq 1$,

$$q_k = \sum_{i=1}^{k} (-1)^{k+i} \binom{k}{i} \frac{(k-1)_{k-i}}{(2k-2)_{k-i}} e_i, \tag{3.10}$$

$$e_k = \sum_{j=1}^{k} \binom{k}{j} \frac{2j-1}{(k+j-1)_{k-j+1}} (k-1)_{k-j} p_j. \tag{3.11}$$

4 Applications

Consider the semigroup $(T(t))_{t\geq 0}$ from Section 2. Then (see [1] and the references therein) there exists a diffusion $(Y_t^x)_{t\geq 0}$ with values in $[0,1]$ such that $Y_0^x = x \in [0,1]$ and

$$Ef(Y_t^x) = T(t)f(x),\ f \in C[0,1],\ x \in [0,1]. \tag{4.1}$$

In particular, the moment of order k of Y_t^x is given by

$$E(Y_t^x)^k = T(t)e_k(x),\ k \geq 0. \tag{4.2}$$

So the results about $T(t)e_k$ can be considered also as results about the moments of Y_t^x.

Example 4.1 Let $a = b = 0$. Then

$$E(Y_t^x) = \frac{1}{2} + \Big(x - \frac{1}{2}\Big)e^{-2t}, \tag{4.3}$$

$$E(Y_t^x)^2 = \frac{1}{3} + \Big(x - \frac{1}{2}\Big)e^{-2t} + \Big(x^2 - x + \frac{1}{6}\Big)e^{-6t}, \tag{4.4}$$

$$Var(Y_t^x) = \frac{1}{12} - \Big(x^2 - x + \frac{1}{4}\Big)e^{-4t} + \Big(x^2 - x + \frac{1}{6}\Big)e^{-6t}. \tag{4.5}$$

Example 4.2 For $a \geq 0,\ b \geq 0$ we have

$$\lim_{t\to\infty} E(Y_t^x)^k = \frac{(k+a)_k}{(k+a+b+1)_k}. \tag{4.6}$$

References

[1] F. Altomare and I.Rasa, On some classes of diffusion equations and related approximation problems, to appear in: *Trends and Applications in Constructive Approximation*, M.G.de Bruin, D.H.Mache and J.Szabados (Eds.), ISNM-International Series of Numerical Mathematics, Vol.**151**, Birkhäuser (2005).

[2] S.Cooper and S.Waldron, The eigenstructure of the Bernstein operator, *J.Approx.Theory* **105**(2000), 133-165.

[3] M.R.Da Silva, Nonnegative order iterates of Bernstein polynomials and their limiting semigroup, *Portugal.Math.* **42**(1985), 225-248.

[4] S.Karlin and Z.Ziegler, Iteration of positive approximation operators, *J.Approx.Theory* **3**(1970), 310-339.

[5] R.P.Kelisky and T.J.Rivlin, Iterates of Bernstein polynomials, *Pacific J.Math.* **21**(1967), 511-520.

[6] I.Rasa, Semigroups associated to Mache operators (II), to appear in:*Trends and Applications in Constructive Approximation*, M.G.de Bruin, D.H.Mache and J.Szabados (Eds.), ISNM-International Series of Numerical Mathematics, Vol.**151**, Birkhäuser (2005).

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Fourier Sums in Sonin-Markov Polynomials *

Giuseppe Mastroianni and Donatella Occorsio

Abstract. In this paper we study the behavior in L^p weighted norm of the Fourier Sums in the Sonin-Markov system.

Mathematical Subject Classification (2000). 41A10, 41A65, 43A50.

Keywords and phrases. Orthogonal Polynomial, Fourier Series

1 Introduction

The expansion of functions in orthogonal systems has interested several authors. We recall the papers [2], [3],[13], [16], [17] in the case of finite intervals and [1], [7], [14],[15] in the case of unbounded intervals.
The behavior of the Fourier Sum in the Sonin-Markov polynomial system has not been discussed yet, and no results are available in the literature connected with such problem. In this paper we fill this gap and prove necessary and sufficient conditions for the uniform boundedness of the previous Fourier sums in suitable L^p weighted space. The result in Theorem 2.1 includes a theorem in [1] and some results in [15]. Nevertheless as in the case of the Fourier sums on unbounded intervals, the domain of the parameter p in the L^p weighted space, is very small, i.e. $\left(\frac{4}{3}, 4\right)$. On the other hand, as a consequence of "finite-infinite polynomial inequalities", it is sufficient

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to approximate a finite section of the function by means of a finite section of the Fourier Sum (see Proposition 2.2). This observation allows to extend the previous Theorem 2.1 to the case $1 < p < \infty$. We will show that this procedure is more convenient than the ordinary polynomial approximation.The proof of the results are very simple since they consists essentially to prove the boundedness of Hilbert transform in weighted L^p spaces with Generalized Jacobi weight on the interval $[-1, 1]$. As consequence, analogous results with respect to Fourier sums in the Laguerre system can follow by a simple transformation. The main results are exposed in Section 2 and the proofs are in Section 3.

2 Main Results

Consider the weight $w_\beta(x) = |x|^\beta e^{-x^2}$ and the corresponding sequence of orthonormal polynomials $\{p_m(w_\beta)_m\}$ with positive leading coefficient i.e.

$$p_m(w_\beta, x) = \gamma_m x^m + \ldots, \quad \gamma_m = \gamma_m(w_\beta) > 0$$

and

$$\int_{\mathbb{R}} p_m(w_\beta) p_n(w_\beta) w_\beta = \delta_{nm}.$$

The polynomials $p_m(w_\beta)$, $m = 0, 1, \ldots$ are known as Sonin-Markov or generalized Hermite polynomials. These polynomials were considered in [5],[6],[9], and many properties are known. To our aim we recall some facts we will use in the sequel. We shall denote by C a positive constant different in different formulas and we shall write $A \sim B$ if a positive constant C exists, with C independent of the parameters of A and B, s.t. $\frac{1}{C}B \leq A \leq CB$.

The recurrence relation is the following:

$$x p_m(w_\beta, x) = \alpha_m p_{m-1}(w_\beta, x) + \alpha_{m+1} p_{m+1}(w_\beta, x),$$

where

$$\alpha_m = \frac{\gamma_{m-1}}{\gamma_m} = \begin{cases} \sqrt{\frac{m}{2}} & m \quad odd, \\ \sqrt{\frac{\beta+m}{2}} & m \quad even. \end{cases} \tag{2.1}$$

The zeros $x_{m,k}(w_\beta)$, $k = 1, 2 \ldots, m$ of $p_m(w_\beta)$ belong to

$$\left(-\sqrt{2m} + \frac{C}{m^{\frac{1}{6}}}, \sqrt{2m} - \frac{C}{m^{\frac{1}{6}}} \right)$$

with $C > 0$ independent of m, and the following estimate

$$p_m^2(w_\beta, x)e^{-x^2}\left(|x| + \frac{1}{\sqrt{m}}\right)^\beta \sqrt{|2m - x^2| + (2m)^{\frac{1}{3}}} \sim \left(\frac{x - x_d}{x_{d\pm1} - x_d}\right)^2 \tag{2.2}$$

holds for $|x| \le \sqrt{2m}\left(1 + \frac{c}{m^{\frac{2}{3}}}\right)$. Here x_d is a node close to x and the constants in "$\sim$" are independent of x and m (see for instance [5], [9]).

Now, with $u(x) = u_\gamma(x) = |x|^\gamma e^{-\frac{x^2}{2}}$, denote by L_u^p, $1 < p < \infty$ the set of all measurable functions f s.t.

$$\|fu\|_p^p = \int_{\mathbb{R}} |fu|^p(x)dx < \infty.$$

The m–th Fourier sum of a function $f \in L_u^p$ in the system of Sonin-Markov polynomials is

$$S_m(w_\beta, f) = \sum_{k=0}^{m-1} c_k p_k(w_\beta), \quad c_k = c_k(f) = \int_{\mathbb{R}} f p_k(w_\beta) w_\beta,$$

assuming that the coefficients c_k are finite.

The behaviour of $S_m(w_\beta) : L_u^p \to L_u^p, u(x) = |x|^\gamma e^{-\frac{x}{2}}, 1 < p < +\infty$, is given by the following theorem.

Theorem 2.1 *For all $f \in L_u^p$ the bound*

$$\|S_m(w_\beta, f)u\|_p \le C\|fu\|_p, \tag{2.3}$$

with C positive constant independent of m and f, is true if and only if $\frac{4}{3} < p < 4$ and

$$\gamma > -\frac{1}{p}, \quad -\frac{1}{p} < \gamma - \frac{\beta}{2} < 1 - \frac{1}{p}, \quad \beta - \gamma > -1 + \frac{1}{p}. \tag{2.4}$$

We observe that in the special case $\beta = \gamma = 0$ the previous theorem was proved in [1] by R. Askey and S. Wainger. The case $\beta = 0$ and $-\frac{1}{p} < \gamma < 1 - \frac{1}{p}$ was considered by B. Muckenoupt in [15]. In this last case is easy to see as the condition $-\frac{1}{p} < \gamma < 1 - \frac{1}{p}$ is strongly restrictive. On the contrary (2.3) is satisfied also for γ "large" if β belongs to the interval $\max\left(2\gamma - 2 + \frac{2}{p}, \gamma - 1 + \frac{1}{p}\right) < \beta < 2\gamma + \frac{2}{p}$ and $\frac{4}{3} < p < 4$. Moreover, (2.4) holding for $p \in (1, \infty) \setminus \left(\frac{4}{3}, 4\right)$, (2.3) can be replaced by

$$\|S_m(w_\beta, f)u\|_p \le Cm^{\frac{1}{3}}\|fu\|_p. \tag{2.5}$$

We also remark that (2.3) is true for a range of p very small and, as consequence, Theorem 2.1 can be applied to a restricted class of functions. Perhaps Fourier sums in Laguerre and Freud polynomials have a similar behaviour (see for instance [7], [14]).

Hence it is natural to modify the previous approximation process. To do this we denote by $\Delta_\theta = \Delta_{m,\theta}$ the characteristic function of the interval $[-\sqrt{2m\theta}, \sqrt{2m\theta}]$ where $\theta \in (0,1)$ is fixed and we define the integer $M = \left[m\sqrt{\frac{\theta}{1+\theta}}\right]$.

Proposition 2.2 *For all $f \in L_u^p$, $1 < p < \infty$, we have:*

$$\|(1-\Delta_{m,\theta})fu\|_p \leq C\left[E_M(f)_{u,p} + e^{-Am}\|fu\|_p\right] \tag{2.6}$$

where $E_M(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)u\|_p$ and the constants C and A depend on θ (fixed) and are independent of m and f.

Then, if we replace the sequence $\{S_m(w_\beta, f)\}_m$ by $\{\Delta_{m,\theta}S_m(w_\beta, \Delta_{m,\theta}f)\}_m$, we can write

$$\begin{aligned} \|[f-\Delta_{m,\theta}S_m(w_\beta,\Delta_{m,\theta}f)]u\|_p &\leq \|(1-\Delta_{m,\theta})fu\|_p + \\ &+ \|[f-S_m(w_\beta,\Delta_{m,\theta}f)]\Delta_{m,\theta}u\|_p \end{aligned} \tag{2.7}$$

and, by (2.6), we have to estimate the second term on the right of (2.7). Therefore, in order to approximate f in L_u^p, we have only to approximate fu by means of $uS_m(w_\beta, \Delta_{\theta,m}f)$ in the interval $[-\sqrt{2m\theta}, \sqrt{2m\theta}]$. A similar idea was used in [8], [9]. The advantage of this procedure is shown in the following

Theorem 2.3 *With the previous notations, let $f \in L_u^p$ and $p \in (1,\infty)$. Then there exists a positive constant C independent of m and f s.t.*

$$\|S_m(w_\beta,\Delta_{m,\theta}f)u\Delta_{m,\theta}\|_p \leq C\|fu\Delta_{m,\theta}\|_p \tag{2.8}$$

if and only if (2.4) holds. Moreover (2.8) implies the estimate

$$\|[f-\Delta_{m,\theta}S_m(w_\beta,\Delta_{m,\theta}f)]u\|_p \leq C\left[E_M(f)_{u,p} + e^{-Am}\|fu\|_p\right]. \tag{2.9}$$

Theorem 2.3 characterizes, for every $p \in (1,\infty)$, the weights w_β and u for which (2.8) is true.
The next theorem shows that the chosen procedure is the best possible.

Theorem 2.4 *We have*

$$\|S_m(w_\beta, \Delta_{m,\theta} f)u\|_p \le C\|fu\Delta_{m,\theta}\|_p \tag{2.10}$$

if and only if (2.4) holds with $p \in (1,4)$ *and* $f \in L^p_u$.
Moreover the inequality

$$\|S_m(w_\beta, f)u\Delta_{m,\theta}\|_p \le C\|fu\|_p \tag{2.11}$$

is satisfied if and only if (2.4) is true with $p > 4/3$. *The constants* C *are independent of* m *and* f.

Theorem 2.4, for $\beta = 0$ and $-\frac{1}{p} < \gamma < 1 - \frac{1}{p}$, reduces to Theorems 3 and 4 in [15].
For completeness we write some estimates of $E_m(f)_{u,p}$, $1 < p < \infty$. For more details the reader can consult [11]. Here we consider some interesting class of functions.

If $f \in L^p_u$ is such that $f^{(r-1)}$ is absolutely continuous in $\mathbb{R}$ ($f^{(r-1)} \in AC(\mathbb{R})$) with $\|f^{(r)}u\|_p < \infty$, $r \ge 1$, then

$$E_m(f)_{u,p} \le \frac{C}{(\sqrt{m})^r}\|f^{(r)}u_{m,\gamma}\|_p$$

where $u_{m,\gamma} = e^{-\frac{x^2}{2}}\left(|x| + \frac{1}{\sqrt{m}}\right)^\gamma$ and C is independent of m and f. The term $\frac{1}{\sqrt{m}}$ in $u_{m,\gamma}$ can be dropped if $-\frac{1}{p} < \gamma < 1 - \frac{1}{p}$.
We can also consider the case of functions that are singular in 0 or the case of functions whose derivatives are singular in 0.
If $f^{(r-1)} \in AC(\mathbb{R} \setminus \{0\})$, $\frac{1}{p} + \gamma > r$, $r \ge 1$ and $\|f^{(r)}u\|_p < \infty$, then we have the estimate

$$E_m(f)_{u,p} \le \frac{C}{(\sqrt{m})^r}(\|f^{(r)}u\|_p + \|fu\|_p). \tag{2.12}$$

In addition, if $\gamma + \frac{1}{p} < r$, $\gamma + \frac{1}{p}$ is not a natural number and $f^{(r-\tau-1)}$, $\tau = \left[\gamma + \frac{1}{p}\right]$ exists in zero with $\|f^{(r)}u\|_p < \infty$, then (2.12) is also true. Finally, (2.12) is not true if $\gamma + \frac{1}{p} \le r$ is an integer.

3 The Proofs

We first prove that

$$\begin{aligned} S_m(w_\beta, f, x) &= d_m\{\alpha_{m-1}[p_m(w_\beta; x)H(F_m f w_\beta, x) \\ &- F_m(x)H(p_{m-1}(w_\beta)f w_\beta, x)] \\ &+ p_{m-1}(w_\beta, x)c_{m-1}\}, \end{aligned} \tag{3.1}$$

where $c_{m-1} = c_{m-1}(f), F_m(x) = p_m(w_\beta, x) - p_{m-2}(w_\beta, x)$, $d_m = \frac{\alpha_m}{\alpha_m + \alpha_{m-1}}$, α_m is defined in (2.1) and $H(g,x) = \int_{\mathbb{R}} \frac{g(t)}{t-x} dt$ is the Hilbert transform of g. Denoting by

$$K_m(w_\beta, x, t) = \alpha_m \frac{p_m(w_\beta, x)p_{m-1}(w_\beta, t) - p_{m-1}(w_\beta, x)p_m(w_\beta, t)}{x - t}$$

the Darboux kernel, we can write

$$K_m(w_\beta; x, t) = \frac{1}{2}[K_m(w_\beta; x, t) + K_{m-1}(w_\beta; x, t) + p_{m-1}(w_\beta, x)p_{m-1}(w_\beta, t)].$$

Since

$$\begin{aligned} K_m(w_\beta; x, t) + K_{m-1}(w_\beta; x, t) &= \alpha_{m-1} \left[\frac{K_m(w_\beta; x, t)}{\alpha_m} \right. \\ &+ \left. \frac{K_{m-1}(w_\beta; x, t)}{\alpha_{m-1}} \right] \\ &+ \frac{\alpha_m - \alpha_{m-1}}{\alpha_m} K_m(w_\beta; x, t) \end{aligned}$$

it follows

$$\begin{aligned} K_m(w_\beta; x, t) &= \frac{\alpha_{m-1}}{2} \left[\frac{K_m(w_\beta; x, t)}{\alpha_m} + \frac{K_{m-1}(w_\beta; x, t)}{\alpha_{m-1}} \right] \\ &+ \frac{\alpha_m - \alpha_{m-1}}{2\alpha_m} K_m(w_\beta; x, t) \\ &+ \frac{1}{2} p_{m-1}(w_\beta, x)p_{m-1}(w_\beta, t), \end{aligned}$$

and

$$\begin{aligned} K_m(w_\beta; x, t) &= d_m \left\{ \left[\frac{K_m(w_\beta; x, t)}{\alpha_m} + \frac{K_{m-1}(w_\beta; x, t)}{\alpha_{m-1}} \right] \alpha_{m-1} \right. \\ &+ \left. p_{m-1}(w_\beta, x)p_{m-1}(w_\beta, t) \right\}, \end{aligned}$$

where $d_m = \frac{\alpha_m}{\alpha_m + \alpha_{m-1}}$.
Since, with $F_m(x) = p_m(w_\beta, x) - p_{m-2}(w_\beta, x)$,

$$\frac{K_m(w_\beta; x, t)}{\alpha_m} + \frac{K_{m-1}(w_\beta; x, t)}{\alpha_{m-1}} = p_{m-1}(w_\beta, t) \frac{F_m(x)}{x - t} - p_{m-1}(w_\beta, x) \frac{F_m(t)}{x - t}$$

we get

$$\begin{aligned} K_m(w_\beta; x, t) &= d_m \alpha_{m-1} \left[p_{m-1}(w_\beta; t) \frac{F_m(x)}{x - t} - p_{m-1}(w_\beta; x) \frac{F_m(t)}{x - t} \right] \\ &+ d_m p_{m-1}(w_\beta, x)p_{m-1}(w_\beta, t). \end{aligned}$$

Since

$$S_m(w_\beta, f, x) = \int_{\mathbb{R}} K_m(w_\beta; x, t) f(t) w_\beta(t) dt,$$

(3.1) follows. □

The following propositions will be useful in the sequel.

Proposition 3.1 *Let be* $w_\beta(x) = v^\beta(x)e^{-x^2}, v^\beta(x) = |x|^\beta$ *and* $\sigma(x) = v^\alpha(x)e^{-\frac{x^2}{2}}, v^\alpha = |x|^\alpha$. *For a fixed* $0 < \theta < 1$, $1 \le p < \infty$ *and* $\alpha - \frac{\beta}{2} > -\frac{1}{p}$, *we have*

$$\|p_m(w_\beta)\sigma\|_{L_p[-\sqrt{2m\theta},\sqrt{2m\theta}]} \ge \frac{C}{\sqrt[4]{m}} \left\| \frac{v^\alpha}{\sqrt{v^\beta}} \right\|_{L^p(-1,1)} \tag{3.2}$$

and

$$\|p_m(w_\beta)\sigma\|_p \ge C(\sqrt{m})^{\alpha-\frac{\beta}{2}+\frac{1}{p}-\frac{1}{2}} \left(\int_{-1}^{1} \frac{|x|^{(\alpha-\frac{\beta}{2})p}}{\sqrt[4]{\left(1-x^2+m^{-\frac{2}{3}}\right)^p}} dx \right)^{\frac{1}{p}}, \tag{3.3}$$

where C *is independent of* m.

Proof. Let $\delta > 0$ be "small". Define $\delta_k = \frac{\delta}{4}\Delta x_k = \frac{\delta}{4}(x_{k+1} - x_k)$, and $I_m = \cup_{1\le k\le m}([x_k - \delta_k, x_k + \delta_k])$. To prove (3.2), set $CI_m = [-1,1]\backslash(I_m \cap [-1,1])$. By (2.2) we get

$$|p_m(w_\beta, x)|\sigma(x) \ge C\frac{|x|^{\alpha-\frac{\beta}{2}}}{\sqrt[4]{m}}, \quad x \in CI_m,$$

and consequently

$$\|p_m(w_\beta)\sigma\|_{L^p[-\sqrt{2m\theta},\sqrt{2m\theta}]} \ge \frac{C}{\sqrt[4]{m}} \left\| \frac{v^\alpha}{\sqrt{v^\beta}} \right\|_{L^p(CI_m)}. \tag{3.4}$$

Since the measure of I_m is bounded by δ, for a suitable δ, we conclude

$$\|p_m(w_\beta)\sigma\|_{L^p[-\sqrt{2m\theta},\sqrt{2m\theta}]} \ge \frac{C}{\sqrt[4]{m}} \left\| \frac{v^\alpha}{\sqrt{v^\beta}} \right\|_{L^p([-1,1])}. \tag{3.5}$$

To prove (3.3), setting $A_m = [-\sqrt{2m}, \sqrt{2m}]$ and $C^*I_m = A_m\backslash I_m$, by (2.2), we have

$$|p_m(w_\beta, x)\sigma(x)| \ge C\frac{|x|^{\alpha-\frac{\beta}{2}}}{\sqrt[4]{2m - x^2 + m^{\frac{1}{3}}}}, \quad x \in C^*I_m.$$

Then

$$\begin{aligned}\|p_m(w_\beta)\sigma\|_p &\geq \|p_m(w_\beta)\sigma\|_{L^p(C^*I_m)}\\ &\geq C\left(\int_{C^*I_m}\frac{|x|^{(\alpha-\frac{\beta}{2})p}}{\sqrt[4]{\left(2m-x^2+m^{\frac{1}{3}}\right)}^p}dx\right)^{\frac{1}{p}}\\ &= C(\sqrt{2m})^{\alpha-\frac{\beta}{2}+\frac{1}{p}-\frac{1}{2}}\left(\int_{A_m^*}\frac{|x|^{(\alpha-\frac{\beta}{2})p}}{\sqrt[4]{\left(1-x^2+m^{-\frac{2}{3}}\right)}^p}dx\right)^{\frac{1}{p}}\end{aligned}$$

where $A_m^*=[-1,1]\backslash B_m^*$, and $B_m^*=\left(\cup_{1\leq k\leq m}\left[\frac{x_k-\delta_k}{\sqrt{2m}},\frac{x_k+\delta_k}{\sqrt{2m}}\right]\right)$ i.e.,

$$\|p_m(w_\beta)\sigma\|_p \geq C(\sqrt{2m})^{\alpha-\frac{\beta}{2}+\frac{1}{p}-\frac{1}{2}}\left\{\int_{-1}^{1}\frac{|x|^{(\alpha-\frac{\beta}{2})p}}{\sqrt[4]{\left(1-x^2+m^{-\frac{2}{3}}\right)}^p}dx\right.$$

$$\left.-\sum_{k=1}^{m}\int_{\frac{x_k-\delta_k}{\sqrt{2m}}}^{\frac{x_k+\delta_k}{\sqrt{2m}}}\frac{|x|^{(\alpha-\frac{\beta}{2})p}}{\sqrt[4]{\left(1-x^2+m^{-\frac{2}{3}}\right)}^p}dx\right\}^{\frac{1}{p}} =: C(\sqrt{2m})^{\alpha-\frac{\beta}{2}+\frac{1}{p}-\frac{1}{2}}\{\bar{I}_1-\bar{I}_2\}^{\frac{1}{p}}.$$

We observe that the measure of B_m^* is δ and $\bar{I}_2$ is finite. So we can choose δ such that $\bar{I}_2=\frac{1}{2}\bar{I}_1$, and (3.3) follows. □

The next proposition follows, with small changes, from a theorem proved by S. W. Jha and D. S. Lubinski (see [7], Theorem 1.1, p.338).

Proposition 3.2 *With $F_m(x)=p_m(w_\beta,x)-p_{m-2}(w_\beta,x)$, the following estimate holds*

$$|F_m(x)|\sqrt{w_\beta(x)}\leq\frac{\mathcal{C}}{\sqrt{m}}\sqrt[4]{|2m-x^2|+(2m)^{\frac{1}{3}}},\quad x\in I\!R. \tag{3.6}$$

Now we can prove the statements of the Section 2.

Proof of Proposition 2.2. We recall some polynomial inequalities. For any polynomial p_m of degree m or less ($p_m\in I\!P_m$) we have [11]:

$$\left(\int_{I\!R}|p_m(x)u(x)|^p dx\right)^{\frac{1}{p}}\leq A\left(\int_{B_m}|p_m(x)u(x)|^p dx\right)^{\frac{1}{p}}, \tag{3.7}$$

where $B_m = [-\sqrt{2m}, \sqrt{2m}]\setminus\left[-\frac{C}{\sqrt{m}}, \frac{C}{\sqrt{m}}\right]$ and A is a positive constant dependent of C.
Moreover, for every $\delta > 0$, it results

$$\left(\int_{\left\{|x|>\sqrt{2m(1+\delta)}\right\}} |p_m(x)u(x)|^p dx\right)^{\frac{1}{p}} \tag{3.8}$$
$$\leq \quad Ce^{-Am}\left(\int_{\mathbb{R}} |p_m(x)u(x)|^p dx\right)^{\frac{1}{p}},$$

where C and A depend on δ and are independent of $\mathbb{P}_m$.
For completeness we recall also that the Mhaskar-Saff number [12] is $a_m = \sqrt{2m+]\gamma[}$ i.e.,

$$\|pu\|_\infty = \|pu\|_{[-a_m,a_m]}, \quad p \in \mathbb{P}_m.$$

Now, for any polynomial P_M of degree $M = \left[m\sqrt{\frac{\theta}{1+\theta}}\right]$, we can write

$$\|f(1-\Delta_{m,\theta})u\|_p \leq \|[f-P_M]u\|_p + \|P_M(1-\Delta_{m,\theta})u\|_p$$

and by (3.8) it is

$$\|u(1-\Delta_{m,\theta})P_M\|_p \leq Ce^{-Am}\|P_M u\|_p$$

taking the infimum on P_M, proposition follows. □

To prove Theorem 2.1, we recall the $\mathcal{A}_p$ weight definition with $1 < p < \infty$. A weight σ is an $\mathcal{A}_p$ weight in $[a,b]$ if and only if, for every interval $I \subset [a,b]$, it results

$$\|\sigma\|_{L^p(I)}\|\sigma^{-1}\|_{L^p(I)} \leq C|I|,$$

where C is independent of $|I| = meas(I)$.
If σ is a generalized Jacobi weight, then $\sigma \in L^p(a,b)$ and $\sigma^{-1} \in L^q(a,b)$, $q = \frac{p}{p-1}$, imply the $\mathcal{A}_p$ condition of σ in (a,b)(see for instance [10]).
We remember also that the $\mathcal{A}_p$ condition is equivalent to the following inequality

$$\left(\int_a^b \left|\sigma(t)\int_a^b \frac{f(x)}{x-t}dx\right|^p dt\right)^{\frac{1}{p}} \leq C\left(\int_a^b |f(x)\sigma(x)|^p dx\right)^{\frac{1}{p}}.$$

Proof of Proposition 2.1. First we prove that (2.4), for $\frac{4}{3} < p < 4$, implies (2.3).

With $B_m = A_m \setminus \left[-\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right] = [-\sqrt{2m}, \sqrt{2m}] \setminus \left[-\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right]$, using (3.7), we have

$$\|S_m(w_\beta, f)u\|_p \le C\|S_m(w_\beta, f)u\|_{L^p(B_m)}.$$

By (3.1), and $\alpha_m \sim \sqrt{m}$, we get

$$\begin{aligned} \|S_m(w_\beta, f)u\|_p &\le \sqrt{m}\left\{\|p_{m-1}(w_\beta)H(fF_m w_\beta)u\|_{L^p(B_m)}\right. \\ &+ \left.\|F_m H(fp_{m-1}(w_\beta)w_\beta)u\|_{L^p(B_m)}\right\} \\ &+ |c_m(f)|\|p_{m-1}(w_\beta)u\|_{L^p(B_m)} \\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{3.9}$$

Consider J_1. With $\delta \in (0,1)$, we write

$$\begin{aligned} J_1 &\le \sqrt{m}\left\{\left(\int_{B_m}\left|p_m(w_\beta,x)\left(\int_{-\sqrt{2m}(1+\delta)}^{\sqrt{2m}(1+\delta)}\frac{(fF_m w_\beta)(t)}{t-x}dt\right)u(x)dx\right|^p\right)^{\frac{1}{p}}\right. \\ &+ \left.\left(\int_{B_m}\left|p_m(w_\beta,x)\left(\int_{|t|>\sqrt{2m}(1+\delta)}\frac{(fF_m w_\beta)(t)}{t-x}dt\right)u(x)dx\right|^p\right)^{\frac{1}{p}}\right\} \\ &=: J_{1,1} + J_{1,2} \end{aligned} \tag{3.10}$$

By (2.2) we have

$$\begin{aligned} |p_m(w_\beta,x)|\sqrt{w_\beta(x)}\sqrt[4]{|2m-x^2|+(2m)^{\frac{1}{3}}} &\le C, \\ \frac{C}{\sqrt{m}} < |x| &\le \sqrt{2m}\left(1+\frac{C}{m^{2/3}}\right), \end{aligned} \tag{3.11}$$

and consequently

$$J_{1,1} \le C\sqrt{m}\left(\int_{B_m}\left|\frac{|x|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{|2m-x^2|+m^{\frac{1}{3}}}}\int_{-\sqrt{2m}(1+\delta)}^{\sqrt{2m}(1+\delta)}\frac{(fF_m w_\beta)(t)}{t-x}dt\right|^p dx\right)^{\frac{1}{p}}$$

$$\le C\sqrt{m}\left(\int_{-\sqrt{2m}(1+\delta)}^{\sqrt{2m}(1+\delta)}\left|\frac{|x|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{|2m-x^2|+m^{\frac{1}{3}}}}\int_{-\sqrt{2m}(1+\delta)}^{\sqrt{2m}(1+\delta)}\frac{(fF_m w_\beta)(t)}{t-x}dt\right|^p dx\right)^{\frac{1}{p}}$$

Setting $x=\sqrt{2m}y$, $t=\sqrt{2m}z$

$$J_{1,1} = C(\sqrt{m})^{\gamma-\frac{\beta}{2}+\frac{1}{p}+\frac{1}{4}} \times \left(\int_{-(1+\delta)}^{(1+\delta)} \left| \frac{|y|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{|1-y^2|+m^{-\frac{2}{3}}}} \int_{-(1+\delta)}^{(1+\delta)} \frac{(fF_m w_\beta)(t\sqrt{2m})}{z-y} dz \right|^p dy \right)^{\frac{1}{p}}.$$

The function $|y|^{\gamma-\frac{\beta}{2}}(|1-y^2|+m^{-\frac{2}{3}})^{-\frac{1}{4}}$ is a generalized Jacobi weight, which by (2.4) and $p<4$, is also an $\mathcal{A}_p$ weight in $[-(1+\delta),(1+\delta)]$. Therefore we have

$$\begin{aligned} J_{1,1} &\le C(\sqrt{m})^{\gamma-\frac{\beta}{2}+\frac{1}{p}+\frac{1}{4}} \left(\int_{-(1+\delta)}^{(1+\delta)} \left| \frac{|y|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{|1-y^2|+m^{-\frac{2}{3}}}} (fF_m w_\beta)(y\sqrt{2m}) \right|^p dy \right)^{\frac{1}{p}} \\ &= C\sqrt{m} \left(\int_{-\sqrt{2m}(1+\delta)}^{\sqrt{2m}(1+\delta)} \left| \frac{|x|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{|2m-x^2|+m^{\frac{1}{3}}}} (fF_m w_\beta)(x) \right|^p dx \right)^{\frac{1}{p}} \\ &\le C \left(\int_{-\sqrt{2m}(1+\delta)}^{\sqrt{2m}(1+\delta)} |f(x)u(x)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

where, in the last inequality, we used estimate (3.6) for $|F_m\sqrt{w_\beta}|$.
To estimate $J_{1,2}$, use $x-t>\delta\sqrt{2m}$ and Hölder inequality to obtain

$$J_{1,2} \le C\|p_{m-1}(w_\beta)u\|_{L^p(B_m)} \|fu\|_p \left\| F_m \frac{w_\beta}{u} \right\|_{L^q(|x|>\sqrt{2m}(1+\delta))}.$$

Now by (3.11) we have

$$\|p_{m-1}(w_\beta)u\|_{L^p(B_m)} \le Cm^{-\frac{1}{12}} \left(\int_{B_m} |x|^{(\gamma-\frac{\beta}{2})p}\right)^{\frac{1}{p}}.$$

Moreover, using (3.6) and (3.8), we get

$$\left\| F_m \frac{w_\beta}{u} \right\|_{L^q(|x|>\sqrt{2m}(1+\delta))} \le C\frac{e^{-Am}}{\sqrt[4]{m}} \left(\int_{B_m} |x|^{(\frac{\beta}{2}-\gamma)q}\right)^{\frac{1}{q}},$$

and, recalling (2.4), we conclude

$$J_{1,2} \le C\|fu\|_p. \tag{3.12}$$

In a similar way we estimate J_2. Finally we consider J_3. Using Hölder inequality, (3.9) and (3.11), we get

$$\begin{aligned} J_3 &\leq C\|fu\|_p \|p_m(w_\beta)u\|_p \left\|p_m \frac{w_\beta}{u}\right\|_{L^q(B_m)} \\ &\leq C\|fu\|_p \left(\int_{-\sqrt{2m}}^{\sqrt{2m}} \left|\frac{|x|^{\frac{\beta}{2}-\gamma}}{\sqrt[4]{2m-x^2}}\right|^q dx\right)^{\frac{1}{q}} \left(\int_{-\sqrt{2m}}^{\sqrt{2m}} \left|\frac{|x|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{2m-x^2}}\right|^p dx\right)^{\frac{1}{p}} \\ &\leq C\|fu\|_p \left(\int_{-1}^{1} \left|\frac{|y|^{\frac{\beta}{2}-\gamma}}{\sqrt[4]{1-y^2}}\right|^q dy\right)^{\frac{1}{q}} \left(\int_{-1}^{1} \left|\frac{|y|^{\gamma-\frac{\beta}{2}}}{\sqrt[4]{1-y^2}}\right|^p dy\right)^{\frac{1}{p}} \end{aligned} \tag{3.13}$$

and taking into account (2.4) and $\frac{4}{3} < p < 4$,

$$J_3 \leq C\|fu\|_p.$$

Now we prove the necessary condition. By (2.3)

$$\|[S_m(w_\beta, f) - S_{m-1}(w_\beta, f)]u\|_p \leq 2C\|fu\|_p,$$

i.e.,

$$\left|\int_{\mathbb{R}} f(x)p_{m-1}(w_\beta, x)w_\beta(x)dx\right| \|p_{m-1}(w_\beta)u\|_p \leq 2C\|fu\|_p$$

and

$$\sup_{\|g\|_p=1} \left|\int_{\mathbb{R}} g(x)p_{m-1}(w_\beta, x)\frac{w_\beta(x)}{u(x)}dx\right| \|p_{m-1}(w_\beta)u\|_p \leq 2C$$

or

$$\left\|p_{m-1}(w_\beta)\frac{w_\beta}{u}\right\|_q \|p_{m-1}(w_\beta)u\|_p \leq C.$$

Then $\beta - \gamma \in L^q$ and by (3.3) we deduce

$$\left(\int_{-1}^{1} \frac{|x|^{(\gamma-\frac{\beta}{2})p}}{\sqrt[4]{\left(1-x^2+m^{-\frac{2}{3}}\right)^p}}dx\right)^{\frac{1}{p}} \left(\int_{-1}^{1} \frac{|x|^{(-\gamma+\frac{\beta}{2})q}}{\sqrt[4]{\left(1-x^2+m^{-\frac{2}{3}}\right)^q}}dx\right)^{\frac{1}{q}} \leq C$$

and consequently it is $\frac{4}{3} < p < 4$ and (2.4) holds. □

In order to prove that (2.5) is true, (2.4) holding with $p \in (1,\infty)\setminus\left(\frac{3}{4}, 4\right)$, we start from (3.10), i.e.

$$J_1 \leq J_{1,1} + J_{1,2}.$$

From (3.11) and (3.6) we get

$$|p_m(w_\beta,x)|\sqrt{w_\beta(x)} \le \frac{C}{m^{\frac{1}{12}}}$$

and

$$|F_m(x)|\sqrt{w_\beta(x)} \le \frac{C}{m^{\frac{1}{4}}}$$

where $\frac{C}{\sqrt{m}} \le |x| \le \sqrt{2m}$. Using the above inequalities we obtain

$$J_{1,1} \le Cm^{\frac{1}{6}}\|fu\|_p \quad \text{and} \quad J_{1,2} \le C\|fu\|_p.$$

Recalling (3.9), the estimate of J_2 is similar to the one of J_1. To evaluate J_3 we start from (3.13) and we deduce

$$\begin{aligned} J_3 &\le C\|fu\|_p\|p_m(w_\beta)u\|_p \left\|p_m\frac{w_\beta}{u}\right\|_q \\ &\le C\|fu\|_p\, m^{-\frac{1}{6}} \left(\int_{-\sqrt{2m}}^{\sqrt{2m}} |x|^{(\gamma-\frac{\beta}{2})q}dx\right)^{\frac{1}{q}} \left(\int_{-\sqrt{2m}}^{\sqrt{2m}} |x|^{(\frac{\beta}{2}-\gamma)p}dx\right)^{\frac{1}{p}} \\ &\le Cm^{\frac{1}{3}}\|fu\|_p. \end{aligned}$$

Then (2.5) is proved.

Proof of Theorem 2.3. The proof of this theorem is simpler. With $1 < p < \infty$, we have

$$\|S_m(w_\beta, f\Delta_{m,\theta})u\Delta_{m,\theta}\|_p = \|S_m(w_\beta, f\Delta_{m,\theta})u\|_{L^p(-\sqrt{2m\theta},\sqrt{2m\theta})}.$$

Since, in the interval $(-\sqrt{2m\theta},\sqrt{2m\theta})$, the function e^{-x^2} is equivalent to a polynomial of degree m (see for instance [4]), we can use a Remez type inequality with respect to the interval $(-\sqrt{2m\theta},\sqrt{2m\theta})$ to obtain

$$\|S_m(w_\beta, f\Delta_{m,\theta})u\Delta_{\theta m}\|_p \le C\|S_m(w_\beta, f\Delta_{m,\theta})u\|_{L^p(C_m)},$$

where $C_m = (-\sqrt{2m\theta},\sqrt{2m\theta})\setminus\left[-\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right]$. Using (3.1), we can write

$$\begin{aligned} \|S_m(w_\beta, f\Delta_{m,\theta})u\Delta_{m,\theta}\|_p &\le |c_{m-1}(\Delta_{m,\theta}f)|\|p_{m-1}(w_\beta)u\|_{L^p(C_m)} \\ &+ C\sqrt{m}\|p_{m-1}(w_\beta)H(\Delta_{m,\theta}fF_mw_\beta)u\|_{L^p(C_m)} \\ &+ C\sqrt{m}\|F_muH(fp_{m-1}(w_\beta)w_\beta)\|_{L^p(C_m)} \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

In the interval C_m, it results

$$\left| p_m(w_\beta, x)\sqrt{w_\beta(x)} \right| \leq \frac{C}{\sqrt[4]{m}}.$$

Then

$$J_1 \leq \|fu\|_p \left\| p_m(w_\beta)\frac{w_\beta}{u} \right\|_{L^q(-\sqrt{2m\theta},\sqrt{2m\theta})} \|p_{m-1}(w_\beta)u\|_{L^p(C_m)}$$

and by (2.4), these norms are bounded. Then we continue as

$$\begin{aligned} J_1 &\leq C\|fu\|_p \left\| p_m(w_\beta)\frac{w_\beta}{u} \right\|_{L^q(C_m)} \|p_{m-1}(w_\beta)u\|_{L^p(C_m)} \\ &\leq \frac{C}{\sqrt[4]{m}}\|fu\|_p \left(\int_{C_m} |x|^{(\frac{\beta}{2}-\gamma)q} dx \right)^{\frac{1}{q}} \left(\int_{C_m} |x|^{(\gamma-\frac{\beta}{2})p} dx \right)^{\frac{1}{p}} \\ &\leq C\|fu\|_p \left(\int_0^1 x^{(\frac{\beta}{2}-\gamma)q} dx \right)^{\frac{1}{q}} \left(\int_0^1 x^{(\gamma-\frac{\beta}{2})p} dx \right)^{\frac{1}{p}} \end{aligned}$$

and, under the condition (2.4) with $1 < p < \infty$, the integrals on the right-hand side are bounded.
Moreover, with $1 < p < \infty$, we obtain

$$J_2 \leq C\sqrt[4]{m} \left(\int_{C_m} \left| |x|^{\gamma-\frac{\beta}{2}} \int_{-\sqrt{2m\theta}}^{\sqrt{2m\theta}} \frac{(fF_m w_\beta)(t)}{t-x} dt \right|^p dx \right)^{\frac{1}{p}}.$$

Proceeding like in the Proof of Theorem 2.1, the function $|x|^{\gamma-\frac{\beta}{2}}$ is an $\mathcal{A}_p$ weight and

$$J_2 \leq C\sqrt[4]{m} \left(\int_{-\sqrt{2m\theta}}^{\sqrt{2m\theta}} \left| |x|^{\gamma-\frac{\beta}{2}} \frac{(F_m w_\beta)(x)}{u(x)} (fu)(x) \right|^p dx \right)^{\frac{1}{p}}.$$

Since by (3.6) follows

$$|F_m\sqrt{w_\beta}|(x) \leq \frac{C}{\sqrt[4]{m}},$$

now we conclude

$$J_2 \leq \|fu\|_p.$$

In a similar way we estimate J_3.

Now we prove that (2.8) implies (2.4). Consider the function $\tilde{f} = f\Delta_{\theta,m-1}$, where $\Delta_{\theta,m-1}$ is the characteristic function on $[-\sqrt{2(m-1)\theta}, \sqrt{2(m-1)\theta}]$. By (2.8)

$$\|S_m(w_\beta, \tilde{f}\Delta_{m,\theta})u\Delta_{m,\theta}\|_p \le C\|\tilde{f}u\Delta_{m,\theta}\|_p = C\|\tilde{f}u\|_p.$$

By applying a Remez-type inequality (see [11])

$$\begin{aligned}\|S_{m-1}(w_\beta, \tilde{f}\Delta_{m,\theta})u\Delta_{m,\theta}\|_p &= \|S_{m-1}(w_\beta, f\Delta_{m-1,\theta})u\Delta_{m,\theta}\|_p \\ &\le C\|S_{m-1}(w_\beta, f\Delta_{m-1,\theta})u\Delta_{m-1,\theta}\|_p \quad (3.14)\\ &\le \|fu\Delta_{m-1,\theta}\|_p = C\|\tilde{f}u\|_p \quad (3.15)\end{aligned}$$

Therefore we have

$$\|[S_m(w_\beta, \tilde{f}\Delta_{m,\theta}) - S_{m-1}(w_\beta, \tilde{f}\Delta_{m,\theta})]u\Delta_{m,\theta}\|_p \le 2C\|\tilde{f}u\|_p$$

$$\left|\int_{\mathbb{R}} \tilde{f}(x)\Delta_{m,\theta}(x)p_{m-1}(w_\beta, x)w_\beta(x)dx\right| \|p_{m-1}(w_\beta)u\Delta_{m,\theta}\|_p \le 2C\|\tilde{f}u\|_p$$

and

$$\sup_{\substack{\|g\|_{L^p([-\sqrt{2m\theta},\sqrt{2m\theta}])}=1 \\ \|p_{m-1}(w_\beta)u\Delta_{m,\theta}\|_p \le 2C,}} \left|\int_{\mathbb{R}} g(x)p_{m-1}(w_\beta, x)\frac{w_\beta(x)}{u(x)}dx\right| \times$$

i.e.,

$$\left\|p_{m-1}(w_\beta)\frac{w_\beta}{u}\Delta_{m,\theta}\right\|_q \|p_{m-1}(w_\beta)u\Delta_{m,\theta}\|_p \le C.$$

Therefore $\beta - \gamma > -\frac{1}{q}$ follows and (3.3) assures that

$$\left(\int_0^1 x^{(\gamma-\frac{\beta}{2})q}dx\right)^{\frac{1}{q}} \left(\int_0^1 x^{(\frac{\beta}{2}-\gamma)p}dx\right)^{\frac{1}{p}} \le C,$$

which implies $-\frac{1}{p} < \frac{\beta}{2} - \gamma < 1 - \frac{1}{p}$. Now we prove (2.9). For any polynomial P_M of degree $M = \left[m\sqrt{\frac{\theta}{1+\theta}}\right]$

$$\begin{aligned}\|[f - \Delta_{m,\theta}S_m(w_\beta, \Delta_{m,\theta}f)]u\|_p &\le \|[f - P_M]u\|_p + \|P_M(1 - \Delta_{m,\theta})u\|_p \\ &+ \|\Delta_{m,\theta}S_m(w_\beta, P_M(1 - \Delta_{m,\theta})u\|_p \quad (3.16)\\ &+ \|\Delta_{m,\theta}S_m(w_\beta, \Delta_{m,\theta}(f - P_M))u\|_p\end{aligned}$$

The first term is dominated by the best approximation error $E_M(f)_{u,p}$ and the last addendum can estimated by (2.8). For the second addendum we use (2.5) and we get

$$\begin{aligned}\|\Delta_{m,\theta}S_m(w_\beta, P_M(1-\Delta_{m,\theta})u\|_p &\leq Cm^{\frac{1}{3}}\|P_M(1-\Delta_{m,\theta})u\|_p \\ &\leq Ce^{-Am}m^{\frac{1}{3}}\|P_M u\|_p \leq Ce^{-Am}m^{\frac{1}{3}}\|fu\|_p \\ &\sim e^{-Am}\|fu\|_p.\end{aligned}$$

The Theorem is completely proved. □

Proof of Theorem 2.4. We prove Theorem 2.4 using the same arguments used in the proofs of the theorems 2.1 and 2.3. We omit the details. □

References

[1] R. Askey and S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, *Amer. J. Math.* **87**(1965), 695–708.

[2] V. M. Badkov, Convergence in the mean and almost everywhere of Fourier series in polynomials that are orthogonal on an interval , *(Russian) Mat. Sb. (N.S.)* **95(137)** ((1974), 229–262, 327.

[3] G.Criscuolo and G. Mastroianni, Fourier and Lagrange operators in some weighted Sobolev-type spaces , *Acta Sci. Math. (Szeged)* **60** (1995).

[4] M. C. De Bonis, G. Mastroianni, M. Viggiano, $K-$functionals, Moduli of Smoothness and Weighted Best Approximation on the semiaxis , *Functions, Series, Operators (L. Leindler, F. Schipp, J. Szabados, eds.) Janos Bolyai Mathematical Society, Budapest, Hungary,* 2002*, Alexits Memorial Conference* (1999), 181–211.

[5] O. Kis, Lagrange interpolation with nodes at the roots of Sonin-Markov polynomials , *(Russian), Acta Math. Acad. Sci. Hungar.* **23** (1972), 389–417.

[6] N. X. Ky, A contribution to the problem of weighted polynomial approximation of the derivative of a function by the derivative of its approximating polynomial , *Scient. Math. Hungar.* **10** (1975), 309–316.

[7] S. W. Jha and D. S. Lubinsky, Necessary and Sufficient Conditions for Mean convergence of Orthogonal Expansions for Freud Weights , *Constr. Approx.* **11**, (1995), 331-363.

[8] G. Mastroianni, G. Monegato, Truncated quadrature rules over $(0, \infty)$ and Nyström type methods , *SIAM J. Num. Anal.* **41** (2003), *no.* 5, 1870–1892.

[9] G. Mastroianni, D. Occorsio, Lagrange interpolation at Sonin-Markov zeros , *Suppl. Rendiconti del Circolo Matematico di Palermo,* **Serie II 68**, (2002) 683-697.

[10] G. Mastroianni, M.G. Russo, Lagrange interpolation in weighted Besov spaces , *Constr. Approx.* **15** (1999), no. 2, 257–289.

[11] G. Mastroianni, J. Szabados, Polynomial approximation on infinite intervals with weights having inner zeros , *Acta Math. Hungar.,* **96** (2002), *no.* 3, 221–258.

[12] H. N. Mhaskar, E. B. Saff, Extremal Problems for Polynomials with Laguerre Weights , *Approx. Theory IV, Academic Press eds.* (1983).

[13] B. Muckenhoupt, Mean convergence of Jacobi series , *Proc. Amer. Math. Soc.* **23** (1969) , 306–310.

[14] B. Muckenhoupt, Mean convergence of Hermite and Laguerre series I , *Trans. Amer. Math. Soc.* **147** (1970), 419–431.

[15] B. Muckenhoupt, Mean convergence of Hermite and Laguerre series II , *Trans. Amer. Math. Soc.* **147** (1970), 433–460.

[16] H. Pollard, The mean convergence of orthogonal series. II , *Trans. Amer. Math. Soc.* **63** (1948), 355-367.

[17] M.G. Russo, Proiettori in spazi funzionali ed equazioni integrali , *Phd thesis University of Naples "Federico II",* (Italy) (1998).

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Regularized Solutions Constructed by Semigroup and Distribution Techniques

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Abstract. Cauchy problems that are not uniformly well-posed are under consideration. Semigroup, distribution, Fourier transform methods, as well as methods of the ill-posed problems theory are investigated from a general point of view; namely, we show that all these methods are the techniques for constructing regularized solutions and regularizing operators.

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1 Introduction

We consider the abstract Cauchy problem

$$u'(t) = Au(t), \quad t \in [0,T),\ T \le \infty, \quad u(0) = f, \tag{1.1}$$

where A is a linear operator on a Banach space X; in particular, we consider a matrix differential operator $A\left(i\frac{\partial}{\partial x}\right)$ so that (1.1) stands for the Cauchy problem

$$\frac{\partial u(x;t)}{\partial t} = A\left(i\frac{\partial}{\partial x}\right)u(x;t), \quad t \in [0;T),\ x \in \mathbb{R}^n, \quad u(x;0) = f(x). \tag{1.2}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

We assume that A does not generate a semigroup of class C_0, or equivalently that (1.1) is not uniformly well–posed.

We recall that the Cauchy problem (1.1) is said to be uniformly well–posed if

$$\forall f \in \mathrm{Dom}\, A,\ \exists!\ u(\cdot) \in C^1([0,\infty), X) \cap C([0,\infty), \mathrm{Dom} A) :$$

$$\sup_{t \le \tau < T} \|u(t)\| \le C_T \|f\| \qquad \text{for some } C_T \ge 0.$$

For treating and solving such problems three approaches based on semigroup, distribution, and regularization methods were pointed out in [16].

To this regard, the following questions arise. What type of peculiarities of the ill–posed problems can each of mentioned methods treat? What type of solutions of abstract Cauchy problem (1.1) and differential one (1.2) can be constructed by these methods? Finally, how to use these techniques for constructing stable approximate solutions?

The present paper is meant to answer the previous questions. Accordingly, in the next sections we study semigroup techniques as well as distribution, ultradistribution, and generalized function ones, and in the last section we construct regularizing operators combining the obtained results with ones from the ill–posed problems theory.

Modern semigroup methods (see, for instance, [1, 5, 7, 16]) are directed to construct families of bounded operators which are more general than semigroups of class C_0, such as integrated, K-convoluted and R-semigroups. They allow to construct solution operators and corresponding solutions of (1.1) for initial data f in subclasses of $\mathrm{Dom}\, A$ which are stable with respect to variations of f by a norm stronger than that of the given space X. We show that the semigroups themselves supply regularized solution operators.

The methods from the abstract distribution and ultradistribution theory [8, 2, 16] provide a generalized (with respect to t) solution of (1.1) for each $f \in X$. In this case, the term 'abstract' means that we deal with X-valued operators instead of ordinary ($\mathbb{C}$-valued) functionals.

Generalized (with respect to 'spatial variables' of A) solutions for (1.1) can be constructed in the Ivanov new generalized functions spaces [10, 16], while, for the differential problem (1.2) in the Gelfand spaces [9]. In the latter, the Fourier transform is successfully applied to explore the problem (1.2) by stressing specific differential features of the problem and allowing to get generalized solutions with respect to x.

The third mentioned approach deals with the methods developed within the regularization theory of ill–posed problems. The methods of regulariz-

ing ill–posed problems allow to construct a solution u_ε of a certain well–posed problem related to (1.1) and depending on a regularizing parameter ε, for each initial data $f_\delta \in X$ given with an error ($\|f_\delta - f\| \le \delta$). The proper co-ordination of the parameters ε and δ ($\varepsilon(\delta) \to_{\delta\to 0} 0$) yields the convergence $u_{\varepsilon(\delta)}(t) \to u(t)$ as $\delta \to 0$. The function $u_{\varepsilon(\delta)}(\cdot)$ is called regularized solution, and is determined by a regularizing operator $\mathbf{R}_\varepsilon(t)$: $\mathbf{R}_\varepsilon(t) f_\delta = u_\varepsilon(t),\ t \in [0;T)$.

In the paper we show that the semigroup techniques as well as the distribution techniques, including the generalized Fourier transform ones for differential problem (1.2), give a way for constructing several regularized (in a broad sense) solutions. Regularization in a broad sense means that we construct a corrected (smoothed out) solution which does not approximate an exact one, generally. In the case of generalized solutions the regularization (smoothing) is due to corresponding test functions, in the case of K-convoluted or R-semigroups it is due to convolution of solution operators with a smooth function K or due to a bounded operator R applied to initial data.

As it was mentioned above, the regularized solutions constructed by these methods are not approximate in the general case. The only solutions to approximate an exact one are those constructed by virtue of regularizing operators [11, 14]:

$$u_{\varepsilon(\delta)}(t) = \mathbf{R}_{\varepsilon(\delta)}(t) f_\delta \to u(t) \quad \text{as} \quad \delta \to 0.$$

We construct regularizing operators for problems (1.2) on the basis of relationship between regularizing operators and R–semigroups [14, 16]. In the paper this regularization will be realized for the so called strongly ill–posed problems such as the backward Cauchy problem ($A = -d^2/dx^2$) and for weakly ill–posed ones such as the Cauchy problem for the hyperbolic equation $u''_{tt}(x,t) = u''_{xx}(x,t)$ considered in the form (1.2) for system with $u_1 = u, u_2 = u'_t$ $\left(A = \begin{bmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}\right)$.

2 Regularization by semigroup methods

We begin with the definition of the most general semigroups, which we call regularized semigroups. This type of bounded operator families on a Banach space X includes the integrated semigroups introduced by Arendt [1], the R–semigroups of Da Prato and Davies–Pang [6, 4, 18], and the Cioranesku–Lumer convoluted semigroups [3] as specific cases.

Definition 2.1 *Let A be a closed linear operator and $R(t), t \in [0,T)$, be bounded linear operators on X. A strongly continuous family of linear bounded operators $\{S(t), t \in [0,T)\} =: S$ such that $S(t)Af = AS(t)f$ for any $f \in \operatorname{Dom} A$ and*

$$S(t)f = A\int_0^t S(s)fds + R(t),\ f \in X, \tag{2.1}$$

is called the R-regularized semigroup generated by A, and A is the generator of S.

If there exist $C > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \le Ce^{\omega t}$ for $t \ge 0$, then S is exponentially bounded, in the case $T < \infty$ it is a local one.

In the special case in which for any $t \in [0,T)$, $R(t)$ coincides with a linear bounded invertible operator R with dense image, S is an R-semigroup. For this case, important for the regularization, another definition of R-semigroups (called in [4] C-semigroups) is given through the 'semigroup relation':

$$S(t+s)R = S(t)S(s), \quad s,t,s+t \in [0,T), \quad S(0) = R.$$

The operator $Af := R^{-1}\lim_{h\to 0} h^{-1}(S(h)f - f)$, where $f \in$Dom A if the limit belongs to $\operatorname{Im} R$, is called the generator of S. (Relations between such definitions are studied in [15, 13]).

It is proved that the generator is densely defined and that for a densely defined operator A the Cauchy problem (1.1) is R–well–posed on $[0,T)$, i.e.

$$\forall f \in R(\operatorname{Dom} A),\ \exists!\ u(\cdot) : \sup_{0\le t\le\tau<T} \|u(t)\| \le\ C_T\|R^{-1}f\|,$$

if and only if A is the generator of an R-semigroup [4]. If, in addition, A is the generator of a semigroup $\{U(t), t \ge 0\}$ of class C_0, then $S(t)f = U(t)Rf = RU(t)f$ $(f \in X)$. In the general case of $U(t)$ being unbounded solution operators, we have $S(t)f = U(t)Rf$. Hence the regularization of solution operators in the R-semigroup approach comes out from replacing f by Rf.

In the other special case in which, for any $t \in [0,T)$, $R(t) = I\int_0^t K(s)ds$, where $K(\cdot)$ is a continuous function, S is called a K-convoluted semigroup. If, in addition, A is the generator of a semigroup $\{U(t), t \ge 0\}$ of class C_0, then the K-convoluted semigroup

$$S(t)f = \int_0^t K(t-s)U(s)fds;$$

specifically, the n-times integrated one $S(t)f = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}U(s)fds$ is the n-th order primitive of U.

In the general case of A being the generator of a K-convoluted semigroup, the solution operators $U(t)$ are unbounded, hence $u = Uf$ is a solution of (1.1) only for f belonging to a suitable subset of $\mathrm{Dom}\, A$ and, as usually for an inhomogeneous equation, for (2.1) with $R(t) = I\int_0^t K(s)ds$ we have $\int_0^t S(s)fds = R * u$, hence $Sf = K * u = (K * U)f$. If $f \in X$, the convolution is defined in an extended sense and Sf supplies with a K-convoluted solution.

Thus, in the case of K-convoluted semigroups we construct the regularized solution $Sf = (U * K)f$ by smoothing solution operators through the convolution with K.

Now we show that this smoothing of solution operators is equivalent to reducing the resolvent increase through the multiplication of the resolvent by corresponding function $\widetilde{K}(\lambda)$, the Laplace transform of K. It follows from the theorems below that the resolvent of the generator of a K-convoluted semigroup exists in a region Λ from a right semi–plane and can exponentially increase: $\|\mathcal{R}_A(\lambda)\| \le Ce^{M(\lambda)}, \lambda \in \Lambda$, where a real-valued function $M(\lambda)$ grows not faster than $|\lambda|^a$ $(0 < a < 1)$ and is defined by the order of decreasing of $\widetilde{K}(\lambda)$.

Theorem 2.2 *Let A be the generator of a K-convoluted semigroup $\{S(t), t \in [0,T)\}$, where K is such that its Laplace transform satisfies the condition:*

$$|\widetilde{K}(\lambda)| = \mathcal{O}_{\substack{|\lambda|\to\infty,\\ \Re\lambda \ge \gamma M(\alpha\lambda)/T}}\left(e^{-M(\alpha\lambda)}\right), \text{ some } \alpha > 0, \gamma \ge 1. \tag{2.2}$$

Then for $\lambda \in \Lambda_{\alpha,\beta,\gamma} := \{\lambda \in \mathbb{C} : \Re\lambda \ge \gamma M(\alpha\lambda)/T + \beta\}$ we have

$$\|\mathcal{R}_A(\lambda)\| \le Ce^{M(\alpha\lambda)}. \tag{2.3}$$

Proof. Let $|K(t)| \le const\, e^{\omega t}$ be such that its Laplace transform $\widetilde{K}(\lambda), \Re\lambda > \omega$ satisfies (2.2). Consider the local Laplace transform $\int_0^t e^{-\lambda\tau}S(\tau)\,d\tau$ of the semigroup S. Then the operator

$$L(\lambda,t) := [\widetilde{K}(\lambda)]^{-1}\int_0^t e^{-\lambda\tau}S(\tau)\,d\tau, \quad t \in [0,T),$$

is bounded and 'nearly' the resolvent: $(\lambda I - A)L(\lambda,t) = I - B(\lambda,t)$, where

$$\|B(\lambda,t)\| \le const\, e^{M(\lambda)-(\Re\lambda-\omega)t}, \text{ and } \|B(\lambda,t)\| \le \delta < 1 \text{ for } \lambda \in \Lambda_{\alpha,\beta,\gamma},$$

$\beta = \beta(T,\omega,\gamma)$ [16]. Hence $\mathcal{R}_A(\lambda) = L(\lambda,t)[(I - B(\lambda,t))]^{-1}, \lambda \in \Lambda_{\alpha,\beta,\gamma}$, and (2.3) holds true. □

Theorem 2.3 *Suppose*

$$\forall \lambda \in \Lambda = \{\lambda \in \mathbb{C} : \ \Re\lambda \geq M(\alpha\lambda)/T\}, \ \|\mathcal{R}_A(\lambda)\| \leq Ce^{M(\alpha\lambda)}, \quad C, \alpha, T > 0.$$

Then A is the generator of a K-convoluted semigroup on $[0,T)$ with K such that $|\widetilde{K}(\lambda)| = \mathcal{O}_{\substack{|\lambda|\to\infty, \\ \Re\lambda \geq \gamma M(\alpha\lambda)/T}} \left(e^{-\gamma M(\alpha\lambda)}\right), \ \gamma > 2$.

Proof. Consider the inverse Laplace transform of $\widetilde{K}(\lambda)\mathcal{R}_A(\lambda)$:

$$\int_{\partial\Lambda} e^{\lambda t}\widetilde{K}(\lambda)\mathcal{R}_A(\lambda)\, d\lambda =: S(t), \tag{2.4}$$

where $\mathcal{R}_A$ and K satisfy the stated hypotheses. Since

$$\|e^{\lambda t}\widetilde{K}(\lambda)\mathcal{R}_A(\lambda)\|\Big|_{\partial\Lambda} \leq Ce^{\Re\lambda t + M(\alpha\lambda) - \gamma M(\alpha\lambda)},$$

the operators $S(t)$ are bounded for $t \in [0,T), \gamma > 2$. It can be verified that $S(t)$ satisfies the convoluted semigroup equation (2.1). Hence S is a K-convoluted semigroup on $[0,T)$ generated by A. □

Thus we see that in the construction of a K–convoluted solution $u = Sf$ the function K is chosen so that the integral (2.4) is convergent (and defines S).

3 Regularization by distribution methods

In the previous section we saw how smoothing functions K and operators R may be used to construct K–convoluted and R–regularized solutions. Now we show that, with given solutions in an extended sense, test functions may be used for constructing regularized solutions.

First we consider generalized (in t) solutions in spaces of abstract distributions and ultradistributions, and show that regularization via test functions corresponds to the regularization in the case of K–convoluted semigroups, where the choice of K conforms to the given spaces of test functions.

Consider a solution $\mathbf{u}$ to the Cauchy problem in the space of the abstract Schwartz distributions $\mathcal{D}'(X) := L(\mathcal{D}, X)$ [8]:

$$\langle P * \mathbf{u}, \varphi\rangle = \langle \delta \otimes f, \varphi\rangle, \quad \varphi \in \mathcal{D}. \tag{3.1}$$

Here $P := \delta' \otimes I - \delta \otimes A \in \mathcal{D}'(L([\mathrm{Dom}A], X)), \langle \delta \otimes A, \varphi\rangle := \langle \delta, \varphi\rangle A$, $[\mathrm{Dom}\, A] := \{f \in \mathrm{Dom}\, A, \|f\|_A := \|f\| + \|Af\|\}$, $\mathbf{u} \in \mathcal{D}'(X)$ and $supp\, \mathbf{u} \subset [0,\infty)$.

It is proved that well-posedness of (3.1) in $\mathcal{D}'(X)$ is equivalent to A being the generator of a local n-times integrated semigroup for some n; generally, when A is the generator of a convoluted semigroup, well-posedness of (3.1) is studied in spaces of ultradistributions [2, 3, 16]. Spaces of test functions for ultradistributions are spaces of infinitely differentiable functions defined in terms of estimates of their derivatives depending on some sequence of positive numbers M_n satisfying the following conditions [12]:

(M.1) $M_n^2 \le M_{n-1}M_{n+1}, \ \ n = 1, 2, \ldots;$

(M.2) $\exists\, \alpha, \beta \in \mathbb{R} : M_n \le \alpha\beta^n \min_{0\le s\le n} M_s M_{n-s};$

(M.3) $\exists\, \gamma \in \mathbb{R} : \sum\limits_{s=n+1}^{\infty} \frac{M_{s-1}}{M_s} \le n\gamma \frac{M_n}{M_{n+1}}.$

Let Ω be a compact set in $\mathbb{R}$ and $h > 0$. We consider the normed space $\mathcal{D}_{\{M_n\},h,\Omega}$ of functions $\varphi \in C^\infty(\mathbb{R})$ with a compact support Ω and estimates

$$\|\varphi^{(n)}\|_{C(\Omega)} = \sup_{t\in\Omega} |\varphi^{(n)}(t)| \le CM_n h^n, \ \ \|\varphi\|_{M_n,h,\Omega} = \sup_{n\in\mathbb{N}} \frac{\|\varphi^{(n)}\|_{C(\Omega)}}{M_n h^n}, \ n \in \mathbb{N}.$$

Definition 3.1 *The space* $\mathcal{D}_{\{M_n\}} := \operatorname{ind}\lim_{\Omega\subset\mathbb{R}} \operatorname{proj}\lim_{h\to 0} \mathcal{D}_{\{M_n\},h,\Omega}$ *is called the space of ultradifferentiable functions of class* M_n*. The dual space* $\mathcal{D}'_{\{M_n\}}$ *is the space of ultradistributions of class* M_n*.*

The following theorem connects the Cauchy problem well-posedness in $\mathcal{D}'_{\{M_n\}}$ with some estimates for the resolvent of A.

Theorem 3.2 *[16] For (3.1) in the space of ultradistributions the following statements are equivalent.*

(D) *there exists an operator-ultradistribution* $U \in \mathcal{D}'_{\{M_n\}}(L(X, [\operatorname{Dom} A]))$ *with* $\operatorname{supp} U \subset [0, \infty)$ *satisfying* $P * U = \delta \otimes I_X, \ U * P = \delta \otimes I_{[\operatorname{Dom} A]}$;

(R) *there exists a region* $\Lambda := \{\lambda \in \mathbb{C} |\ \Re\lambda \ge \gamma M(\alpha\lambda) + \beta\}, \alpha, \gamma, \beta > 0$ *such that (2.3) holds true for* $\lambda \in \Lambda$*.*

In this case $\mathbf{u} = Uf$ is a unique solution to (3.1) and the function M, which is said to be associated to $\{M_n\}$, is defined as $M(\sigma) = \sup_{n\in\mathbb{N}} \ln \frac{\sigma^n M_0}{M_n}, \sigma \in (0, \infty)$.

Thus, in case (3.1) is well-posed in an ultradistribution space, the estimates for $\mathcal{R}_A(\lambda)$ are the same as those for the resolvent of the generator of a K-convoluted semigroup with function K such that

$$|\widetilde{K}(\lambda)| \le Ce^{-M(\alpha\lambda)}, \quad \lambda \in \Lambda, \tag{3.2}$$

with associated function M. Here the regularization of solution distribution U is realized via applying it to test functions from $\mathcal{D}_{\{M_n\}}$ or equivalently via applying $\mathcal{R}_A(\cdot)$ to test functions with the same growth at infinity as $\widetilde{K}(\cdot)$ defined by the growth order of the resolvent. Thus, we have shown how to choose test function spaces (and corresponding distribution spaces) in order to construct regularization via generalized (in t) solutions.

Now we consider regularization of solutions to (1.1) via test functions in 'variables of A', especially regularization of solutions to (1.2) via test functions in x. In case the Cauchy problem (1.1) is ill–posed, such spaces were introduced by Ivanov [10, 16]. These spaces are constructed in such a way that instead of differential operators used to be 'multiplicators' in the spaces considered above, a certain unbounded operator P (especially, giving a solution of an ill-posed problem in X) serves such a 'multiplicator'.

Let H be a Hilbert space and P a self-adjoint operator generating an orthonormal basis of eigenvectors $\{\alpha_k\}$ corresponding to the eigenvalues $|\lambda_1| \le |\lambda_2| \le \cdots$. We denote by $\mathcal{H}_+$ the class of such operators if $\lambda_k \to \infty$. The domain of the operator is dense in H, but, in general, it does not coincide with H. The same happens for $\mathrm{Dom}\, P^k$. The following normed and countably normed spaces

$$\mathcal{P}_k := \left\{ \varphi \in \mathrm{Dom} P^k \subset H : \quad \|\varphi\|_k = \sum_{i=0}^{k} \|P^i u\|_H \right\}, \quad k = 0, 1, 2, ...$$

and $\mathcal{P}_\infty := \mathrm{proj}\, \lim_k \mathcal{P}_k$ are used to construct generalized solutions to (1.1):

$$\forall \varphi \in \Phi, \ \langle du/dt, \varphi \rangle = \langle Au, \varphi \rangle, \quad \langle u(0), \varphi \rangle = \langle f, \varphi \rangle, \tag{3.3}$$

where $\Phi = \mathcal{P}_k$ $(\mathcal{P}_\infty)$. Spaces of new generalized functions Φ' are defined as dual spaces of Φ. The space $\mathcal{P}_\infty$ extends the Schwartz space of infinitely differentiable test functions, while $\mathcal{P}_k$ generalize Sobolev spaces: if P is equal to the differentiation operator defined on $H = L_2(G)$, then $\mathcal{P}_k$ coincides with $W_2^k(G) \equiv H_k(G)$, and its dual space coincides with $H_{-k}(G)$. This is the reason why we denote the dual spaces by $\mathcal{P}_{-k}$ and $\mathcal{P}_{-\infty}$, respectively.

Note that $\mathcal{P}_\infty$ (and hence all $\mathcal{P}_k$) are not empty, since they contain all eigenvectors α_n of P, and the construction using $\mathrm{Dom}\, P^k$ can be applied in the case of Banach spaces as well.

Besides $\mathcal{P}_k$ and $\mathcal{P}_{-k}$ we also consider the spaces $\mathcal{J}_k[\gamma_i]$ and $\mathcal{J}_{-k}[\gamma_i]$, where $\{\gamma_i\}$ is a given numerical sequence such that $|\gamma_1| \le |\gamma_2| \le \ldots$.

The elements of the spaces $\mathcal{J}_k[\gamma_i]$ and $\mathcal{J}_{-k}[\gamma_i]$ are defined in the form of Fourier series with coefficients that have corresponding decreasing or

increasing order depending on $\{\gamma_i\}$. Such spaces can coincide with those already introduced, but both these approaches turn out to be useful for constructing generalized solutions.

Let $\{\alpha_n\}$ be an orthonormal system in H, then

$$\mathcal{J}_\infty[\gamma_i] := \left\{ \varphi = \sum_{n=1}^{\infty} \tilde{\varphi}_n \alpha_n \in H : \ \forall k \geq 1, \ \sum_{n=1}^{\infty} |\tilde{\varphi}_n|^2 |\gamma_n|^{2k} =: \|\varphi\|_k^2 < \infty \right\},$$

and the corresponding space $\mathcal{J}_{-\infty}[\gamma_i]$ of the new generalized functions f is defined in the form of series $f = \sum_{n=1}^{\infty} \tilde{f}_n \alpha_n$ whose coefficients are subject to the condition

$$\exists k_0 \geq 1 \quad \sum_{n=1}^{\infty} |\tilde{f}_n|^2 |\gamma_n|^{-2k_0} < \infty. \tag{3.4}$$

Spaces $\mathcal{J}_{-k}[\gamma_i]$ are such that $\mathcal{J}_{-\infty}[\gamma_i] = \bigcup_{k=1}^{k=\infty} \mathcal{J}_{-k}[\gamma_i]$, i.e. Fourier coefficients for $f \in \mathcal{J}_{-k}[\gamma_i]$ are subject to (3.4) with $k_0 = k$.

The following results on generalized well–posedness and on construction of regularized solutions take place in the spaces above introduced.

Theorem 3.3 *[17] Suppose that $A \in \mathcal{H}_+$. Then for any $f \in \mathcal{J}_{-k}[e^{\lambda_k T}]$ ($\mathcal{J}_{-\infty}[e^{\lambda_k T}]$) there exists a unique solution u of (3.3), and it is stable in the space $\mathcal{J}_{-k-1}[e^{\lambda_k T}]$ ($\mathcal{J}_{-\infty}[e^{\lambda_k T}]$). In this case $\mathcal{J}_{-k}[e^{\lambda_k T}] = \mathcal{P}_{-k}$ with $P = e^{TA}$, $u = e^{tA} f$, and A is the generator of an R-semigroup with $R = P^{-1}$.*

Note that if we apply the preceding theorem to each element of the family $P_\varepsilon = I + \varepsilon e^{TA}$ ($\varepsilon > 0$) instead of P, then operators of semigroups $\{S_\varepsilon(t) = e^{tA}(I + \varepsilon e^{TA})^{-1}, t \in [0, T)\}$ coincide with the regularizing operators obtained by the ABC (auxiliary boundary conditions) method, while if we consider $P_\varepsilon = e^{\varepsilon T A^2}$, then the corresponding $S_\varepsilon(t)$ coincide with the regularizing operators obtained by a modified quasireversion method [14, 16].

Now we consider some solutions of the differential Cauchy problem (1.2) regularized by test functions.

Let Φ be a space of test functions $\varphi(x), x \in \mathbb{R}^n$, and $\widetilde{\Phi}$ the space of their corresponding Fourier transforms $\widetilde{\varphi}(\sigma), \sigma \in \mathbb{R}^n$. We apply the generalized Fourier transform to the Cauchy problem (1.2):

$$\frac{d\widetilde{u}(\sigma, t)}{dt} = A(\sigma)\widetilde{u}(\sigma, t), \quad \widetilde{u}(\sigma, 0) = \widetilde{f}(\sigma), \quad \sigma \in \mathbb{R}^n, \ \widetilde{f} \in \widetilde{\Phi}', \tag{3.5}$$

and obtain $\widetilde{u}(\sigma, t) = e^{tA(\sigma)}\widetilde{f}(\sigma)$. Here the spaces Φ and Φ' have to be searched to construct generalized (in x) solutions to (1.2) and corresponding

regularization via test functions or via smoothing operators of convolution. In [9] the spaces are chosen so that $e^{tA(\cdot)}$ is a multiplicator in $\widetilde{\Phi}$.

The behavior of $e^{tA(\cdot)}$ is closely related to $\lambda_k(s), s = \sigma + i\tau, k = 1, \dots m$, the system of eigenvalues of the operator-matrix $A(s)$ (of order m) [9]:

$$|e^{t\lambda_k(s)}| = e^{t\Re\lambda_k(s)} \leq \|e^{tA(s)}\|$$

$$\leq M\left(1 + 2t\|A(s)\| + \dots + 2t^{m-1}\|A(s)\|^{m-1}\right) e^{t\max\Re\lambda_k(s)}.$$

According to the behavior of λ_k, all systems in (1.2) are divided into three classes. (I): *Petrovsky correct* if $\Re\lambda_k(\sigma) \leq C$ for some $C > 0$, (II): *conditionally correct* if $\Re\Lambda_k(\sigma) \leq a|\sigma|^h + b$ for some $a > 0, h < 1, b \in \mathbb{R}$, and (III): the remaining systems are called *incorrect*.

It is shown that the spaces Φ and $\widetilde{\Phi}$ may be taken of type W_M^Q and $\mathcal{S}_\alpha^\beta$, where the parameters of the spaces are chosen in dependence on the classes, that is in dependence on the behavior of $e^{tA(\cdot)}$ [9].

To summarize, in the case of generalized (in x) solutions to (1.2) we can obtain the regularization via multiplication of $\tilde{u}(\cdot, t) = e^{tA(\cdot)}\tilde{f}(\cdot)$ by test functions from the corresponding $\widetilde{\Phi}$-spaces, or equivalently via multiplication of $u(\cdot, t)$ by test functions from Φ. Besides the regularization by test functions, we actually may obtain the same regularization by multiplying $e^{tA(\sigma)}$ or $\tilde{f}(\sigma)$ with a function $\tilde{K}(\sigma), \sigma \in \mathbb{R}^n$, with the same decreasing order of the test functions from $\widetilde{\Phi}$. This is equivalent to the construction of an R-semigroup, where R is the bounded smoothing operator of convolution (in x) with function K.

Acting this way, in the section below we construct some examples of regularizing operators via R-semigroups.

4 Construction of regularizing operators

In sections 2 and 3 we investigated the regularization of some ill-posed problems via distribution and semigroup techniques (in a broad sense, as it was explained in the introduction). In doing so we have shown that the function K used for the construction of the K-convoluted (in t) semigroups can be taken so that its Laplace transform satisfies the estimate (3.2), where M is defined by test functions used in (3.1) or by the resolvent growth. A similar situation holds when we choose the operators R for constructing R-semigroups for (1.2): R is taken as the convolution operator with K such that its Fourier transform $\tilde{K}$ behaves at infinity as the test functions $\tilde{\varphi} \in \tilde{\Phi}$ do, where $\tilde{\Phi}$ is constructed in dependence on the behavior of $e^{tA(\cdot)}$.

Now we construct regularizing operators for ill-posed problems (1.2) on the basis of their relationship with the regularized semigroups. These regularizing operators using the differential character of the problems take into account different behavior of e^{tA}. Several regularizing operators for abstract ill-posed problems (1.1) via K-convoluted and R-semigroups are constructed in [11, 14, 16, 17].

Consider the Cauchy problem (1.1), which is not uniformly well-posed. Suppose, in agreement with the theory of ill-posed problems, that for certain f there exists a solution u and f_δ $(\|f_\delta - f\|_X \leq \delta)$ is given. As indicated in the introduction, approximate solutions to the Cauchy problem (as $\delta \to 0$) will be constructed through various regularizing operators.

The following theorem establishes the relationship between the existence of the regularizing operators and the R-semigroups.

Theorem 4.1 *[14, 16] Let $-A$ generate a strongly continuous semigroup on a Banach space X; then the following statements are equivalent:*

(i) *A is the generator of a local R_ε-semigroup $\{S_\varepsilon(t), t \in [0,T)\}$, where the operator R_ε is convergent to the identity operator as $\varepsilon \to 0$ on* Dom$A$*;*

(ii) *for (1.1) there exist bounded linear regularizing operators $\mathbf{R}_\varepsilon(t), t \in [0,T)$; they are invertible and commute with A.*

The condition that $-A$ generates a semigroup of class C_0 holds for ill-posed problems of the back heat type (strongly ill–posed problems). The differential problems (1.2) with conditionally correct or incorrect systems may possess this property. The following theorem shows that for ill-posed Cauchy problems with Petrovsky correct systems (weakly ill–posed problems) we can also use R_ε-semigroups for constructing the regularizing operators.

Theorem 4.2 *[16, 17] Let A be the generator of a local R_ε-semigroup $\{S_\varepsilon(t), t \in [0,T)\}$ with operator R_ε convergent to the identity operator as $\varepsilon \to 0$ on Dom A. Then for (1.1) there exist linear bounded regularizing operators $\mathbf{R}_\varepsilon(t)$ commuting with A.*

Regularizing operators for (1.1) based on these theorems are constructed and their properties such as convergence, degree of convergence, and optimality are investigated in [14, 11, 16]. Now we prove that common type of operators R_ε can be used for regularization of a wide class of ill-posed differential problems (1.2). To this end we consider vector-functions $u(t,\cdot)$ and $f(\cdot)$ as functions in $X = L_2^m(\mathbb{R}^n) := L_2(\mathbb{R}^n) \times \cdots \times L_2(\mathbb{R}^n)$ $(m \geq 1)$. In this case, the Plancherel theorem applies, and the Fourier transforms $\tilde{u}(t,\cdot)$ and $\tilde{f}(\cdot)$ are also vector-functions with coordinates in $L_2(\mathbb{R}^n)$. Hence we

obtain the Fourier transformed Cauchy problem (3.5) in $L_2^m(\mathbb{R}^n)$ as well. Due to the Plancherel equality, estimates for the solution operators $e^{tA(\cdot)}$ to (3.5) coincide with the estimates for the solution operators $U(t)$ to (1.2).

For Petrovsky correct and conditionally correct systems we obtain the following general result on regularization of (1.2).

Theorem 4.3 *Let $A = A\left(i\frac{\partial}{\partial x}\right)$ in (1.2) be a differential operator–matrix of order r and Δ be the Laplace operator in $X = L_2^m(\mathbb{R}^n)$. In the case of a Petrovsky correct system, R_ε-semigroups with generator A and*

$$R_\varepsilon = \frac{1}{\varepsilon^\beta}\mathcal{R}_\Delta^\beta(1/\varepsilon), \quad \beta > r(m-1)/2,\ \varepsilon > 0,$$

form regularizing operators for corresponding weakly ill-posed Cauchy problem (1.2). In the case of a conditionally correct system, $R_\varepsilon(t)$-semigroups with generator A and $R_\varepsilon = U_\Delta(\varepsilon)$, where $\{U_\Delta(t), t \geq 0\}$ is the semigroup generated by Δ, form regularizing operators for corresponding strongly ill-posed Cauchy problem (1.2).

Proof. The R_ε-semigroups are constructed as follows:

$$(S_\varepsilon^1(t)f)(x) = \int_{\mathbb{R}^n} \frac{e^{i(\sigma,x)}e^{tA(\sigma)}\tilde{f}(\sigma)}{\left(1+\varepsilon|\sigma|^2\right)^\beta}d\sigma,\ t \in [0,T),\ \varepsilon > 0,\ \beta > r(m-1)/2, \tag{4.1}$$

$$(S_\varepsilon^2(t)f)(x) = \int_{\mathbb{R}^n} e^{i(\sigma,x)}e^{tA(\sigma)}e^{-\varepsilon\sigma^2}\tilde{f}(\sigma)d\sigma,\ t \in [0,T), \quad \varepsilon > 0. \tag{4.2}$$

The convergence of the integrals with indicated values of parameters and equations (2.1) with $R = R_\varepsilon$ (or the relation: $S_\varepsilon(t+\tau)R_\varepsilon = S_\varepsilon(t)S_\varepsilon(\tau)$) can be easily checked. The convergence

$$\frac{1}{\varepsilon^\beta}\mathcal{R}_\Delta^\beta(1/\varepsilon) \to f, \qquad U_\Delta(\varepsilon)f \to_{\varepsilon\to 0} f \quad \text{as } \varepsilon \to 0$$

on $\operatorname{Dom}\Delta$ (and on X) follows from the property of the semigroup $\{U_\Delta(t), t \geq 0\}$ to be of class C_0 in $X = L_2^m(\mathbb{R}^n)$ [16]. □

Regularized solutions or regularizing algorithm $\mathbf{R}_\varepsilon^1(t)f_\delta := S_\varepsilon^1(t)f_\delta$ corresponding to (4.1) are constructed as solutions of the following well-posed boundary problems depending on $\varepsilon > 0$:

$$\frac{\partial u(x;t)}{\partial t} = A\left(i\frac{\partial}{\partial x}\right)u(t,x),\ x \in \mathbb{R}^n,\ t \in [0,T), \quad (I-\varepsilon\Delta)^\beta u(0,x) = f_\delta(x), \tag{4.3}$$

where the coefficient β which satisfies the inequality $\beta > r(m-1)/2$ is taken among the integers. The regularizing algorithm $\mathbf{R}^2_\varepsilon(t)f_\delta := S^2_\varepsilon(t)f_\delta$ corresponding to (4.2) leads to the following well-posed problems:

$$\frac{\partial v(t,x)}{\partial t} = \Delta v(t,x),\ x \in \mathbb{R}^n,\ 0 \le t \le \varepsilon, \quad v(0,x) = f_\delta(x), \tag{4.4}$$

$$\frac{\partial u(x;t)}{\partial t} = A\left(i\frac{\partial}{\partial x}\right)u(t,x),\ x \in \mathbb{R}^n,\ t \in [0,T), \quad u(0,x) = v(\varepsilon,x). \tag{4.5}$$

Both of these algorithms consist of two steps. For (4.3) the first step is the smoothing of initial data with the help of the operator $R_\varepsilon = \frac{1}{\varepsilon^\beta}R^\beta_\Delta(1/\varepsilon)$, i.e., solving the well–posed elliptic problem $(I-\varepsilon\Delta)^\beta v(x) = f_\delta(x), x \in \mathbb{R}^n$; the second step consists of solving (1.2) with these smoothed initial data. Here the very important fact is that various weakly ill-posed problems have the common first step, to solve the elliptic problem, as a part of the regularization. For (4.4)–(4.5) the first step consists of smoothing of initial data by finding $v(t,x)$ to (4.4) as $t = \varepsilon$, the second step is solving (1.2) with these smoothed initial data $v(\varepsilon,x)$. Here various strongly ill-posed problems also have the common first step as a part of the regularization.

The important questions related to the suggested regularizing algorithms such as numerical realization of problems (4.3), (4.4)–(4.5), convergence of regularized solutions to an exact one, degree of convergence, and optimality of the regularizing operators are the matter of further investigations.

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References

[1] Arendt W, Batty Ch., et all, *Vector-valued Laplace transform and Cauchy problems*, Birkhauser Verlag: Basel–Boston–Berlin, 2001.

[2] Chazarain J., Problemes de Cauchy Abstraits et Applications a quelques Problemes Mixtes. *J. Funct. Anal.* **7**, 1971, 386–446.

[3] Cioranescu I. and Lumer G., On K-convoluted semigroups. Recent developments in evolution equations, *Pitman Research Notes in Math.*, Ser. **324**, 1995.

[4] Davies E.B. and Pang M.M., The Cauchy Problem and a Generalisation of the Hille–Yosida Theorem. *Proc. London Math. Soc.,* **55**, 1987, 181–208.

[5] De Laubenfels R., *Existence families, functional calculi and evolution equations*, Springer Verlag, 1994.

[6] Da Prato G., Semigruppi regolarizzabili, *Ricerche Mat.*, **15**, 1966, 223–248.

[7] Engel K.–J. and Nagel R., *One–parameter semigroups for linear evolution equations*, Graduate Texts in Math., **194**, Springer–Verlag, New York, Inc., 1999.

[8] Fattorini H.O., *The Cauchy problem*, Addison-Wesley, Reading, Mass., 1983.

[9] Gel'fand I.M. and Shilov G.E., *Spaces of test and generalezed functions*, Academic Press, New York, 1967. *Some questions of the theory of differential equations*, Academic Press, New York, 1968.

[10] Ivanov V.K. and Melnikova I.V., New generalized functions and weak well-posedness of operator problems. *Soviet Math. Dokl.*, **43**, No.2, 1991, 22–26.

[11] Ivanov V.K., Melnikova I.V. and Filinkov A.I., *Differential-operator equations and ill-posed problems*, Nauka, Moscow, 1995, Russ.

[12] Komatsu H., Ultradistributions, I. Structure theorems and characterization, *J. Fac. Sci. Univ. Tokyo*, **20**, No.1, 1973, 25–106.

[13] Kuo Ch.-Ch. and Shaw S.-Y., Abstract Cauchy problems assosiated with local C-semigroups, *Semigroups of operators: Theory and applications*, Opt. Soft. Inc., New York, 2002, 158–168.

[14] Melnikova I.V., General theory of ill-posed Cauchy problem *J. Inverse and Ill–Posed Problems*, **2**, 1995, 149–171.

[15] Melnikova I.V., Cauchy problem for inclusion in Banach and ditributions spaces *Sibirian Math. J.* , **42**, No 4, 2001, 892–910.

[16] Melnikova I.V. and Filinkov A.I., *The Cauchy problem. Three approaches*, Monographs and Surveys in Pure and Applied Mathematics, **120**, London-New York: Chapman&Hall, 2001.

[17] Melnikova I.V. and Anufrieva U.A., *Peculiarities and regularization of ill–posed Cauchy problems with differential operators, Fundamental Mathematics. Mordem problems*, VINITI, Moscow, **12–13**, 2004, to appear. Engl. transl. in J. Math. Sci.

[18] Tanaka N. and Okazawa N., Local C-semigroups and local integrated semigroups *Proc. London Math. Soc.*, **61**, 1990, 63–90.

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Some Convergence Properties of the Walsh-Kaczmarz System with respect to the Marcinkiewicz Means

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Abstract. In this paper we investigate the two-dimensional Marcinkiewicz kernels. We give a good decomposition and the exact values of the Marcinkiewicz kernels.

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Keywords and phrases. Marcinkiewicz means, a.e. convergence, Walsh group, Walsh-Kaczmarz system, two-variable functions.

1 Introduction

In 1948 A. A. Šneider [14] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 F. Schipp [11] and Wo-Sang Young [15] proved that the Walsh-Kaczmarz system is a convergence system. V. A. Skvorcov in 1981 [13] showed that the Fejér means converges uniformly to f for any continuous functions f. G. Gát (1998 [4, 6]) proved, for any integrable functions, that the Fejér means converges almost everywhere to the function.

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First we give a brief introduction to the theory of dyadic analysis[12, 1].

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2. Let

$$G := \underset{k=0}{\overset{\infty}{\times}} Z_2,$$

G is called the Walsh group. The elements of G can be represented by a sequence $x = (x_0, x_1, ..., x_k, ...)$ where $x_k \in \{0, 1\}$ $(k \in \mathbf{N})(\mathbf{N} := \{0, 1, ...\}, \mathbf{P} := \mathbf{N}\backslash\{0\})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the measure (denoted by μ) and the topology are the product measure and topology. Consequently, G is a compact Abelian group. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1}...)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Furthermore, let $L^p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|$). The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \ (i \in \mathbf{N}),$$

where only a finite number of n_i's defferent from zero. Let the order of n be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$.

Define the Walsh-Paley functions by

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Let the Walsh-Kaczmarz functions be defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Walsh-Paley system is $\omega := (\omega_n : n \in \mathbf{N})$ and the Walsh-Kaczmarz system is $\kappa := (\kappa_n : n \in \mathbf{N})$. It is well known that

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_k : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{N}$ and $\kappa_0 = \omega_0$.

A relation between Walsh-Kaczmarz functions and Walsh-Paley functions was given by V. A. Skvorcov in the following way (see [13]). Let the transformation $\tau_A : G \to G$ be defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{N}$. We have that

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

Define the Fourier coefficients, the partial sums of Fourier series, the Dirichlet kernels, the Fejér means and the Fejér kernels:

$$\hat{f}^{\alpha}(n) := \int_G f\bar{\alpha}_n, \ S_n^{\alpha} f := \sum_{k=0}^{n-1} \hat{f}^{\alpha}(k)\alpha_k,$$

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k, \quad \sigma_n^{\alpha} f := \frac{1}{n}\sum_{k=0}^{n} S_k^{\alpha} f, \quad K_n^{\alpha} := \frac{1}{n}\sum_{k=0}^{n} D_k^{\alpha},$$

where $\alpha_n = \omega_n$ or κ_n $(n \in \mathbf{P})$. $D_0^{\alpha} := 0, K_0^{\alpha} := 0$.

Next, we introduce some notation with respect to the theory of two-dimensional system. Let the two-dimensional Walsh group be $G \times G$ and the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, Dirichlet kernels, the Marcinkiewicz means and Marcinkiewicz kernels be defined as:

$$\hat{f}^{\alpha}(n_1, n_2) := \int_{G\times G} f\alpha_{n_1}\alpha_{n_2} d\mu,$$

$$S_{n_1,n_2}^{\alpha} f(x^1, x^2) := \sum_{k=0}^{n_1-1}\sum_{l=0}^{n_2-1} \hat{f}^{\alpha}(k, l)\alpha_k(x^1)\alpha_l(x^2),$$

$$D_{n_1,n_2}^{\alpha}(x^1, x^2) := D_{n_1}^{\alpha}(x^1) D_{n_2}^{\alpha}(x^2),$$

$$\mathcal{M}_n^{\alpha} f := \frac{1}{n}\sum_{k=0}^{n} S_{k,k}^{\alpha} f, \quad \mathcal{K}_n^{\alpha} := \frac{1}{n}\sum_{k=0}^{n} D_{k,k}^{\alpha}.$$

where $\alpha_n = \omega_n$ or κ_n $(n \in \mathbf{P})$. It is well-known that

$$S_{\underline{n}}^{\alpha} f(\underline{y}) = \int_{G\times G} f(\underline{x}+\underline{y}) D_{\underline{n}}^{\alpha}(\underline{x}) d\mu(\underline{x}),$$
$$\mathcal{M}_n^{\alpha} f(\underline{y}) = \int_{G\times G} f(\underline{x}+\underline{y}) \mathcal{K}_n^{\alpha}(\underline{x}) d\mu(\underline{x}),$$
$$(\underline{n} = (n_1, n_2), \underline{y} \in G \times G,\ f \in L^1(G\times G)).$$

In 1939 J. Marcinkiewicz [10] proved for two-dimensional trigonometric system that $\mathcal{M}_n f \to f$ *a.e.* $(n \to \infty)$ for all $f \in L\log L([0,2\pi]^2)$. L. V. Zhizhiashvili (1968 [16]) improved this result for $f \in L([0,2\pi]^2)$. Dyachenko [3] proved this result for dimensions greater than 2. In 2003 U. Goginava proved [9] the same result with respect to the d-dimensional Walsh-Paley system. In 2004 G. Gát generalized [5] this result with respect to two-dimensional bounded Vilenkin systems. Motivated by the work of G. Gát and U. Goginava we prove the following theorem.

Theorem 1.1 *For all* $f \in L^1(G\times G)$

$$\mathcal{M}_n^{\kappa} f \to f \quad a.e.$$

relation holds.

2 The proofs

The first step to prove the theorem is to make a good decomposition of Marcinkiewicz kernels.

Lemma 2.1 *For* $k \in \mathbf{P}$ *and* $(x^1, x^2) \in G \times G$ *holds*

$$\begin{aligned}
k\mathcal{K}_k^{\kappa}(x^1,x^2) &= 1 + \sum_{j=0}^{|k|-1} 2^j D_{2^j,2^j}(x^1,x^2) + \sum_{j=0}^{|k|-1} 2^j D_{2^j}(x^1) r_j(x^2) K_{2^j}^{\omega}(\tau_j(x^2)) \\
&+ \sum_{j=0}^{|k|-1} 2^j D_{2^j}(x^2) r_j(x^1) K_{2^j}^{\omega}(\tau_j(x^1)) + \sum_{j=0}^{|k|-1} 2^j r_j(x^1+x^2) \mathcal{K}_{2^j}^{\omega}(\tau_j(x^1),\tau_j(x^2)) \\
&+ (k-2^{|k|})(D_{2^{|k|},2^{|k|}}(x^1,x^2) + D_{2^{|k|}}(x^1) r_{|k|}(x^2) K_{k-2^{|k|}}^{\omega}(\tau_{|k|}(x^2)) \\
&+ D_{2^{|k|}}(x^2) r_{|k|}(x^1) K_{k-2^{|k|}}^{\omega}(\tau_{|k|}(x^1)) \\
&+ r_{|k|}(x^1+x^2) \mathcal{K}_{k-2^{|k|}}^{\omega}(\tau_{|k|}(x^1),\tau_{|k|}(x^2))).
\end{aligned}$$

Proof. For $1 \le j \le 2^n$ and $x \in G$ we have

$$D^{\kappa}_{2^n+j}(x) = D_{2^n}(x) + r_n(x) D^{\omega}_j(\tau_n(x))$$

Let $k = 2^n + m,\ 1 \le m \le 2^n$ and $(x^1, x^2) \in G \times G$. Then

$$\begin{aligned}
& k\mathcal{K}^{\kappa}_k(x^1,x^2) = (2^n+m)\mathcal{K}^{\kappa}_{2^n+m}(x^1,x^2) = 2^n\mathcal{K}^{\kappa}_{2^n}(x^1,x^2) \\
+ & \sum_{j=1}^{m} D^{\kappa}_{2^n+j,2^n+j}(x^1,x^2) = 2^n\mathcal{K}^{\kappa}_{2^n}(x^1,x^2) \\
+ & \sum_{j=1}^{m} \left(D_{2^n}(x^1) + r_n(x) D^{\omega}_j(\tau_n(x^1))\right)\left(D_{2^n}(x^2) + r_n(x^2) D^{\omega}_j(\tau_n(x^2))\right) \\
= & 2^n\mathcal{K}^{\kappa}_{2^n}(x^1,x^2) + mD_{2^n}(x^1)D_{2^n}(x^2) + D_{2^n}(x^1) r_n(x^2) \sum_{j=1}^{m} D^{\omega}_j(\tau_n(x^2)) \\
+ & D_{2^n}(x^2) r_n(x^1) \sum_{j=1}^{m} D^{\omega}_j(\tau_n(x^1)) + r_n(x^1+x^2) \sum_{j=1}^{m} D^{\omega}_j(\tau_n(x^1)) D^{\omega}_j(\tau_n(x^2)) \\
= & 2^n\mathcal{K}^{\kappa}_{2^n}(x^1,x^2) + mD_{2^n,2^n}(x^1,x^2) + mD_{2^n}(x^1) r_n(x^2) K^{\omega}_m(\tau_n(x^2)) \\
+ & mD_{2^n}(x^2) r_n(x^1) K^{\omega}_m(\tau_n(x^1)) + mr_n(x^1+x^2)\mathcal{K}^{\omega}_m(\tau_n(x^1),\tau_n(x^2)).
\end{aligned}$$

In particular, for $m = 2^n$ we find that

$$\begin{aligned}
& 2^{n+1}\mathcal{K}^{\kappa}_{2^{n+1}}(x^1,x^2) = 2^n\mathcal{K}^{\kappa}_{2^n}(x^1,x^2) + 2^n D_{2^n,2^n}(x^1,x^2) \\
+ & 2^n D_{2^n}(x^1) r_n(x^2) K^{\omega}_{2^n}(\tau_n(x^2)) + 2^n D_{2^n}(x^2) r_n(x^1) K^{\omega}_{2^n}(\tau_n(x^1)) \\
+ & 2^n r_n(x^1+x^2)\mathcal{K}^{\omega}_{2^n}(\tau_n(x^1),\tau_n(x^2)).
\end{aligned}$$

Applying successively this formula, we get that

$$\begin{aligned}
& 2^n\mathcal{K}^{\kappa}_{2^n}(x^1,x^2) \\
= & 1 + \sum_{j=0}^{n-1} 2^j D_{2^j,2^j}(x^1,x^2) + \sum_{j=0}^{n-1} 2^j D_{2^j}(x^1) r_j(x^2) K^{\omega}_{2^j}(\tau_j(x^2)) \\
+ & \sum_{j=0}^{n-1} 2^j D_{2^j}(x^2) r_j(x^1) K^{\omega}_{2^j}(\tau_j(x^1)) + \sum_{j=0}^{n-1} 2^j r_j(x^1+x^2)\mathcal{K}^{\omega}_{2^j}(\tau_j(x^1),\tau_j(x^2)).
\end{aligned}$$

It completes the proof of lemma. □

Let

$$\mathcal{K}^{\alpha}_{a,b} := \sum_{j=a}^{a+b-1} D^{\alpha}_{j,j},$$

and $n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i$ $(n, s \in \mathbf{N})$. By simple calculations we get

$$n\mathcal{K}_n^{\alpha} = \sum_{s=0}^{|n|} n_s \mathcal{K}^{\alpha}_{n^{(s+1)},2^s} + D^{\alpha}_{n,n}.$$

The second step to prove the theorem, as the decomposition lemma shows, is to investigate the two-dimensional Marcinkiewicz kernels with respect to Walsh-Paley system (because G. Gát gave [5] only estimations for the kernels, but in our proof we need the exact values of the Marcinkiewicz kernels see [4, 6]). We will use the following lemma (see [4]).

Lemma 2.2 (G. Gát) *Suppose that $s, t, n \in \mathbf{N}$ and $x \in I_t \backslash I_{t+1}$. If $t < s \leq |n|$, then*

$$K^{\omega}_{n^{(s+1)},2^s}(x) = \begin{cases} 0, & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x) 2^{s+t-1}, & \text{if } x - x_t e_t \in I_s. \end{cases}$$

Now we will investigate $\mathcal{K}^{\omega}_{n^{(s+1)},2^s}$.

Lemma 2.3 *Suppose that $s, t, n \in \mathbf{N}$, $(x^1, x^2) \in I_{|n|+1} \times (I_t \backslash I_{t+1})$. If $s \leq t \leq |n|$, then*

$$\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2)\right| \leq c 2^{s+t}(n^{(s+1)} + 2^s).$$

If $t < s \leq |n|$ then we have

$$\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2) =$$

$$\begin{cases} 0 & \text{if } \exists l, t < t+l < s, x^2 - x_t^2 e_t - e_{t+l} \notin I_s, x_{t+l}^2 \neq 0, \\ \omega_{n^{(s+1)}}(x^2) 2^{2t+s+l-2} & \text{if } \exists l, t < t+l < s, x^2 - x_t^2 e_t - e_{t+l} \in I_s, x_{t+l}^2 \neq 0, \\ \omega_{n^{(s+1)}}(x^2) 2^{t-2} n(s,t) & \text{if } x^2 - x_t^2 e_t \in I_s, \end{cases}$$

where $n(s,t) = [n^{(s+1)} 2^{s+1} - 2^t(2^s - 2^{t-1} + \frac{1}{2}) - 2^s(2^s - 2)]$

Proof. If $s \leq t \leq |n|$, then for all $j \in \mathbf{N}$ we have $|D_j^{\omega}(x^2)| \leq c \sum_{k=0}^{t} 2^k \leq c 2^t$ and

$$\begin{aligned} \left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2)\right| &\leq \sum_{j=n^{(s+1)}}^{n^{(s+1)}+2^s-1} \left|D_j^{\omega}(x^1) D_j^{\omega}(x^2)\right| = \sum_{j=n^{(s+1)}}^{n^{(s+1)}+2^s-1} j |D_j^{\omega}(x^2)| \\ &\leq c \sum_{j=n^{(s+1)}}^{n^{(s+1)}+2^s-1} j 2^t \leq c \sum_{l=0}^{2^s-1} (n^{(s+1)} + l) 2^t \\ &\leq c 2^{s+t}(n^{(s+1)} + 2^s). \end{aligned}$$

If $t < s \leq |n|$, then we have

$$D^{\omega}_{n^{(s+1)}+j}(x^2) = \omega_{n^{(s+1)}+j}(x^2)\left(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t\right)$$

and

$$\begin{aligned}\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1,x^2) &= \sum_{j=n^{(s+1)}}^{n^{(s+1)}+2^s-1} jD^{\omega}_j(x^2) = \sum_{j=0}^{2^s-1}(n^{(s+1)}+j)D^{\omega}_{n^{(s+1)}+j}(x^2)\\ &= \sum_{j=0}^{2^s-1} n^{(s+1)}D^{\omega}_{n^{(s+1)}+j}(x^2) + \sum_{j=0}^{2^s-1} jD^{\omega}_{n^{(s+1)}+j}(x^2)\\ &=: \sum_1 + \sum_2.\end{aligned}$$

By the help of G. Gát's Lemma we get that

$$\sum_1 = \begin{cases} 0 & \text{if } x^2 - x_t^2 e_t \notin I_s \\ n^{(s+1)}\omega_{n^{(s+1)}}(x^2)2^{s+t-1} & \text{if } x^2 - x_t^2 e_t \in I_s. \end{cases}$$

Now we will investigate $\sum_2$

$$\begin{aligned}\sum_2 &= \omega_{n^{(s+1)}}(x^2)\sum_{j=0}^{2^s-1} j\omega_j(x^2)\left(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t\right)\\ &= \omega_{n^{(s+1)}}(x^2)\left(\sum_{j=0}^{2^s-1} j\omega_j(x^2)\sum_{k=0}^{t-1} j_k 2^k - \sum_{j=0}^{2^s-1} j\omega_j(x^2) j_t 2^t\right)\\ &= \omega_{n^{(s+1)}}(x^2)\left(\sum_{2,1} - \sum_{2,2}\right)\end{aligned}$$

If $x^2 - x_t^2 e_t \in I_s$, then we have

$$\begin{aligned}\sum_{2,1} &= \sum_{\substack{j_i=0\\ i=0,\dots,s-1\\ i\neq t, j_t=0}}^{1} j\sum_{k=0}^{t-1} j_k 2^k - \sum_{\substack{j_i=0\\ i=0,\dots,s-1\\ i\neq t, j_t=1}}^{1} j\sum_{k=0}^{t-1} j_k 2^k = -\sum_{\substack{j_i=0\\ i=0,\dots,s-1\\ i\neq t}}^{1} 2^t\sum_{k=0}^{t-1} j_k 2^k\\ &= -2^t 2^{s-t-1}\sum_{\substack{j_i=0\\ i=0,\dots,t-1}}^{1}\sum_{k=0}^{t-1} j_k 2^k = -2^{s-1}\frac{2^t(2^t-1)}{2} = -2^{s+t-2}(2^t-1),\end{aligned}$$

$$\sum_{2,2} = 2^t \sum_{\substack{j=0 \\ j_t=1}}^{2^s-1} j = 2^t \sum_{j=2^t}^{2^s-1} j = 2^t \frac{(2^t+2^s-1)}{2}(2^s-2^t).$$

Thus,

$$\sum_{2} = -\omega_{n^{(s+1)}}(x^2)2^{t-2}[2^t(2^s-2^{t-1}+\frac{1}{2})+2^s(2^s-2)].$$

If $x^2 - x_t^2 e_t \notin I_s$ then there is a coordinate $x_{t+l_1}^2 \neq 0 (t < t+l_1 \leq s)$. We have two cases. If there exists another coordinate $x_{t+l_2}^2 \neq 0 (t < t+l_2 \leq s)$ different from $x_{t+l_1}^2$ or not. First, let be given two coordinates $x_{t+l_1}^2 \neq 0$ and $x_{t+l_2}^2 \neq 0 (l_1 \neq l_2)$ then

$$\sum_{2} = \omega_{n^{(s+1)}}(x^2) \sum_{\substack{j_i=0 \\ i\neq t+l_1 \\ i\neq t+l_2}}^{1} \sum_{j_{t+l_1}=0}^{1} \sum_{j_{t+l_2}=0}^{1} \left(j_{t+l_1}2^{t+l_1} + j_{t+l_2}2^{t+l_2} + \phi_1 \right) \omega_j(x^2)\phi_2$$

where the function ϕ_1 and ϕ_2 does not depend on j_{t+l_1} and j_{t+l_2}. Consequently, we can write

$$\sum_{j_{t+l_1}=0}^{1} (j_{t+l_2}2^{t+l_2} + \phi_1)\omega_j(x^2)\phi_2 = 0$$

for all j_{t+l_2} and

$$\sum_{j_{t+l_2}=0}^{1} j_{t+l_1}2^{t+l_1}\omega_j(x^2)\phi_2 = 0$$

for all j_{t+l_1}. These give that $\sum_2 = 0$. Second, let x_{t+l}^2 be the only one coordinate which is not zero $(t < t+l \leq s)$. Then

$$\sum_{2,1} = \sum_{\substack{j_i=0 \\ i\in\{0,\ldots,s-1\} \\ i\neq t+l,t}}^{1} \sum_{j_{t+l}=0}^{1} \sum_{j_t=0}^{1} \left(j_t 2^t + j_{t+l}2^{t+l} + \phi_1 \right) \omega_j(x^2)\phi_2 = 0$$

where the function ϕ_1 and ϕ_2 does not depend on j_t and j_{t+l}, but for $\sum\limits_{2,2}$ we have

$$\begin{aligned}\sum_{2,2} &= \sum_{\substack{j_i=0\\ i\in\{0,\dots,s-1\}\\ i\neq t, j_t=1}}^{1} j\omega_j(x^2)2^t = 2^t \sum_{\substack{j_i=0\\ i\in\{0,\dots,s-1\}\\ i\neq t,t+l\\ j_t=1}}^{1} \sum_{j_{t+l}=0}^{1} j\omega_j(x^2)\\ &= 2^t \sum_{\substack{j_i=0\\ i\in\{0,\dots,s-1\}\\ i\neq t,t+l\\ j_t=1}}^{1} \sum_{j_{t+l}=0}^{1} (2^t + j_{t+l}2^{t+l} + \phi)\omega_j(x^2) = 2^t \sum_{\substack{j_i=0\\ i\in\{0,\dots,s-1\}\\ i\neq t,t+l\\ j_t=1}}^{1} 2^{t+l}\\ &= 2^{2t+s+l-2},\end{aligned}$$

where the function ϕ does not depend on j_t and j_{t+l}. □

Corollary 2.4 *Let $A, t, l \in \mathbf{N}$, $(x^1, x^2) \in I_A \times (I_t \backslash I_{t+1})$ and $t < t+l < A$. Then*

$$\mathcal{K}^{\omega}_{2^A}(x^1, x^2) =$$

$$\begin{cases} 0 & \text{if } \exists l, t < t+l < A, x^2 - x_t^2 e_t - e_{t+l} \notin I_A, x_{t+l}^2 \neq 0,\\ 2^{2t+l-2} & \text{if } \exists l, t < t+l < A, x^2 - x_t^2 e_t - e_{t+l} \in I_A, x_{t+l}^2 \neq 0,\\ 2^{t-2} n(A,t) & \text{if } x^2 - x_t^2 e_t \in I_A, \end{cases}$$

where $n(A,t) = [-\frac{2^t}{2^A}(2^A - 2^{t-1} + \frac{1}{2}) - (2^A - 2)]$.
If $(x^1, x^2) \in I_A \times I_A$, *then*

$$\mathcal{K}^{\omega}_{2^A}(x^1, x^2) = \frac{(2^A+1)(2^{A+1}+1)}{6}.$$

Proof. If $(x^1, x^2) \in I_A \times (I_t \backslash I_{t+1})$ for some $t < A$, then by

$$\begin{aligned} 2^A \mathcal{K}^{\omega}_{2^A}(x^1, x^2) &= \sum_{j=0}^{2^A} D^{\omega}_{j,j}(x^1, x^2) = \sum_{j=0}^{2^A-1} D^{\omega}_{j,j}(x^1, x^2) + D^{\omega}_{2^A,2^A}(x^1, x^2)\\ &= \mathcal{K}^{\omega}_{0,2^A}(x^1, x^2) + D^{\omega}_{2^A,2^A}(x^1, x^2) = \mathcal{K}^{\omega}_{(2^A)^{(A+1)},2^A}(x^1, x^2)\end{aligned}$$

and by Lemma 2 (where $s = A$) we have that this Corollary holds. □

Lemma 2.5 *Let $s, t^1, t^2, n \in \mathbf{N}$. Suppose that $t^1 \leq t^2 < |n|$ and $(x^1, x^2) \in (I_{t^1} \setminus I_{t^1+1}) \times (I_{t^2} \setminus I_{t^2+1})$.*

If $s \leq t^1 \leq t^2 \leq |n|$, or $t^1 < s \leq t^2 \leq |n|$ then

$$\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2)\right| \leq c2^{t^1+t^2+s}.$$

If $t^1 \leq t^2 < s \leq |n|$ then

$$\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2) =$$

$$\begin{cases} 0 & \text{if } \exists i \in B_1, x^1_i \neq x^2_i, \\ 0 & \text{if } \forall i \in B_1, x^1_i = x^2_i, \exists l \in B_2, x^1 - e_{t^1} - e_l \notin I_{t^2+1}, x^1_l = 1, \\ -\omega_{n^{(s+1)}}(x^1 + x^2)2^{s+t^1+l-2} & \text{if } \forall i \in B_1, x^1_i = x^2_i, \exists l \in B_2, x^1 - e_{t^1} - e_l \in I_{t^2+1}, x^1_l = 1, \\ \omega_{n^{(s+1)}}(x^1 + x^2)2^{s+2t^1-1} & \text{if } x^1 - e_{t^1} \in I_{t^2+1}, (\forall i \in B_1, x^1_i = x^2_i), \end{cases}$$

where $B_1 = \{t^2 + 1, ..., s-1\}$, $B_2 = \{t^1 + 1, ..., t^2\}$.

Proof. If $s \leq t^1 \leq t^2 \leq |n|$ or $t^1 < s \leq t^2 \leq |n|$, then for all $k \in \mathbf{N}$ we have that $|D^{\omega}_k(x^1)| \leq c\sum_{j=0}^{t^1} 2^j \leq c2^{t^1}$ and $|D^{\omega}_k(x^2)| \leq c2^{t^2}$. Thus,

$$\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2)\right| \leq \sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} |D^{\omega}_k(x^1) D^{\omega}_k(x^2)| \leq c2^{t^1+t^2+s}.$$

If $t^1 \leq t^2 < s \leq |n|$, then

$$\begin{aligned} \mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1, x^2) &= \sum_{j=0}^{2^s-1} D^{\omega}_{n^{(s+1)}+j}(x^1) D^{\omega}_{n^{(s+1)}+j}(x^2) \\ &= \omega_{n^{(s+1)}}(x^1 + x^2) \sum_{j=0}^{2^s-1} \omega_{j^{(t^1)}}(x^1)\omega_{j^{(t^2)}}(x^2) g_j(t^1) g_j(t^2) \\ &= \omega_{n^{(s+1)}}(x^1 + x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\}}}^{1} g_j(t^1) g_j(t^2) \sum_{\substack{j_i=0 \\ i\in\{t^2+1,...,s-1\}}}^{1} \omega_{j^{(t^1)}}(x^1)\omega_{j^{(t^2)}}(x^2) \end{aligned}$$

where $g_j(t^1) = \sum_{k=0}^{t^1-1} j_k 2^k - j_{t^1} 2^{t^1}, g_j(t^2) = \sum_{k=0}^{t^2-1} j_k 2^k - j_{t^2} 2^{t^2}$. Now we will investigate $\sum_{j_i=0, i\in\{t^2+1,...,s-1\}}^{1} \omega_{j^{(t^1)}}(x^1)\omega_{j^{(t^2)}}(x^2)$.

$$\sum_{j_i=0}^{1} r^{j_i}_i(x^1) r^{j_i}_i(x^2) = \begin{cases} 0, & \text{if } x^1_i + x^2_i = 1 \\ 2, & \text{if } x^1_i + x^2_i = 0 \end{cases} \quad (i = t^2 + 1, ..., s-1).$$

This implies that $\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1,x^2)$ can be different from zero only in the case $x^1_i = x^2_i (i = t^2+1,...,s-1)$. Suppose this. If among $x^1_{t^1+k},...,x^1_{t^2}$ there are two coordinates, which are not zero, then $\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1,x^2) = 0$ (see [5]).

This means that if $\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1,x^2) \neq 0$, then among $x^1_{t^1+1},...,x^1_{t^2}$ can be only one coordinate which is different from zero. Let this coordinate be $x^1_{t^1+l} = 1$ for some $l \in \{1,...,t^2-t^1\}$. (If $l \in \{1,...,t^2-t^1\}$ then $l^\star := l + t^1 \in B_2$.)

$$
\begin{aligned}
& \mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1,x^2) = \\
= \;& \omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\}}}^{1} g_j(t^1)g_j(t^2) \sum_{\substack{j_i=0 \\ i\in\{t^2+1,...,s-1\}}}^{1} \omega_{j^{(t^1)}}(x^1)\omega_{j^{(t^2)}}(x^2) \\
= \;& \omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\}}}^{1} g_j(t^1)g_j(t^2)\omega_{j^{(t^1)}}(x^1)2^{s-t^2-1} \\
= \;& 2^{s-t^2-1}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\} \\ i\neq t^1+l}}^{1} g_j(t^1) \sum_{j_{t^1+l}=0}^{1} \omega_{j^{(t^1)}}(x^1)g_j(t^2) \\
= \;& -2^{s+t^1+l-t^2-1}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\} \\ i\neq t^1+l}}^{1} g_j(t^1)\omega_{j^{(t^1)}}(x^1) \\
= \;& -2^{s+l-2}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^1\}}}^{1} \left(\sum_{k=0}^{t^1-1} j_k 2^k - j_{t^1}2^{t^1}\right)\omega_{j^{(t^1)}}(x^1) \\
= \;& -2^{s+l-2}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^1-1\}}}^{1} 2^{t^1} = -2^{s+l+2t^1-2}\omega_{n^{(s+1)}}(x^1+x^2) \\
= \;& -2^{s+l^\star+t^1-2}\omega_{n^{(s+1)}}(x^1+x^2).
\end{aligned}
$$

Set $x_i^1 = x_i^2$ for all $i = t^2 + 1, ..., s-1$ and $x_i^1 = 0$ for all $i = t^1 + 1, ..., t^2$ that is $x^1 - e_{t^1} \in I_{t^2+1}$.

$$\begin{aligned}
&\mathcal{K}^{\omega}_{n^{(s+1)},2^s}(x^1,x^2) = \\
&= \omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\}}}^{1} g_j(t^1)g_j(t^2)\omega_{j^{(t^1)}}(x^1)2^{s-t^2-1} \\
&= 2^{s-t^2-1}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\} \\ i\neq t^1}}^{1} \sum_{j_{t^1}=0}^{1} g_j(t^1)g_j(t^2)\omega_{j^{(t^1)}}(x^1) \\
&= 2^{s-t^2-1}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\} \\ i\neq t^1}}^{1} 2^{t^1}\left[\sum_{\substack{k=0 \\ k\neq t^1}}^{t^2-1} j_k 2^k - j_{t^2}2^{t^2} + 2^{t^1} - (\sum_{k=0}^{t^1-1} j_k 2^k - 2^{t^1})\right] \\
&= 2^{s+t^1-t^2-1}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{0,...,t^2\} \\ i\neq t^1}}^{1} \left[\sum_{k=t^1+1}^{t^2-1} j_k 2^k + 2^{t^1+1} - j_{t^2}2^{t^2}\right] \\
&= 2^{s+2t^1-t^2-1}\omega_{n^{(s+1)}}(x^1+x^2) \sum_{\substack{j_i=0 \\ i\in\{t^1+1,...,t^2\} \\ i\neq t^1}}^{1} \left[\sum_{k=t^1+1}^{t^2-1} j_k 2^k + 2^{t^1+1} - j_{t^2}2^{t^2}\right] \\
&= \omega_{n^{(s+1)}}(x^1+x^2)2^{s+2t^1-1}.
\end{aligned}$$

□

Corollary 2.6 *Let* $A, t^1, t^2 \in \mathbf{N}$, $t^1 \leq t^2 < A$, $(x^1,x^2) \in (I_{t^1}\backslash I_{t^1+1}) \times (I_{t^2}\backslash I_{t^2+1})$. *Then*

$$\mathcal{K}^{\omega}_{2^A}(x^1,x^2) =$$

$$\begin{cases}
0 & \text{if } \exists i \in B_1, x_i^1 \neq x_i^2, \\
0 & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists l \in B_2, x^1 - e_{t^1} - e_l \notin I_{t^2+1}, x_l^1 = 1, \\
2^{t^1+l-2} & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists l \in B_2, x^1 - e_{t^1} - e_l \in I_{t^2+1}, x_l^1 = 1, \\
2^{2t^1-1} & \text{if } x^1 - e_{t^1} \in I_{t^2+1}, (\forall i \in B_1, x_i^1 = x_i^2),
\end{cases}$$

where $B_1 = \{t^2 + 1, ..., A - 1\}$, $B_2 = \{t^1 + 1, ..., t^2\}$.

The proof of the Theorem is based on the lemmas and corollaries proved above and will be published in another journal.

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References

[1] G.H. Agaev, N.Ja. Vilenkin, G.M. Dzhafarli, and A.I. Rubinstein, *Multiplicative systems of functions and harmonic analysis on* 0*-dimensional groups,* Izd. ("ELM"), Baku, (1981), (Russian).

[2] I. Blahota and G. Gát Pointwise convergence of double Vilenkin-Fejér means, *Studia Sci. Math. Hungar.* **36,** (2000), 49-63.

[3] M.I. Dyachenko, Distribution function inegualities for martingales, *Soobshch. Akad. Nauk Gruzii* **131,** (1988), 261-263.

[4] G. Gát, On $(C, 1)$ summability of integrable functions with respect to the Walsh-Kaczmarz system, *Studia Math.* **130,** (1998), 135-148.

[5] G. Gát, Convergence of Marcinkiewicz means of integrable functions with respect to two-dimensional Vilenkin systems, *Georgian Mathematical Journal* **11,** (2004), No. 3, 467-478.

[6] G. Gát and K. Nagy, Cesáro summability of the character system of the p-series field in the Kaczmarz rearrangement, *Analysis Mathematica* **28,** (2002), 1-23.

[7] G. Gát and R. Toledo, L^p-norm convergence of series in compact totally disconected groups, *Acta Math. Hungar.* **22,** (1996), 13-24.

[8] U. Goginava, On the convergence and summability of N-dimensional Fourier series with respect to the Walsh-Paley systems in the spaces $L^p([0,1]^N), p \in [1, \infty]$, *Georgian Math. J.* **7,** (2000), 53-72.

[9] U. Goginava, Almost everywhere summability of multiple Fourier series, *Math. Anal. and Appl.* **287,** (2003), no. 1, 90-100.

[10] J. Marcinkiewicz, Sur une methode remarquable de sommation des series doublefes de Fourier, *Ann. Scuola Norm. Sup. Pisa* **8,** (1939), 149-160.

[11] F. Schipp, Pointwise convergence of expansions with respect to certain product systems, *Analysis Math.* **2,** (1976), 63-75.

[12] F. Schipp, W. R. Wade, P. Simon, and J. Pál, *Walsh Series. An Introduction to Dyadic Harmonic Analysis,* Adam Hilger (Bristol-New York 1990).

[13] V. A. Skvorcov, On Fourier series with respect to the Walsh-Kaczmarz system, *Analysis Math.* **7,** (1981), 141-150.

[14] A. A. Šneider, On series with respect to the Walsh functions with monotone coefficients, *Izv. Akad. Nauk SSSR Ser. Math.* **12,** (1948), 179-192.

[15] W. S. Young, On the a.e. convergence of Walsh-Kaczmarz-Fourier series, *Proc. Amer. Math. Soc.* **44,** (1974), 353-358.

[16] L.V. Zhizhiasvili, Generalization of a theorem of Marcinkiewicz, *Izv. Akad. nauk USSR Ser Math.* **32,** (1968), 1112-1122.

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A New Class of Approximation Spaces

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Abstract. We study approximation spaces with respect to the norm $\|\{a_n(f,X)\}\|_A$, where the sequence space A is "too close" to l_∞, so that such spaces cannot be studied by classical methods. Using a special representation for A, we prove most of standard theorems, such as equivalent norms, embeddings, representation, iteration and interpolation.

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1 Introduction

Let X be a quasi-Banach space and let $\{G_n\}$, $n = 1, 2, \ldots$, be an *approximation family* of subsets from X, that is, arbitrary sets having the properties

a) $G_1 \subset G_2 \subset \ldots \subset X, \quad G_0 = \{0\}$,
b) $\lambda G_n \subseteq G_n$ for all $\lambda \in I\!R$ and all n,
c) $G_m + G_n \subseteq G_{m+n}$ for all m, n.

The *approximation numbers* of arbitrary $f \in X$ are defined as

$$a_n(f,X) = \inf_{g \in G_{n-1}} \|f - g\|_X, \tag{1.1}$$

so that $a_1(f,X) = \|f\|_X$. Evidently, $\{a_n(f,X)\}$ is a nonincreasing sequence of nonnegative numbers.

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The main idea of approximation theory is to classify the elements $f \in X$ in accordance with the properties of their approximation numbers. In particular, we can construct various *approximation spaces* of f, requiring that $\{a_n(f,X)\} \in A$ for special types of sequence spaces A. A general description of such an approach was given by Y.Brudnyi [2] in 1977 and then realized for the case

$$\|\{a_n\}\|_A = \Big(\sum_{n=1}^{\infty}\big(n^{\theta-1/p}a_n\big)^p\Big)^{1/p} \tag{1.2}$$

in the joint paper [3]. Analogous results were obtained by A.Pietsch [9] in 1981 (the last paper contains also new interesting applications of approximation spaces to harmonic analysis and operator theory).

Since any sequence $\{a_n(f,X)\}$ is nonincreasing, it makes sense to use only rearrangement-invariant spaces A, i.e., having the properties

i) $|a_n| \le |b_n|$ for all $n \implies \|\{a_n\}\|_A \le \|\{b_n\}\|_A$,

ii) $\|\{a_n\}\|_A = \|\{a_n^*\}\|_A$, where $\{a_n^*\}$ stands for the nonincreasing rearrangement of $\{|a_n|\}$.

For example, the paper [7] is dealing with the so-called Lorentz-Zygmund spaces $l^{pr}(\log l)^{\alpha}$ having the norm

$$\|\{a_n\}\|_{p,r,\alpha} = \Big(\sum_{n=1}^{\infty}\big(n^{1/p-1/r}(1+\ln n)^{\alpha}a_n^*\big)^r\Big)^{1/r} \tag{1.3}$$

for the cases when $p < \infty$. This class of spaces is larger than that from the paper [9], however only two problems (embeddings and comparison with other spaces) are considered. At the same time, the paper [9] gives a large spectrum of applications and fundamental properties of corresponding approximation spaces such as representation, reiteration, transformation and other theorems.

It turns out that all results from [9] can be generalized to a rather large class of *ultrasymmetric spaces* $A = l_{\varphi,E}$ with the (quasi)norms $\|\{a_n\}\|_A = \|\{\varphi(n)a_n^*\}\|_{\widetilde{E}}$. Here $\widetilde{E}$ denotes arbitrary sequence space which is rearrangement-invariant with respect to the atomic measure $\mu(n) = 1/n$, and φ is arbitrary positive increasing function such that its extension indices

$$\pi_\varphi = \lim_{s\to 0}\frac{\ln m_\varphi(s)}{\ln s}, \quad \rho_\varphi = \lim_{s\to\infty}\frac{\ln m_\varphi(s)}{\ln s}, \quad \text{where } m_\varphi(s) = \sup_t \frac{\varphi_E(ts)}{\varphi_E(t)},$$

satisfy the conditions $\pi_\varphi > 0$, $\rho_\varphi < \infty$. This class includes the Lorentz-Zygmund spaces as a partial case — it is enough to take $\varphi(n) = n^{1/p}(1+$

$\ln n)^\alpha$ and $\widetilde{E} = \widetilde{l}_r$, where

$$\|\{a_n\}\|_{\widetilde{l}_r} = \Big(\sum_{n=1}^{\infty} \frac{1}{n}|a_n|^r\Big)^{1/r}.$$

A full exposition of the above-mentioned results is given in [11]. Unfortunately, the restriction $\pi_\varphi > 0$ is unavoidable there (it is analogous to the inequality $p < \infty$, required in [9] and [7]). At the same time, some consequent papers (such as [5], [6], [8]) consider the Lorentz-Zygmund approximation spaces with $p = \infty$, using a special way for representation of their functions. This representation is based on particular properties of the $\widetilde{l}_r$-spaces and cannot be generalized to arbitrary spaces $\widetilde{E}$ — the change of exponent of logarithmic factor cannot be reproduced as change of function $\varphi(n)$.

In the present paper we show that the desired generalization can be obtained by using another class of parameter-spaces instead of $\widetilde{E}$. Namely, we should take $\|\{a_n\}\|_A = \|\{\varphi(n)a_n^*\}\|_{\overline{E}}$ with arbitrary space $\overline{E}$ rearrangement-invariant with respect to the atomic measure $\frac{1}{n\ln n}$. In result, the excessive logarithmic factor enters into measure and the remaining part obeys all needed requirements in general case. The parameter-function φ also will be taken more general than the simple $(1 + \ln n)^\alpha$ as in Lorentz-Zygmund spaces — we admit arbitrary positive function with the property $\varphi(n) \sim \varphi(n^2)$.

Throughout the paper we do not differentiate spaces with equivalent norms or quasinorms and use the same letters C, C_1 etc. for various constants (as a rule, these constants depend only on parameter functions φ, ψ etc.). We write $a \sim b$ if $(1/C)a \le b \le Ca$ and $a \ll b$ (correspondingly, $a \gg b$) if $a \le Cb$ ($a \ge Cb$).

2 Embeddings and equivalent norms

In what follows we shall use only such rearrangement-invariant spaces E which are interpolation[1] between l_1 and l_∞, that is, can be obtained, using some interpolation functor: $E = \mathcal{F}(l_1, l_\infty)$. The same functor will be used for defining the space $\overline{E} = \mathcal{F}(\bar{l}_1, l_\infty)$, where

$$\|\{a_n\}\|_{\bar{l}_1} = |a_1| + \sum_{n=2}^{\infty} \frac{1}{n\ln n}|a_n|.$$

[1]Recall that a space E is called interpolation between spaces E_0, E_1 if any linear operator, bounded on E_0, E_1, is also bounded on E.

Denote by $X_{\varphi,E}$ the approximation space with the quasinorm

$$\|f\|_{X_{\varphi,E}} := \|\{\varphi(n)a_n(f,X)\}\|_{\overline{E}}\,, \tag{2.1}$$

which can be readily proved to be a quasi-Banach space. It is also plain that the relations $\varphi(n) \geq \psi(n)$ and $E \hookrightarrow F$ imply an embedding $X_{\varphi,E} \hookrightarrow X_{\psi,F}$. Some stronger relation between parameter-functions allows us to dispose of the requirements to parameter-spaces.

Theorem 2.1 *Let*

$$\int_2^\infty \frac{\psi(t)}{t\varphi(t)\ln t}\,dt < \infty. \tag{2.2}$$

Then $X_{\varphi,E} \hookrightarrow X_{\psi,F}$ independently of spaces E, F.

Proof. The properties of parameter-functions φ and parameter-spaces $\overline{E}$ provide the existence of constants $C_1(\varphi,E)$ and $C_2(\varphi,E)$ such that, for any nonincreasing sequence $\{a_n\}$,

$$C_1(\varphi,E)\|\{\varphi(n)a_n\}\|_{l_\infty} \leq \|\{\varphi(n)a_n\}\|_{\overline{E}} \leq C_2(\varphi,E)\|\{\varphi(n)a_n\}\|_{\bar{l}_1}.$$

Therefore, if $f \in X_{\varphi,E}$ with unit norm, then $\varphi(n)a_n(f,X) \leq 1/C_1(\varphi,E)$ for each $n = 1,2,\ldots$ and thus

$$\|f\|_{X_{\psi,F}} \leq C_2(\psi,F)\|\{\psi(n)a_n(f,X)\}\|_{\bar{l}_1} \leq \frac{C_2(\psi,F)}{C_1(\varphi,E)}\|\psi(n)/\varphi(n)\|_{\bar{l}_1} < \infty,$$

whenever (2.2) is fulfilled. □

More subtle embedding assertions with more delicate interplay between parameter-functions and parameter-spaces can be obtained if to use another (equivalent) form of the quasinorm in spaces $X_{\varphi,E}$. We define the numbers

$$\nu_n = 2^{(2^n)}, \qquad n = 0,1,\ldots$$

Theorem 2.2 *Let $\varphi(n^2) \sim \varphi(n)$. Then the quasinorm (2.1) is equivalent to*

$$\|f\|^{\exp}_{X_{\varphi,E}} := \varphi(1)a_1(f,X) + \|\{\varphi(\nu_n)a_{\nu_n}(f,X)\}\|_E \tag{2.3}$$

with equivalence constant depending only on φ.

Proof. For any sequence $a = \{a_n\}$, we define a linear operator T by the rule: $(Ta)_n = \varphi(n)a_n$ for $n = 1,2$ and $(Ta)_n = \varphi(n)a_{\nu_n}$ when $n = \nu_{k-1}+1,\ldots,\nu_k$

for any $k = 1, 2, \dots$ Let us estimate the norms of this operator on the spaces $l_\infty, \bar{l}_1$. We get

$$\|Ta\|_{l_\infty} \le \max\left\{\varphi(1)|a_1|,\ \varphi(2)|a_2|,\ \sup_{k\ge 1}\varphi(\nu_k)|a_{\nu_k}|\right\}, \tag{2.4}$$

$$\|Ta\|_{\bar{l}_1} \le \varphi(1)|a_1| + \frac{\varphi(2)}{2\ln 2}|a_2| + \sum_{k=1}^{\infty}\left(\sum_{n=\nu_{k-1}+1}^{\nu_k}\frac{\varphi(n)}{n\ln n}\right)|a_{\nu_k}|.$$

But for any $k \ge 1$,

$$\sum_{n=\nu_{k-1}+1}^{\nu_k}\frac{\varphi(n)}{n\ln n} \le \int_{\nu_{k-1}}^{\nu_k}\frac{\varphi(t)}{t\ln t}\,dt \le \varphi(\nu_k)\,\ln 2,$$

so that

$$\|Ta\|_{\bar{l}_1} \le \varphi(1)|a_1| + \sum_{k=0}^{\infty}\varphi(\nu_k)|a_{\nu_k}|. \tag{2.5}$$

Define now a function $\delta(n)$ equal to 1 when either $n = 1$ or $n = \nu_k$ with some k; for other values let $\delta(n) = 0$. For arbitrary sequence space E, we denote by $E(\delta)$ a space with the weighted norm $\|\{a_n\}\|_{E(\delta)} = \|\{\delta(n)a_n\}\|_E$. Using these definitions, we rewrite the inequalities (2.4) and (2.5) as

$$\|Ta\|_{l_\infty} \le \|\{\varphi(n)a_n\}\|_{l_\infty(\delta)}, \qquad \|Ta\|_{\bar{l}_1} \le \|\{\varphi(n)a_n\}\|_{l_1(\delta)},$$

and by interpolation property of spaces E and $\overline{E}$, obtain that

$$\|Ta\|_{\overline{E}} \ll \|\{\varphi(n)a_n\}\|_{E(\delta)} \sim \varphi(1)|a_1| + \|\{\varphi(\nu_n)a_{\nu_n}(f,X)\}\|_E.$$

Returning to the sequence of approximation numbers $a_n = a_n(f, X)$, we may use its decreasing, so that $(Ta)_n \ge \varphi(n)a_n$ for any n. Therefore

$$\|\{\varphi(n)a_n(f,X)\}\|_{\overline{E}} \ll \varphi(1)a_1(f,X) + \|\{\varphi(\nu_n)a_{\nu_n}(f,X)\}\|_E. \tag{2.6}$$

In order to prove the opposite inequality, we define another operator S such that

$$(Sa)_1 = \varphi(1)a_1, \quad (Sa)_2 = \frac{\varphi(2)}{2\ln 2}\,a_2, \quad (Sa)_{n+2} = \sum_{k=\nu_{n-1}+1}^{\nu_n}\frac{\varphi(k)a_k}{k\ln k}$$

for $n \ge 1$. Then immediately $\|Sa\|_{l_1} = \|\{\varphi(n)a_n\}\|_{\bar{l}_1}$. On the other hand,

$$\sum_{k=\nu_{n-1}+1}^{\nu_n}\frac{\varphi(k)}{k\ln k} \le \varphi(\nu_n) = \varphi(\nu_{n-1}^2) \sim \varphi(\nu_{n-1}),$$

hence

$$\|Sa\|_{l_\infty} \ll \sup\{\varphi(1)|a_1|,\ \varphi(2)|a_2|,\ \dots,\ \max_{\nu_{n-1}+1\le k\le\nu_n}\varphi(k)|a_k|,\dots\}$$

$$= \|\{\varphi(n)a_n\}\|_{l_\infty}.$$

We obtained that S is a bounded operator from $\bar{l}_1(\varphi)$ into l_1 and from $l_\infty(\varphi)$ into l_∞. Applying again interpolation properties of spaces E and $\overline{E}$, we prove that $\|Sa\|_E \le \|\{\varphi(n)a_n\}\|_{\overline{E}}$ for any sequence $a=\{a_n\}$.

Let now a sequence $\{a_n\}$ be nonnegative and decreasing. Then, for every $n\ge 1$,

$$(Sa)_{n+2} \ge a_{\nu_n}\int_{\nu_{n-1}}^{\nu_n-1}\frac{\varphi(t)}{t\ln t}\,dt \ge \frac{1}{2}\,\varphi(\nu_{n-1})a_{\nu_n}.$$

But $\varphi(\nu_{n-1})\sim\varphi(\nu_{n-1}^8)=\varphi(\nu_{n+2})$, hence $(Sa)_{n+2}\gg\varphi(\nu_{n+2})a_{\nu_{n+2}}$. Since for $n=0$ such an inequality is also true, we obtain that

$$\|\{\varphi(n)a_n(f,X)\}\|_{\overline{E}} \gg \varphi(1)a_1(f,X)+\|\{\varphi(\nu_n)a_{\nu_n}(f,X)\}\|_E,$$

and the required equivalence of quasinorms is proved. □

The equivalent quasinorm (2.3) is convenient for stating some new conditions for embeddings of approximation spaces one into another.

Theorem 2.3 *Let $r(n)=\psi(\nu_n)/\varphi(\nu_n)$ and let E^d be the cone of nonnegative and nonincreasing sequences from the space E. Let, as usual, r^* means the nonincreasing rearrangement of the sequence r. If the linear operator $Qa=\{r^*(n)a_n\}$ is bounded from E^d to F, then $X_{\varphi,E}\hookrightarrow X_{\psi,F}$.*

Proof. We will use an inequality $\|ab\|_A\le 2\|a^*b^*\|_A$ which holds for any rearrangement-invariant space A. It gives that

$$\|f\|_{X_{\psi,F}}-\psi(1)\|f\|_X \sim \|\{\psi(\nu_n)a_{\nu_n}(f,X)\}\|_F \le 2\|\{r^*(n)c^*(n)\}\|_F,$$

where $\{c^*(n)\}$ is the nonincreasing rearrangement of the sequence $\{c(n)\}=\{\varphi(\nu_n)a_{\nu_n}(f,X)\}$. The properties of operator Q imply that

$$\|f\|_{X_{\psi,F}}-\psi(1)\|f\|_X \ll \|\{c^*(n)\}\|_E$$

$$= \|\{\varphi(\nu_n)a_{\nu_n}(f,X)\}\|_E \sim \|f\|_{X_{\varphi,E}}-\varphi(1)\|f\|_X\,,$$

and we are done. □

Example 2.4 *Let $E = l_p$, $F = l_q$, $p > q$ and let $\psi(t)/\varphi(t) \sim (\ln \ln t)^{\varepsilon}$ with some $\varepsilon < 0$ for sufficiently big t. Then $Qa \sim \{n^{\varepsilon} a_n\}$ and by the Hölder inequality we obtain that*

$$\|Qa\|_{l_q} \sim \Big(\sum_{n=1}^{\infty} (n^{\varepsilon} a_n)^q\Big)^{1/q} \le \Big(\sum_{n=1}^{\infty} a_n^p\Big)^{1/pq} \Big(\sum_{n=1}^{\infty} n^{\varepsilon qp/(p-q)}\Big)^{(p-1)/pq} < \infty$$

for any $a \in E^d$, if $\varepsilon pq/(p-q) < -1$, that is, $\varepsilon < 1/p - 1/q$. Hence, for any such ε, we obtain that $X_{\varphi,E} \hookrightarrow X_{\psi,F}$. Remark that for $\varepsilon \ge -1$ this result does not follow from Theorem 2.1.

3 Representation and iterations

One of the basic properties of approximation spaces is a special representation of their functions via functions from the sets G_n. An analogous assertion can be stated in our class of spaces as well.

Theorem 3.1 *Any function $f \in X_{\varphi,E}$ can be represented in X as $f = \sum g_n$ for some $g_n \in G_{\nu_n}$, $n = 0, 1, \dots$, such that $\{\varphi(\nu_n)\|g_n\|_X\} \in E$ and*

$$\|f\|_{X_{\varphi,E}} \ge C(X,\varphi) \|\{\varphi(\nu_n)\|g_n\|_X\}\|_E \,. \tag{3.1}$$

Proof. Let $f \in X_{\varphi,E}$ be given. For each $n = 0, 1, \dots$, define $f_n \in G_{\nu_n - 1}$ such that $\|f - f_n\|_X \le 2\, a_{\nu_n}(f, X)$. Set now $g_0 = 0$, $g_1 = f_0$ and $g_n = f_{n-1} - f_{n-2}$ for any $n \ge 2$. Since $G_{\nu_{n-1}-1} + G_{\nu_{n-2}-1} \subseteq G_{\nu_n}$, all g_n satisfy the required conditions and also

$$\|f - \sum_{n=0}^{k} g_n\|_X = \|f - f_{k-1}\|_X \le 2\, a_{\nu_{k-1}} \to 0$$

as $k \to \infty$ (due to definition of the space $X_{\varphi,E}$), that is, the series $\sum g_n$ converges to f in X as needed.

Furthermore, for any $n \ge 2$,

$$\|g_n\|_X \le C(X)\, (\|f - f_{n-1}\|_X + \|f - f_{n-2}\|_X) \le 4C(X)\, a_{\nu_{n-1}}(f, X),$$

hence

$$\varphi(\nu_n)\|g_n\|_X \le 4C(X)\, \varphi(\nu_n)\, a_{\nu_{n-1}}(f, X) \le C(X,\varphi)\, \varphi(\nu_{n-1})\, a_{\nu_{n-1}}(f, X).$$

Due to symmetricity of the space E, this implies that

$$\|\{\varphi(\nu_n)\|g_n\|_X\}\|_E \le C(X,\varphi)\,\|\{\varphi(\nu_n)\,a_{\nu_n}(f,X)\}\|_E \ll \|f\|_{X_{\varphi,E}},$$

as desired. □

A stronger version of the representation theorem in previously studied approximation spaces asserts that the required representation exists *only* for functions $f \in X_{\varphi,E}$. The condition $\varphi(t) \sim \varphi(t^2)$ is insufficient for such a strengthening of Theorem 3.1, which could fail for functions φ increasing too slow (see an analogous example in [6]). That is why we need an additional condition, restricting class of functions φ "from below" and excluding, e.g., such functions as $\varphi(t) = \ln\ln t$.

We say that $\varphi(t)$ is a *quasi-logarithmic* function, if $\varphi(t) \sim \varphi(t^2)$ and the lower extension index π of the function $\varphi(e^t)$ is positive. The last implies existence of two positive numbers C, ε such that

$$\varphi(\nu_m)/\varphi(\nu_n) \le C\,2^{\varepsilon(m-n)}, \qquad \text{for any } n \ge m \ge 0. \tag{3.2}$$

Theorem 3.2 *If a function φ is quasi-logarithmic, then the space $X_{\varphi,E}$ contains all elements $f \in X$, representable as $f = \sum g_n$, $g_n \in G_{\nu_n}$, such that $\{\varphi(\nu_n)\|g_n\|_X\} \in E$. Moreover, the quasinorm of f in $X_{\varphi,E}$ is equivalent to*

$$\|f\|^{\rm rep}_{X_{\varphi,E}} := \inf \|\{\varphi(\nu_n)\|g_n\|_X\}\|_E\,, \tag{3.3}$$

where infimum is taken over all possible representations of f.

Proof. Let $f = \sum g_n$ be an admissible representation. Taking into account that $\sum_{k=0}^{n-1} g_k \in G_2 + \cdots + G_{\nu_{n-1}} \subseteq G_{\nu_n - 1}$, we obtain immediately that

$$a_{\nu_n}(f,X) \le \|f - \sum_{k=0}^{n-1} g_k\|_X \le \sum_{k=n}^{\infty} \|g_k\|_X.$$

Therefore it is enough to prove the following assertion.

Lemma 3.3 *Let $b_n \ge 0$, $n \in \mathbb{N}_0$ and let $\{\varphi(\nu_n)b_n\} \in E$. If $c_n = \sum_{k=n}^{\infty} b_k$ then $\{\varphi(\nu_n)c_n\} \in E$ for any quasi-logarithmic function $\varphi(t)$.*

Proof. In fact, we have to prove that the linear operator $Q\{b_n\} = \{c_n\}$ is bounded on the weight space $E(\varphi(\nu_n))$ for any symmetric sequence space E. We shall do this, using interpolation property of E.

Let $b_n \in l_\infty(\varphi(\nu_n))$ with unit norm, then $b_n \le 1/\varphi(2^n)$. By inequality (3.2) we obtain that

$$c_n \le \sum_{k=n}^{\infty} \frac{1}{\varphi(\nu_k)} \le \frac{C}{\varphi(\nu_n)} \sum_{k=n}^{\infty} 2^{\varepsilon(n-k)} \le \frac{C_1}{\varphi(\nu_n)},$$

so that $\{\varphi(2^n)c_n\} \in l_\infty$ with norm not exceeding C_1.

If now $\{b_n\} \in l_1(\varphi(\nu_n))$ then

$$\|\{\varphi(\nu_n)c_n\}\|_{l_1} = \sum_{n=0}^{\infty} \varphi(\nu_n) \sum_{k=n}^{\infty} b_k = \sum_{k=0}^{\infty} b_k \sum_{n=0}^{k} \varphi(\nu_n).$$

But

$$\sum_{n=0}^{k} \varphi(\nu_n) \le C\,\varphi(\nu_k) \sum_{n=0}^{k} 2^{\varepsilon(n-k)} \le C_1 \varphi(\nu_k),$$

so that

$$\|\{\varphi(\nu_n)c_n\}\|_{l_1} \le C_1 \sum_{k=0}^{\infty} \varphi(\nu_k)\, b_k.$$

Thus the operator Q is bounded on the both spaces $l_1(\varphi(\nu_n))$ and $l_\infty(\varphi(\nu_n))$, and the required assertion follows from the fact that the space $E(\varphi(\nu_n))$ has the same interpolation properties as the space E. □

In order to finish the proof of Theorem 3.2, we should now put $b_n = \|g_n\|_X$, recall that $a_{\nu_n}(f, X) \le c_n$ and apply Lemma3.3. This gives us that $\{\varphi(\nu_n)a_{\nu_n}(f, X)\} \in E$, that is, $f \in X_{\varphi,E}$, as was asserted. □

For the further discussion, we need two special estimates, usually mentioned in approximation theory as Jackson and Bernstein inequalities. Since any rearrangement-invariant sequence space is embedded into l_∞, that is, $\sup |a_n| \le C\|\{a_n\}\|_E$, Theorem 2.2 immediately gives that

$$a_{\nu_n} \le \frac{C}{\varphi(\nu_n)} \|f\|_{X_{\varphi,E}}. \tag{3.4}$$

The second inequality is stated only for quasi-logarithmic functions and follows from Theorem 3.2. Namely, taking arbitrary function $g \in G_{\nu_n}$, we may apply this theorem to the element $f = g$, considering this equality as one of possible representations for the f. This implies that

$$\|g\|_{X_{\varphi,E}} \le C\,\varphi(\nu_n)\,\|g\|_X \qquad \text{for any } g \in G_{\nu_n}. \tag{3.5}$$

The definition of approximation spaces implies that $G_n \subset X_{\varphi,E}$ for any n and any parameters φ, E. Therefore we can define the approximation numbers with respect to the space $X_{\varphi,E}$ instead of X:

$$a_n(f, X_{\varphi,E}) = \inf_{g \in G_{n-1}} \|f - g\|_{X_{\varphi,E}},$$

and construct the *iterated* approximation spaces $(X_{\varphi,E})_{\psi,F}$. Our aim is to compare these new spaces with initial ones.

First of all, let us compare the approximation numbers.

Lemma 3.4 *If $\varphi(t) \sim \varphi(t^2)$, there exists a constant $C = C(\varphi)$ such that*

$$a_{\nu_{n+1}}(f, X) \le \frac{C}{\varphi(\nu_n)}\, a_{\nu_n}(f, X_{\varphi,E}). \tag{3.6}$$

Proof. Let two elements $g_1, g_2 \in G_{\nu_n - 1}$ be chosen such that, for some $\varepsilon > 0$,

$$\|f - g_1\|_{X_{\varphi,E}} \le a_{\nu_n}(f, X_{\varphi,E}) + \varepsilon\,,$$

$$\|(f - g_1) - g_2\|_X \le a_{\nu_n}(f - g_1, X) + \varepsilon\,.$$

Then $g_1 + g_2 \in G_{2(\nu_n - 1)} \subset G_{\nu_{n+1}-1}$, hence by (3.4)

$$\begin{aligned} a_{\nu_{n+1}}(f, X) &\le \|f - (g_1 + g_2)\|_X \le a_{\nu_n}(f - g_1, X) + \varepsilon \\ &\le \frac{C}{\varphi(\nu_n)}\, \|f - g_1\|_{X_{\varphi,E}} + \varepsilon \le \frac{C}{\varphi(\nu_n)}\, (a_{\nu_n}(f, X_{\varphi,E}) + \varepsilon) + \varepsilon\,. \end{aligned}$$

Since ε is arbitrary, this leads to (3.6). □

Theorem 3.5 *If $\varphi(t) \sim \varphi(t^2)$, $\psi(t) \sim \psi(t^2)$, then $(X_{\varphi,E})_{\psi,F} \hookrightarrow X_{\varphi\psi,F}$ for any parameter spaces E, F. If, in addition, the functions φ, ψ are quasi-logarithmic, then $(X_{\varphi,E})_{\psi,F} = X_{\varphi\psi,F}$.*

Proof. If $f \in (X_{\varphi,E})_{\psi,F}$ then by Theorem 2.2 and Lemma3.4

$$\begin{aligned} \|f\|_{(X_{\varphi,E})_{\psi,F}} &\sim \psi(1) a_1(f, X_{\varphi,E}) + \|\{\psi(\nu_n)\, a_{\nu_n}(f, X_{\varphi,E})\}\|_F \\ &\gg \psi(1)\varphi(1) a_1(f, X) + \|\{\psi(\nu_n)\varphi(\nu_n)\, a_{\nu_{n+1}}(f, X)\}\|_F. \end{aligned}$$

Having in mind that $\psi(\nu_n)\varphi(\nu_n) \sim \psi(\nu_{n+1})\varphi(\nu_{n+1})$ and $\psi(1)\varphi(1) a_1(f, X) \gg \psi(2)\varphi(2) a_2(f, X)$, we arrive at $\|f\|_{(X_{\varphi,E})_{\psi,F}} \gg \|f\|_{X_{\varphi\psi,F}}$, and the first assertion of the theorem is proved.

Let now the functions φ, ψ be quasi-logarithmic, so that we may use the inequality (3.5). If $f \in X_{\varphi\psi,F}$ then, by Theorem 3.2, there exists a representation $f = \sum g_n$, $g_n \in G_{\nu_n}$, converging in X and such that

$$\|\{\psi(\nu_n)\varphi(\nu_n)\|g_n\|_X\}\|_F \ll \|f\|_{X_{\varphi\psi,F}}.$$

In particular, due to the inequality (3.5), this implies that for any $n = 0, 1, \ldots$

$$\|g_n\|_{X_{\varphi,E}} \le C\,\varphi(\nu_n)\|g_n\|_X \ll \frac{1}{\psi(\nu_n)}\,\|f\|_{X_{\varphi\psi,F}}.$$

But the inequality (3.2) implies that

$$\sum_{n=0}^{\infty} \frac{1}{\psi(\nu_n)} \le \frac{C}{\psi(\nu_0)} \sum_{n=0}^{\infty} 2^{-\varepsilon n} < \infty,$$

hence the series $\sum g_n$ converges to f in the space $X_{\varphi,E}$ as well. Moreover,

$$\{\psi(\nu_n)\|g_n\|_{X_{\varphi,E}}\} \ll \{\psi(\nu_n)\varphi(\nu_n)\|g_n\|_X\} \in F,$$

which means that $f \in (X_{\varphi,E})_{\psi,F}$ with inequality $\|f\|_{(X_{\varphi,E})_{\psi,F}} \ll \|f\|_{X_{\varphi\psi,F}}$, and the second assertion of the theorem is also proved. □

4 Interpolation

Theorem 3.5 may be considered as stating stability of our new class of approximation spaces with respect to iteration. In this section we will show the same property of such spaces with respect to interpolation. We consider only K-method of interpolation which for any quasi-Banach couple [2] X, Y defines interpolation spaces $(X, Y)^K_\Phi$ with the (quasi)norm $\|f\| = \|K(t, f, X, Y)\|_\Phi$. Here Φ may be arbitrary Banach function space containing the function $\min(1, t)$ and $K(t, f, X, Y)$ is the K-functional of J. Peetre:

$$K(t, f, X, Y) = \inf_{f=f_1+f_2} \big(\|f_1\|_X + t\,\|f_2\|_Y\big).$$

Sometimes (e.g., when X, Y are sequence spaces) it is more convenient to use discretization of the parameter space Φ (see, e.g., [4], p. 358). In

[2] Recall that two spaces form a couple if they both are continuously embedded into the same linear topological Hausdorff space.

particular, we may take $\Phi = l_{\psi,E}$ with $\rho_\psi < 1$, obtaining for the case of $Y \hookrightarrow X$ the (quasi)norm of $(X,Y)^K_\Phi$ in equivalent form

$$\|f\|_{K,\psi,E} := \|\{\psi(q^n)\, K(q^{-n}, f, X, Y)\}\|_E\,, \tag{4.1}$$

with an arbitrary $q > 1$.

As a basic one, we consider the couple (X, Y_p), where X is the given quasi-Banach space and $Y_p = X_{\varphi_0,E_0}$ with special parameters $E_0 = l_1$ and $\varphi_0(t) = (\ln t)^p$ for $t \ge 2$. Let us show that any approximation space $X_{\varphi,E}$ with quasi-logarithmic parameter-function $\varphi(t)$ is intermediate in this couple if p is sufficiently large, namely greater than the upper extension index ρ for the function $\varphi(e^t)$. Indeed, a simple computation gives that this $\rho \le \ln c(\varphi)/\ln 2$, where $c(\varphi) = \sup_{t>1} \varphi(t^2)/\varphi(t)$, hence ρ is finite for any quasi-logarithmic function. If $p > \rho$ then $\varphi(e^t) \le C\, t^p$ with some $C = C(p,\varphi)$, that is, $\varphi(t) \le C\,(\ln t)^p$. Together with trivial embedding $l_1 \hookrightarrow E$ this proves that $Y_p \hookrightarrow X_{\varphi,E} \hookrightarrow X$.

Lemma 4.1 *For any $n = 0, 1, \dots$ and any $f \in X$,*

$$(1/C)\, a_{\nu_n}(f,X) \le K(2^{-np}, f, X, Y_p)$$

$$\le C\Big(2^{-np}\|f\|_X + \sum_{k=0}^{n} 2^{p(k-n)} a_{\nu_k}(f,X)\Big). \tag{4.2}$$

Proof. Taking arbitrary $h \in Y_p$, we get

$$a_{\nu_n}(f,X) = \inf_{g \in G_{\nu_n - 1}} \|f-g\|_X \ll \|f-h\|_X + \inf_{g \in G_{\nu_n-1}} \|h-g\|_X$$

$$\ll \|f-h\|_X + a_{\nu_n}(h,X).$$

By the inequality (3.4), we obtain that $a_{\nu_n}(h,X) \le (C/2^{np})\, \|h\|_{Y_p}$, whence

$$a_{\nu_n}(f,X) \ll \|f-h\|_X + 2^{-np}\|h\|_{Y_p}.$$

Passing to infimum over all $h \in Y_p$, we obtain the first inequality in (4.2).

To prove the second inequality, for any given $n = 0, 1, \dots$, let us define a function $g_n \in G_{\nu_n - 1}$ such that $\|f - g_n\|_X \le 2a_{\nu_n}(f,X)$, and then put $f_1 = g_0$, $f_n = g_{n-1} - g_{n-2}$ $(n \ge 2)$. We obtain that

$$\|f_1\|_X \ll \|f-g_0\|_X + \|f\|_X \le 2a_2(f,X) + \|f\|_X \ll \|f\|_X,$$

$$\|f_n\|_X \ll \|f-g_{n-1}\|_X + \|f-g_{n-2}\|_X \ll a_{\nu_{n-1}}(f,X), \quad \text{for all } n \ge 2.$$

Since $g_n \in Y_p$ for each n, we may write that

$$K(2^{-np}, f, X, Y_p) \le \|f - g_n\|_X + 2^{-np}\|g_n\|_{Y_p} \le 2\, a_{\nu_n}(f, X) + 2^{-np}\|g_n\|_{Y_p},$$

and then, by the inequality (3.5),

$$\|g_n\|_{Y_p} = \Big\| \sum_{k=1}^{n+1} f_k \Big\|_{Y_p} \ll \|f_1\|_X + \sum_{k=2}^{n+1} 2^{kp}\|f_k\|_X \ll \|f\|_X + \sum_{k=1}^{n} 2^{kp} a_{\nu_k}(f, X).$$

In result,

$$K(2^{-np}, f, X, Y_p) \ll a_{\nu_n}(f, X) + 2^{-np}\Big(\|f\|_X + \sum_{k=1}^{n} 2^{kp} a_{\nu_k}(f, X)\Big),$$

and we are done. □

Theorem 4.2 *Let ρ be the upper extension index for a function $\varphi(e^t)$ and let $\psi(\ln^p t) = \varphi(t)$ for some $p > \rho$. Then*

$$X_{\varphi,E} = (X, Y_p)^K_\Phi, \qquad \textit{where} \;\; \Phi = l_{\psi,E}.$$

Proof. We will use the quasinorm (4.1) with $Y = Y_p$ and $q = 2^p$. The theorem will be proved if we show that $\|f\|_{K,\psi,E} \sim \|f\|_{X_{\varphi,E}}$. Using the first inequality in (4.2), we obtain immediately that

$$\|f\|_{K,\psi,E} \gg \|\{\psi(2^{np})\, a_{\nu_n}(f, X)\}\|_E \sim \|\{\varphi(\nu_n)\, a_{\nu_n}(f, X)\}\|_E.$$

At the same time, $(X, Y_p)^K_\Phi \hookrightarrow X$ and thus $\|f\|_{K,\psi,E} \gg \|f\|_X = a_1(f, X)$. Therefore, by Theorem 2.2, we obtain that

$$\|f\|_{K,\psi,E} \gg \|f\|_{X_{\varphi,E}} \quad \Longrightarrow \quad (X, Y_p)^K_\Phi \hookrightarrow X_{\varphi,E}\,.$$

The second inequality from (4.2) gives that

$$\|f\|_{K,\psi,E} \ll \|\{2^{-np}\varphi(\nu_n)\}\|_E \|f\|_X + \Big\|\Big\{\varphi(\nu_n) \sum_{k=0}^{n} 2^{p(k-n)} a_{\nu_k}(f, X)\Big\}\Big\|_E\,. \tag{4.3}$$

The definition of ψ implies that $\rho_\psi = \rho/p$, hence, for any $\varepsilon > 0$ and any $k \le n$,

$$\psi(2^{np}) \le \psi(2^{kp})\, m_\psi(2^{p(n-k)}) \le C(\varepsilon)\, 2^{p(n-k)((\rho/p)+\varepsilon)} \psi(2^{kp}).$$

Taking $\varepsilon = (p-\rho)/(2p)$, we obtain that

$$\varphi(\nu_n) \sim \psi(2^{np}) \ll 2^{p(n-k)(1-\varepsilon)}\psi(2^{kp}) \sim 2^{p(n-k)(1-\varepsilon)}\varphi(\nu_k),$$

whence $2^{-np}\varphi(\nu_n) \ll 2^{-\varepsilon np}\varphi(\nu_0)$ for all n, so that the first summand in (4.3) is no greater than $C\,\|f\|_X \sim \varphi(1)a_1(f,X)$.

At the same time,

$$\|f\|_{K,\psi,E} \ll \varphi(1)a_1(f,X) + \Big\|\Big\{\sum_{k=0}^{n} 2^{\varepsilon p(k-n)}\varphi(\nu_k)a_{\nu_k}(f,X)\Big\}\Big\|_E,$$

and, for finishing the proof, we need the following

Lemma 4.3 *The linear operator $Pa = b$, defined by the equalities*

$$b_n = \sum_{k=0}^{n} 2^{\varepsilon p(k-n)}a_k, \qquad n = 0, 1, \dots,$$

is bounded in any symmetric sequence space E.

Proof. As usual, it is enough to show that P is bounded in l_1 and l_∞. We have that

$$\|Pa\|_{l_1} \le \sum_{n=0}^{\infty}\sum_{k=0}^{n} 2^{\varepsilon p(k-n)}|a_k| = \sum_{k=0}^{\infty}|a_k|\sum_{n=k}^{\infty} 2^{-\varepsilon p(n-k)} \le C\sum_{k=0}^{\infty}|a_k|,$$

and also

$$\|Pa\|_{l_\infty} \le \sup_n \sum_{k=0}^{n} 2^{\varepsilon p(k-n)}|a_k| \le \sup_n\Big(\max_{k\le n}|a_k|\cdot\sum_{k=0}^{n} 2^{-\varepsilon p(n-k)}\Big) \le C\sup_n|a_n|,$$

that was required. □

Returning to the proof of Theorem 4.2, we obtain that

$$\|f\|_{K,\psi,E} \ll \varphi(1)a_1(f,X) + \|\{\varphi(\nu_n)a_{\nu_n}(f,X)\}\|_E \sim \|f\|_{X_{\varphi,E}},$$

and the theorem is proved. □

Corollary 4.4 *Let a number p be greater than the upper extension indices of functions $\varphi_0(e^t)$, $\varphi_1(e^t)$ and let $\psi(\ln^p t) = \varphi(t)$, $\psi_i(\ln^p t) = \varphi_i(t)$, $i = 0, 1$. A space $X_{\varphi,E}$ is interpolation one between spaces X_{φ_0,E_0} and X_{φ_1,E_1} if*

and only if the ultrasymmetric space $l_{\psi,E}$ is interpolation between the spaces l_{ψ_0,E_0} and l_{ψ_1,E_1}. Moreover, for any real interpolation functor $\mathcal{F}$, one has

$$\mathcal{F}(X_{\varphi_0,E_0}, X_{\varphi_1,E_1}) = (X, Y_p)^K_\Psi, \qquad \Psi = \mathcal{F}(l_{\psi_0,E_0}, l_{\psi_1,E_1}).$$

This assertion follows immediately from Theorem 4.2, using general *reiteration* theorems of interpolation (see, e.g., [4], p. 357).

For application of Corollary 4.4, we should solve a very difficult problem: when a given ultrasymmetric space $l_{\psi,E}$ is interpolation between two other such spaces l_{ψ_0,E_0} and l_{ψ_1,E_1} (recall that all parameters ψ, ψ_i are now arbitrary positive increasing functions). The simplest case is when the parameter ψ is the same for all three spaces. Namely, a space $l_{\psi,E}$ is interpolation between l_{ψ,E_0} and l_{ψ,E_1} if and only if the space E is interpolation between E_0 and E_1 (interpolation properties of rearrangement-invariant spaces are nowadays well investigated). For the simultaneous change of all parameters, we have only some partial results. For instance, we can prove that if $\rho_{\varphi_0} < \pi_\varphi \le \rho_\varphi < \pi_{\varphi_1}$ then

$$(X_{\varphi_0,E_0}, X_{\varphi_1,E_1})_{\theta,q} = X_{\varphi,l_q}, \quad \varphi(t) = (\varphi_0(t)^{1-\theta}(\varphi_1(t)^\theta,$$

whenever $0 < \theta < 1,\ 1 \le q \le \infty$ (for the definition and properties of the (θ, q)-method, see, e.g., [1], Section 3.1).

The most general of the known interpolation theorems for ultrasymmetric function spaces is apparently Theorem 5.1 from [10]; it is easy to show that it remains true for the sequence spaces too. One of the consequent assertions for the approximation spaces is as follows.

Corollary 4.5 *Let the functions $\psi_0(t)$, $\psi_1(t)$ (defined as before) and $\sigma(t)$ be such that $\pi_{\psi_1/\psi_0} > 0$, $0 < \pi_\sigma \le \rho_\sigma < 1$ and let $\varphi = \varphi_0\sigma(\varphi_1/\varphi_0)$. Then the space $X_{\varphi,E}$ is interpolation between the spaces X_{φ_0,E_0} and X_{φ_1,E_1} independently of parameter-spaces E, E_0, E_1; moreover,*

$$X_{\varphi,E} = (X_{\varphi_0,E_0}, X_{\varphi_1,E_1})^K_\Psi, \quad \Psi = l_{\sigma,E}.$$

References

[1] J. Bergh and J. Löfström, Interpolation spaces. An Introduction, Grundleheren der mathematische Wissenschaften 223, Springer, Berlin–Heidelberg–New York, 1976.

[2] Ju. A. Brudnyi,Approximation spaces (Russian), in "Geometry of Linear Spaces and Operator Theory", pp. 3–30, Jaroslavl, 1977.

[3] Ju. A. Brudnyi and N. Ja. Krugljak, About a family of approximation spaces (Russian), in "Theory of functions of several real variables", pp. 15–42, Jaroslavl, 1978.

[4] Ju. A. Brudnyi and N. Ja. Krugljak, Interpolation functors and interpolation spaces. Vol. 1, North-Holland, Amsterdam, 1991.

[5] F. Cobos and M. Milman, On a limit class of approximation spaces, Numer. Funct. Anal. Optim. 11 (1990), 11–31.

[6] F. Cobos and I. Resina, Representation theorems for some operator ideals, J. Lond. Math. Soc. 39 (1989), 324–334.

[7] R. A. DeVore, S. D. Riemenschneider and R. C. Sharpley, Weak interpolation in Banach spaces, J. Funct. Anal. 33 (1979), 58–94.

[8] F. Feher and G. Gräßler, On an extremal scale of approximation spaces, J. Comp. Anal. Appl. 3 (2001), 95–108.

[9] A. Pietsch, Approximation spaces, J. Approx. Theory 32 (1981), 115–134.

[10] E. Pustylnik, Ultrasymmetric spaces, J. Lond. Math. Soc. 68 (2003), 165–182.

[11] E. Pustylnik, Ultrasymmetric sequence spaces in approximation theory, 20 p. (*submitted*).

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Quasi-Inverses and Approximation with Min-Max Operators

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Abstract. The Semi-group of Min-Max Operators, as used for Nonlinear Smoothing, or Multiresolution Analysis, generally lacks inverses. This corresponds to the fact that smoothers can only destroy information selectively. Given this constraint, it is important that a procedure exists for selecting those with minimal damage. Total Variation is a natural norm in this secondary approximation problem.

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1 Introduction

We consider sequences of real numbers generated by some measurement and consider these to be in the vectorspace ℓ_1 of real sequences with absolutely convergent sums, since they generally have finite support. We define the min- and max-operators by;

Definition 1.1 $(\bigvee x)_i := \max\{x_i, x_{i+1}\}$ *and* $(\bigwedge x)_i := \min\{x_{i-1}, x_i\}$, *for each sequence* x.

Definition 1.2 $\mathcal{C} = \{A : \ A$ *is a composition of* $\bigvee$ *and* $\bigwedge\}$ $\mathcal{L} = \{A : \ A \in \mathcal{C}, \ A^2 = A\}$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Wild and Rohwer (5) show that basic partial ordering can be established in $\mathcal{C}$, using the trivial, inequalities $\bigwedge^n \leq I = \bigwedge^0 = \bigvee^0 \leq \bigvee^n$. Although less obvious inequalities like $\bigvee^n \bigwedge^m \leq I \leq \bigvee^m \bigwedge^n$ and$\bigwedge^n \bigvee^n \bigvee^m \bigwedge^m \leq \bigvee^m \bigwedge^m \bigwedge^n \bigvee^n$ are provable, there are incomparable compositions, yielding only a partial ordering. Also $\bigwedge$ and $\bigvee$ define a so-called Galois connection and all are Smoothers, in the sense that they are axes- and scale independent, or that;

($\bigvee(\gamma x) = \gamma \bigvee(x)$, for $\gamma \geq 0$, $\bigvee(x + c) = \bigvee x + c$, c a constant sequence and $(\bigvee x)_{i+n} = (E^n \bigvee x)_i$, where $(Ex)_i = x_{i+1}$ defines the (forward-) shift operator.

It is clear that a composition of smoothers is a smoother, and further arguments by Wild (5) yield the equivalence of the following statements for compositions of $\bigvee$ and $\bigwedge$, which clearly form a semi-group of smoothers with $I = \bigvee^0 = \bigwedge^0$, as identity.

Theorem 1.3 *(Wild) For all $x \in \mathcal{C}$ the following are equivalent:*

(a) A is idempotent ($A^2 = A$).

(b) A is the product of equally many $\bigvee$ and $\bigwedge$.

(c) A is the product of basic smoothers of the type $L_n = \bigvee^n \bigwedge^n$ and $U_n = \bigwedge^n \bigvee^n$, for example $U_5 L_3 U_6 L_1$.

We choose as norms in the space ℓ_1 the usual p-norms, particularly the $p = 1$ and $p = 2$-norms, and the Total Variation semi-norm, which becomes a norm if we restrict ourselves to the space ℓ_1. The latter turns out to be a natural norm for some reasons that have been argued (3). There are further reasons, as becomes evident.

Definition 1.4 $T(x) = \sum_{i=-\infty}^{\infty} |\triangle x_i|$, *with $\triangle x_i = x_{i+1} - x_i$, and $\triangle$ the forward difference operator.*

Clearly $T(\cdot)$ is a seminorm on X with $T(x) \leq 2||x||_1$, and $T(x) = 0 \Leftrightarrow x = 0$ if $x \in \ell_1$, so that it becomes a norm and in particular it can be shown that, for each sequence $x \in \ell_1$, the total variation is reduced by each operator in $\mathcal{C}$. This is proved in (3) by the following theorem.

Theorem 1.5 $T(x) = T(\bigvee x) + T(\bigwedge \bigvee x - x) = T(\bigwedge x) + 2||(\bigvee \bigwedge x - x)||_1$ *and* $T(x) = T(\bigwedge x) + T(\bigvee \bigwedge x - x) = T(\bigvee x) + 2||\bigwedge \bigvee x - x||_1$.

This important Variation Reducing property, as $T(x) \leq T(\bigvee x)$ and $T(x) \leq T(\bigwedge x)$, clearly holds for any of the operators in $\mathcal{C}$, as it leads immediately to the following theorem;

Theorem 1.6 *If $A, B \in \mathcal{C}$, then $T(BAx) \leq T(Ax),\ \forall x \in \ell_1$.*

It is clear that all compositions of $\bigvee$ and $\bigwedge$ are variation reducing, which is often a heuristic prerequisite for a "smoother", as Total Variation is implicitly widely accepted as a measure of smoothness. Noting that $\bigvee$ and $\bigwedge$ are syntone (monotone), in that, $x \geq y \Rightarrow Ax \geq Ay$ for a syntone operator A, and that a large class of their compositions are idempotent, we see that these satisfy the axioms of Morphological Filters (6). Of these special compositions many have the additional, arguably important, property of co-idempotence (3) in that, if A is co-idempotent then x is additively separated into Ax and $(I-A)x$ in a consistent fashion so that $A(Ax) = Ax$ and $(I-A)^2 = I - A$.

A whole theory of (non-linear-) smoothing results. A very useful result is a Multiresolution Analysis with "Pulses", where a sequence is decomposed into a sum of basic sequences of pulses in progressively more restrictive smoothness classes, characterised by local Monotonicity (4).

Smoothing can then be considered as a mapping into classes of "smoother" sequences. These mappings can also be shown to be "near-optional" in the usual p-norms in the sense of being comparable to best approximations in the specific smoothness class (2). The question being addressed here is whether there is not a natural norm, in which selected smoothers are optimal. The contention is that the Total Variation yields a natural norm and that, where some operators suggest themselves for a specific application, there are natural "(quasi-inverses)" that can be employed to yield a minimal damage, in that the residual is minimal in Total Variation.

2 The Basic Smoothers $\bigvee$ and $\bigwedge$

Suppose x is a sequence of correlated measurements contaminated by some "occasional" single impulses, in that an arbitrary large single upward outlier is added. The simplest operator (with smallest support) that will remove such outliers is the operator $\bigwedge$, which will simply replace the contaminated number x_i by its neighbour x_{i+1}. Thus we separate x into $\bigwedge x$ (the signal) and $(I-\bigwedge)x$ (the noise). If the measurements are samplings of a "smooth" function f, the inevitable damage elsewhere is not much. Provided the sampling rate is sufficient and $|f'|$ is small the error is less than $h|f'|$ where

h is the sampling interval. Considering a sequence with no actual impulses, which is equivalent to considering a uncontaminated section, as the operators have local support, we can consider $(I - \bigwedge)x$ as the actual error incurred by the smoothing. Was there a better smoother $B\bigwedge$ available in the class to obtain less da mage, in that $T((I - B\bigwedge)x) < T((I - \bigwedge)x)$ for some B and every sequence $x \in X$? A first step is the following.

Theorem 2.1 *$T((I - \bigvee\bigwedge)x) \leq T((I - \bigwedge)x)$ and $T(I - \bigwedge\bigvee)x) \leq T((I - \bigvee)x)$.*

Proof. Let x be a sequence in $X = \ell_1$. By theorem 1.5 it follows that $T(x) = T(\bigwedge x) + T(\bigvee\bigwedge x - x)$. But $T(x) \leq T(\bigwedge x) + T(\bigwedge x - x)$, by the triangle inequality for the seminorm, so that

$$T(\bigwedge x - x) \geq T(\bigvee\bigwedge x - x).$$

The other half is proved similarly. □

Corollary 2.2 *$||(I - \bigvee\bigwedge)x||_1 \leq ||(I - \bigwedge)x||_1$ and $||(I - \bigwedge\bigvee)x||_1 \leq ||(I - \bigvee)x||_1$.*

Proof. $T(y) \leq 2||y||_1$, and therefore $T(I - \bigwedge)x \leq 2||(I - \bigwedge)x||_1$ and $T(I - \bigvee)x \leq 2||(I - \bigvee)x||_1$. Also $T(I - \bigwedge)x \geq T(I - \bigvee\bigwedge)x = 2||(I - \bigvee\bigwedge)x||_1$ and $T(I - \bigvee)x \geq T(I - \bigwedge\bigvee)x = 2||(I - \bigwedge\bigvee)x||_1$ by theorem 2.1.

In practice the difference between $||\bigwedge x - x||_1$ and $||\bigvee\bigwedge x - x||_1$ can be considerable, as the following example illustrates. □

Example 2.3 *The simple example of a centred linear B-spline $f(x) = (1 - |x|)(|x| < 1)$ sampled at intervals of $\frac{1}{n}$ yields $x_i = (1 - |\frac{i}{n}|)(|\frac{i}{n}| < 1)$,*

$$(\bigwedge x)_i = \begin{cases} \left(1 - \frac{i}{n}\right) & 0 \leq \frac{i}{n} \leq 1 \\ \left(1 + \frac{i-1}{n}\right) & (-1 < \frac{i}{n} \leq 0) \\ 0, & \textit{elsewhere,} \end{cases}$$

and

$$\bigwedge x - x)_i = \begin{cases} \frac{1}{n}, & \textit{for} \quad -1 < i \leq 0 \\ 0, & \textit{elsewhere.} \end{cases}$$

Therefore $(\bigvee\bigwedge x - x)_i = -\frac{\delta_{i0}}{n}$ *where* δ_{i0} *is the Kronecker-delta.*

Thus $T(x) = 2$, $T(\bigvee\bigwedge x - x) = \frac{2}{n}$, $T(\bigvee\bigwedge x) = 2 - \frac{2}{n} = T(\bigwedge x)$ and $T(\bigwedge x - x) = \frac{2}{n}$, but $||\bigvee\bigwedge x - x||_1 = \frac{1}{n}$ and $||\bigwedge x - x||_1 = 1$.

We note that $T(\bigvee\bigwedge x) = T(\bigwedge x)$, which will always be the case as $T(\bigwedge x) = T(\bigvee\bigwedge x) + T(\bigwedge\bigvee\bigwedge x - \bigwedge x)$, from theorem 1.5, but $\bigwedge\bigvee\bigwedge = \bigwedge$ for any Galois connection so that $T(\bigwedge\bigvee\bigwedge x - \bigwedge x) = T(0) = 0$. This suggest a crucial further comparison, as it implies that $T(x) = T(U_1 x) + T((I - U_1)x)$. This norm conservation can be compared to the theorem of Pythagoras in the ℓ_2 norms;

$$||x||_2^2 = ||Px||_2^2 + ||(I - P)x||_2^2, \quad \text{if} \quad P \quad \text{is a projection.}$$

Definition 2.4 *An operator A is Variation Preserving (VP) if*
$$T(x) = T(Ax) + T(Ax - x),\ \forall x \in \ell_1.$$

Besides U_1, does the Variation Preservation hold for other compositions of $\bigvee$ and $\bigwedge$? Clearly it does not hold for $\bigvee$ or $\bigwedge$ themselves, as the above example demonstrates. The argument however extends to U_n and L_n, and also all compositions of L_n and U_m since the following theorem holds (3).

Theorem 2.5 $T(x) = T(U_n x) + T(I - U_n)x$
$= T(L_n x) + T(I - L_n)x, \forall x \in \ell_1.$

Theorem 2.6 *If A and B are Variation Preserving (VP), then AB is VP.*

Proof. $T(x) = T(Ax) + T(Ax - x) = T(BAx) + T(BAx - Ax) + T(Ax - x) \geq T(BAx) + T(BAx - x)$, by the triangle inequality on the last two terms.

But also $T(x) \leq T(BAx) + T(BAx - x)$, by the triangle inequality. This proves the theorem. □

Corollary 2.7 *Compositions of Variation Preserving operators are Variation Preserving, and therefore all operators in the class* $\mathcal{L}$ *are Variation Preserving.*

Theorem 2.8 *Let A be any min-max operator. Then*
$T(A\bigwedge x - x) \geq T(\bigvee\bigwedge x - x)$ *and* $T(A\bigvee x - x)$
$\geq T(\bigwedge\bigvee x - x)$.

Proof. $T(x) \leq T(A \bigwedge x) + T(A \bigwedge x - x)$ by the triangle inequality Therefore $T(A \bigwedge x - x) \geq T(x) - T(A \bigwedge x) \geq T(x) - T(\bigwedge x)$, since A is variation reducing $= T(\bigvee \bigwedge x - x)$, by theorem 1.5.

The theorem now follows by a similar proof for the other part. □

We thus define a quasi-inverse operator of an operator S in the class of min-max operators $\mathcal{C}$, and note that $\bigwedge$ and $\bigvee$ have such quasi-inverses.

3 Quasi-inverses of general min-max operators.

Definition 3.1 *Let $S \in \mathcal{C}$. Then $S^* \in \mathcal{C}$ is a (left-) quasi-inverse operator to S in $\mathcal{C}$ iff $||(S^*S - I)|| \leq ||AS - I||$ for all $A \in \mathcal{C}$, where $||A||$ is the induced norm: $||A|| = \sup_{x \neq 0} \frac{||A(x)||}{||x||}$.*

Corollary 3.2 *$\bigwedge$ is a (left-) quasi-inverse of $\bigvee$ and $\bigvee$ is a (left-) quasi-inverse of $\bigwedge$.*

Of interest is whether such a quasi-inverse is unique. We note that from the example of $x_i = \delta_{i0}$ that $T(\bigwedge \bigvee \bigwedge x - x) = T(\bigwedge x - x) > T(\bigvee \bigwedge x - x)$, for at least one sequence x. Similarly, a simple example suffices to establish $T(\bigvee \bigvee \bigwedge x - x) > T(\bigvee \bigwedge x - x)$, for some x. An example is the sequence $x_i = -\delta_{i0}$ (the Kronecker-delta), when $(\bigvee x)_i = 0$, $T(\bigvee \bigwedge x - x) = 0$ and $\bigvee \bigwedge x = x$. But we have $T(\bigvee \bigvee \bigwedge x - x) = T(\bigvee x - x) = 2$.

The quasi-inverse are not unique. An example is the case of $\bigwedge$, since $\bigvee$ and $(\bigvee \bigwedge)^n \bigvee$ are all quasi-inverses. There are however in essence only two; $\bigvee$ and $\bigvee \bigwedge \bigvee$. The reasoning is clear; following $\bigwedge$ by $\bigvee$ or $\bigvee \bigwedge \bigvee$ yields $\bigvee \bigwedge$ and $\bigvee \bigwedge \bigvee \bigwedge$ respectively, but these are equal, since $L_1 = \bigvee \bigwedge$ is idempotent. (It is instructive to review argument for the idempotence; $(\bigvee \bigwedge) \bigvee \bigwedge \leq I \bigvee \bigwedge$, since $\bigvee \bigwedge \leq I$ and $\bigvee (\bigwedge \bigvee) \bigwedge \geq \bigvee I \bigwedge$, since $\bigwedge \bigvee \geq I$ and $\bigvee$ is syntone.) This raises the question, are quasi-inverses general and are they unique?

If we want to remove single upward (downward-) impulses from a sequence we would naturally prefer $\bigvee \bigwedge$ to $\bigwedge$ (and $\bigwedge \bigvee$ to $\bigvee$), as they are minimal in destructiveness in general, and simplest for the purpose. If we expect both upward and downward single pulses we would also naturally select either $L_1 = \bigvee \bigwedge$ followed by $U_1 = \bigwedge \bigvee$ or U_1 followed by L_1. Are these optimal? They certainly both remove single isolated impulses and replace them with neighbouring values, but may differ slightly in output. This corresponds to a well known ambiguity in the interpretation of an

impulse (1). As becomes clear, L_1U_1 and U_1L_1 are themselves minimal in damage and thus I is a quasi-inverse for both, but again this is not unique since, for example, $IL_1U_1 = L_1^n(L_1U_1) = U_1^n(L_1U_1) = (L_1U_1)^nL_1U_1$, for all values of n. Selecting the simplest seems to make it unique.

To generalise the ideas we investigate the proof of theorem 1, which uses the argument that if A is an arbitrary finite product of operators $\bigvee$ and $\bigwedge$, then A is of one of the following normal forms (5);

(i) $A = U_{n_1}L_{m_1}U_{n_2}L_{m_2}\dots \quad (m_1 > m_2 > \cdots, n_1 > n_2 > \cdots)$

(ii) $A = \bigvee^s \bigwedge^t U_{n_1}, L_{m_1}U_{n_2}L_{m_2}\dots \quad (m_1 > m_2 > \cdots, n_1 > n_2 > \cdots)$ $(0 \le s < t)$.

(iii) $A = L_{m_1}U_{n_1}L_{m_2}U_{n_2}\cdots \quad (m_1 > m_2 > \cdots, n_1 > n_2 > \cdots)$.

(iv) $A = \bigwedge^t \bigvee^s L_{m_1}U_{n_1}L_{m_2}U_{n_2}\cdots (m_1 > m_2 > \cdots, n_1 > n_2 > \cdots)$ $(0 \le t < s)$.

We also need the following Lemma.

Lemma 3.3 *(i)* $\bigvee^n \bigwedge^n \bigvee^n = \bigvee^n$ *and* $\bigwedge^n \bigvee^n \bigwedge^n = \bigwedge^n$, *for all* n.
(ii) $T(\bigwedge^k \bigvee^n x) = T(\bigvee^n x)$ *and* $(T(\bigvee^k \bigwedge^n x) = T(\bigwedge^n x)$, *for* $k \le n$, *and* $x \in \ell_1$.

Proof. (i) $(\bigvee^n \bigwedge^n \bigvee^n) = L_n \bigvee^n \le I \bigvee^n = \bigvee^n$, since $L_n \le I$. Also, $\bigvee^n(\bigwedge^n \bigvee^n) = \bigvee^n U_n \ge \bigvee^n I = \bigvee^n$, since $U_n \ge I$ and $\bigvee$ is syntone. Therefore $\bigvee^n \bigwedge^n \bigvee^n = \bigvee^n$, and similarly $\bigwedge^n \bigvee^n \bigwedge^n = \bigvee^n$.

(ii) $T(\bigvee^n x) \ge T(\bigwedge^k \bigvee^n x) \ge T(\bigvee^n \bigwedge^n \bigvee^n x) = T(\bigvee^n x)$; since $\bigwedge$ and $\bigvee$ are variation reducing. □

Theorem 3.4 *Let* $A \in \mathcal{L}$. *Then* I *is a (left-) quasi-inverse of* A *in terms of the* T*-norm.*

Proof. Let $B \in \mathcal{C}$. By the triangle inequality, $T(BAx - x) \ge T(x) - T(BAx) \ge T(x) - T(Ax) = T(Ax - x)$, since A is Variation Preserving by the corollary of Theorem 2.6. Thus $T(BA - I) \ge T(A - I)$ and the theorem is proved. □

Theorem 3.5 *Let* $A \in \mathcal{C}$. *Then* A *has a (left-) quasi-inverse in terms of the* T*-norm.*

Proof. We may assume that $A \notin \mathcal{L}$, as otherwise theorem 3.4 holds. By the theorem of Wild we consider the case where $A = \bigwedge^k \bigvee^n B$, with $B \in \mathcal{L}$ and

$0 \leq k < n$. Let $A^* = \bigwedge^{n-k}$. Then $A^*A \in \mathcal{L}$, since $A^* \bigwedge^k \bigvee^n = U_n$. From the lemma we have that $T(\bigwedge^i \bigvee^n x) = T(\bigwedge^n \bigvee^n x)$, for each $x \in \ell_1$.

Hence $T(Ax-x) \geq T(x)-T(Ax) = T(x)-T(\bigwedge^n \bigvee^n Bx) = T(\bigwedge^n \bigvee^n Bx-x)$.

Thus the theorem is proved for the particular A. A similar proof for the case where $A = \bigvee^k \bigwedge^n B$, with $B \in \mathcal{L}$ and $0 \leq k < n$ completes the proof. □

4 Uniqueness of the Quasi-inverses.

The foregoing proves that for each min-max operator Q there is an operator Q^* such that $T(Q^*Qx - x) \leq T(XQx - x)$ for each sequence $x \in \ell_1$, and therefore $T(Q^*Q - I) \leq T(XQ - I)$ in the induced norm. Uniqueness is not automatic as is clear when $B \in \mathcal{L}$ and I and B are (left-) quasi-inverses of B, as $B^2 = B$ (5). Clearly however I is the simplest of these, having the smallest index number of all such quasi-inverses.

Definition 4.1 *The index number $i(A)$ is the sum of exponents of $\bigvee$ and $\bigwedge$ in the normal form of A.*

Example 4.2 *(a)* $i(I) = i(\bigvee^0 \bigwedge^0) = 0$
(b) $i(A) = 2(\sum m_i + \sum n_i) + s + t$ *where*

$$A = \bigvee^{s} \bigwedge^{t} U_{n_1} L_{m_1} U_{n_2} L_{m_2} \cdots m_1 > m_2 > \cdots, n_1 > n_2 \cdots, (0 \leq s < t).$$

It is clear, by selecting an example, that the uniqueness of the quasi-inverse with smallest index will be general for $Q \in \mathcal{C}$.

Example 4.3 *Let A be the example in (b) above. Choosing a block pulse amplitude -1 and width $m = \max\{n_1, m_1\} + 1$ by $x = \left\{ x_i = \sum_{j=0}^{m} \delta_{ji} \right\}$ we see that $B = U_{n_1} L_{m_1} U_{n_2} L_{m_2} \ldots$ will preserve the sequence x as it is m-monotone. $\bigwedge^t$ will widen the pulse and $\bigvee^s$ will narrow it leaving the sequence $\bigvee^s \bigwedge^t x - x$ with $T(\bigvee^s \bigwedge^t x - x) = 2$, and composition with $\bigvee^{t-s}$ will leave $T(\bigvee^t \bigwedge^t x - x) = 0$. There clearly is none simpler than $\bigvee^{t-s}$.*

We also note that the following theorem holds.

Theorem 4.4 *Let $A, B \in \mathcal{L}$. Then the following statements are equivalent.*

(a) $T(AB) = T(B)$.
(b) $T(AB - I) = T(B - I)$.
(c) $AB = B$.

Proof. $A, B \in \mathcal{L}$ implies that $T(x) = T(Bx) + T(Bx - x)$, $T(x) = T(Ax) + T(Ax - x)$ and $T(x) = T(ABx) + T(ABx - x), \forall x \in \ell_1$

Therefore $T(Bx) = T(ABx) \Leftrightarrow T(ABx - x) = T(Bx - x),\ \forall x \in \ell_1$ and $T(Bx) = T(ABx) \Leftrightarrow T(ABx - Bx) = 0,\ \forall x \in \ell_1 \Leftrightarrow ABx - Bx = 0,\ \forall x \in \ell_1$ □

5 Minimising Damage with Presmoothing

Nonlinear Smoothers like the median, and compositions of these, were introduced and popularised by experts in smoothing at Bell Laboratories. The idea was to reduce the amount of "impulsive noise" before this does damage in linear smoothers that are subsequently used in the analysis of measurements or data. The heuristic wisdom was; first apply simple selectors ("presmooth") for removing impulsive noise and then apply the standard linear operators approximate to the type of signal expected. This is easy to rationalise, as the response of a linear operator to an impulse is characteristic of such an operator, and in general spreads the "energy" of such an impulse. The Heissenberg uncertaintly principle, for instance, guarantees a wide spread in the frequency content of the Fourier Transform if the impulse is brief. We could thus seek to apply the idea of presmoothing here, instead of recovering damage after applying a selected smoother S from the class of min-max operators. Given $S \in \mathcal{C}$, is there a (right-) quasi-inverse $\tilde{S}$ in $\mathcal{C}$ such that $||S\tilde{S} - I|| \leq ||SA - I||$, for each $A \in \mathcal{C}$? Or, firstly is there an operator B such that $||SBx - x|| \leq ||Sx - x||$? Strangely enough the problem is quite different from the case of reducing the damage with post-smoothing. We see this already with the simplest cases of $\bigwedge$ and $\bigvee$. Consider the case of $S = \bigwedge$, chosen to remove single upward impulses. Presmoothing with $\bigvee$ is the natural attempt and the first question is; is $T(\bigwedge \bigvee x - x) \leq T(\bigwedge x - x)$ valid? Even this is not easy.

Noting that $T(\bigvee^{n+1} x - x) \geq T(\bigvee x - x)$, presmoothing with $\bigvee^n$ does not help, since it is the same as post smoothing with $\bigvee^n$. Since $T(\bigwedge \bigvee \bigwedge x - x) = T(\bigwedge x - x)$, $\bigvee \bigwedge$ is no better than I as a presmoother. In fact no operator can improve on $\bigwedge \bigvee$ or $\bigvee \bigwedge$, but there seems to be no simple argument. We could attempt to prove; If A is a min-max operator and

$x \in \ell_1$, then $T(\bigwedge Ax - x) \geq T(\bigwedge \bigvee x - x)$. We note that there are two nontrivial possibilities:

a) If $A = B \bigvee$ then $T(\bigwedge B \bigvee x - x) \geq T(\bigwedge \bigvee x - x)$, by theorem 7.

b) The case where $A = C \bigwedge$ yields a failure, as the following example illustrates

Let $x_i = 2$, for $i = -1, 1$
0, elsewhere.

Then $T(U_1 x - x) = 6$ and $T(L_1 x - x) = 4$. Since $U_1 L_1 x = L_1 x$ we get $T(U_1 L_1 x - x) < T(U_1 x - x)$. Thus the choice $A = \bigvee \bigvee \bigwedge$ has $T(\bigwedge Ax - x) = T(\bigwedge \bigvee \bigvee \bigwedge x - x \leq T(\bigwedge \bigvee x - x)$. However the sequece $y - x$ reverses the inequality. The (right-) quasi-inverse therefore is more elusive than the (left-) quasi-inverse. The example also shows that it does not help going to the 1-norm, as then again
$||U_1 L_1 x - x||_1 = 2$ and $||U_1 x - x|| = 3$, whereas
$||U_1 L_1 y - y||_1 = 3$ and $||U_1 y - y||_1 = 2$.

As we can expect similar problems for more complicated min-max operators, we must abandon the idea in its present form.

6 Comparison with Median Smoothers.

It is well known that the simple and popular three point median smoother M_1, defined by $(M_1 x)_i = \text{ median}\{x_{i-1}, x_i, x_{i+1}\}$ also removes single isolated impulses from data. It is not in $\mathcal{C}$, but we do have that $U_1 L_1 \leq M_1 \leq L_1 U_1$ (1). We may well compare the destructiveness of the median with the other two, and for that we use the following lemma.

Lemma 6.1 $M_1 = U_1 + L_1 - I$

Proof. Let x be any sequence. $(M_1 x)_i = \text{ median}\{x_{i-1}, x_i, x_{i+1}\}, (U_1(x)_i = \min\{\max\{x_{i-1} x_i\}, \max\{x_i, x_{i+1}\}$ and $(L_1 x)_i = \max\{\min\{x_{i-1}, x_i\},$ $\min\{x_i, x_{i+1}\}$ The elements involved can only be ordered in 6 different ways. They are monotone, convex or concave. Investigating the three cases proves the lemma. □

It must be noted that $Q_n = U_n + L_n - I$ form smoothers for each $n \geq 0$, and in fact $L_n U_n \geq Q_n \geq U_n L_n$, but $Q_2 \neq M_2$ and Q_3 is not even a selector. The proofs that follow therefore cannot easily extend to larger n. However we can prove that M_1 is variation reducing. (The author has not picked up any proof before, but is sure there must be one.)

Theorem 6.2 *M_1 is variation reducing or: $T(M_1x) \leq T(x),\ \forall x \in \ell_1$.*

Proof. $T(M_1x) = T(U_1+L_1-I)x \leq T(U_1x)+T((I-L_1)x)$, by the triangle inequality $= T(x) - T(U_1x - x) + T(I - L_1)x$, since U_1 is vp. Similarly $T(M_1x) \leq T(x) - T(L_1 - I)x + T(I - U_1)x$.
Therefore $T(M_1, x) \leq T(x) - |T(I - U_1x) - T(I - L_1)x| \leq T(x)$. □

Although M_1 is Variation reducing, a few examples with pieces of Nyquist frequency suffice to show that the reduction is often very slow compared to L_1U_1 and U_1L_1. They generally are very similar in noise removal, as $L_1U_1 \geq M_1 \geq U_1L_1$.

Of relevance here is the approximation error: $T(M_1x - x)$, compared to those of min-max operators. With the help of the lemma the following is easy, and we can simplify the notation with $M = M_1, L = L_1$ and $U = U_1$.

Theorem 6.3 *If $(Mx)_i \neq (LUx)_i$ then $(Mx)_i = (ULx)_i$, for each $x \in \ell_1$.*

Proof. If $(Mx)_i \neq (LUx)_i$ then $(LUx)_i > (Mx)_i$, since $M \leq LU$.

But then

$$(Ux)_i \geq (LUx)_i > (Mx)_i \text{ and therefore}$$
$$(Ux)_i = \min\{\max\{x_{i-1}, x_i\}, \max\{x_i, x_{i+1}\}\} > (Mx)_i.$$

Since $(Mx)_i = \min\{\max\{x_{i-1}, x_i\},\ \max\{x_{i-1}, x_{i+1}\}, \max\{x_i, x_{i+1}\}\}$, it follows that $(Mx)_i$ can only be
$\max\{x_{i-1}, x_{i+1}\}$, and $x_i > x_{i-1}, x_{i+1}$, since the other maxima are larger. Therefore $(Lx)_i = \max\{\min\{x_{i-1}, x_i\}, \min\{x_i, x_{i+1}\}\} = \max\{x_{i-1}, x_{i+1}\}$.

Since $(Lx)_i = (Mx)_i$, $U(Lx)_i \geq (Mx)_i$ and $UL \leq M$, it follows that $(ULx)_i = (Mx)_i$. □

Using this theorem we can now prove the following

Theorem 6.4 *For each $x \in \ell_1$,*

(a) $T(Mx) \geq T(LUx), T(ULx)$,
(b) $T(Mx - x) \geq T(LUx - x),\ T(ULx - x)$.

Proof. Let $T = T(LU) = \sum_{i=-\infty}^{\infty} |LUx_{i+1} - LUx_i|$ and $J = \{j_\ell\}$ the subsequence of points such that $(Mx)_k \neq (LUx)_k$ for $k \in J$.

Since Mx cannot differ from LUx at consecutive points, $j_{i+1} - j_i > 1$. T can be separated into two sums such that $T = T_d + T_s$, where T_d contains the terms that contain an index from J and T_s the other terms.

Clearly $T_m = T(Mx)$ can similarly be separated into two such sums and T_s is the same in both. However if $k \in J$ there are two terms in the sum containing the index k.

Let $a_k = |LUx_{k+1} - LUx_k| + |LUx_k - LUx_{k-1}|$ be the such a sum. Since LUx is 1-monotone the three consecutive elements are monotone (5), and

$a_k = |LUx_{k+1} - LUx_{k-1}|$. But $a_k \leq |LUx_{k+1} - Mx_k| + |Mx_k - LUx_{k-1}|$ (triangle inequality) and $a_k = |Mx_{k+1} - Mx_k| + |Mx_k - Mx_{k-1}|$, since $Mx_{k+1} = LUx_{k+1}$ and $Mx_{k-1} = LUx_{k-1}$, as $k-1 \notin J$ and $k+1 \notin J$.

Summing over the indexes in J yields:

$T(LUx) \leq \sum_{k \in J} (|Mx_{k+1} - Mx_k| + |Mx_k - Mx_{k-1}|) + T_s = T(Mx)$.

A similar argument for UL completes the proof for (a).

The proof for $I - LU$ is as the proof above, with $I - LU$ replacing LU, up to the inequality involving a_k. Since $(I - LU)x$ is not necessarily 1-monotone the previous argument cannot be used here. However, since $Mx_k < LUx_k$, it follows that $x_k > x_{k-1}, x_{k+1}$, by the argument in the proof of theorem 6.3 above.

It therefore follows that

$$\begin{aligned} \tilde{a_k} &= |(I - LU)x_{k+1} - (I - LU)x_k| + |(I - LU)x_{k-1} - (I - LU)x_k| \\ &\leq |(I - LU)x_{k+1} - (I - M)x_k| + |(I - LU)x_{k-1} - (I - M)x_k| \end{aligned}$$

as $LU \geq M$ and $(I - LU)x_{k+1}, (I - LU)x_{k-1} \leq (I - LU)x_k$, since $I - LU$ is neighbour trend preserving (5). (It preserves the order between consecutive elements).

Thus $T(LUx - x) =$

$\sum_{k \in T} |(L_1U_1 - I)x_{k-1} - (L_1U_1 - I)x_k| + |(L_1U_1 - I)x_{k+1} - (LU = I)x_k| + S,$

where S contains the equivalent terms with no index in J.

The first sum is bounded by

$$\begin{aligned} &\sum_{K \in J} |(I - LU)x_{k-1} - (I - M)x_k| + |(I - LU)x_{k+1} - (I - M)x_k| \\ = &\sum_{K \in J} |(I - M)x_{k-1} - (I - M)x_k| + |(I - M)x_{k+1} - (I - M)x_k|. \end{aligned}$$

In S all the values LUx_i can be replaced by Mx_i, and summing the right-hand side thus yields precisely $T(Mx - x)$. This completes the proof of the theorem. □

It is interesting to note that in, any norm, the triangle inequality yields

$$||Mx - x|| \leq ||Ux - x|| + ||Lx - x||.$$

Comparing M_n, for general n, seems very complicated and hardly worth the effort. Simulations increasingly favour selected equivalent min-max operators with respect to damage done to the signal.

7 Conclusion

In the class of min-max operators each element has a unique simplest, left quasi-inverse with respect to the Total Variation norm in ℓ_1, implying that, for a given smoothing task of mapping onto n-monotone sequences (removing n-impulses), or more selected smoothness classes (5), the idempotent operators are optimal in damage control. Two important research areas concern the questions whether the same results hold for more usual p-norms, and whether any useful results for right-inverses exist.

References

1. C.H. ROHWER and L.M. TOERIEN, Locally Monotone Robust Approximation of Sequences, *Journal of Computational and Applied Mathematics* **36** (1991), 399-408.
2. C.H. ROHWER, Fast Approximation with Locally Monotone Sequences, *Proceedings 4th FAAT 2000 Conference, Maratea, Suppl. Rend. Circ. Mat. Palermo*, Series II numero 68 - anno 2002.
3. C.H. ROHWER, Variation Reduction and $LULU$-smoothing, *Quaestiones Mathematicae* **25** (2002), 163-176.
4. C.H. ROHWER, Multiresolution Analysis with Pulses, *Advanced Problems in Constructive Approximation* (Editors: M.D. Buhmann, D.A. Mache), ISNA Vol 142, 165-186, Birkhäuser Verlag Basel, 2002.
5. C.H. ROHWER and M. WILD, Natural Alternative for One Dimensional Median Filtering, *Quaestiones Mathematicae* **25** (2002), 135-162.
6. J. SERRA, *Image Analysis and Mathematical Morphology,* Acad. Press, London 1982.

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A Tauberian Theorem for Convolution Semigroups

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Abstract. A theorem on the asymptotic behaviour of the n-times convolution $U * \mu^{*n}$ of a probabilty measure μ with a Banach space-valued L^∞-function U is extended by studying the convolution $U * \mu_t$, where $\{\mu_t : t > 0\}$ is a semigroup of nonsingular probability measures. Suppose that either U is uniformly continuous or all μ_t are absolutely continuous, and assume that the mappings $t \mapsto \int_{-n}^{n} U(\cdot - w)d\mu_t(w)$ and $t \mapsto \mu_t([-n, n])$ are measurable for all $n \in \mathbb{N}$. It turns out that the L^∞-valued Cesàro limit $\lim_{r\to\infty} \frac{1}{r}\int_0^r U * \mu_t dt$ exists if and only if the L^∞-limit $\lim_{t\to\infty} U * \mu_t$ exists itself. As a consequence, results for the stability of subordinated semigroups and for equibounded holomorphic C_0-semigroups are obtained.

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1 Introduction

Let μ be a probability measure on the Borel sets of $\mathbb{R}$, that is not singular with respect to the Lebesgue measure λ. For $n \in \mathbb{N}$, the powers μ^{*n} are defined by $\mu^{*1} := \mu$ and $\mu^{*(n+1)}(B) := (\mu * \mu^{*n})(B) = \int_{-\infty}^{\infty} \mu^{*n}(B-u)d\mu(u)$ (B Borel measurable).
On $L^p(\mathbb{R})$, $(1 \le p \le \infty)$, the convolution operator L_μ is defined by

$$(L_\mu f)(v) = (f * \mu)(v) = \int_{-\infty}^{\infty} f(v - r)d\mu(r)$$

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For $p < \infty$, the asymptotic behaviour of its iterates $L_\mu^n f = f * \mu^{*n}$ is well-known; it has been investigated generally for the case that $\mathbb{R}$ is replaced by any locally compact group G (see e.g. [3], [4], [8], [9], [10]).

In [13], the corresponding $\|.\|_\infty$-convergence of $U * \mu^{*n}$ was considered for the case $G = \mathbb{R}$, where $U \in L^\infty(\mathbb{R}, \mathcal{X})$ and $\mathcal{X}$ is a Banach space. In particular, the following theorem was proved; here, I is the identity operator on $L^\infty(\mathbb{R}, \mathcal{X})$, and $\mathcal{N}(I-L_\mu)$ and $\mathcal{R}(I-L_\mu)$ denote the kernel and the range of $I - L_\mu$, respectively. The limits in (i) and (ii) as well as the closures in (iii) and (iv) are taken with respect to $\|.\|_\infty$:

Theorem 1.1 *Let μ be a probability measure that is not absolutely continuous with respect to the Lebesgue measure λ. For $U \in L^\infty(\mathbb{R}, \mathcal{X})$, the following assertions are equivalent:*

(i) $\lim\limits_{n\to\infty} U * \mu^{*n}$ *exists*

(ii) $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=0}^{n} U * \mu^{*k}$ *exists (where $U * \mu^{*0} := U$)*

(iii) $U \in \mathcal{N}(I - L_\mu) \oplus \overline{\mathcal{R}(I - L_\mu)}$

(iv) $U \in \{V \in L^\infty(\mathbb{R}, \mathcal{X}) : V \text{ const.a.e.}\} \oplus \overline{\{V \in L^\infty(\mathbb{R}, \mathcal{X}) : V \text{ int.bdd.}\}}$

Here, a function is called "integral-bounded" (abbreviated by "int.bdd.") if there exists $L \geq 0$ such that, for any $T \in \mathbb{R}$, $\| \int_0^T U(x)dx\| \leq L$.
Moreover, the decomposition sets in (iii) and (iv) coincide, and if U does have the listed properties, then the limits in (i) and (ii) equal U's constant part given by the decomposition in (iii) and (iv).

In particular, the limit is independent of the particular choice of the probability measure μ.

Note that the equivalence $(ii) \Leftrightarrow (iii)$ is an immediate consequence of the mean ergodic theorem.
The implication $(ii) \Rightarrow (i)$ is a Tauberian theorem, and one may wonder, considering measures with continuous parameter under appropriate conditions, if an analoguous implication holds with respect to Cesàro means with continuous parameter: Let $\{\mu_t : t > 0\}$ be a semigroup of probability measures, i.e.

$$\forall s, t > 0 : \ \mu_s * \mu_t = \mu_{s+t}.$$

Examples of such semigroups are those defined by $d\mu_t(v) = f_t(v)dv$, where $(f_t)_t$ is one of the following families of functions; see [5], [2], [15]; however, in what follows, the measures are not necessarily absolutely continuous.

Examples 1.2 *a) For $t > 0$ and fixed $\alpha > 0$, the Gamma densities of order α are defined by* $g_{\alpha,t}(v) := \begin{cases} 0 & \text{if } v \le 0 \\ \frac{1}{\Gamma(t)}\,\alpha^t\, v^{t-1} e^{-\alpha v} & \text{if } v > 0 \end{cases}$.

b) For $t > 0$ and $\alpha \in (0, 2]$, the generalized Weierstraß densities are defined by their Fourier transform

$$\forall v \in \mathbb{R}: \quad \int_{-\infty}^{\infty} e^{-ivs} w_{\alpha,t}(s)\, ds = e^{-t|v|^\alpha}.$$

For $\alpha = 1$ one obtains the Cauchy-Poisson densities and for $\alpha = 2$ the Gauß-Weierstraß densities, i.e.

$$w_{1,t}(v) = \frac{t}{\pi(t^2 + v^2)} \quad \text{and} \quad w_{2,t}(v) = \frac{1}{2\sqrt{\pi t}}\, e^{-v^2/4t}$$

c) Similarly, for $t > 0$ and $\alpha \in (0, 1)$, the Lévy densities are defined on $\mathbb{R}_{>0}$ by their Laplace transform $\forall \mathfrak{Re} z \ge 0: \ \int_0^\infty e^{-zv} f_{\alpha,t}(v) dv = e^{-tz^\alpha}$, and $f_{\alpha,t}(v) := 0$ for $v \le 0$.

The Cesàro limit of the $L^\infty(\mathbb{R}, \mathcal{X})$-valued function $t \mapsto U * \mu_t$ is defined by

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r (U * \mu_t) dt, \qquad (*)$$

provided that the integrands are measurable and that the limit exists; recall that a function is called measurable, if it is the pointwise almost everywhere limit of simple functions.

It is the purpose of this article to investigate if, in a similar manner as in Theorem 1.1, the existence of the Cesàro limit $(*)$ is sufficient for the existence of $\lim\limits_{t\to\infty} U * \mu_t$ (which is equivalent to the existence of $\lim\limits_{n\to\infty} U * \mu_n = \lim\limits_{n\to\infty} U * (\mu_1)^{*n}$). In this case, for such convolution processes, further assertions can be added to the equivalences given by Theorem 1.1.

Suppose that the functions $t \mapsto \mu_t([-n, n])$ and $t \mapsto \int_{-n}^{n} U(\cdot - w) d\mu_t(w)$ are measurable for all $n \in \mathbb{N}$. As a pointwise limit of measurable functions then also $t \mapsto U * \mu_t$ is measurable. It is shown in Theorem 3.1 that the answer of the above problem is affirmative, if $U \in BUC(\mathbb{R}, \mathcal{X})$ (i.e. bounded and uniformly continuous) or if all μ_t are absolutely continuous with respect to the Lebesgue measure. In this case, $\int_{-n}^{n} U(\,\cdot - w) d\mu_t(w) \in BUC(\mathbb{R}, \mathcal{X})$ for all $t > 0$, thus the above L^∞-measurability implies BUC-measurability. This yields the product measurability needed to apply Fubini's theorem in the proof of Theorem 3.1. So far I do not know if the latter assumption

may be omitted.

By applying the results to the special choice that $U(t) = T(|t|)x$ is the extended orbit of an element $x \in \mathcal{X}$ under an equibounded C_0-semigroup $T(t)$ (i.e. $\sup_{t\geq 0} \|T(t)\| < \infty$), we obtain results about the asymptotic behaviour of subordinated semigroups, in particular of equibounded holomorphic semigroups (Theorem 4.2).

2 Preliminaries

Let $\{\mu_t : t > 0\}$ be a semigroup of probability measures on $\mathbb{R}$. Throughout this paper, assume that no μ_t is singular with respect to the Lebesgue measure λ. Let $\mathcal{X}$ be a Banach space, and let $U \in L^\infty(\mathbb{R}, \mathcal{X})$. (Here and in the following, "L^∞" as well as "a.e." refers to the Lebesgue measure.) W.l.o.g. suppose that U is Borel measurable. The convolution $U * \mu_t$ is defined by the Bochner integral

$$(U * \mu_t)(\cdot) = \int_{-\infty}^{\infty} U(\cdot - w) d\mu_t(w).$$

Note that $U * \mu_t \in L^\infty(\mathbb{R}, \mathcal{X})$ with $\|U * \mu_t\|_\infty \leq \|U\|_\infty$. If μ_t is absolutely continuous or if U is bounded and uniformly continuous, then $U * \mu_t \in BUC(\mathbb{R}, \mathcal{X})$.

The first Lemma shows that, in order to observe the asymptotic behaviour of the process $t \mapsto U * \mu_t$, it is sufficient to consider the convolutions with the discrete powers of a member μ_b of the semigroup; the limits are taken with respect to $\|\cdot\|_\infty$.

Lemma 2.1 *If $U \in L^\infty(\mathbb{R}, \mathcal{X})$, then $\lim_{t\to\infty} U * \mu_t$ exists if and only if there are $a \geq 0$, $b > 0$ such that $\lim_{n\to\infty} U * \mu_{a+nb}$ exists.*

Proof. Let $V = \lim_{n\to\infty} U * \mu_{a+bn} = \lim_{n\to\infty} (U * \mu_a) * (\mu_b)^{*n}$. Then V is a.e. constant by Theorem 1.1, and thus $\lim_{n\to\infty} (U - V) * \mu_{a+bn} \equiv 0$. The claim follows as for all $n \in \mathbb{N}$ and $s \geq 0$

$$\big\|\big(U * \mu_{a+bn+s}\big) - V\big\|_\infty = \big\|(U - V) * \mu_{a+bn+s}\big\|_\infty \leq \big\|(U - V) * \mu_{a+bn}\big\|_\infty.$$

3 Asymptotic behaviour of the process $U * \mu_t$

The following theorem contains a Tauberian result saying that the convergence of the considered means already implies the corresponding conver-

gence of the convolution process - provided a measurability condition on the probability measures holds true.
The proof makes use of the characterization of those U for which the discrete Cesàro mean $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{n} U * \mu^{*k}$ exists (see the implication $(ii) \Rightarrow (iv)$ in Theorem 1.1). All limits are taken with respect to $\|\cdot\|_\infty$; recall the definitions given before Theorem 1.1.

Theorem 3.1 *Let $\{\mu_t : t > 0\}$ be a convolution semigroup of probability densities that are not singular with respect to the Lebesgue measure λ. Let $U \in L^\infty(\mathbb{R}, \mathcal{X})$ and suppose that either $U \in BUC(\mathbb{R}, \mathcal{X})$ or all μ_t are absolutely continuous with respect to λ. Assume that for all $n \in \mathbb{N}$, the mappings $t \mapsto \mu_t([-n, n])$ and $t \mapsto \int_{-n}^{n} U(\cdot - w) d\mu_t(w)$ are measurable. Then the following assertions are equivalent:*

(i) $\lim_{t\to\infty} U * \mu_t$ *exists*

(ii) $\lim_{n\to\infty} U * \mu_n$ *exists*

(iii) the Abelian mean $\lim_{\lambda\to 0+} \lambda \int_0^\infty e^{-\lambda t}(U * \mu_t)\, dt$ *exists*

(iv) the Cesàro mean $\lim_{r\to\infty} \frac{1}{r} \int_0^r U * \mu_t\, dt$ *exists*

(v) the discrete Cesàro mean $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} U * \mu_k$ *exists (where, for convenience, $U * \mu_0 := U$)*

(vi) $U \in \{V \in L^\infty(\mathbb{R}, \mathcal{X}) : V \text{ const.a.e.}\} \oplus \overline{\{V \in L^\infty(\mathbb{R}, \mathcal{X}) : V \text{ int.bdd.}\}}$

(vii) $U \in \mathcal{N}(I - L_{\mu_1}) \oplus \overline{\mathcal{R}(I - L_{\mu_1})}$

In case that U has one and hence all of these properties, all limits coincide and equal the constant function obtained by the decomposition of U in (vi) resp. (vii).

Proof. First of all, the Bochner integrals in (iii), (iv) are well-defined since for all $t > 0$, $\lim_{n\to\infty} \int_{-n}^{n} U(\cdot - w) d\mu_t(w) = U * \mu_t$, and thus $t \mapsto U * \mu_t$ is measurable.
The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii)$ are given by Lemma 2.1 and Theorem 1.1. The implication $(i) \Rightarrow (iv)$ is obvious. For the equality of Abelian mean and the Cesàro mean (iv) of bounded functions see [1].
We shall now prove $(iv) \Rightarrow (v)$. Suppose that U has the property (iv) and

let $S : \begin{cases} \mathbb{R}_{>0} \to L^\infty(\mathbb{R}, \mathcal{X}) \\ t \mapsto \sum_{k=0}^{\infty} \chi_{[k,k+1)}(t) U * \mu_k \end{cases}$, where $\chi_{[k,k+1)}$ denotes the characteristic function of the interval $[k, k+1)$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{n+1} \int_0^{n+1} \big(U * \mu_t - S(t)\big) dt &= \frac{1}{n+1} \sum_{k=0}^{n} \int_k^{k+1} \Big(U * \mu_t - U * \mu_k\Big) dt \\ (\bullet) \qquad &= \frac{1}{n+1} \sum_{k=0}^{n} \Big(\int_0^1 U * \mu_t \; dt - U \Big) * \mu_k \end{aligned}$$

We will show that $\int_0^1 U * \mu_t \; dt - U$ can be approximated in $\|.\|_\infty$ by integral bounded functions; by Theorem 1.1 this yields that the mean $(\bullet)$ tends to the nullfunction. Then (v) follows from (iv), since

$$\lim_{n\to\infty} \frac{1}{n+1} \int_0^{n+1} U * \mu_t \; dt = \lim_{n\to\infty} \frac{1}{n+1} \int_0^{n+1} S(t) dt = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} U * \mu_k.$$

In order to prove the mentioned approximability, for $v \in \mathbb{R}$ let

$$\begin{aligned} U_n(v) &:= \int_0^1 \mu_t\big(\mathbb{R}\backslash[-n,n]\big) dt \cdot U(v) \quad \text{and} \\ V_n(v) &:= \int_0^1 \Big[(U * \mu_t)(v) - \int_{\mathbb{R}\backslash[-n,n]} U(v-w) d\mu_t(w) \Big] dt. \end{aligned}$$

By assumption, these functions are well-defined for each $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} \|U_n\|_\infty = 0$ holds by Lebesgue's dominated convergence theorem, while $\lim_{n\to\infty} V_n = \int_0^1 U * \mu_t \; dt$. Therefore,

$$\lim_{n\to\infty} (U_n + V_n - U) = \int_0^1 U * \mu_t \; dt - U \quad \text{with respect to } \|.\|_\infty.$$

Finally, each $U_n + V_n - U$ is integral bounded: Indeed, the measurability of the BUC-valued function $t \mapsto \int_{-n}^{n} U(\cdot - w) d\mu_t(w)$ yields that the functions

$(t,v) \mapsto \int\limits_{\mathbb{R}\setminus[-n,n]} U(v-w)d\mu_t(w)$ and $(t,v) \mapsto (U * \mu_t)(v)$ are measurable. Therefore, by Fubini's Theorem, for any $T \in \mathbb{R}$

$$
\begin{aligned}
&\| \int_0^T (U_n + V_n - U)(v)dv \| \\
&\quad = \Big\| \int_0^T \Big\{ \int_0^1 \int_{\mathbb{R}\setminus[-n,n]} d\mu_t(w)dt \cdot U(v) - \int_0^1 \int_{\mathbb{R}\setminus[-n,n]} U(v-w)d\mu_t(w)dt \Big\} dv \\
&\qquad\qquad + \int_0^T \Big\{ \int_0^1 (\mu_t * U)(v)dt - U(v) \Big\} dv \Big\| \\
&\quad = \Big\| \int_0^1 \int_{\mathbb{R}\setminus[-n,n]} \Big(\int_0^T U(v)dv - \int_0^T U(v-w)dv \Big) d\mu_t(w)dt \\
&\qquad\qquad + \int_0^1 \int_{-\infty}^{\infty} \Big(\int_0^T U(v-w)dv - \int_0^T U(v)dv \Big) d\mu_t(w)dt \Big\| \\
&\quad = \Big\| \int_0^1 \int_{-n}^{n} \Big(\int_{-w}^{T-w} U(s)ds - \int_0^T U(s)ds \Big) d\mu_t(w)dt \Big\| \\
&\quad = \Big\| \int_0^1 \int_{-n}^{n} \Big(\int_{-w}^{0} U(s)ds - \int_{T-w}^{T} U(s)ds \Big) d\mu_t(w)dt \Big\| \\
&\quad \leq 2\|U\|_\infty \int_0^1 \int_{-n}^{n} |w| d\mu_t(w)dt \leq 2\|U\|_\infty n, \qquad \text{independently of } T.
\end{aligned}
$$

This completes the proof. □

Remark 3.2 Let $M(\mathbb{R})$ denote the space of complex Borel measures on $\mathbb{R}$, endowed with the variation norm. Suppose that the mapping $H : t \mapsto \mu_t$ is measurable. Then the equivalences $(i) - (vii)$ of the above Theorem hold for all $U \in BUC(\mathbb{R}, \mathcal{X})$: Indeed, since all linear operators

$$
F_n : \begin{cases} M(\mathbb{R}) \to \mathbb{C} \\ \mu \mapsto \mu([-n,n]) \end{cases}, \qquad G_n : \begin{cases} M(\mathbb{R}) \to M(\mathbb{R}) \\ \mu \mapsto \mu|_{[-n,n]} = \mu(\cdot \cap [-n,n]) \end{cases} \quad \text{and}
$$

$$M_U : \begin{cases} M(\mathbb{R}) \to BUC(\mathbb{R}, \mathcal{X}) \\ \mu \mapsto U * \mu \end{cases} \quad \text{are continuous, the mappings}$$

$$F_n \circ H : t \mapsto \mu_t([-n, n]) \quad \text{and} \quad M_U \circ G_n \circ H : t \mapsto \int_{-n}^{n} U(\cdot - w) d\mu_t(w)$$

are measurable for all $n \in \mathbb{N}$.

Similarly, if each μ_t has a density $f_t \in L^1(\mathbb{R})$, and if $H : t \mapsto f_t$ is measurable, then the conditions $(i) - (vii)$ hold for all $U \in L^\infty(\mathbb{R}, \mathcal{X})$.

Finally, it is easy to see that, if

$$\forall \delta > 0 : \lim_{t \to 0+} \mu_t([-\delta, \delta]) = 1 \qquad (\diamond)$$

then the assertions in Theroem 3.1 are fulfilled for all $U \in BUC(\mathbb{R}, \mathcal{X})$, and for all $U \in L^\infty(\mathbb{R}, \mathcal{X})$ if all μ_t are absolutely continuous. Condition $(\diamond)$ means that the semigroup $\{\mu_t : t \geq 0\}$ (with $\mu_0(B) = 1 :\Leftrightarrow 0 \in B$) is vaguely continuous. Note that this is the case for all convolution semigroups given by Example 1.2: □

However, in general the measurability conditions are not redundant. The following example is due to L. Baringhaus from the University of Hannover:

Example 3.3 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-measurable solution of the functional equality $\varphi(s + t) = \varphi(s) + \varphi(t)$ (see [7, p.116] for its existence). For $t > 0$ define $b_t : \mathbb{R} \to \mathbb{R}_{>0}$ by

$$b_t(v) := \frac{1}{\sqrt{2\pi t}} \, e^{-\frac{(v - \varphi(t))^2}{2t}}$$

and let μ_t be the measure with density b_t. Then $(\mu_t)_t$ is a convolution semigroup. If $U = \chi_{[0,\infty)}$, then the $BUC(\mathbb{R}, \mathcal{X})$-valued mapping $t \mapsto U * \mu_t$ is not measurable: Indeed, suppose it is. Then so is

$$\begin{aligned} S : t \mapsto \int_{-\infty}^{\infty} U(-v) b_t(v) dv &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{0} e^{-\frac{(v - \varphi(t))^2}{2t}} dv \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{\varphi(t)}{\sqrt{t}}} e^{-y^2/2} dy = \Phi(-\frac{\varphi(t)}{\sqrt{t}}), \end{aligned}$$

where Φ is the standard normal distribution function. Since its inversion Φ^{-1} is continuous, the measurability of φ follows, a contradiction.

Since BUC is a closed subspace of L^∞, the mapping is not L^∞-measurable either. □

Example 3.4 Basing on the same idea, by taking a particular φ we see that also the measurability of the functions $t \mapsto \mu_t([-n,n])$ may fail for a convolution semigroup $(\mu_t)_t$: Let B be a basis of $\mathbb{R}$ over $\mathbb{Q}$. Take $a_0 \in B$. It is shown in [7] that by $\varphi(a_0) := 1$ and $\forall a \in B\backslash\{a_0\} : \varphi(a) := 0$, a nonmeasurable solution of the above functional equality is determined. Moreover, it follows from [7, p.102] that

$$\{t > 0 : \varphi(t) = 0\} \text{ is not measurable.} \qquad (\star)$$

With this φ now define b_t as above. Since

$$\int_{-n}^{n} b_t(v)dv = \frac{1}{\sqrt{2\pi}} \int_{\frac{-n-\varphi(t)}{\sqrt{t}}}^{\frac{n-\varphi(t)}{\sqrt{t}}} e^{-y^2/2}dy,$$

the measurability of $t \mapsto \mu_t([-n,n])$ would yield the measurability of

$$\Big\{t > 0 : \int_{\frac{-n-\varphi(t)}{\sqrt{t}}}^{\frac{n-\varphi(t)}{\sqrt{t}}} e^{-y^2/2}dy = \int_{\frac{-n}{\sqrt{t}}}^{\frac{n}{\sqrt{t}}} e^{-y^2/2}dy\Big\} = \{t > 0 : \varphi(t) = 0\}$$

in contradiction to $(\star)$. □

If U is bounded *everywhere* on $\mathbb{R}$, then, by the above proof of Theorem 3.1, $\frac{1}{n+1}\sum_{k=0}^{n}\Big(\int_0^1 U * \mu_t dt - U\Big) * \mu_k$ converges to the nullfunction, uniformly on the whole real axis. Therefore, if

$$\lim_{t\to\infty} \frac{1}{r}\int_0^r (U * \mu_t)(s_0)dt =: c$$

exists for a single value s_0, then

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n}(U * \mu_k)(s_0) = c.$$

Similar to Theorem 3.1, this fact allows that another theorem from [13] can be supplemented by Cesàro limits with continuous parameter. This is done in the following section.

4 An application to equibounded C_0-semigroups

The L^∞-function U is now supposed to be the (uniformly continuous) extended orbit $t \mapsto T(|t|)x$ of an equibounded C_0-semigroup $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(\mathcal{X})$ with generator A, applied to the element $x \in \mathcal{X}$. Then the following holds true; here, B denotes the derivative operator in $L^\infty(\mathbb{R}, \mathcal{X})$ with kernel $\mathcal{N}(B)$ and range $\mathcal{R}(B)$:

Theorem 4.1 *Let $\{\mu_t : t > 0\}$ be a convolution semigroup of nonsingular probability measures. Let $x \in \mathcal{X}$ and denote $T_x(t) := T(|t|)x$. Suppose that the mappings $t \mapsto \mu_t([-n, n])$ and $t \mapsto \int_{-n}^{n} T_x(\cdot - s)d\mu_t(s)$ are measurable for all $n \in \mathbb{N}$. Then the following assertions are equivalent*

(i) $\lim_{t\to\infty} T_x * \mu_t$ *exists in* $L^\infty(\mathbb{R}, \mathcal{X})$

(ii) $\lim_{t\to\infty} \int_{-\infty}^{\infty} T_x(s)d\mu_t(s)$ *exists in* $\mathcal{X}$

(iii) $\lim_{r\to\infty} \frac{1}{r} \int_0^r T_x * \mu_t \, dt$ *exists in* $L^\infty(\mathbb{R}, \mathcal{X})$

(iv) $\lim_{r\to\infty} \frac{1}{r} \int_0^r \int_{-\infty}^{\infty} T_x(s)d\mu_t(s)dt$ *exists in* $\mathcal{X}$

(v) $\lim_{r\to\infty} \frac{1}{r} \int_0^r T(t)x dt$ *exists*

(vi) $x \in \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$

(vii) $T_x \in \mathcal{N}(B) \oplus \overline{\mathcal{R}(B)}$,

and the limits in L^∞ equal the constant function whose value is the limit obtained in (ii) and (iv). The limit operator in (i) (resp. (iii)) is the projection onto $\mathcal{N}(B) = \{V \in L^\infty(\mathbb{R}, \mathcal{X}) : V$ const.a.e.$\}$ along $\overline{\mathcal{R}(B)} = \{V \in L^\infty(\mathbb{R}, \mathcal{X}) : V$ int.bdd.$\}$, and the limit operator in (ii) (resp. (iv), (v)) is the projection onto the kernel of A along the closure of its range.

Proof. The equivalence $(v) \Leftrightarrow (vi)$ is given by the well-known ergodic theorem for equibounded semigroups. Moreover, it is shown in [13] that (vi) holds if and only if $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (T_x * \mu_k)(0)$ exists. Therefore, $(iv) \Leftrightarrow (vi)$ follows by the remark at the end of the last section. The remaining implications are either obvious or given by Theorem 3.1, the representations of $\mathcal{N}(B)$ and $\overline{\mathcal{R}(B)}$ are proved in [13]. □

Note that further equivalences concerning Abelian mean and discrete Cesàro mean can be added to the assertions in Theorem 4.1: see [13].

By Remark 3.2, the measurability assumptions in Theorem 4.1 are fulfilled if $\lim_{t\to 0+} \mu_t([-\delta,\delta]) = 1$ holds for all $\delta > 0$. If furthermore $\mu_t\big((-\infty,0)\big) = 0$ for all $t > 0$, or if $(T(t))$ extends to a C_0-group, then

$$W(t) : x \mapsto \int_{-\infty}^{\infty} T(s)x d\mu_t(s) \quad \text{for } t > 0, \qquad W(0)x = x$$

defines a new C_0-semigroup on $\mathcal{X}$ (which then is called a "subordinated semigroup").
The equivalence $(ii) \Leftrightarrow (v)$ in Theorem 4.1 shows that the asymptotic behaviour of $W(t)$ is related to the ergodic behaviour of $T(t)$. Further evaluation of Theorem 4.1 then yields ergodic results for such subordinated C_0-semigroups of operators, in particular the following theorem for equibounded holomorphic C_0-semigroups; see [13] for details.

Theorem 4.2 *Let $\{T(t) : t \geq 0\}$ be an equibounded holomorphic C_0-semigroup with generator A. Let $x \in \mathcal{X}$.*

a) If there are $a \geq 0, b > 0$ such that $\lim_{n\to\infty} T(a+bn)x =: c \in \mathcal{X}$ exists, then $\lim_{t\to\infty} T(t)x = c$.

b) The following are equivalent

(i) $\lim_{t\to\infty} T(t)x$ exists

(ii) x is ergodic with respect to $T(t)$ (i.e. $\lim_{r\to\infty} \frac{1}{r}\int_0^r T(t)x dt$ exists)

(iii) x is totally ergodic with respect to $T(t)$ (i.e. $\lim_{r\to\infty} \frac{1}{r}\int_0^r e^{-i\eta t}T(t)x dt$ exists for all $\eta \in \mathbb{R}$)

In this case, writing $x = x_1 + x_2$ with $x_1 \in \mathcal{N}(A)$ and $x_2 \in \overline{\mathcal{R}(A)}$, the limits in (i) and (ii) both equal x_1, and $\lim_{r\to\infty} \frac{1}{r}\int_0^r e^{-i\eta t}T(t)x dt = 0$ holds for all $\eta \in \mathbb{R}\backslash\{0\}$.
If $\mathcal{X}$ is reflexive, then the assertions $(i) - (iii)$ hold.

c) $T(t)$ is strongly stable (i.e. $\lim_{t\to\infty} T(t)x = 0$ for all $x \in \mathcal{X}$) if and only if $\overline{\mathcal{R}(A)} = \mathcal{X}$.

d) The orbit $t \mapsto T(t)x$ is either injective or constant.

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References

[1] W. Arendt, C.J.K. Batty, M.Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems,* Monographs in Mathematics **96,** Birkhäuser, 2001

[2] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation I,* Birkhäuser, 1971

[3] Y. Derriennic, M.Lin, Sur le comportement des puissances de convolution d'une probabilité, *Ann. Inst. Henri Poincaré* **20** (1984) No.2, 127-132.

[4] Y. Derriennic and M. Lin, Convergence of iterates of averages of certain operator representations and of convolution powers, *J. Funct. Anal.* **85** (1989), 86-102.

[5] W. Feller, *An Introduction to Probability Theory and its Applications II,* J. Wiley & Sons, Inc., 1966

[6] G. B. Folland, *Real Analysis,* John Wiley and Sons, 2nd. Ed. 1999

[7] H. Hahn and A. Rosenthal, *Set Functions,* University of New Mexico Press, 1948

[8] M. Lin, Ergodic properties of an operator obtained from a continuous representation, *Ann. Inst. Henri Poincaré* **13** (1977), Nr.4, 321-331.

[9] M. Lin and R. Wittmann, Convergence of representation averages and of convolution powers, *Israel J. Math.* **88** (1994), 125-157.

[10] M. Lin and R. Wittmann, Averages of unitary representations and weak mixing of random walks, *Studia Math.* **114** (2) (1995), 127-145.

[11] R.L. Schilling, On the domain of the generator of a subordinated semigroup, *J. Potential Theory-ICPT 94 (Kouty 1994)* deGruyter, Berlin 1996, 449-462.

[12] M. Schmalmack, Zum asymptotischen Verhalten von Faltungshalbgruppen, Dissertation, Hannover 2003, http://edok01.tib.uni-hannover.de/edoks/e01dh03/361122926.pdf

[13] M. Schmalmack, Asymptotic behaviour of convolution semigroups, to appear in *Semigroup Forum*

[14] U. Westphal, A generalized version of the Abelian mean ergodic theorem with rates for semigroup operators and fractional powers of infinitesimal generators, *Results in Mathematics* **34** (1998), 381-394.

[15] K. Yosida, *Functional Analysis,* Grundlehren math. Wiss. **123**, Springer-Verlag, Berlin 1965

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Shift Strategies for Polynomial Expansions

Michael-Ralf Skrzipek

Abstract. Let $v_n \in \Pi_n$ be a polynomial, expanded in terms of p_ν,

$$v_n = \sum_{\nu=0}^{n} a_{\nu,n} p_\nu \ ,$$

where the elements of $\{p_\nu \in \Pi_\nu\}_{\nu\geq 0}$ satisfy a three term recurrence relation. By a k-shift of the Fourier- and recurrence coefficients we obtain polynomials

$$v_{n-m}^{[k]} = \sum_{\nu=0}^{n-m} a_{\nu+k,n} p_\nu^{(k)} \in \Pi_{n-m} \ , \quad 0 \leq k \leq m \leq n \ ,$$

which generalize the associated (orthogonal) polynomials $\{p_\nu^{(k)} \in \Pi_\nu\}_{\nu\geq 0}$ of order $k \geq 0$. We present some properties and applications of them. Especially we introduce a decomposition strategy which can be used in numerical applications and in signal processing.

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1 Introduction

In this article we consider polynomial expansions with shifts. We use some concepts which are known from the theory of Fourier- and wavelet analysis. Thus our results have applications in both theories. For example, in

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theorems 2.1 and 2.2 we derive decompositions which can be used similarly as the 'derive and conquer strategy' known from the FFT algorithm. We obtain them by using recurrence relations and shift strategies, i.e. techniques, known from the wavelet analysis. The basis of our work is the theory of polynomials, satisfying a three term recurrence relation, as orthogonal polynomials do, and the corresponding associated polynomials of order k.

Let $\{p_\nu\}_{\nu\geq 0}$ be a sequence of polynomials, satisfying the recurrence

$$\begin{aligned} p_{-1}(x) &= 0\ , \quad p_0(x) := 1\ , \\ p_{n+2}(x) &= (\alpha_{n+1}x - \beta_{n+1})p_{n+1}(x) - \gamma_{n+1}p_n(x)\ , \qquad n \geq -1\ , \end{aligned} \tag{1.1}$$

with $\alpha_n \neq 0 \neq \gamma_n$ for all $n \geq 0$. Then these polynomials are orthogonal with respect to a quasi-definite linear functional $\mathcal{L}$, defined on the space Π of all algebraic polynomials (cf. [2]). Furthermore, γ_0 can be chosen arbitrarily. For simplicity we choose $\gamma_0 := \mathcal{L}(p_0^2)$. Shifting the recurrence coefficients in (1.1), we obtain a sequence of so called k-associated polynomials $\{p_n^{(k)}\}_{n\geq 0}$, satisfying

$$\begin{aligned} p_{-1}^{(k)}(x) &= 0\ , \quad p_0^{(k)}(x) = 1\ , \\ p_{n+2}^{(k)}(x) &= (\alpha_{n+k+1}x - \beta_{n+k+1})p_{n+1}^{(k)}(x) - \gamma_{n+k+1}p_n^{(k)}(x)\ , \\ &\qquad\qquad n \geq -1, \quad k \in \mathbb{N}_0\ . \end{aligned} \tag{1.2}$$

Obviously we have $p_n^{(0)} = p_n$.

Besides (1.2), associated polynomials satisfy a 'dual' recurrence relation [3, 5, 8]

$$\begin{aligned} p_{-1}^{(k+2)}(x) &= 0, \qquad p_0^{(k+1)}(x) = 1\ , \\ p_n^{(k)}(x) &= (\alpha_k x - \beta_k)p_{n-1}^{(k+1)}(x) - \gamma_{k+1}p_{n-2}^{(k+2)}(x)\ , \\ &\qquad\qquad k \in \mathbb{N}_0\ , \quad n = 0, 1, 2, \ldots \end{aligned} \tag{1.3}$$

Here, polynomials with different polynomial degrees and degree (order) of association are combined. Note, that the recurrence coefficients in this formula are independent of n. Besides the recurrences (1.2), (1.3), an explicit representation is given by

$$p_{n-m}^{(m)} = \frac{1}{\alpha_{m-1}\mathcal{L}(p_{m-1}^2)}\mathcal{L}\left([\cdot, y; p_n]p_{m-1}(y)\right)\ , \quad 1 \leq m \leq n\ , \tag{1.4}$$

where $\mathcal{L}$ is applied on y, and

$$[x, y; f] := \begin{cases} \frac{f(x)-f(y)}{x-y}\ , & x \neq y\ , \\ f'(x)\ , & x = y\ , \end{cases}$$

denotes the first divided difference of a differentiable function f. Especially we have [8]

$$\sum_{k=0}^{n} \alpha_k p_k(x) p_{n-k}^{(k+1)}(y) = [x, y; p_{n+1}] \ , \quad n \in \mathbb{N}_0 \ . \tag{1.5}$$

This formula can be used to prove (1.4).

Let v_n be a polynomial of degree n,

$$v_n = \sum_{\nu=0}^{n} a_{\nu,n} p_\nu \ . \tag{1.6}$$

Then we define the (normalized) *(generalized) associated polynomial* $v_{n-m}^{[m]}$ *of order* m by

$$\begin{aligned} v_{n-m}^{[m]} \quad := \quad & \frac{1}{\alpha_{m-1} \mathcal{L}(p_{m-1}^2)} \mathcal{L}\left(\frac{v_n(\cdot) - v_n(y)}{\cdot - y} p_{m-1}(y) \right), \\ & 1 \le m \le n \ . \end{aligned} \tag{1.7}$$

That the right side is a polynomial of degree $n - m$ can be easily shown by using (1.6) and the orthogonality of the p_ν with respect to $\mathcal{L}$. More precisely, it can be shown that

$$v_{n-m}^{[m]} \quad = \quad \sum_{\nu=0}^{n-m} a_{\nu+m,n} p_\nu^{(m)} \ , \quad 0 \le m \le n \ . \tag{1.8}$$

Obviously there hold $p_k^{(m)} = p_k^{[m]} \ , \quad 0 \le m \le n$. From (1.8) we define a little bit more general

$$v_{n-m}^{[k]} = \sum_{\nu=0}^{n-m} a_{\nu+k,n} p_\nu^{[k]} \ , \quad 0 \le k \le m \le n \ .$$

In particular, $v_{n-m}^{[0]} = v_{n-m}$ is the $(n-m)$th partial sum of v_n in (1.6). Because n is fixed and v_n is a given polynomial, there is no confusion in this notation.

Usually, evaluation of v_n, given as in (1.6), for a fixed point x is done by the Clenshaw algorithm. With $a_{n+1}^{\{1\}} := a_{n+2}^{\{1\}} := 0$ we define

$$\begin{aligned} a_{n-\nu}^{\{1\}} \quad := \quad & a_{n-\nu,n} + (\alpha_{n-\nu} x - \beta_{n-\nu}) a_{n-\nu+1}^{\{1\}} - \gamma_{n-\nu+1} a_{n-\nu+2}^{\{1\}} \\ & \text{for} \quad \nu = 0, \ldots, n \ , \end{aligned} \tag{1.9}$$

and obtain $a_0^{\{1\}} = v_n(x)$. If we allow x to be not only fixed but also variable, the $a_{n-\nu}^{\{1\}}$ are polynomials. Comparing the recurrences (1.9) and (2.4) from below we see that $a_m^{\{1\}} = v_{n-m}^{[m]},\ m = 0, \ldots, n$.

As a generalization of (1.5) we get

$$\begin{aligned}
[x, y; v_n] &= \sum_{\nu=0}^{n} a_{\nu,n}[x, y; p_\nu] \\
&= \sum_{\nu=0}^{n} a_{\nu,n} \sum_{\mu=0}^{\nu-1} \alpha_\nu p_\nu(x) p_{\nu-\mu-1}^{(\mu+1)}(y) \\
&= \sum_{\mu=0}^{n-1} \alpha_\mu p_\mu(x) \left(\sum_{\nu=\mu+1}^{n} a_{\nu,n} p_{\nu-1-\mu}^{(\mu+1)}(y) \right) \\
&= \sum_{\mu=0}^{n-1} \alpha_\mu p_\mu(x) \left(\sum_{\nu=0}^{n-(\mu+1)} a_{\nu+\mu+1,n} p_\nu^{(\mu+1)}(y) \right) \\
&= \sum_{\mu=0}^{n-1} \alpha_\mu p_\mu(x) v_{n-(\mu+1)}^{[\mu+1]}(y) \ . \qquad (1.10)
\end{aligned}$$

It is remarkable that the sum is symmetric with respect to x and y because $[x, y; v_n]$ has this property.

Since $v_{n-(\mu+1)}^{[\mu+1]} = a_{\mu+1}^{\{1\}},\ 0 \le \mu \le n-1$, we get $v_n'(x)$ after evaluating $v_n(x)$ by using (1.9) and then restarting the Clenshaw algorithm for evaluating the polynomial $\sum_{\mu=0}^{n-1} a_{\mu,n}^{(1)} p_\mu(x)$ at x, where $a_{\mu,n}^{(1)} := \alpha_\mu a_{\mu+1}^{\{1\}}(x)$ for this fixed x. Repeating this proceeding we get a 'complete' Clenshaw algorithm for evaluating $\frac{1}{k!}\frac{d^k}{dx^k} v_n(x),\ k = 0, \ldots, n$, as we have done in a similar manner in [7].

2 A Generalized Recurrence Formula and a Decomposition of a Polynomial

For the (inverse) FFT-algorithm, polynomials

$$\left(c_\mu^{(N+1)}(f) \right)(x) := \sum_{\lambda=0}^{N} f_\lambda x^\lambda \ , \quad \mu = 0, \ldots, N \ ,$$

are evaluated at $x = \exp\left(\frac{2\pi \mathrm{i}}{N+1}\right)^\mu$ where $f = \{f_\lambda\}_{\lambda \in \mathbb{Z}}$ denotes an $(N+1)$-periodic sequence. This can be done by a recursive process for which these

polynomials are decomposed in a first step recursively

$$\begin{aligned}\left(c_\mu^{(N+1)}(f)\right)(x) &= [f_0 + f_2x^2 + f_4x^4 + \ldots] + x[f_1 + f_3x^2 + f_5x^4 + \ldots] \\ &= \left([f_0 + f_4x^4 + f_8x^8 + \ldots] + x^2[f_2 + f_6x^4 + f_{10}x^8 + \ldots]\right) \\ &+x\left([f_1 + f_5x^4 + f_9x^8 + \ldots] + x^2[f_3 + f_7x^4 + f_{11}x^8 + \ldots]\right) \\ &= \ldots\end{aligned}$$

In each decomposition step, we have a separation in sums of products of monomials and polynomials which contain some of the f_λ.

For polynomials $p_{\nu+s}^{(m)}$ such a decomposition is given by a generalization of the three term recurrence relations (1.2) and (1.3) which is given by

$$p_{\nu+s}^{(m)} = p_\nu^{(s+m)} p_s^{(m)} - \gamma_{s+m} p_{\nu-1}^{(s+m+1)} p_{s-1}^{(m)} \ , \quad m, \nu, s \in \mathbb{N}_0 \ , \tag{2.1}$$

(cf. [1, Lemma 2.5 for the monic case]).

In the next theorem we formulate an analogue relation for $v_{n-m}^{[m]}$. We see that $p_s^{(m)}$ and $\gamma_{s+m}p_{s-1}^{(m)}$ correspond to the monomials in case of monomial expansions, and $v_{n-m-s}^{[s+m]}$, $v_{n-(m+s+1)}^{[m+s+1]}$ to the parts which contain the Fourier coefficients. For a given s this is a decomposition of the high frequency part $v_{n-m}^{[m]} - v_{s-1}^{[m]}$ of $v_{n-m}^{[m]}$. But we can restart this decomposition with $v_{s-1}^{[m]}$ to get a complete decomposition of v_n which we give in (2.5).

Theorem 2.1 *For $0 \le s \le n-m-1$ we have*

$$v_{n-m}^{[m]} = p_s^{(m)} v_{n-m-s}^{[s+m]} - \gamma_{s+m} p_{s-1}^{(m)} v_{n-(m+s+1)}^{[m+s+1]} + v_{s-1}^{[m]} \ , \tag{2.2}$$

where $v_{-1}^{[m]} := 0$.

Proof. Using (2.1) we get

$$\begin{aligned} v_{n-m}^{[m]} &= v_{s-1}^{[m]} + \sum_{\nu=s}^{n-m} a_{\nu+m,n} p_\nu^{(m)} \\ &= v_{s-1}^{[m]} + \sum_{\nu=0}^{n-m-s} a_{\nu+m+s,n} p_{\nu+s}^{(m)} \\ &= v_{s-1}^{[m]} + \sum_{\nu=0}^{n-m-s} a_{\nu+m+s,n} \left[p_\nu^{(s+m)} p_s^{(m)} - \gamma_{s+m} p_{\nu-1}^{(s+m+1)} p_{s-1}^{(m)}\right] \\ &= v_{s-1}^{[m]} + p_s^{(m)} v_{n-m-s}^{[s+m]} - \gamma_{s+m} p_{s-1}^{(m)} v_{n-(m+s+1)}^{[m+s+1]} \ , \\ &\qquad 0 \le s \le n-m-1 \ . \end{aligned}$$

□

For $m = 0$ we have

$$v_n \;=\; p_s v_{n-s}^{[s]} - \gamma_s p_{s-1} v_{n-(s+1)}^{[s+1]} + v_{s-1} \;, \tag{2.3}$$

Because $v_n - v_{s-1} = \sum_{\nu=s}^{n} a_\nu p_\nu$ we have an alternative representation for the 'high frequency part' of v_n:

$$v_n - v_{s-1} \;=\; p_s v_{n-s}^{[s]} - \gamma_s p_{s-1} v_{n-(s+1)}^{[s+1]} \;, \tag{2.4}$$

From a numerical point of view, this representation is more suitable than the classical Fourier expansion. If, for example, v_n is a partial sum of the expansion of a function v and n a large number, then the $|a_\nu|$ are usually small for $\nu \geq s$ where s is sufficiently large. Furthermore the corresponding p_ν oscillate with high frequencies. These things make the polynomial sensitive against noising and it is difficult to evaluate it in a numerical stable manner. Depending on the chosen s in (2.4) there appear only the 'lower degree' polynomials $v_{n-s}^{[s]}$, $v_{n-(s+1)}^{[s+1]}$ which are less oscillating.

The factor $p_s v_{n-s}^{[s]}$ can be interpreted as follows: The p_s are basic oscillations which are independent of v_n and the $v_{n-s}^{[s]}$ contain information about the examined function v_n. The same is valid for $p_{s-1} v_{n-(s+1)}^{[s+1]}$.

When we evaluate $v_{n-s}^{[s]}(x)$ by the Clenshaw algorithm we calculate $v_0^{[n-s]}(x), \ldots, v_{n-s-1}^{[s+1]}(x)$, $v_{n-s}^{[s]}(x)$, too, cf. (1.9). Thus, no additional work is necessary to get $v_{n-s-1}^{[s+1]}(x) = a_1^{\{1\}}(x)$.

If $n-s$ is too large then $v_n - v_{n-s}$ can be decomposed recursively. From this we get a complete decomposition of the high frequency part of v_n or of v_n itself.

Theorem 2.2 *Let v_n be given as in (1.6) with $v_{-1} := 0$ and $\{s_\nu\}_{\nu=0,\ldots,k+1}$ be a sequence of integers, $0 \leq s_1 < s_1 < \ldots < s_k < n =: s_{k+1} - 1$. Then v_n has the representation*

$$v_n \;=\; v_{s_1-1} + \sum_{\mu=1}^{k} \left(p_{s_\mu} v_{s_{\mu+1}-s_\mu-1}^{[s_\mu]} - \gamma_{s_\mu} p_{s_\mu-1} v_{s_{\mu+1}-s_\mu-2}^{[s_\mu+1]} \right) \;. \tag{2.5}$$

Proof. We start with a decomposition of $v_n - v_{s_1-1}$. With s_2, $s_1 < s_2 < n$, we write

$$v_n = v_{s_2-1} + \sum_{\nu=s_2}^{n} a_{\nu,n} p_\nu \;.$$

From (2.3) we have

$$\sum_{\nu=s_2}^{n} a_{\nu,n}p_\nu = v_{n-s_2}^{[s_2]}p_{s_2} - \gamma_{s_2}v_{n-(s_2+1)}^{[s_2+1]}p_{s_2-1}$$

which describes the high frequency part $v_n - v_{s_2-1}$ and

$$v_{s_2-1} = v_{s_1-1} + \sum_{\nu=s_1}^{s_2-1} a_{\nu,n}p_\nu$$

is the low frequency part of $v_n - v_{s_1-1}$, where, as in (2.3),

$$\sum_{\nu=s_1}^{s_2-1} a_{\nu,n}p_\nu = p_{s_1}v_{s_2-s_1-1}^{[s_1]} - \gamma_{s_1}p_{s_1-1}v_{s_2-s_1-2}^{[s_1+1]} \ .$$

Analogously we can repeat this proceeding for a given sequence $0 \le s_1 < s_2 < \ldots < s_k < n =: s_{k+1} - 1$ and obtain the decomposition

$$\begin{aligned} v_n &= v_{s_1-1} + \sum_{\mu=1}^{k-1}\sum_{\nu=s_\mu}^{s_{\mu+1}-1} a_{\nu,n}p_\nu + \sum_{\nu=s_k}^{n} a_{\nu,n}p_\nu \\ &= v_{s_1-1} + \sum_{\mu=1}^{k-1}\left(p_{s_\mu}v_{s_{\mu+1}-s_\mu-1}^{[s_\mu]} - \gamma_{s_\mu}p_{s_\mu-1}v_{s_{\mu+1}-s_\mu-2}^{[s_\mu+1]}\right) \\ &\quad + \left(v_{n-s_k}^{[s_k]}p_{s_k} - \gamma_{s_k}v_{n-(s_k+1)}^{[s_k+1]}p_{s_k-1}\right) \\ &= v_{s_1-1} + \sum_{\mu=1}^{k}\left(p_{s_\mu}v_{s_{\mu+1}-s_\mu-1}^{[s_\mu]} - \gamma_{s_\mu}p_{s_\mu-1}v_{s_{\mu+1}-s_\mu-2}^{[s_\mu+1]}\right) \ . \end{aligned}$$

□

The sum on the right side is a decomposition of $v_n - v_{s_1-1}$ with respect to a given scale $\{s_1, \ldots, s_{k+1}\}$ resp. fineness $\Delta_\nu := s_{\nu+1} - s_\nu,\ \nu = 1, \ldots, k$. If $s_1 = 0$ then we have a complete decomposition of v_n. Otherwise we have a decomposition of the high frequency part of v_n.

Representation (2.4) is suited for 'divide and conquer' strategies. If we choose e. g. $s = \lfloor n/2 \rfloor$ then we can refine the decomposition for the low frequency part $v_{n-\lfloor n/2 \rfloor}$ or for $v_{n-\lfloor n/2 \rfloor}^{[\lfloor n/2 \rfloor]}$ to get insights in what happens for the high frequency part ('projected' on $\Pi_{n-\lfloor n/2 \rfloor}$). (2.5) is suited for decompositions for which we have an a priori given sequence $\{s_\nu\}_{\nu=1,\ldots,k+1}$ of separating frequencies.

Example 2.3 *We consider*

$$v_{2n}(x) := \frac{4}{\pi}\left[\frac{1}{2} + \sum_{\nu=1}^{n} \frac{(-1)^{\nu+1}}{4\nu^2 - 1} T_{2\nu}(x)\right] ,$$

the $(2n)$*th partial sum of* $|x|$, $x \in [-1, 1]$. *For* $n = 16$ *it looks visually in* $[-1, 1]$ *like* $|x|$ *except that in* 0 *the peak is blunted. A possible sequence of decomposed polynomials is given by* $v_{32} \to v_{16}^{[16]} \to v_8^{[24]} \to v_4^{[28]} \to v_2^{[30]} \to v_1^{[31]}$.

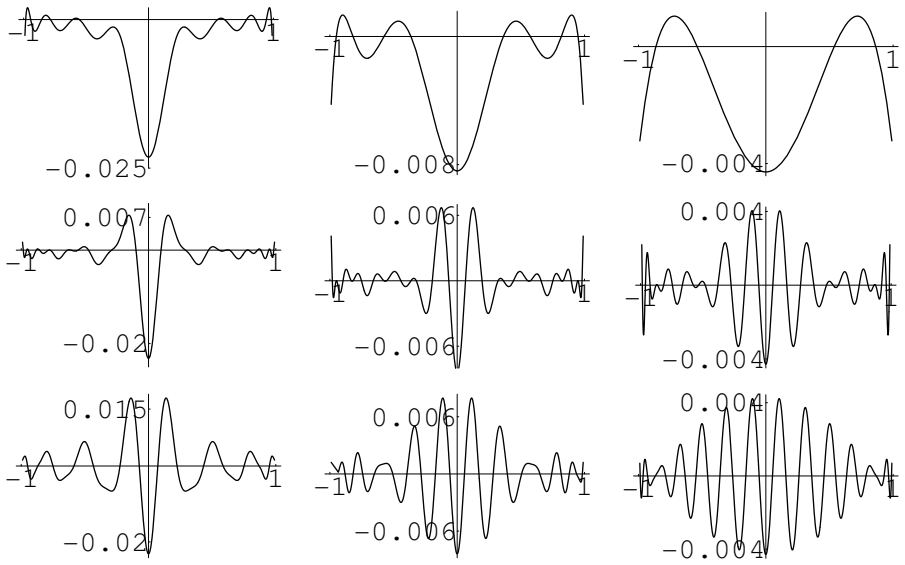


Figure 1: A decomposition of v_{32} for $s = 16, 24, 28$.

In figure 1 we have (from left to right) in the first line $v_{16}^{[16]}$, $v_8^{[24]}$, $v_4^{[28]}$, *in the second line* $T_{16}v_{16}^{[16]}$, $T_{24}v_8^{[24]}$, $T_{28}v_4^{[28]}$, *and in the third line some high frequency parts of* v_{32}, *namely* $v_{32} - v_{15}$, $v_{32} - v_{23}$, $v_{32} - v_{27}$. *We see that the peak of* v_{32} *is recognized by the associated polynomials in the first line although there are only low degree polynomials. In the second line we use the approximations* $T_s v_{32-s}^{[s]} \in \Pi_{32}$ *from (2.4) of* $v_{32} - v_{s-1}$, *given in the third line. Here, due to high oscillations, the peak at zero is only recognized if* s *is sufficient small. The same hold for the high frequency parts of* v_{32}, *given in the third line.*

Example 2.4 *Consider the* $(2n-1)$*th partial sum of* $\operatorname{sign}(x)$ *in* $[-1,1]$,

$$v_{2n-1}(x) = \frac{4}{\pi}\sum_{\nu=1}^{n}\frac{(-1)^{\nu-1}}{2\nu-1}T_{2\nu-1}(x)\ .$$

For $n = 32$ *some associated polynomials for varying* s *are shown in figure 2. We see that the jump in zero of the* sign *function will be recognized by a*

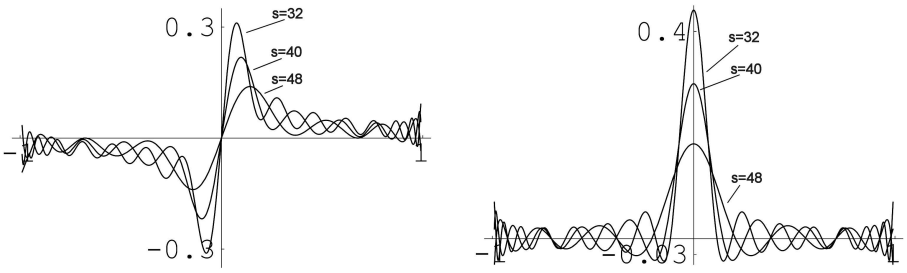


Figure 2: Graphs of $v_{63-s}^{[s]}$ (left) and $v_{63-s-1}^{[s+1]}$ (right) for $s = 32, 40, 48$.

higher amplitude near zero of the corresponding associated polynomials for increasing polynomial degrees $n - s$.

3 Association with Respect to Different Moment Functionals

Until now we have assumed that $v_n = v_n(\cdot;\mathcal{L})$ is expanded in terms of polynomials which are orthogonal with respect to the moment functional $\mathcal{L}$ with respect to which we associate. Now, let $\{\tilde{p}_\nu\}_{\nu\geq 0}$ be a sequence of polynomials, orthogonal with respect to a moment functional $\tilde{\mathcal{L}}$ and satisfying a recurrence relation as in (1.1) but with recurrence coefficients $\tilde{\alpha}_\nu,\ \tilde{\beta}_\nu,\ \tilde{\gamma}_\nu$, with $\tilde{\alpha}_\nu \neq 0 \neq \tilde{\gamma}_\nu$ for all $\nu \geq 0$. The mth associated polynomial $v_{n-m}^{[m]}(x;\tilde{\mathcal{L}}),\ 1 \leq m \leq n$, of v_n with respect to $\tilde{\mathcal{L}}$ is given by

$$v_{n-m}^{[m]}(x;\tilde{\mathcal{L}}) \ := \ \frac{1}{\tilde{\alpha}_{m-1}\tilde{\mathcal{L}}(\tilde{p}_{m-1}^2)}\tilde{\mathcal{L}}([x,y;v_n]\tilde{p}_{m-1}(y))\ , \qquad (3.1)$$

where $\tilde{\mathcal{L}}$ works on y. In general we have $v_{n-m}^{[m]}(\cdot;\tilde{\mathcal{L}}) \neq v_{n-m}^{[m]}(\cdot;\mathcal{L})$ for $\tilde{\mathcal{L}} \neq \mathcal{L}$. If v_n is given as in (1.6) we have for $m \geq 1$

$$\begin{aligned}
\tilde{\mathcal{L}}([x,y;v_n]\tilde{p}_{m-1}(y)) &= \sum_{\nu=0}^{n} a_{\nu,n}\tilde{\mathcal{L}}([x,y;p_\nu]\tilde{p}_{m-1}(y)) \\
&= \sum_{\nu=0}^{n} a_{\nu,n}\tilde{\mathcal{L}}\left(\sum_{\mu=0}^{\nu-1}\alpha_\mu p_\mu(y)p_{\nu-\mu-1}^{(\mu+1)}(x)\tilde{p}_{m-1}(y)\right) \qquad (3.2)\\
&= \sum_{\nu=0}^{n} a_{\nu,n}\sum_{\mu=0}^{\nu-1}\alpha_\mu p_{\nu-\mu-1}^{(\mu+1)}(x)s_{\mu,m-1} \ ,
\end{aligned}$$

where $s_{\mu,m-1} := \tilde{\mathcal{L}}(p_\mu\tilde{p}_{m-1})$ and $p_{\nu-\mu-1}^{(\mu+1)} = p_{\nu-\mu-1}^{(\mu+1)}(\cdot;\mathcal{L})$. Because $s_{\mu,m-1} = 0$ for $\mu < m-1$ we can start the summation on μ beginning with $\mu = m-1$. Since $\nu \geq \mu \geq m-1$, the sum on ν requires only $\nu \geq m$ and we get

$$\begin{aligned}
\tilde{\mathcal{L}}([x,y;v_n]\tilde{p}_{m-1}(y)) &= \sum_{\nu=m}^{n} a_{\nu,n}\sum_{\mu=m-1}^{\nu-1}\alpha_\mu p_{\nu-\mu-1}^{(\mu+1)}(x)s_{\mu,m-1} \\
&= \sum_{\nu=0}^{n-m} a_{\nu+m,n}\sum_{\mu=0}^{\nu}\alpha_{\mu+m-1}p_{\nu-\mu}^{(\mu+m)}(x)s_{\mu+m-1,m-1} \\
&= \sum_{\mu=0}^{n-m}\alpha_{\mu+m-1}s_{\mu+m-1,m-1}\sum_{\nu=\mu}^{n-m} a_{\nu+m,n}p_{\nu-\mu}^{(\mu+m)}(x) \ .
\end{aligned}$$

From (1.8) we have $\sum_{\nu=0}^{n-m-\mu} a_{\nu+\mu+m,n}p_\nu^{(\mu+m)}(x) = v_{n-(m+\mu)}^{[\mu+m]}(x;\mathcal{L})$ and we get

$$\begin{aligned}
v_{n-m}^{[m]}(x;\tilde{\mathcal{L}}) &= \frac{1}{\tilde{\alpha}_{m-1}\tilde{\mathcal{L}}(\tilde{p}_{m-1}^2)}\sum_{\mu=0}^{n}\alpha_{\mu+m-1}s_{\mu+m-1,m-1}v_{n-(m+\mu)}^{[\mu+m]}(x;\mathcal{L}) \\
&= \frac{1}{\tilde{\alpha}_{m-1}\tilde{\mathcal{L}}(\tilde{p}_{m-1}^2)}\sum_{\mu=m}^{n+m}\alpha_{\mu-1}s_{\mu-1,m-1}v_{n-\mu}^{[\mu]}(x;\mathcal{L}) \ . \qquad (3.3)
\end{aligned}$$

Theorem 3.1 *Let $\{\tilde{p}_\nu\}_{\nu\in\mathbb{N}_0}$ be a sequence of polynomials, orthogonal with respect to a moment functional $\tilde{\mathcal{L}}$, satisfying a recurrence relation as in (1.1) but with recurrence coefficients $\tilde{\alpha}_\nu$, $\tilde{\beta}_\nu$, $\tilde{\gamma}_\nu$, where $\tilde{\alpha}_\nu \neq 0 \neq \tilde{\gamma}_\nu$ for all $\nu \geq 0$. We assume that v_n is given as in (1.6). Then*

$$v_{n-m}^{[m]}(x;\tilde{\mathcal{L}}) = \frac{1}{\tilde{\alpha}_{m-1}\tilde{\mathcal{L}}(\tilde{p}_{m-1}^2)}\sum_{\mu=0}^{n-m}\alpha_\mu\left(\sum_{\nu=\mu}^{n-m} a_{\nu+m,n}r_{\nu-\mu-1+m,m-1}^{(\mu+1)}\right)p_\mu(x) \ , \qquad (3.4)$$

where $r_{i,k}^{(j)} := \tilde{\mathcal{L}}(p_i^{(j)}\tilde{p}_k)$, $p_i^{(j)} = p_i^{(j)}(\cdot;\mathcal{L})$ *and* $n \geq m \geq 1$.

Proof. By the symmetry of $[x, y; p_\nu]$ with respect to x and y we see as in (3.2)

$$\begin{aligned}\tilde{\mathcal{L}}([x,y;v_n]\tilde{p}_{m-1}(y)) &= \sum_{\nu=0}^{n} a_{\nu,n}\tilde{\mathcal{L}}\left(\sum_{\mu=0}^{\nu-1}\alpha_\mu p_\mu(x)p_{\nu-\mu-1}^{(\mu+1)}(y)\tilde{p}_{m-1}(y)\right)\\ &= \sum_{\nu=0}^{n} a_{\nu,n}\sum_{\mu=0}^{\nu-1}\alpha_\mu p_\mu(x) r_{n-(\mu+1),m-1}^{(\mu+1)} \ .\end{aligned}$$

Because $r_{\nu-(\mu+1),m-1}^{(\mu+1)} = 0$ for $\nu - \mu < m$ and $m \geq 1$ we have

$$\begin{aligned}\tilde{\mathcal{L}}([x,y;v_n]\tilde{p}_{m-1}(y)) &= \sum_{\nu=0}^{n}\sum_{\mu=0}^{\nu-m}\alpha_\mu p_\mu(x)a_{\nu,n}r_{\nu-(\mu+1),m-1}^{(\mu+1)}\\ &= \sum_{\mu=0}^{n-m}\alpha_\mu p_\mu(x)\sum_{\nu=\mu+m}^{n} a_{\nu,n}r_{\nu-(\mu+1),m-1}^{(\mu+1)}\\ &= \sum_{\mu=0}^{n-m}\alpha_\mu\left(\sum_{\nu=\mu}^{n-m} a_{\nu+m,n}r_{\nu-\mu-1+m,m-1}^{(\mu+1)}\right)p_\mu(x) \ .\end{aligned}$$

□

We remark that the modified moments $s_{\mu,\nu}$ resp. $r_{\mu-\nu,k}^{(\nu)}$ are independent of v_n and that we can calculate them recursively (cf. [4]). Furthermore, by using (1.10), we see immediately that

$$\tilde{\mathcal{L}}(v_{n-(\mu+1)}^{[\mu+1]}(\cdot;\mathcal{L})\tilde{p}_{m-1}) = \sum_{\nu=\mu}^{n-m} a_{\nu+m,n}r_{\nu-\mu-1+m,m-1}^{(\mu+1)} \ .$$

References

[1] S. Belmehdi, *On the associated polynomials*, J. Comput. Appl. Math. **32** (1990), 311–319.

[2] T. S. Chihara, An Introduction to Orthogonal Polynomials, *Gordon and Breach*, N. Y., London, Paris, 1978.

[3] D. J. Dickinson, *On certain polynomials associated with orthogonal polynomials,* Boll. del. Unione Math. Ital. **13** (1958), 116–124.

[4] W. Gautschi, *On the construction of Gaussian quadrature rules from modified moments*, Math. Comp. **24** (1970), 245 – 260.

[5] G. Palamà, *Polinomi più generali di altri classici e dei loro associati, e relazioni tra essi. Funzioni di seconda specie*, Riv. Mat. Univ. Parma **4** (1953), 363–386.

[6] H. E. Salzer, *A recurrence scheme for converting from one orthogonal expansion into another*, Comm. ACM **16** (1973), 705–707.

[7] M.-R. Skrzipek, *Polynomial evaluation and associated polynomials*, Numer. Math. **79** (1998), 601–613.

[8] W. Van Assche, *Orthogonal polynomials, associated polynomials and functions of the second kind*, J. Comput. Appl. Math. **37** (1991), 237–249.

[9] J. Wimp, Computation with Recurrence Relations, *Pitman Advanced Publishing Program*, Boston, 1984.

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Some Results on the Structure of Partial *-Algebras

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Abstract. After reviewing the main facts of the theory of locally convex partial *-algebras, the structure of partial *-algebras in terms of certain C*-seminorms is analyzed. The outcome is some results on the continuity of bounded *-representations of a Banach partial *-algebra.

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1 Introduction

Several results on the theory of abstract partial *-algebras and their operator realizations have been obtained since when, more than two decades ago, they were introduced by Antoine and Karwowski [2, 3] (for a complete account of the theory we refer to the monograph [5]). However the interplay between the algebraic structure of a partial *-algebra and its topological structure has begun to be investigated only recently [1, 5]. Lately, Antoine and the present author [7] have undertaken a study on Banach partial *-algebras considering as a first point the problem of giving an appropriate definition of this object in order to make the topological structure to fit as much a possible with the algebraic one. A preliminary study of certain structure properties of a Banach partial *-algebra has also been given in that paper.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In this paper we continue our study of Banach partial *-algebras, focusing our attention on the continuity of bounded *-representations. A we shall see, a key role in this investigation is played by certain *unbounded C*-seminorms* (roughly speaking a seminorm p enjoying the C*-property $p(a^*a) = p(a)^2$ whenever the product a^*a is well-defined). As shown in [8, 9, 10], they reveal to be crucial for the understanding of the structure of a Banach partial *-algebra.

After reviewing in Section 2 the basic definitions and facts of the theory of partial *-algebras (algebraic setting, representations, GNS construction, locally convex and Banach partial *-algebras), we give in Section 3 some new results concerning the structure of a partial *-algebra $\mathfrak{A}$. For doing this we introduce some unbounded C*-seminorms on $\mathfrak{A}$ related, on one hand, to the sesquilinear forms that allow a GNS construction for $\mathfrak{A}$ (*biweights*) and, on the other hand, to the family of bounded representations of $\mathfrak{A}$. Comparisons of these unbounded C*-seminorms give (as it happens for the corresponding objects in the theory of Banach *-algebras) a deeper insight on the structure of $\mathfrak{A}$. The main results are Theorem 3.8, which essentially states that, under quite reasonable assumption, all the unbounded C*-seminorms introduced before coincide, and Corollary 3.10, where a sufficient condition for the continuity of a (regular) bounded *-representation of $\mathfrak{A}$ is given.

2 A survey on the theory of partial *-algebras

In order to keep this paper sufficiently self-contained, we give in this Section a quick overview of the basic definitions and facts on partial *-algebras from the elementary algebraic aspects up to the definition of Banach partial *-algebra. For more details we refer to [5, 7].

2.1 The algebraic structure

A *partial *-algebra* is a complex vector space $\mathfrak{A}$, endowed with an involution $x \mapsto x^*$ (that is, a bijection such that $x^{**} = x$) and a partial multiplication defined by a set $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$, $\forall\, \lambda, \mu \in \mathbb{C}$;

(iii) for any $(x, y) \in \Gamma$, there is defined a product $xy \in \mathfrak{A}$, which is distributive w.r. to the addition and satisfies the relation $(xy)^* = y^*x^*$.

Notice that the partial multiplication is *not* required to be associative (and often it is not). We shall assume the partial *-algebra $\mathfrak{A}$ contains a unit e,

i.e., $e^* = e$, $(e, x) \in \Gamma$, $\forall\, x \in \mathfrak{A}$, and $ex = xe = x$, $\forall\, x \in \mathfrak{A}$. (If $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial *-algebra with unit, in the standard fashion [6].)

Given the defining set Γ, spaces of multipliers are defined in the obvious way:

$$\begin{aligned}(x, y) \in \Gamma \quad &\Leftrightarrow \quad x \in L(y) \text{ or } x \text{ is a left multiplier of } y \\ &\Leftrightarrow \quad y \in R(x) \text{ or } y \text{ is a right multiplier of } x.\end{aligned}$$

For any subset $\mathfrak{N} \subset \mathfrak{A}$, we write

$$L\mathfrak{N} = \bigcap_{x \in \mathfrak{N}} L(x), \quad R\mathfrak{N} = \bigcap_{x \in \mathfrak{N}} R(x),$$

and, of course, the involution exchanges the two:

$$(L\mathfrak{N})^* = R\mathfrak{N}^*, \quad (R\mathfrak{N})^* = L\mathfrak{N}^*.$$

Clearly all these multiplier spaces are vector subspaces of $\mathfrak{A}$, containing e.

The partial *-algebra is *abelian* if $L(x) = R(x)$, $\forall\, x \in \mathfrak{A}$, and then $xy = yx$, $\forall\, x \in L(y)$. In that case, we write simply for the multiplier spaces $L(x) = R(x) \equiv M(x)$, $L\mathfrak{N} = R\mathfrak{N} \equiv M\mathfrak{N}$ $(\mathfrak{N} \subset \mathfrak{A})$.

Let us denote by $\mathcal{F}^L$, resp. $\mathcal{F}^R$, the set of all LR-closed, resp. RL-closed, subspaces of $\mathfrak{A}$:

$$\mathcal{F}^L = \{\mathfrak{N} \subset \mathfrak{A} : \mathfrak{N} = LR\mathfrak{N}\}, \quad \mathcal{F}^R = \{\mathfrak{N} \subset \mathfrak{A} : \mathfrak{N} = RL\mathfrak{N}\},$$

both ordered by inclusion. Then the set $\mathcal{F}^R$, ordered by inclusion, is a complete lattice with lattice operations

$$\mathfrak{M} \wedge \mathfrak{N} = \mathfrak{M} \cap \mathfrak{N}, \quad \mathfrak{M} \vee \mathfrak{N} = RL(\mathfrak{M} + \mathfrak{N}).$$

The largest element is $\mathfrak{A}$, the smallest $R\mathfrak{A}$. A corresponding result holds for $\mathcal{F}^L$, exchanging L and R.

Very relevant examples of partial *-algebras, are provided by partial *-algebras of closable operators in a Hilbert space (partial O*-algebras) that are also needed to set up a representation theory: indeed a *-representation of a partial *-algebra $\mathfrak{A}$ is a **-homomorphism* of $\mathfrak{A}$ into some partial O*-algebra. Here a **-homomorphism* of a partial *-algebra $\mathfrak{A}$ into another one $\mathfrak{B}$ is a linear map $\rho : \mathfrak{A} \to \mathfrak{B}$ such that (i) $\rho(x^*) = \rho(x)^*$ for each $x \in \mathfrak{A}$, and (ii) whenever $x \in L(y)$ in $\mathfrak{A}$, then $\rho(x) \in L(\rho(y))$ in $\mathfrak{B}$ and

$\rho(x)\rho(y) = \rho(xy)$. The map ρ is a **-isomorphism* if it is a bijection and $\rho^{-1} : \mathfrak{B} \to \mathfrak{A}$ is also a *-homomorphism.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^*{\restriction}\mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\square\, X_2 = {X_1}^\dagger X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, iff $X_2\mathcal{D} \subset \mathcal{D}({X_1}^\dagger)$ and $X_1\mathcal{D} \subset \mathcal{D}(X_2^*)$ (we write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$). When we regard $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ as a partial *-algebra with those operations, we denote it by $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D},\mathcal{H})$.

A *partial O*-algebra* on $\mathcal{D}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D},\mathcal{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D},\mathcal{H})$, containing the identity and such that $X^\dagger \in \mathfrak{M}$ whenever $X \in \mathfrak{M}$ and $X_1 \,\square\, X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_2 \in R^{\mathrm{w}}(X_1)$.

A *-representation of a partial *-algebra $\mathfrak{A}$ is a *-homomorphism of $\mathfrak{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D},\mathcal{H})$, for some pair $\mathcal{D} \subset \mathcal{H}$, that is, a linear map $\pi : \mathfrak{A} \to \mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D},\mathcal{H})$ such that : (i) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathfrak{A}$; (ii) $x \in L(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^{\mathrm{w}}(\pi(y))$ and $\pi(x)\,\square\,\pi(y) = \pi(xy)$.

2.2 The GNS construction

The familiar Gel'fand–Naĭmark–Segal (GNS) construction for *-algebras can be extended to partial *-algebras. The lack of an everywhere defined multiplication imposes, in this case, the use of certain positive sesquilinear forms (instead of positive linear fuctionals) as starting point.

Let $\mathfrak{A}$ be a partial *-algebra. Let φ be sesquilinear form on $D(\varphi)\times D(\varphi)$, where $D(\varphi)$ a subspace of $\mathfrak{A}$. The form φ is said to be *positive* if $\varphi(x,x) \geqslant 0, \forall\, x \in D(\varphi)$. Then, we have:

$$\begin{aligned} \varphi(x,y) &= \overline{\varphi(y,x))},\ \forall\, x,y \in D(\varphi) \\ |\varphi(x,y)|^2 &\leqslant \varphi(x,x)\,\varphi(y,y),\ \forall\, x,y \in D(\varphi), \end{aligned}$$

and hence

$$N_\varphi \equiv \{x \in D(\varphi); \varphi(x,x) = 0\} = \{x \in D(\varphi); \varphi(x,y) = 0 \text{ for all } y \in D(\varphi)\ \},$$

and so N_φ is a subspace of $\mathfrak{A}$. For each $x \in D(\varphi)$, we denote by $\lambda_\varphi(x)$ the coset of $D(\varphi)/N_\varphi$ which contains x, and define an inner product $\langle\cdot|\cdot\rangle$ on

$\lambda_\varphi(D(\varphi)) = D(\varphi)/N_\varphi$ by

$$\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x,y),\ x,y \in D(\varphi).$$

We denote by $\mathcal{H}_\varphi$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(D(\varphi))$.

Definition 2.1 – Let φ be a positive sesquilinear form on $D(\varphi) \times D(\varphi)$. A *core* for φ is a subspace $B(\varphi)$ of $D(\varphi)$ such that

(i) $B(\varphi) \subset R\mathfrak{A}$

(ii) $\{ax; a \in \mathfrak{A}, x \in B(\varphi)\} \subset \mathcal{D}(\varphi)$

(iii) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$

(iv) $\varphi(ax,y) = \varphi(x,a^*y), \quad \forall\, a \in \mathfrak{A}, \forall\, x,y \in B(\varphi)$

(v) $\varphi(a^*x,by) = \varphi(x,(ab)y), \quad \forall\, a \in L(b), \forall\, x,y \in B(\varphi).$

We denote by $\mathcal{B}_\varphi$ the set of all cores $B(\varphi)$ for φ and call *biweight* on $\mathfrak{A}$ a positive sesquilinear form φ on $D(\varphi) \times D(\varphi)$ such that $\mathcal{B}_\varphi \neq \emptyset$ [4]. If $\mathfrak{A}$ has a unit e, we say that a biweight φ, with core $B(\varphi)$, is a *state* if $\varphi(e,e) = 1$.

For biweights a GNS-like construction can be performed, as shown in the next proposition.

Proposition 2.2 *Let φ be a biweight on $\mathfrak{A}$ with a core $B(\varphi)$. Put*

$$\pi^\circ_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathfrak{A}, x \in B(\varphi).$$

*Then π°_φ is a *-representation of $\mathfrak{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\lambda_\varphi(B(\varphi)), \mathcal{H}_\varphi)$.*

Denote by π^B_φ the closure of π°_φ. The triple $(\pi^B_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ is called the *GNS construction* for the biweight φ on $\mathfrak{A}$ with core $B(\varphi)$.

The dependence of the above construction on the core has been analyzed in detail in [5].

2.3 Locally convex and Banach partial *-algebras

2.3.1 Basic definitions

Let $\mathfrak{A}$ be a partial *-algebra with unit and assume it carries a locally convex, Hausdorff, topology τ, which makes it into a locally convex vector space $\mathfrak{A}[\tau]$.

Definition 2.3 – Let $\mathfrak{A}[\tau]$ be a partial *-algebra, which is a locally convex vector space for the locally convex topology τ. Then $\mathfrak{A}[\tau]$ is called a *locally convex partial *-algebra* if the following two conditions are satisfied:

(i) the involution $x \mapsto x^*$ is τ-continuous;

(ii) the maps $x \mapsto ax$ and $x \mapsto xb$ are τ-continuous for all $a \in L\mathfrak{A}$ and $b \in R\mathfrak{A}$.

This definition seems natural, in the sense that it makes the topological structure determined by τ compatible with the multiplier structure of $\mathfrak{A}$. Indeed, the partial *-algebraic structure of $\mathfrak{A}$ is completely characterized by its spaces of left, resp. right, multipliers. Thus, once a topology on $\mathfrak{A}$ is given, it automatically defines appropriate and, in a sense, *natural* topologies on all spaces of multipliers.

Let $\mathfrak{M} \in \mathcal{F}^R$. To every $a \in L\mathfrak{M}$, one may associate a linear map L_a from $\mathfrak{M}$ into $\mathfrak{A}$:

$$L_a(x) = ax, \quad x \in \mathfrak{M},\ a \in L\mathfrak{M}\,.$$

Then we define the topology $\rho_{\mathfrak{M}}$ on $\mathfrak{M}$ as the weakest locally convex topology on $\mathfrak{M}$ such that all maps L_a, $a \in L\mathfrak{M}$, are continuous from $\mathfrak{M}$ into $\mathfrak{A}[\tau]$. This topology (which is a projective topology) is clearly Hausdorff. In the same way, the topology $\lambda_{\mathfrak{N}}$ on $\mathfrak{N} \in \mathcal{F}^L$ is the weakest locally convex topology on $\mathfrak{N}$ such that all maps $R_b : x \mapsto xb$, $x \in \mathfrak{N}$, $b \in R\mathfrak{N}$, are continuous from $\mathfrak{N}$ into $\mathfrak{A}[\tau]$.

As for $\mathfrak{A}$ itself, it is easy to see that both $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$ are, in general, finer than τ. The continuity conditions required in Definition 2.3 imply that they are in fact equivalent.

Let us now consider multiplier spaces $\mathfrak{M} \in \mathcal{F}^R$. If $\mathfrak{M}_1 \subset \mathfrak{M}_2$, it is easily seen that the embedding is continuous but it has not, in general, dense range. A full knowledge of the whole lattice $\mathcal{F}^R$ is often difficult and sometimes impossible. For this reason it is useful to introduce the notion of *generating family*, that is, a subset $\mathcal{I}^R$ of $\mathcal{F}^R$ such that

(i) $R\mathfrak{A} \in \mathcal{I}^R$ and $\mathfrak{A} \in \mathcal{I}^R$, and

(ii) $x \in L(y)$ iff $\exists\, \mathfrak{M} \in \mathcal{I}^R$ such that $y \in \mathfrak{M}, x \in L\mathfrak{M}$.

A generating family for $\mathcal{F}^L$ is defined in a similar way. Clearly, if $\mathcal{I}^R$ is a generating family for $\mathcal{F}^R$, $\mathcal{I}^L = L\mathcal{I}^R = \{L\mathfrak{M} : \mathfrak{M} \in \mathcal{I}^R\}$ is generating for $\mathcal{F}^L$. Moreover, if $\mathcal{I}^R$ is generating, the *complete* lattice generated by $\mathcal{I}^R$ is $\mathcal{F}^R$ itself. It turn also out that if $R\mathfrak{A}$ is dense in $\mathfrak{M}[\rho_{\mathfrak{M}}]$ for every $\mathfrak{M} \in \mathcal{I}^R$, then $\mathfrak{M}_1$ is dense in $\mathfrak{M}_2[\rho_{\mathfrak{M}_2}]$, for any pair $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{F}^R$ such that $\mathfrak{M}_1 \subset \mathfrak{M}_2$. For details and proofs see [1, 5].

The locally convex partial *-algebra $\mathfrak{A}[\tau]$ is said to be *tight*, if, in addition to the two conditions given in Definition 2.3, the following condition also holds:

(iii) there is a generating family $\mathcal{J}^R$ for $\mathcal{F}^R$ such that $R\mathfrak{A}$ is dense in $\mathfrak{M}[\rho_{\mathfrak{M}}]$, $\forall \mathfrak{M} \in \mathcal{J}^R$.

Example 2.4 The simplest example is that of a *locally convex quasi *-algebra*, which is defined as follows. Let $\mathfrak{A}_o$ be a locally convex *-algebra such that the multiplication is separately, but not jointly, continuous, and $\mathfrak{A}_o$ is not complete. Then the completion $\mathfrak{A}$ of $\mathfrak{A}_o$ is, in general, only a partial *-algebra: the product xy is defined only if either x or y belongs to $\mathfrak{A}_o$ by an obvious limit process. Clearly, $(\mathfrak{A}, \mathfrak{A}_o)$ is a (trivial) partial *-algebra with $L\mathfrak{A} = R\mathfrak{A} = \mathfrak{A}_o$ and $\mathfrak{A}_o$ is dense in $\mathfrak{A}$. Thus every locally convex quasi *-algebra is a tight locally convex partial *-algebra.

2.3.2 Banach partial *-algebras

Assume now that $\mathfrak{A}$ is a Banach space for a norm $\|\cdot\|$, satisfying

(b1) $\|x\| = \|x^*\|$, $\forall\, x \in \mathfrak{A}$;

(b2) $\forall\, a \in L\mathfrak{A}$, $\exists$ a constant $\gamma_a > 0$ such that $\|ax\| \leqslant \gamma_a \|x\|$, $\forall\, x \in \mathfrak{A}$.

Then $\mathfrak{A}[\|\cdot\|]$ is a locally convex partial *-algebra and both $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$ are equivalent to the norm topology on $\mathfrak{A}$.

We will now describe the topologies of the spaces of multipliers. Let us consider an arbitrary matching pair $\mathfrak{M} \in \mathcal{F}^R$, $L\mathfrak{M} \in \mathcal{F}^L$. Apart from $\rho_{\mathfrak{M}}$ and $\lambda_{L\mathfrak{M}}$, other topologies can be defined on $\mathfrak{M}$ and $L\mathfrak{M}$, respectively, starting from the fact that, as above, $L\mathfrak{M}$ can be identified with a space of linear maps from $\mathfrak{M}$ into $\mathfrak{A}$. Let $\mathcal{G}$ be a bounded subset of $\mathfrak{M}[\rho_{\mathfrak{M}}]$ and $a \in L\mathfrak{M}$. We put

$$\|a\|_{\mathcal{G}} = \sup_{x \in \mathcal{G}} \|ax\|.$$

This family of seminorms endows $L\mathfrak{M}$ with a topology $\Lambda_{L\mathfrak{M}}$ finer than $\lambda_{L\mathfrak{M}}$.

One defines in a similar way a topology $\mathrm{P}_{\mathfrak{M}}$ on $\mathfrak{M}$. In general these topologies are neither normable, nor Fréchet.

A norm $\|\cdot\|_{\mathfrak{M}}$ on $\mathfrak{M} \in \mathcal{F}^R$ is called *admissible* if $\rho_{\mathfrak{M}} \preceq \|\cdot\|_{\mathfrak{M}} \preceq \mathrm{P}_{\mathfrak{M}}$. Similarly, a norm $\|\cdot\|_{L\mathfrak{M}}$ on $L\mathfrak{M} \in \mathcal{F}^L$ is admissible whenever $\lambda_{L\mathfrak{M}} \preceq \|\cdot\|_{L\mathfrak{M}} \preceq \Lambda_{L\mathfrak{M}}$.

Let now $\|\cdot\|_{\mathfrak{M}}$ be an admissible norm on $\mathfrak{M} \in \mathcal{F}^R$. Then $L_a : \mathfrak{M}[\|\cdot\|_{\mathfrak{M}}] \to \mathfrak{A}[\|\cdot\|]$ is continuous, i.e.,

$$\|L_a x\| = \|ax\| \leqslant \gamma \|x\|_{\mathfrak{M}}, \quad x \in \mathfrak{M}$$

Then we can define a norm $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$ on $L\mathfrak{M}$ by

$$\|a\|^{\diamond}_{L\mathfrak{M}} = \sup_{\|x\|_{\mathfrak{M}}\leqslant 1} \|ax\|.$$

Since the unit ball of $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is bounded in $\mathfrak{M}[\rho_{\mathfrak{M}}]$, it follows that $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$ is admissible too. In the same way, we can define a new norm $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ on $\mathfrak{M}$ by

$$\|x\|^{\diamond\diamond}_{\mathfrak{M}} = \sup_{\|a\|^{\diamond}_{L\mathfrak{M}}\leqslant 1} \|ax\|.$$

It is easily seen that $\|x\|^{\diamond\diamond}_{\mathfrak{M}} \leqslant \|x\|_{\mathfrak{M}}$, for every $x \in \mathfrak{M}$ and that $\|x\|^{\diamond\diamond}_{\mathfrak{M}}$ is admissible. Moreover, the following relations hold:

(a) $\|x\|^{\diamond\diamond}_{\mathfrak{M}} \leqslant \|x\|_{\mathfrak{M}}$, $\forall\, x \in \mathfrak{M}$;

(b) $\|x\|^{\diamond\diamond\diamond}_{L\mathfrak{M}} = \|x\|^{\diamond}_{L\mathfrak{M}}$, $\forall\, x \in L\mathfrak{M}$.

The norm $\|\cdot\|_{\mathfrak{M}}$ on $\mathfrak{M}$ is said to be *reproducing* if $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}} \sim \|\cdot\|_{\mathfrak{M}}$.
The inequality

$$\|ax\| \leq \|a\|^{\diamond}_{L\mathfrak{M}}\|x\|^{\diamond\diamond}_{\mathfrak{M}}, \quad \forall a \in L\mathfrak{M}, x \in \mathfrak{M}$$

motivates both the notion of admissible norm and the following definition.

Definition 2.5 Let $\mathfrak{A}[\tau]$ be a locally convex partial *-algebra, and $\mathcal{I}^R$ a generating family. $\mathfrak{A}[\tau]$ is said to be a *normed partial *-algebra* if

(i) τ is a norm topology, with norm $\|\cdot\|$;

(ii) the involution $x \mapsto x^*$ is isometric on $\mathfrak{A}[\|\cdot\|]$;

(iii) each space $\mathfrak{M} \in \mathcal{I}^R$ is a Banach space for a reproducing norm $\|\cdot\|_{\mathfrak{M}}$.

$\mathfrak{A}[\tau]$ is called a *Banach partial *-algebra* if, in addition,

(iv) $\mathfrak{A}[\tau]$ is a Banach space.

Example 2.6 A simple example of a tight Banach partial *-algebra can be constructed as in Example 2.4 endowing $\mathfrak{A}_o$ with a norm topology where the multiplication is not jointly continuous (we name this situation as *Banach quasi *-algebras*). Many concrete examples can be realized in this way: think, for instance, of $L^p(\mathbb{R})$ as completion of $C_o(\mathbb{R})$ w.r.to the L^p-norm.

3 C*-seminorms and continuous *-representations

In [7] conditions for the continuity of bounded *-representations of a Banach partial *-algebra were discussed. The assumptions made there are however rather strong. For this reason a further study is needed. This is the scope of this Section where we present some new results in this direction.

We denote with $BW(\mathfrak{A})$ the set of all biweights of $\mathfrak{A}$. Moreover, if $\mathfrak{B}$ is a subspace of $R\mathfrak{A}$ we call $BW(\mathfrak{A};\mathfrak{B})$ the following subset of $BW(\mathfrak{A})$:

$$BW(\mathfrak{A};\mathfrak{B}) = \{\varphi \in BW(\mathfrak{A}) : \mathfrak{B} \subset \mathcal{D}(\varphi),\ \mathfrak{B} \text{ is a core for } \varphi\}.$$

If $\mathfrak{B} \subset R\mathfrak{A}$, we put

$$p_{\mathfrak{B}}(a)^2 = \sup\{\varphi(ax, ax) : \varphi \in BW(\mathfrak{A};\mathfrak{B}), x \in \mathfrak{B}, \varphi(x,x) = 1\}$$

and

$$\mathcal{D}(p_{\mathfrak{B}}) = \{a \in \mathfrak{A} : p_{\mathfrak{B}}(a) < \infty\}.$$

If $BW(\mathfrak{A};\mathfrak{B}) = \{0\}$, we put $p_{\mathfrak{B}}(a) = 0$ for every $a \in \mathfrak{A}$ and $\mathcal{D}(p_{\mathfrak{B}}) = \mathfrak{A}$. Then the following results may be found in [10]:

(1) $\mathcal{D}(p_{\mathfrak{B}})$ is a partial *-subalgebra of $\mathfrak{A}$;
(2) $p_{\mathfrak{B}}$ is an unbounded C*-seminorm on $\mathcal{D}(p_{\mathfrak{B}})$, i.e., $p_{\mathfrak{B}}(a^*a) = p_{\mathfrak{B}}(a)^2$, for every $a \in \mathcal{D}(p_{\mathfrak{B}})$ and $a^* \in L(a)$;
(3) the set $N(p_{\mathfrak{B}}) = \{a \in \mathfrak{A} :\ p_{\mathfrak{B}}(a) = 0\}$ is a *partial *-ideal* in the sense that if $b \in \mathfrak{A}$, $a \in N(p_{\mathfrak{B}})$ and ba is well-defined, then $ba \in N(p_{\mathfrak{B}})$.

Proposition 3.1 *The GNS representation π_φ^B corresponding to a biweight φ with domain $D(\varphi)$ and core $B(\varphi)$ is bounded if, and only if, φ is admissible on $B(\varphi)$, i.e., for every $a \in \mathfrak{A}$ there exists $\gamma_a > 0$ such that*

$$\varphi(ax, ax) \leq \gamma_a \varphi(x,x), \quad \forall x \in B(\varphi). \tag{3.1}$$

The notion of admissibility given above depends, of course, on the existence of a *core of admissibility* $B(\varphi)$. In principle if we choose a different core $B_1(\varphi)$, then φ need not satisfy condition (3.1) when $x \in B_1(\varphi)$.

Vector sesquilinear forms will play a relevant role in what follows.

Let π be a *-representation of the partial *-algebra $\mathfrak{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D},\mathcal{H})$ and $\xi \in \mathcal{H}$. Define

$$\mathcal{D}(\varphi_\xi^\pi) = \{a \in \mathfrak{A} : \xi \in D(\pi(a^*)^*)\}$$
$$\varphi_\xi^\pi(a,b) = \langle \pi(a^*)^*\xi | \pi(b^*)^*\xi\rangle.$$

Then φ_ξ^π is a sesquilinear form on $\mathcal{D}(\varphi_\xi^\pi) \times \mathcal{D}(\varphi_\xi^\pi)$.

If $\xi \in \mathcal{D}$, we put

$$B(\varphi_\xi^\pi) = \{x \in R\mathfrak{A} : \pi(x)\xi \in \mathcal{D}^{**}(\pi)\}.$$

If $\xi \in \mathcal{H} \setminus \mathcal{D}$ we define $B(\varphi_\xi^\pi)$ as the linear span of the set:

$$B_0(\varphi_\xi^\pi) = \{x \in R\mathfrak{A} : \xi \in D(\overline{\pi(x)}),\ \overline{\pi(x)}\xi \in \mathcal{D}\}.$$

Then, if $\pi(B(\varphi_\xi^\pi))\xi$ is dense in $\overline{\pi(D(\varphi_\xi^\pi))\xi}$, φ_ξ^π is a biweight on $\mathfrak{A}$. A proof of this statement can be done as in [5, Example 9.1.12].

Definition 3.2 A *-representation π of a partial *-algebra $\mathfrak{A}$ is said to be *nonsingular* if there exists $\xi \in \mathcal{H}$ such that φ_ξ^π is a biweight on $\mathfrak{A}$. The *-representation π is said to be *regular* if φ_ξ^π is a biweight on $\mathfrak{A}$, for every $\xi \in \mathcal{H}$.

Remark 3.3 If φ is a biweight with domain $D(\varphi)$ and core $B(\varphi)$ with $e \in B(\varphi)$, then the GNS representation π_φ^B is nonsingular since $\lambda_\varphi(B(\varphi)) = \pi_\varphi^B(B(\varphi))\lambda_\varphi(e)$ is dense in $\mathcal{H}_\varphi$.

If π is a *bounded* *-representation of $\mathfrak{A}$, then $\mathcal{D}(\varphi_\xi^\pi) = \mathfrak{A}$ and we may choose, without loss of generality, $B(\varphi_\xi^\pi) = R\mathfrak{A}$ since, in this case, if $B(\varphi_\xi^\pi) \subset R\mathfrak{A}$ is a core for φ_ξ^π, so is also $R\mathfrak{A}$.

Proposition 3.4 *Let $\mathfrak{A}$ be a Banach partial *-algebra and assume that $R\mathfrak{A}$ is dense in $\mathfrak{A}$. Then every continuous bounded representation is regular.*

Proof. Let π be a continuous bounded representation and $\xi \in \mathcal{H}$. Then there exists $\gamma > 0$ such that

$$\|\pi(a)\| \leq \gamma\|a\|, \quad \forall a \in \mathfrak{A}.$$

If $a \in \mathfrak{A}$, there exists a sequence $\{x_n\} \subset R\mathfrak{A}$ such that $\|a - x_n\| \to 0$. Then we have

$$\begin{aligned} \|\lambda_{\varphi_\xi^\pi}(a) - \lambda_{\varphi_\xi^\pi}(x_n)\|^2 &= \varphi_\xi^\pi(a - x_n, a - x_n) \\ &= \|\pi(a - x_n)\xi\|^2 \leq \gamma^2\|a - x_n\|^2\|\xi\| \to 0. \end{aligned}$$

□

Let $\mathrm{BRep}(\mathfrak{A})$ denote the set of all bounded *-representation of $\mathfrak{A}$ and define

$$q(a) = \sup_{\pi \in \mathrm{BRep}(\mathfrak{A})} \|\pi(a)\|$$

on the set

$$\mathcal{D}(q) = \{a \in \mathfrak{A} : q(a) < \infty\}.$$

Then it is easily seen that $\mathcal{D}(q)$ is a partial *-subalgebra of $\mathfrak{A}$ and q is an *unbounded C*-seminorm* on $\mathfrak{A}$, that is: $q(ab) \leq q(a)q(b)$ for $a, b \in \mathfrak{A}$ with $a \in L(b)$; $q(a^*a) = q(a)^2$ for $a \in \mathfrak{A}$ with $a^* \in L(a)$ [8, 10].

If φ is admissible on the core $B(\varphi)$, we get, for every $a \in \mathcal{D}(q)$, $x \in B(\varphi)$:

$$\begin{aligned} \varphi(ax, ax) &= \langle \pi_\varphi^B(a)\lambda_\varphi(x) | \pi_\varphi^B(a)\lambda_\varphi(x)\rangle \\ &= \|\pi_\varphi^B(a)\lambda_\varphi(x)\|^2 \leq \|\pi_\varphi^B(a)\|^2 \|\lambda_\varphi(x)\|^2 \leq q(a)^2\varphi(x,x). \end{aligned} \tag{3.2}$$

Let

$$\mathrm{BRep}_{ad}(\mathfrak{A}) = \{\pi \in \mathrm{BRep}(\mathfrak{A}) : \pi = \pi_\varphi^B,\ \varphi \text{ is admissible on } B(\varphi)\}.$$

Then an inequality similar to (3.2) holds, as is easy to see, if we replace q with the unbounded C*-seminorm q_{ad} defined by

$$q_{ad}(a) = \sup_{\varphi \in \mathrm{BRep}_{ad}(\mathfrak{A})} \|\pi_\varphi^B(a)\|$$

on the partial *-subalgebra

$$\mathcal{D}(q_{ad}) = \{a \in \mathfrak{A} : q_{ad}(a) < \infty\}.$$

It is readily checked that

$$\mathcal{D}(q) \subseteq \mathcal{D}(q_{ad}) \quad \text{and} \quad q_{ad}(a) \leq q(a), \quad \forall a \in \mathcal{D}(q).$$

Let $\mathfrak{B}$ be a subspace of $R\mathfrak{A}$ and $\varphi \in BW(\mathfrak{A}; \mathfrak{B})$. Then by the definition itself of $p_{\mathfrak{B}}$ we get [10]:

$$\varphi(ax, ax) \leq p_{\mathfrak{B}}(a)^2\varphi(x,x), \quad \forall a \in \mathcal{D}(p_{\mathfrak{B}}), x \in \mathcal{B}. \tag{3.3}$$

Then, if $\mathcal{D}(p_{\mathfrak{B}}) = \mathfrak{A}$, φ is admissible on $\mathfrak{B}$ and since

$$\|\pi_\varphi^{\mathfrak{B}}(a)\|^2 = \sup\{\varphi(ax, ax) : x \in \mathfrak{B},\ \varphi(x,x) = 1\}$$

we get

$$p_{\mathfrak{B}}(a) = \sup_{\varphi \in BW(\mathfrak{A};\mathfrak{B})} \|\pi_\varphi^{\mathfrak{B}}(a)\|, \quad \forall a \in \mathfrak{A}.$$

If $e \in \mathfrak{B}$, from Remark 3.3 and from the admissibility of any $\varphi \in BW(\mathfrak{A}; \mathfrak{B})$, it follows that

$$p_{\mathfrak{B}}(a) \leq \sup_{\pi \in \mathrm{BRep}_{ad}(\mathfrak{A})} \|\pi(a)\| = q_{ad}(a)$$

for every $a \in \mathfrak{A}$ for which the r.h.s. is finite. We have then proved the following

Lemma 3.5 *For any subspace $\mathfrak{B} \subset R\mathfrak{A}$ such that $e \in \mathfrak{B}$ and $\mathcal{D}(p_{\mathfrak{B}}) = \mathfrak{A}$ the following inequality holds:*

$$p_{\mathfrak{B}}(a) \leq q_{ad}(a), \quad \forall a \in \mathcal{D}(q_{ad}).$$

Remark 3.6 If $e \notin \mathfrak{B}$ a similar inequality holds, with the sup on the r.h.s. taken over *all* bounded representations of $\mathfrak{A}$.

Lemma 3.7 *Let π be a regular bounded *-representation of $\mathfrak{A}$. Then*

$$\|\pi(a)\| \leq p_{R\mathfrak{A}}(a), \quad \forall a \in \mathcal{D}(p_{R\mathfrak{A}}).$$

Proof. Since π is regular, then for each $\xi \in \mathcal{H}$, $\|\xi\| = 1$, φ_ξ^π is a biweight with domain $\mathfrak{A}$ and core $R\mathfrak{A}$ satisfying $\varphi_\xi^\pi(e, e) = 1$. Then we have, for every $a \in \mathcal{D}(p_{R\mathfrak{A}})$,

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi | \pi(a)\xi \rangle = \varphi_\xi^\pi(a, a) = \varphi_\xi^\pi(ae, ae) \leq p_{R\mathfrak{A}}(a)$$

which, clearly, implies the statement. □

We can now prove the main result of this paper.

Theorem 3.8 *Let $\mathfrak{A}$ be a partial *-algebra and assume that*

(a) $\mathcal{D}(p_{R\mathfrak{A}}) = \mathfrak{A}$.

(b) $R\mathfrak{A}$ *is $p_{R\mathfrak{A}}$-dense in $\mathfrak{A}$ (i.e. for each $a \in \mathfrak{A}$ there exists a sequence $\{x_n\} \subset R\mathfrak{A}$ such that $p_{R\mathfrak{A}}(a - x_n) \to 0$).*

Then the following statements are equivalent.

*(i) Every bounded *-representation π of $\mathfrak{A}$ is regular.*

(ii) Then there exists a C-seminorm q on $\mathfrak{A}$ such that:*

(ii.1) Every $\varphi \in BW(\mathfrak{A}; R\mathfrak{A})$ is admissible and

$$\varphi(ax, ax) \leq q(a)^2 \varphi(x, x), \quad a \in \mathfrak{A}.$$

*(ii.2) If π is a bounded *-representation of $\mathfrak{A}$, then*

$$\|\pi(a)\| \leq q(a), \quad \forall a \in \mathfrak{A}.$$

(ii.3) One has

$$q(a) = p_{R\mathfrak{A}}(a) = \sup_{\pi \in \mathrm{BRep}(\mathfrak{A})} \|\pi(a)\|, \quad \forall a \in \mathfrak{A}.$$

Proof. (i)⇒ (ii) We define q as before by

$$q(a) = \sup_{\pi \in \mathrm{BRep}(\mathfrak{A})} \|\pi(a)\|, \quad a \in \mathcal{D}(q).$$

We show that

$$\mathcal{D}(q) = \mathfrak{A} \quad \text{and} \quad q(a) = p_{R\mathfrak{A}}(a), \ \ \forall a \in \mathfrak{A}.$$

Indeed, from Lemma 3.7, we have

$$\|\pi(a)\| \leq p_{R\mathfrak{A}}(a), \ \ \forall a \in \mathcal{D}(p_{R\mathfrak{A}}) = \mathfrak{A}.$$

Since any bounded *-representation π is regular, taking the sup over $\pi \in \mathrm{BRep}(\mathfrak{A})$ on the l.h.s. we obtain that q is everywhere defined and $q(a) \leq p_{R\mathfrak{A}}(a)$, for every $a \in \mathfrak{A}$. Therefore

$$q(a) \leq p_{R\mathfrak{A}}(a) \leq q_{ad}(a) \leq q(a), \quad \forall a \in \mathfrak{A}$$

which also proves (ii.3). Indeed to complete the proof of (ii.1) and (ii.2) it is sufficient to apply the inequality (3.3) with $\mathfrak{B} = R\mathfrak{A}$, Lemma 3.5 and Lemma 3.7.
(i)⇒ (ii) Let π be a bounded *-representation and $\xi \in \mathcal{H}$. Then, by (ii.2) and (ii.3)

$$\|\pi(a)\| \leq p_{R\mathfrak{A}}(a), \ \ \forall a \in \mathfrak{A}.$$

If $a \in \mathfrak{A}$, by the assumption, there exists a sequence $\{x_n\} \subset R\mathfrak{A}$ such that $p_{R\mathfrak{A}}(a - x_n) \to 0$). Then we have

$$\begin{aligned} \|\lambda_{\varphi^\pi_\xi}(a) - \lambda_{\varphi^\pi_\xi}(x_n)\|^2 &= \varphi^\pi_\xi(a - x_n, a - x_n) \\ &= \|\pi(a - x_n)\xi\|^2 \leq p_{R\mathfrak{A}}(a - x_n)^2 \|\xi\|^2 \to 0. \end{aligned}$$

Therefore, π is regular. □

Remark 3.9 We notice that the density condition (b) is not needed in the proof of (i) ⇒ (ii).

Corollary 3.10 *Let $\mathfrak{A}$ be a Banach partial *-algebra with norm $\|\cdot\|$ and assume that for every biweight $\varphi \in BW(\mathfrak{A}; R\mathfrak{A})$ the following inequality holds:*

$$\varphi(ax, ax) \leq \|a\|\varphi(x, x), \quad \forall a \in \mathfrak{A}, x \in R\mathfrak{A}.$$

*Then every regular bounded *-representation of $\mathfrak{A}$ is continuous from $\mathfrak{A}[\|\ \|]$ into $\mathcal{B}(\mathcal{H})[\|\ \|]$.*

Proof. The assumption implies that $q(a) \leq \|a\|$ for every $a \in \mathfrak{A}$. The statement then follows from (ii) of Theorem 3.8 and from Remark 3.9. □

Remark 3.11 This result about the continuity of *regular* bounded *-representation has been given in [7, Proposition 5.18] with a different proof.

References

[1] J-P. Antoine, F. Bagarello and C. Trapani "Topological partial *-algebras: Basic properties and examples", Reviews Math. Phys. **11** (1999) 267–302

[2] J-P. Antoine and W. Karwowski "Partial *-algebras of closed operators", in *"Quantum Theory of Particles and Fields"*, pp. 13–30; B. Jancewicz and J. Lukierski (eds.), World Scientific, Singapore, 1983

[3] J-P. Antoine and W. Karwowski "Partial *-algebras of closed linear operators in Hilbert space", Publ. RIMS, Kyoto Univ. **21** (1985) 205–236; Add./Err. ibid. **22** (1986) 507–511

[4] J-P. Antoine, A. Inoue and C. Trapani "Biweights on partial *-algebras", J. Math. Anal. and Appl. **242** (2000) 164–190

[5] J-P. Antoine, A. Inoue, and C. Trapani *"Partial *-Algebras and Their Operator Realizations"*, Mathematics and Its Applications, vol. 553, Kluwer, Dordrecht, NL, 2002

[6] J-P. Antoine and F. Mathot "Partial *-algebras of closed operators and their commutants. I. General structure", Ann. Inst. H. Poincaré **46** (1987) 299–324

[7] J-P. Antoine and C. Trapani "Banach partial *-algebras: Basic properties, representations, spectrum" preprint UCL-IPT-03-14, Louvain-la-Neuve

[8] F. Bagarello, A. Inoue and C. Trapani, "Unbounded C*-seminorms on partial *-algebras" , Zeit. Anal. Anwen. **20** (2001), 295-314

[9] C. Trapani, "Some seminorms on quasi *-algebras", Studia Matematica **158** (2003) 99 -115; *Erratum/Addendum*, Studia Math. **160** (2004), 101-101

[10] C. Trapani and F. Tschinke, "Unbounded C*-seminorms and biweights on partial *-algebras", preprint Dip. Matematica e Applicazioni n.201, Agosto 2003

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Auxiliary Seminorms and the Structure of a CQ*-Algebra

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Abstract. After reviewing the main facts of the theory of CQ*-algebras, we give some new results on the structure of proper CQ*-algebras using some seminorms defined by certain families of positive sesquilinear forms.

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1 Introduction

A CQ*-algebra can be thought, roughly speaking, as the completion of a given C*-algebra $\mathfrak{A}_0$ with respect to a weaker norm, with certain coupling properties of the two norms. There are mainly two interesting situations: the first one occurs when the involution of the C*-algebra extends to the completion $\mathfrak{X}$ of $\mathfrak{A}_0$, which is, therefore, a Banach quasi *-algebra containing a dense C*-algebra: in this case we speak of a *proper* CQ*-algebra).

On the other hand, if $\mathfrak{X}$ is endowed with an isometric involution different from that of $\mathfrak{A}_0$, then the structure becomes more involved. There is also a third naturally defined involution and the whole structure reminds of the Tomita-Takesaki theory (this connection has been studied in [7]).

Both cases have been investigated, under several points of view, in a series of previous papers [2]-[7], [13]-[17] to that we refer for a detailed analysis.

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In [4], the notion of *-semisimplicity for CQ*-algebras has been introduced and studied with the help of a particular class of positive bounded sesquilinear forms on $\mathfrak{X}$. *-Semisimple CQ*-algebra enjoy a series of properties (existence of auxiliary norms, possibility of defining a well behaved spectrum, possibility of extending the multiplication, etc.) and for this reason they may be considered as a reasonably good generalization of the notion of C*-algebra in the partial algebraic setting [1]. In particular, in [3] the commutative case was examined in detail and it was proved that any commutative *-semisimple CQ*-algebra can be realized as a CQ*-algebra of functions.

In a recent paper [5], Bagarello and the authors have considered the following problem: given a *-semisimple non-commutative proper CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ and the universal *-representation of $\mathfrak{A}_0$, defined via the Gel'fand–Naĭmark theorem, can $\mathfrak{X}$ be realized as a CQ*-algebra of operators affiliated with the von Neumann algebra $\pi(\mathfrak{A}_0)''$? This question, similar to that posed for the commutative case in [3], finds the natural environment where to be discussed in the framework of the Segal-Nelson theory [11, 9] of non-commutative integration and has received a positive answer (at least under certain circumstances).

In this paper, after a short overview of the main results obtained on this subject, we continue our study of the structure properties of a proper CQ*-algebra. The main tool is constituted by certain seminorms that all closely remind of the Gel'fand–Naĭmark seminorm on a Banach *-algebra (and automatically coincide with it in that case). The study of their relationship gives rather detailed information on the structure of a proper CQ*-algebra.

2 CQ*-algebras: an overview

In this Section we summarize the basic definitions and the main results on CQ*-algebras.

2.1 The general case

Let $\mathfrak{A}_\#$ be a C*-algebra, with involution $\#$ and norm $\| \ \|_\#$, and $\mathfrak{X}[\| \ \|]$ a left Banach $\mathfrak{A}_\#$-module. This means, in particular, that there is a bounded bilinear map

$$(a, x) \to ax$$

from $\mathfrak{A}_\# \times \mathfrak{X}$ into $\mathfrak{X}$ such that

$$(a_1 a_2)x = a_1(a_2 x), \quad \forall a_1, a_2 \in \mathfrak{A}_\#, x \in \mathfrak{X}.$$

We will always assume that the following inequality holds:

$$\|ax\| \leq \|a\|_{\#}\|x\|, \quad \forall x \in \mathfrak{X}, a \in \mathfrak{A}_{\#}.$$

This implies, as shown in [17], that

$$\|a\|_{\#} = \sup_{x \in \mathfrak{X}; \|x\| \leq 1} \|ax\|, \quad a \in \mathfrak{A}_{\#}.$$

A left Banach $\mathfrak{A}_{\#}$-module $\mathfrak{X}$ is called a CQ*-algebra if

(i) in $\mathfrak{X}$ an involution * is defined and $\|x^*\| = \|x\|$ for every $x \in \mathfrak{X}$;

(ii) $\mathfrak{A}_{\#}$ is a $\| \ \|$-dense subspace of $\mathfrak{X}$ and the module left-multiplication extends the multiplication of $\mathfrak{A}_{\#}$;

(iii) $\mathfrak{A}_{\#} \cap \mathfrak{A}_{\#}^*$ is a *-algebra and it is dense in $\mathfrak{A}_{\#}$ with respect to $\| \ \|_{\#}$.

A CQ*-algebra is denoted with $(\mathfrak{X}, *, \mathfrak{A}_{\#}, {}_{\#})$ to underline the different involutions.

If $(\mathfrak{X}, *, \mathfrak{A}_{\#}, {}_{\#})$ is a CQ*-algebra, it is natural to define on $\mathfrak{A}_{\#}^*$ an involution $\flat$ by:

$$a^{\flat} := a^{*\#*}, \quad a \in \mathfrak{A}_{\#}^*$$

and the norm

$$\|a\|_{\flat} := \|a^*\|_{\#}, \quad a \in \mathfrak{A}_{\#}^*. \tag{2.1}$$

It is easy to check that $\|a^{\flat}a\|_{\flat} = \|a\|_{\flat}^2$, for every $a \in \mathfrak{A}_{\#}^*$, so that $\mathfrak{A}_{\flat} := \mathfrak{A}_{\#}^*$ is a C*-algebra, also contained in $\mathfrak{X}$. Moreover, $\mathfrak{X}$ can be made into a right Banach $\mathfrak{A}_{\flat}$-module by defining

$$xb := (b^*x^*)^*, \quad \forall x \in \mathfrak{X}, b \in \mathfrak{A}_{\flat}.$$

By the definition itself, * is an isometric anti-isomorphism of $\mathfrak{A}_{\#}$ onto $\mathfrak{A}_{\flat}$.

If $* = {}_{\#}$ on $\mathfrak{A}_{\#}$, the CQ*-algebra is called *proper* and denoted simply as $(\mathfrak{X}, \mathfrak{A}_0)$, with $\mathfrak{A}_0 := \mathfrak{A}_{\#} = \mathfrak{A}_{\flat}$.

If $(\mathfrak{X}, *, \mathfrak{A}_{\#}, {}_{\#})$ is a CQ*-algebra, then $(\mathfrak{X}, \mathfrak{A}_0)$, with $\mathfrak{A}_0 = \mathfrak{A}_{\#} \cap \mathfrak{A}_{\#}^*$ is a *Banach quasi *-algebra*.

We remind that a *quasi *-algebra* [10] is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{X}$ and $\mathfrak{X}$ is a $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. The *unit* of $(\mathfrak{X}, \mathfrak{A}_0)$ is an element $e \in \mathfrak{A}_0$ such that $xe = ex = x$, for every $x \in \mathfrak{X}$. A *quasi-unit* u is an element of $\mathfrak{X}$ such that $ua = au = a$ for every $a \in \mathfrak{A}_0$. A quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is called a Banach quasi *-algebra if $\mathfrak{X}$ is endowed with a norm topology such that the involution is

isometric and the multiplications $x \mapsto xa$, $x \mapsto ax$, $a \in \mathfrak{A}_0$ are bounded linear maps.

The following proposition [6] provides a simple way to construct examples of CQ*-algebras.

Proposition 2.1 *Let $\mathfrak{A}_\#$ be a C*-algebra, with norm $\| \|_\#$ and involution $\#$; let $\| \|$ be another norm on $\mathfrak{A}_\#$, weaker than $\| \|_\#$ and such that*

(i) $\| ab \| \leq \| a \| \| b \|_\# \qquad \forall a, b \in \mathfrak{A}_\#$

*(ii) there exists a $\| \|_\#$-dense subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}_\#$ where an involution * (which makes of $\mathfrak{A}_0$ a *-algebra) is defined with the property*

$$\| a^* \| = \| a \|, \quad \forall a \in \mathfrak{A}_0$$

then, if $\mathfrak{X}$ denotes the $\| \|$-completion of $\mathfrak{A}_\#$, $(\mathfrak{X}, *, \mathfrak{A}_\#, \#)$ is a CQ-algebra.*

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with norm $\| \|$ and involution *. Assume that on $\mathfrak{A}_0$ a second involution $^\#$ and a second norm $\| \|_\#$ are defined and that the following conditions are satisfied:

(a.1) $\|a^\# a\|_\# = \|a\|_\#^2, \quad \forall a \in \mathfrak{A}_0$

(a.2) $\|a\| \leq \|a\|_\#, \quad \forall a \in \mathfrak{A}_0$

(a.3) $\|ax\| \leq \|a\|_\| x\|, \quad \forall a \in \mathfrak{A}_0, x \in \mathfrak{X}.$

Then by (a.2), the identity map $i : \mathfrak{A}_0[\| \|_\#] \to \mathfrak{A}_0[\| \|]$ has a continuous extension $\hat{i}$ from the completion $\mathfrak{A}_\#$ of $\mathfrak{A}_0[\| \|_\#]$ ($\mathfrak{A}_\#$ is, of course, a C*-algebra) into $\mathfrak{X}[\| \|]$. If $\hat{i}$ is injective, then $\mathfrak{A}_\#$ is (identified with) a subspace of $\mathfrak{X}$. A necessary and sufficient condition for this to happen is that

(a.4) the two norms $\| \|$ and $\| \|_\#$ are compatible in the following sense [8]: for any sequence $\{x_n\} \subset \mathfrak{A}_0$ such that $\|x_n\| \to 0$ and $x_n \to x$ in $\mathfrak{A}_\#[\| \|_\#]$, $x = 0$ results, i.e. if $\hat{i}^{-1} : \mathfrak{A}[\| \|] \to \mathfrak{A}_\#[\| \|_\#]$ is closable.

One can easily prove that $\mathfrak{X}$ is a left Banach $\mathfrak{A}_\#$-module. Also in this case one can define on $\mathfrak{A}_0$ a third involution $\flat$ by:

$$a^\flat := a^{*\#*}, \quad a \in \mathfrak{A}_0$$

and the norm

$$\|a\|_\flat := \|a^*\|_\#, \quad a \in \mathfrak{A}_0. \tag{2.2}$$

Then the completion $\mathfrak{A}_\flat$ of $\mathfrak{A}_0[\| \|_\flat]$ is a C*-algebra. Moreover, $\mathfrak{X}$ is a right Banach $\mathfrak{A}_\flat$-module, isometrically anti-isomorphic, by *, to $\mathfrak{A}_\#$.

Definition 2.2 *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *algebra with norm $\| \ \|$ and involution *.*

*(i) Assume that on $\mathfrak{A}_0$ a second involution $^\#$ and a norm $\| \ \|_\#$, satisfying the conditions (a.1)-(a.3) above, are defined. Then we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a strict pre CQ *-algebra.*

(ii) If $(\mathfrak{X}, \mathfrak{A}_0)$ is a strict pre CQ-algebra with $\mathfrak{A}_\# \subset \mathfrak{X}$ [i.e., up to an identification, (a.4) holds], we say that $(\mathfrak{X}, *, \mathfrak{A}_\#, \#)$ is a strict CQ*-algebra.*

If $\# = *$ on $\mathfrak{A}_0$, the above structures are named proper.

If $(\mathfrak{X}, \mathfrak{A}_0)$ is a strict pre CQ*-algebra then a norm can be defined on $\mathfrak{A}_0$ by:

$$\|a\|_0 = \max\{\|a\|_\#, \|a\|_\flat\}.$$

Remark 2.3 Strict CQ*-algebras have been considered in [7]. Every strict CQ*-algebra is a CQ*-algebra, but the converse is not true in general.

Definition 2.4 Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a (strict pre) CQ*-algebra. We denote with $\mathcal{P}(\mathfrak{X})$ the set of all sesquilinear forms φ on $\mathfrak{X} \times \mathfrak{X}$ with the following properties

(i) $\varphi(x, x) \geq 0 \quad \forall x \in \mathfrak{X}$

(ii) $\varphi(xa, b) = \varphi(a, x^* b) \quad \forall x \in \mathfrak{X}, \ \forall a, b \in \mathfrak{A}_0$

(iii) there exists $\gamma > 0$ such that $|\varphi(x, y)| \leq \gamma \| x \| \| y \| \quad \forall x, y \in \mathfrak{X}$.

We put

$$\|\varphi\| := \sup_{\|x\|=\|y\|=1} |\varphi(x, y)| = \sup_{\|x\|=1} \varphi(x, x).$$

We denote with $\mathcal{S}(\mathfrak{X})$ the following set

$$\mathcal{S}(\mathfrak{X}) = \{\varphi \in \mathcal{P}(\mathfrak{X}) : \|\varphi\| \leq 1\}.$$

The (strict pre) CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is called **-semisimple* if $x \in \mathfrak{X}$, $\varphi(x, x) = 0$, for every $\varphi \in \mathcal{S}(\mathfrak{X})$, implies $x = 0$.

Similarly, we denote with $\mathcal{P}_\#(\mathfrak{X})$ the set of all sesquilinear forms φ on $\mathfrak{X} \times \mathfrak{X}$ satisfying (i) and (iii) above, but (ii) is replaced by the following

(ii)$_\#$: $\varphi(ax, y) = \varphi(x, a^\# y) \quad \forall x, y \in \mathfrak{X}, \ \forall a \in \mathfrak{A}_\#$,

and we put

$$\mathcal{S}_\#(\mathfrak{X}) = \{\varphi \in \mathcal{P}_\#(\mathfrak{X}) : \|\varphi\| \leq 1\}.$$

The positivity of $\varphi \in \mathcal{P}_{\#}(\mathfrak{X})$ or $\varphi \in \mathcal{P}(\mathfrak{X})$ implies that:

$$\begin{aligned} \varphi(x,y) &= \overline{\varphi(y,x))},\ \forall\, x,y \in \mathfrak{X} \\ |\varphi(x,y)|^2 &\leqslant \varphi(x,x)\,\varphi(y,y),\ \forall\, x,y \in \mathfrak{X}, \end{aligned}$$

and hence

$$N_\varphi \equiv \{x \in \mathfrak{X} : \varphi(x,x) = 0\} = \{x \in \mathfrak{X} : \varphi(x,y) = 0 \text{ for all } y \in \mathfrak{X}\ \},$$

and so N_φ is a subspace of $\mathfrak{A}$. For each $x \in \mathfrak{X}$, we denote by $\lambda_\varphi(x)$ the coset of $\mathfrak{X}/N_\varphi$ which contains x, and define an inner product $\langle\cdot|\cdot\rangle$ on $\lambda_\varphi(\mathfrak{X}) = \mathfrak{X}/N_\varphi$ by

$$\langle\lambda_\varphi(x)|\lambda_\varphi(y)\rangle = \varphi(x,y),\ x,y \in \mathfrak{X}.$$

We denote by $\mathcal{H}_\varphi$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{X})$.

Each $\varphi \in \mathcal{P}_{\#}(\mathfrak{X})$ can be used to define a bounded $_{\#}$-representation ρ_φ of $\mathfrak{A}_{\#}$ by

$$\rho_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathfrak{A}_{\#},\ x \in \mathfrak{X},$$

and extending (we maintain the same notation for this extension) by continuity to the whole $\mathcal{H}_\varphi$. We refer to ρ_φ as to the $_{\#}$-representation of $\mathfrak{A}_{\#}$ *associated* with φ.

The presence of three different involutions defined on a dense subalgebra of a (strict) CQ*-algebra, reminds of course the Tomita - Takesaki theory. The main results on the structure of a CQ*-algebra point exactly in this direction and they answer to the natural question as to whether a strict pre CQ*-algebra admits a representation that produces in natural way a Tomita-Takesaki theory. The following example is quite enlightening.

Example 2.5 Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a strict pre CQ*-algebra whose norm is Hilbertian. We assume that

$$\langle ax|y\rangle = \langle x|a^{\#}y\rangle, \quad \forall a \in \mathfrak{A}_0,\ x,y \in \mathfrak{X}. \tag{2.3}$$

Then $\mathfrak{A}_0$ is *left Hilbert algebra* [12, 7]. For any $a \in \mathfrak{A}_0$ we denote by $\pi_{\mathfrak{A}_0}(a)$ the map $x \in \mathfrak{X} \to ax \in \mathfrak{X}$; clearly, $\pi_{\mathfrak{A}_0}$ is a $*$-representation of $\mathfrak{A}_0$ on $\mathfrak{X}$. We denote by $S_{\mathfrak{A}_0}$ the closure of the involution $^{\#}$. Let $S_{\mathfrak{A}_0} = J_{\mathfrak{A}_0}\Delta_{\mathfrak{A}_0}^{\frac{1}{2}}$ be the polar decomposition of $S_{\mathfrak{A}_0}$. Then $J_{\mathfrak{A}_0}$ is an isometric involution

on $\mathfrak{X}$ and $\Delta_{\mathfrak{A}_0}$ is a non-singular positive self-adjoint operator in $\mathfrak{X}$ such that $S_{\mathfrak{A}_0} = J_{\mathfrak{A}_0}\Delta_{\mathfrak{A}_0}^{\frac{1}{2}} = \Delta_{\mathfrak{A}_0}^{-\frac{1}{2}}J_{\mathfrak{A}_0}$ and $S^*_{\mathfrak{A}_0} = J_{\mathfrak{A}_0}\Delta_{\mathfrak{A}_0}^{-\frac{1}{2}} = \Delta_{\mathfrak{A}_0}^{\frac{1}{2}}J_{\mathfrak{A}_0}$. The operator $J_{\mathfrak{A}_0}$ is called the *modular conjugation* of $\mathfrak{A}_0$ and $\Delta_{\mathfrak{A}_0}$ is called the *modular operator* of $\mathfrak{A}_0$. The commutant $\mathfrak{A}_0'$ of $\mathfrak{A}_0$ is defined as follows: for any $y \in \mathcal{D}(S^*_{\mathfrak{A}_0})$ we put $\pi'_{\mathfrak{A}_0}(y)a = \pi_{\mathfrak{A}_0}(a)y$, $a \in \mathfrak{A}_0$ and put $\mathfrak{A}_0' = \{y \in \mathcal{D}(S^*_{\mathfrak{A}_0}); \pi'_{\mathfrak{A}_0}(y)$ is bounded $\}$. Then $\mathfrak{A}_0'$ is a left Hilbert algebra in $\mathfrak{X}$ with involution $S^*_{\mathfrak{A}_0}$ and multiplication $y_1y_2 \equiv \pi'_{\mathfrak{A}_0}(y_2)y_1$. Similarly, the commutant $\mathfrak{A}_0''$ of $\mathfrak{A}_0'$ is defined by $\mathfrak{A}_0'' = \{x \in \mathcal{D}(S_{\mathfrak{A}_0});\ y \to xy$ is continuous, $y \in \mathfrak{A}_0'\}$. For any $x \in \mathfrak{A}_0''$ we denote by $\pi_{\mathfrak{A}_0}(x)$ the unique continuous linear operator on $\mathfrak{X}$ such that $\pi_{\mathfrak{A}_0}(x)y = \pi'_{\mathfrak{A}_0}(y)x, y \in \mathfrak{A}_0'$. Then $\mathfrak{A}_0''$ is a left Hilbert algebra in $\mathfrak{X}$ with involution $S_{\mathfrak{A}_0}$ and multiplication $x_1x_2 \equiv \pi_{\mathfrak{A}_0}(x_1)x_2$ containing $\mathfrak{A}_0$. The left Hilbert algebra $\mathfrak{A}_0$ is said to be *full* if $\mathfrak{A}_0 = \mathfrak{A}_0''$. From the Tomita fundamental theorem it follows that $J_{\mathfrak{A}_0}\pi_{\mathfrak{A}_0}(\mathfrak{A}_0)''J_{\mathfrak{A}_0} = \pi_{\mathfrak{A}_0}(\mathfrak{A}_0)'$ and $\Delta^{it}_{\mathfrak{A}_0}\pi_{\mathfrak{A}_0}(\mathfrak{A}_0)''\Delta^{-it}_{\mathfrak{A}_0} = \pi_{\mathfrak{A}_0}(\mathfrak{A}_0)'', \forall t \in \mathbb{R}$. In general, the modular conjugation $J_{\mathfrak{A}_0}$ may differ from the involution * of $\mathfrak{X}$. In [7], we defined as *standard* the case where $J_{\mathfrak{A}_0} =^*$ and it has been proved that $\mathfrak{X}$ is standard if, and only if, $\langle a^{\#}|a^*\rangle \geq 0$, for every $a \in \mathfrak{A}_0$. The standardness condition is also equivalent to the following statements: *there exists a von Neumann algebra $\mathfrak{M}$ on $\mathfrak{X}$ with a cyclic and separating vector.*

The previous example gives a hint for the following

Definition 2.6 Let $\mathfrak{X}$ be a left Banach $\mathfrak{A}_{\#}$-module. A standard representation of $\mathfrak{X}$ in Hilbert space $\mathcal{H}$ consists of a linear map $\Phi : \mathfrak{X} \to \mathcal{H}$ and a $^{\#}$-representation of $\mathfrak{A}_{\#}$, $\pi : \mathfrak{A}_{\#} \to \mathcal{B}(\mathcal{H})$ such that

(i) $\Phi(ax) = \pi(a)\Phi(x), \quad \forall a \in \mathfrak{A}_{\#}, x \in \mathfrak{X}$

(ii) the von Neumann algebra $\pi(\mathfrak{A}_{\#})''$ has a cyclic and separating vector $\xi_0 \in \mathcal{H}$.

The following result, proven in [17], generalizes a statement given in [7].

Proposition 2.7 *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a strict pre CQ*-algebra with quasi-unit u and $\varphi \in \mathcal{S}_{\#}(\mathfrak{X})$ and ρ_φ the $^{\#}$-representation of $\mathfrak{A}_{\#}$ in $\mathcal{H}_\varphi$ associated with φ. Assume that*

$$\varphi(x, y) = \varphi(y^*, x^*), \quad \forall x, y \in \mathfrak{X}.$$

Then the triple $(\rho_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ for φ is standard representation of $\mathfrak{X}$.

2.2 The proper case

The study of the proper case, i.e., $_{\#} = *$ on $\mathfrak{A}_0$ has been mostly performed in [4, 3, 5].

A proper pre CQ*-algebra can be constructed as follows: let $\mathfrak{A}_0$ be a *-algebra with a C*-norm $\| \ \|_0$. Suppose that in $\mathfrak{A}_0$ a second norm $\| \ \|$ is defined, with the properties:

(a) $\|a^*\| = \|a\|, \quad \forall a \in \mathfrak{A}_0$;

(b) $\|ab\| \leq \|a\|\|b\|_0, \quad \forall a, b \in \mathfrak{A}_0$.

Then the couple $(\mathfrak{X}, \mathfrak{A}_0)$, with $\mathfrak{X}$ the $\| \ \|$-completion of $\mathfrak{A}_0$, is a proper (pre) CQ*-algebra. Any proper (pre) CQ*-algebra, on the other hand, can be obtained with this procedure. Any commutative CQ*-algebra is necessarily proper [2].

Examples of proper (pre) CQ*-algebras are easily found among spaces of functions. For instance $(L^p(\mathbb{R}), C_0(\mathbb{R}))$ is a proper pre CQ*-algebra, while $(L^p(I), C(I))$ is a proper CQ*-algebra. They are *-semisimple if, and only if, $p \geq 2$. In [3], in particular, it has been shown that any *-semisimple commutative CQ*-algebra can be realized as a CQ*-algebra of measurable functions on an appropriate measure space.

The analysis of the structure properties in the non-commutative case has been performed in [5]. We summarize here the main results.

Let

$$\mathcal{T}(\mathfrak{X}) = \{\varphi \in \mathcal{S}(\mathfrak{X}) : \ \varphi(x, x) = \varphi(x^*, x^*), \ \forall x \in \mathfrak{X}\}.$$

A proper CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is said to be *strongly *-semisimple* if $\varphi(x, x) = 0$ $(x \in \mathfrak{X})$ for every $\varphi \in \mathcal{T}(\mathfrak{X})$ implies $x = 0$.
If $(\mathfrak{X}, \mathfrak{A}_0)$ is strongly *-semisimple, we say it is *strongly regular* if

$$\|x\| = \sup_{\varphi \in \mathcal{T}(\mathfrak{X})} \varphi(x, x)^{1/2}.$$

If $\mathfrak{M}$ is a von Neumann possessing a sufficient family of normal finite traces $\{\tau_\alpha\}_{\alpha \in \mathcal{I}}$, with $\tau_\alpha(I) \leq C$ for all α then we denote with $\mathfrak{M}_2$ the strongly *-semisimple CQ*-algebra obtained by completing $\mathfrak{M}$ with respect to the norm

$$\|A\|_{2,\mathcal{I}} := \sup_\alpha \tau_\alpha(A^*A)^{1/2}, \quad A \in \mathfrak{M}.$$

As shown in [5], $\mathfrak{M}_2$ consists of *measurable* operators in the sense of Segal [11] and Nelson [9]. This concept is, as is known, the cornerstone of the theory of *non-commutative integration*.

Then we have proven the following representation theorem.

Theorem 2.8 *Let* $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ *be a proper strongly *-semisimple CQ*-algebra with unit* e, π *the universal representation of* $\mathfrak{A}_0$ *and let* $\pi(\mathfrak{A}_0) := \mathfrak{M}$ *be a von Neumann algebra. Then there exists a monomorphism*

$$\Phi : x \in \mathfrak{X} \to \Phi(x) := \widetilde{X} \in \mathfrak{M}_2$$

with the following properties:

(i) Φ *extends the isometry* $\pi : \mathfrak{A}_0 \hookrightarrow \mathcal{B}(\mathcal{H})$ *given by the Gel'fand–Naĭmark theorem;*

(ii) $\Phi(x^*) = \Phi(x)^*, \quad \forall x \in \mathfrak{X}$

(iii) $\Phi(xy) = \Phi(x)\Phi(y)$ *for every* $x, y \in \mathfrak{X}$ *such that* $x \in \mathfrak{A}_0$ *or* $y \in \mathfrak{A}_0$

(iv) if $(\mathfrak{X}, \mathfrak{A}_0)$ *is strongly regular, then* Φ *is an isometric *-isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{M}_2$.

Then $\mathfrak{X}$ *can be identified with a space of operators affiliated with* $\mathfrak{M}$.

3 More results on the structure of a proper CQ*-algebra

In this Section we will discuss some further structure property of proper CQ*-algebras.

3.1 Representations

Families of sesquilinear forms as we have seen play a relevant role for the study of the structure of CQ*-algebras or more in general of Banach C*-modules. The main reason is that they give rise to *representations* with operators acting in Hilbert space.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^*{\restriction}\mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\square\, X_2 = {X_1}^\dagger{}^* X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, iff $X_2\mathcal{D} \subset \mathcal{D}({X_1}^\dagger{}^*)$ and ${X_1}^*\mathcal{D} \subset \mathcal{D}(X_2{}^*)$ (we write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$). When we regard $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ as a *partial *-algebra* [1] with those operations, we denote it by $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$.

A *partial O*-algebra* on $\mathcal{D}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$, containing the identity and such that $X^\dagger \in \mathfrak{M}$

whenever $X \in \mathfrak{M}$ and $X_1 \,\square\, X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_2 \in R^{\mathrm{w}}(X_1)$.

A *Hilbert space *-representation* of the proper CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is a *-homomorphism of $\mathfrak{X}$ into $\mathcal{L}^{\dagger}_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$, for some pair $\mathcal{D} \subset \mathcal{H}$, that is, a linear map $\pi : \mathfrak{X} \to \mathcal{L}^{\dagger}_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$ such that : (i) $\pi(x^*) = \pi(x)^{\dagger}$ for every $x \in \mathfrak{X}$; (ii) if $x \in \mathfrak{X}$ and $a \in \mathfrak{A}_0$ then $\pi(x) \in L^{\mathrm{w}}(\pi(a))$ and $\pi(x) \,\square\, \pi(a) = \pi(xa)$.

First of all, we will remind the adaptation of the Gel'fand–Naĭmark–Segal (GNS) construction to the present situation.

Proposition 3.1 *Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Put*

$$\pi^{\circ}_{\varphi}(x)\lambda_{\varphi}(a) = \lambda_{\varphi}(xa), \quad x \in \mathfrak{X}, a \in \mathfrak{A}_0.$$

*Then π°_{φ} is a *-representation of $\mathfrak{X}$ into $\mathcal{L}^{\dagger}_{\mathrm{w}}(\lambda_{\varphi}(\mathfrak{A}_0), \mathcal{H}_{\varphi})$.*

Denote by π_{φ} the closure of π°_{φ}. Then $\overline{\pi_{\varphi}(\mathfrak{A}_0)} \subset \mathcal{B}(\mathcal{H}_{\varphi})$ and $\pi_{\varphi}(\mathfrak{A}_0)\mathcal{D}(\pi_{\varphi}) = \mathcal{D}(\pi_{\varphi})$. The triple $(\pi_{\varphi}, \lambda_{\varphi}, \mathcal{H}_{\varphi})$ is called the *GNS construction* for φ and we refer to π_{φ} as to the GNS representation of $\mathfrak{X}$ *constructed* from φ.

Proposition 3.2 *The *-representation π_{φ} is bounded if, and only if, φ is admissible, i.e, for every $a \in \mathfrak{A}_0$ there exists $\gamma_x > 0$ such that*

$$\varphi(xa, xa) \leq \gamma_x \varphi(a, a), \quad \forall a \in \mathfrak{A}_0.$$

3.2 Auxiliary seminorms on CQ*-algebras

We put

$$p(x) = \sup_{\varphi \in \mathcal{S}(\mathfrak{X})} \varphi(x, x)^{1/2}$$

then p is a seminorm on $\mathfrak{X}$ with $p(x) \leq \|x\|$ for every $x \in \mathfrak{X}$.

As shown in [15], the set $\mathcal{S}(\mathfrak{X})$ is convex and has extreme elements whose closed convex hull is exactly $\mathcal{S}(\mathfrak{X})$. For this reason, if $E\mathcal{S}(\mathfrak{X})$ denotes the set of extreme elements of $\mathcal{S}(\mathfrak{X})$

$$p(x) = \sup_{\varphi \in E\mathcal{S}(\mathfrak{X})} \varphi(x, x)^{1/2} \leq \sup_{\|\varphi\|=1} \varphi(x, x)^{1/2}$$

because if $\varphi \in E\mathcal{S}(\mathfrak{X})$ then $\|\varphi\| = 1$.
Moreover,

$$\sup_{\|\varphi\|=1} \varphi(x, x)^{1/2} \leq \sup_{\varphi \in \mathcal{S}(\mathfrak{X})} \varphi(x, x)^{1/2}$$

thus we finally get the equality

$$p(x) = \sup_{\|\varphi\|=1} \varphi(x,x)^{1/2}.$$

Without loss of generality, we may assume that $\|e\| = 1$. Then, for any $\varphi \in \mathcal{S}(\mathfrak{X})$, we obviously have: $\varphi(e,e) \leq \|e\|^2 = 1$.

We now pose the following question: is it possible that $\varphi(e,e) = \|\varphi\|$, for every $\varphi \in \mathcal{P}(\mathfrak{X})$, as it happens for the corresponding objects on a Banach *-algebra? For answering this question we introduce the following seminorm:

$$q(x) = \sup_{\varphi(e,e)=1} \varphi(x,x)^{1/2}$$

and

$$D(q) = \{x \in \mathfrak{X} : \, q(x) < \infty\}.$$

Moreover we put

$$q'(x) = \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\, a \in \mathfrak{A}_0,\, \varphi(a,a) = 1\}$$

and

$$D(q') = \{x \in \mathfrak{X} : \, q'(x) < \infty\}.$$

Lemma 3.3 *$D(q) = D(q')$ and $q(x) = q'(x)$ for every $x \in D(q)$.*

Proof. It is clear that if $x \in D(q')$ then we have

$$\begin{aligned} q(x) &= \sup_{\varphi(e,e)=1} \varphi(x,x)^{1/2} \\ &\leq \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\, a \in \mathfrak{A}_0,\, \varphi(a,a) = 1\} < \infty. \end{aligned}$$

Therefore $D(q') \subseteq D(q)$ and

$$q(x) \leq q'(x), \quad \forall x \in D(q').$$

On the other hand it is easily seen that if $\varphi \in \mathcal{P}(\mathfrak{X})$, then, for each $a \in \mathfrak{A}_0$, $\varphi_a \in \mathcal{P}(\mathfrak{X})$ where $\varphi_a(x,y) = \varphi(xa,ya)$ for any $x, y \in \mathfrak{X}$. Clearly $\varphi_a(e,e) = \varphi(a,a)$. Thus,

$$\sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\, a \in \mathfrak{A}_0,\, \varphi(a,a) = 1\} \leq \sup_{\varphi(e,e)=1} \varphi(x,x)^{1/2}.$$

So, if $x \in D(q)$ then $x \in D(q')$ and $q'(x) \leq q(x)$. □

Now, we have

$$p(x) \leq \sup_{\varphi(e,e)\leq 1} \varphi(x,x)^{1/2} = \sup_{\varphi(e,e)=1} \varphi(x,x)^{1/2} = q(x), \quad \forall x \in D(q).$$

Indeed, $q(x) \leq \sup_{\varphi(e,e)\leq 1} \varphi(x,x)^{1/2}$ is trivial. Conversely, if $\varphi(e,e) < 1$ (we may suppose $\varphi(e,e) > 0$) then we put $\psi(x,y) = \frac{\varphi(x,y)}{\varphi(e,e)}$, $x, y \in \mathfrak{X}$ and get the desired inequality.

Now we want to explore what happens if $\varphi(e,e) = \|\varphi\|$, for an arbitrary $\varphi \in \mathcal{P}(\mathfrak{X})$.

Clearly, in this case, $D(q) = \mathfrak{X}$ and $p(x) = q(x)$, for every $x \in \mathfrak{X}$.

Theorem 3.4 *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a proper CQ*-algebra. The following statements are equivalent.*

(i) For each $\varphi \in \mathcal{P}(\mathfrak{X})$, $\varphi(e,e) = \|\varphi\|$.

(ii) For each $\varphi \in \mathcal{P}(\mathfrak{X})$,

$$\varphi(xa, xa) \leq \|x\|^2 \varphi(a,a), \quad \forall x \in \mathfrak{X}, a \in \mathfrak{A}_0.$$

(iii) The GNS representation π_φ of $\mathfrak{X}$ constructed from any $\varphi \in \mathcal{P}(\mathfrak{X})$ is bounded and continuous; moreover

$$\|\pi_\varphi(x)\| \leq \|x\|, \quad \forall x \in \mathfrak{X}.$$

Proof. (i) $\Rightarrow$ (ii): From the definition itself of $q(x)$, for any $\varphi \in \mathcal{P}(\mathfrak{X})$ we get

$$\varphi(xa, xa) \leq q(x)^2 \varphi(a,a), \quad \forall x \in \mathfrak{X}, a \in \mathfrak{A}_0.$$

But, as seen before, the assumption implies that $q(x) = p(x) \leq \|x\|$ for every $x \in \mathfrak{X}$ and then the statement follows.
(ii) $\Rightarrow$ (iii): Let π_φ be the GNS representation constructed from $\varphi \in \mathcal{P}(\mathfrak{X})$. Then we have, for every $x \in \mathfrak{X}$ and $a \in \mathfrak{A}_0$,

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) \leq \|x\|^2 \varphi(a,a) = \|x\|^2 \|\lambda_\varphi(a)\|^2.$$

Therefore

$$\|\pi_\varphi(x)\| \leq \|x\|, \quad \forall x \in \mathfrak{X}.$$

(iii) $\Rightarrow$ (i): Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Then, for every $x \in \mathfrak{X}$ and $a \in \mathfrak{A}_0$, we have

$$\varphi(xa, xa) = \|\pi_\varphi(x)\lambda_\varphi(a)\|^2 \leq \|x\|^2 \varphi(a,a)$$

which implies, for $a = e$

$$\varphi(x,x) \leq \|x\|^2 \varphi(e,e), \quad \forall x \in \mathfrak{X}.$$

From this inequality it follows that $\varphi(e,e) \geq \|\varphi\|$. Since one always has $\varphi(e,e) \leq \|\varphi\|$, the desired equality follows. □

Let $\mathcal{Q}(\mathfrak{X})$ denote the set of all positive sesquilinear forms φ on $\mathfrak{X} \times \mathfrak{X}$ such that

$$\varphi(xa,b) = \varphi(a,x^*b), \quad \forall x \in \mathfrak{X};\ a,b \in \mathfrak{A}_0.$$

Then $\mathcal{Q}(\mathfrak{X}) \supseteq \mathcal{P}(\mathfrak{X}) = \{\varphi \in \mathcal{Q}(\mathfrak{X}) : \varphi \text{ is bounded}\}$.

Then we have

Proposition 3.5 *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a proper CQ*-algebra. Consider the following statements:*

(a) Each $\varphi \in \mathcal{Q}(\mathfrak{X})$ is bounded and $\|\varphi\| = \varphi(e,e)$.

(b) For every $x \in \mathfrak{X}$ one has

$$\begin{aligned} p(x) &= \sup_{\varphi \in \mathcal{Q}(\mathfrak{X});\varphi(e,e)=1} \|\pi_\varphi(x)\| \\ &= \sup\{\|\pi(x)\| : \pi \text{ is a bounded *-representation of } (\mathfrak{X}, \mathfrak{A}_0)\}. \end{aligned}$$

*(c) Each *-representation π of $(\mathfrak{X}, \mathfrak{A}_0)$, satisfying $\pi(\mathfrak{A}_0)\mathcal{D}(\pi) \subseteq \mathcal{D}(\pi)$, is bounded and moreover*

$$\|\overline{\pi(x)}\| \leq \|x\|, \quad \forall x \in \mathfrak{X}.$$

Then:

$$(a) \Longleftrightarrow (b) \Longrightarrow (c).$$

Proof. (a)⇒(b): The assumption implies that $\mathcal{Q}(\mathfrak{X}) = \mathcal{P}(\mathfrak{X})$ and the condition $\|\varphi\| = \varphi(e,e)$ implies that $p(x) = q(x)$. Therefore, for every $x \in \mathfrak{X}$:

$$p(x) = \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{Q}(\mathfrak{X}),\ a \in \mathfrak{A}_0,\ \varphi(a,a) = 1\} = \sup_{\varphi \in \mathcal{Q}(\mathfrak{X})} \|\pi_\varphi(x)\|.$$

This inequality also implies that

$$p(x) \leq \sup\{\|\pi(x)\| : \pi \text{ is a bounded *-representation of } (\mathfrak{X}, \mathfrak{A}_0)\}.$$

On the other hand, if π is a bounded *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ and $\xi \in \mathcal{H}$, we can define

$$\varphi_\xi(x,y) = \langle \pi(x)\xi | \pi(y)\xi \rangle.$$

Then we have

$$\|\pi(x)\xi\|^2 = \|\pi_{\varphi_\xi}(x)\lambda_{\varphi_\xi}(e)\|^2 \leq \|\pi_{\varphi_\xi}(x)\|^2\|\xi\|^2.$$

Thus $\|\pi(x)\| \leq \|\pi_{\varphi_\xi}(x)\|$, for every $x \in \mathfrak{X}$. This clearly implies the statement.
(b)⇒(a): The assumption implies

$$\varphi(x,x) = \|\pi_\varphi(x)\lambda_\varphi(e)\|^2 \leq p(x)^2\varphi(e,e) \leq \|x\|^2\varphi(e,e), \quad \forall x \in \mathfrak{X}.$$

This implies that φ is bounded and that $\|\varphi\| \leq \varphi(e,e)$. But one always has $\varphi(e,e) \leq \|\varphi\|$. The equality then follows.
(a)⇒(c): Let π be a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ and $\xi \in \mathcal{D}(\pi)$. We define, as before,

$$\varphi_\xi(x,y) = \langle \pi(x)\xi | \pi(y)\xi \rangle.$$

Then it is easily seen that $\varphi_\xi \in \mathcal{Q}(\mathfrak{X})$. By the assumption we also get $\varphi_\xi \in \mathcal{P}(\mathfrak{X})$. Then by (iii) of Theorem 3.4, the corresponding GNS representation π_{φ_ξ} is bounded and $\|\pi_{\varphi_\xi}(x)\| \leq \|x\|$ for each $x \in \mathfrak{X}$.
Then we have, for each $x \in \mathfrak{X}$,

$$\|\pi(x)\xi\|^2 = \|\pi_{\varphi_\xi}(x)\lambda_{\varphi_\xi}(e)\|^2 \leq \|x\|^2\|\lambda_{\varphi_\xi}(e)\|^2 = \|x\|^2\varphi_\xi(e,e) = \|x\|^2\|\xi\|^2.$$

Thus π is bounded and $\|\overline{\pi(x)}\| \leq \|x\|$ for each $x \in \mathfrak{X}$.
This concludes the proof.

□

As we have seen the condition $\|\varphi\| = \varphi(e,e)$, for each $\varphi \in \mathcal{P}(\mathfrak{X})$, has very relevant consequences on the structure of $(\mathfrak{X}, \mathfrak{A}_0)$. We leave open the question of finding examples where this condition is satisfied or, alternatively, proving that it forces $\mathfrak{X}$ to be a Banach *-algebra.

References

[1] J-P. Antoine , A. Inoue , and C. Trapani *"Partial *-Algebras and Their Operator Realizations"*, Mathematics and Its Applications, vol. 553, Kluwer, Dordrecht, NL, 2002

[2] F.Bagarello and C.Trapani, States and representations of CQ* -algebras, Ann. Inst. H. Poincaré **61**, 103-133(1994)

[3] F.Bagarello and C.Trapani, L^p- spaces as quasi *-algebras, J.Math. Anal. Appl . **197**, 810-824 (1996)

[4] F.Bagarello and C.Trapani, CQ*-algebras: structure properties Publ. RIMS, Kyoto Univ. **32**, 85-116 (1996)

[5] F.Bagarello, C.Trapani and S.Triolo, "CQ*-algebras and non-commutative integration", preprint Dip. Matematica e Applicazioni 254, Agosto 2004.

[6] F.Bagarello and C.Trapani, Morphisms of certain Banach C*-modules, Publ. RIMS, Kyoto Univ. **36** (2000) 681-705

[7] F.Bagarello, A.Inoue and C.Trapani, Some classes of topological quasi *-algebras, Proc. Amer. Math. Soc., **129** (2001), 2973-2980.

[8] I.M.Gel'fand and N.Ya.Vilenkin, *Generalized functions Vol.4*, Academic Press, New York and London, 1964

[9] E. Nelson, Note on non-commutative integration, J. Funct. Anal., **15** (1974) 103-116

[10] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Akademie-Verlag, Berlin, 1990.

[11] I. E. Segal, A noncommutative extension of abstract integration, *Ann. Math.* **57** (1953), 401–457.

[12] S. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Editura Academiei, Bucharest and Abacus Press, Tunbridge Wells, Kent, 1979.

[13] C. Trapani, Some results on CQ*-algebras of Operators, Proc. Int. Workshop on Operator Theory, Cefalù, Italy, July 14-19, 1997; Suppl. Rend. Circ. Mat. Palermo **56**, 199-205 (1998)

[14] C. Trapani, CQ*-algebras of operators and application to Quantum Models, in Proceedings of the Second ISAAC Conference, Fukuoka, Japan, Volume 1, 679-685, Kluwer (2000).

[15] C. Trapani, Sesquilinear forms on certain Banach C*-modules, Proceedings of the IV International Conference on Functional Analysis and Approximation Theory, Suppl. Ren. Circ. Mat. Palermo., **68** (2002) 855-864

[16] C. Trapani, CQ*-algebras of operators: density properties, in "Operator Theory and Banach Algebras", edited by M.Chidami, R.Curto, M.Mbekhta, F.H.Vasilescu and J.Zemanek , Theta, Bucharest 2003

[17] C. Trapani and S.Triolo, Representations of certain Banach C*-modules, pre-print Dip. Matematica e Applicazioni n.242, Marzo 2004, to appear in Mediterranean J. Math.

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K-Functionals Related to Semigroups of Operators

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Abstract. Let $(X, \|\cdot\|)$ be a Banach space; let $\{T(t) : t \geq 0\}$ be an equibounded $(\mathcal{C}_0)$-semigroup of linear operators with infinitesimal generator A. The purpose of the paper is to characterize the K-functional with respect to $(X, D((-A)^\alpha))$ where $(-A)^\alpha$ is a suitable fractional power of A. This is achieved for all $\alpha > 0$ via appropriate truncated hypersingular integrals, thus extending an earlier result by the authors. A functional calculus due to L. Schwartz also allows analogs for n-parameter groups. An approach via fractional differences is indicated. In the setting of quantitative approximation theory, some applications concerning optimal and non-optimal rates of convergence are considered.

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1 Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space, $B : D(B) \to X$ be a closed linear operator with domain $D(B) \subset X$, dense in X. The K-functional with respect to X and $D(B)$, introduced by J. Peetre 1963, is defined by

$$K(t, f; X, D(B)) := \inf_{g \in D(B)} \Big(\|f - g\| + t \|Bg\| \Big), \qquad t > 0,\ f \in X.$$

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Now we link the idea of the K-functional with the notion of an equibounded $(\mathcal{C}_0)$-semigroup of linear operators $\{T(t) : t \geq 0\}$ from X into itself, defined by the properties

$$\begin{aligned} T(t_1)T(t_2) &= T(t_1+t_2) \text{ for all } t_1, t_2 \geq 0, \qquad T(0) = I, \\ \|T(t)\| &\leq M \text{ with a constant } M \text{ independent of } t \geq 0, \\ \lim_{t\to 0+} \|T(t)f - f\| &= 0 \text{ for each } f \in X \quad ((\mathcal{C}_0)\text{-property}). \end{aligned}$$

Its infinitesimal generator A, given via the strong limit

$$Af := \text{s-}\lim_{t\to 0+} \frac{T(t)-I}{t} f\,, \qquad A^k = A(A^{k-1})\,, \tag{1.1}$$

is a closed linear operator with dense domain $D(A)$. It is a classical result due to J. Peetre that for each $r \in \mathbb{N}$

$$K(t^r, f; X, D(A^r)) \approx \sup_{0<s<t} \| [T(s)-I]^r f \|\,, \quad t > 0,\ f \in X\,. \tag{1.2}$$

If $\{T(t) : t \geq 0\}$ is holomorphic, this characterization was improved by Ditzian and Ivanov [7] to

$$K(t^r, f; X, D(A^r)) \approx \| [T(t)-I]^r f \|\,, \quad t > 0,\ f \in X\,,\ r \in \mathbb{N}. \tag{1.3}$$

Here we mainly want to generalize the relation (1.2) to K-functionals associated with the couple $(X, D((-A)^\alpha)$, $\alpha > 0$, by means of truncated hypersingular integrals. As a motivation for dealing with this problem, let us indicate by means of a simple example how the K-functional may be used as some kind of interface in quantitative approximation theory; in this connection let us mention that, when deriving the quantitative approximation behavior of the Bernstein polynomials, DeVore and Lorentz [5, p. 322] attribute to the K-functional the property of being "an essential tool". Therefore, it is clear that its different characterizations are of interest.

Example 1.1 *Let $X_{2\pi} = L^p_{2\pi}$, $1 \leq p < \infty$, or $C_{2\pi}$ with standard norm and choose two semigroups: On the one hand take the semigroup of left translations $\{T(t)\}$, defined by $T(t)f(x) := f(x+t)$, on the other the holomorphic semigroup $\{W_t\}$ generated by the Weierstrass integral*

$$W_t(f;x) := \sum_{k=-\infty}^{\infty} e^{-tk^2} \widehat{f}_k e^{ikx}, \qquad \widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx, \quad k \in \mathbb{Z}.$$

Then the associated infinitesimal generators are $A_T = d/dx$ *and* $A_W = d^2/dx^2$ *with domains*

$$D(A_T) = \{f \in X_{2\pi} : f \in AC_{2\pi},\, f' \in X_{2\pi}\},$$

$$D(A_W) = \{f \in X_{2\pi} : f, f' \in AC_{2\pi},\, f'' \in X_{2\pi}\},$$

respectively (in this context see, e.g., [3, Chap. I]). Hence $A_T^2 = A_W$, *and the preceding characterizations give the well-known result*

$$\begin{aligned}\|W_t(f) - f\|_{X_{2\pi}} &\approx K(t, f; X_{2\pi}, D(A_W)) = K(t, f; X_{2\pi}, D(A_T^2)) \\ &\approx \sup_{s\le\sqrt{t}} \|f(\cdot + 2s) - 2f(\cdot + s) + f(\cdot)\|_{X_{2\pi}} =: \omega_2(\sqrt{t}, f; X_{2\pi}).\end{aligned}$$

This is a precise description of the approximation behavior of the Weierstrass means of f *via the second order modulus of smoothness of the function* f. *E.g., an elementary argument shows that* $\omega_2(\sqrt{t}, f; X_{2\pi}) = o(t)$ *implies* $f = const.$; *thus, the Weierstrass means can approximate non-trivial* f *at most with an optimal order* $O(t)$, *and all elements* $f \in X_{2\pi}$ *for which* $\|W_t(f) - f\|_{X_{2\pi}} = O(t)$ *holds are characterized by* $\omega_2(t, f; X_{2\pi}) = O(t^2)$.

In the following we write $g(t) \lesssim h(t)$ on an interval $I \subset \mathbb{R}$ if for some constant $c \ge 0$ we have $g(t) \le c\,h(t)$ for all $t \in I$; analogously $g(t) \gtrsim h(t)$ is explained and we use the notation $g(t) \approx h(t)$ if $g \lesssim h$ and $g \gtrsim h$ on I. Generic constants will be denoted by c.

Before stating our main result we have to explain how the fractional power of A is defined. The intuitively simplest way is an approach in the sense of Grünwald-Letnikov (see [12, p. 373] and [18]):

$$(-A)^\alpha f := \text{s-}\lim_{t\to 0+} \frac{[I - T(t)]^\alpha}{t^\alpha} f\,, \quad \alpha > 0, \tag{1.4}$$

for those $f \in X$, for which the limit exists, i.e., for $f \in D((-A)^\alpha)$, where

$$[I - T(t)]^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} T(jt), \qquad \binom{\beta}{j} = \prod_{k=1}^{j} \frac{\beta - k + 1}{k}, \ \beta \in \mathbb{R}.$$

Using a functional calculus, based on the Laplace transform and presented, e.g., in [19] along the lines of a more abstract one due to L. Schwartz, it was shown in [15] that

$$K(t^\alpha, f; X, D((-A)^\alpha)) \approx \sup_{0<s<t} \|[I - T(s)]^\alpha f\| =: \omega_\alpha(t, f)\,, \quad t > 0. \tag{1.5}$$

If $\{T(t) : t \geq 0\}$ is holomorphic, the "sup" can be dropped also in the fractional case, thus

$$K(t^\alpha, f; X, D((-A)^\alpha)) \approx \|[I - T(t)]^\alpha f\|, \quad t > 0,\ f \in X,\ \alpha > 0.$$

We mention some elementary properties of the abstract modulus of smoothness in (1.5)

$$\omega_\alpha(t, f) \lesssim \omega_\beta(t, f), \qquad 0 < \beta < \alpha, \tag{1.6}$$

$$\omega_\alpha(\lambda t, f) \lesssim \max\{1, \lambda^\alpha\}\, \omega_\alpha(t, f) \qquad \text{for all } \lambda > 0, \tag{1.7}$$

$$\frac{\omega_\alpha(s, f)}{s^\alpha} \lesssim \frac{\omega_\alpha(t, f)}{t^\alpha}, \qquad s > t. \tag{1.8}$$

Clearly (1.6) is true since $[I - T(s)]^\alpha = [I - T(s)]^{\alpha-\beta}[I - T(s)]^\beta$, and $\|[I - T(s)]^{\alpha-\beta}\| \lesssim (1 + M)^{\alpha-\beta}$ for all $s \geq 0$. The two remaining assertions immediately follow from the corresponding properties of the K-functional since, trivially, $K((\lambda t)^\alpha, f) \leq \max\{1, \lambda^\alpha\}\, K(t^\alpha, f)$ and $t^{-\alpha} K(t^\alpha, f)$ is monotone decreasing.

There is an equivalent definition of $(-A)^\alpha$ coming from a Marchaud-type representation (see [12, pp. 118], [17], and [19, Sec. 3]), namely

$$(-A)^\alpha f = \text{s-}\lim_{\varepsilon \to 0+} C_{\alpha,r}^{-1} \int_\varepsilon^\infty u^{-\alpha-1}[I - T(u)]^r f\, du, \quad f \in D((-A)^\alpha), \tag{1.9}$$

where $0 < \alpha < r \in \mathbb{N}$ and $C_{\alpha,r} = \int_0^\infty u^{-\alpha-1}(1 - e^{-u})^r du$ is a normalizing constant. We observe that $f \in D(A^r)$, $r \in \mathbb{N}$, implies $f \in D((-A)^\alpha)$, $0 < \alpha < r$, and that in this case

$$(-A)^\alpha f = C_{\alpha,r}^{-1} \int_0^\infty u^{-\alpha-1}[I - T(u)]^r f\, du \tag{1.10}$$

exists as a Bochner integral. In view of (1.4) and (1.5) our main result suggests itself from (1.9).

Theorem 1.2 *Let $(X, \|\cdot\|)$ be a Banach space, $\{T(t) : t \geq 0\}$ be an equibounded $(\mathcal{C}_0)$-semigroup, $D((-A)^\alpha))$, $\alpha > 0$, be as above. If $0 < \alpha < r$, $r \in \mathbb{N}$, then, for all $f \in X$ and $t > 0$,*

$$K(t^\alpha, f; X, D((-A)^\alpha)) \approx \sup_{0<s<t} s^\alpha \left\| \int_s^\infty u^{-\alpha-1}[I - T(u)]^r f\, du \right\|. \tag{1.11}$$

Remark 1.3 *(a) On the one hand, the result (1.11) is inspired by a two-sided estimate for the approximation behavior of the Riesz means in the $L^p(\mathbb{R}^n)$-norm, $p \geq 1$, due to Trigub [9], on the other by the description of the latter via the K-functional in [14].*

(b) We note that Theorem 1.2 extends an earlier result of the authors [15] from the range $0 < \alpha < 1$ or $\alpha \in \mathbb{N}$ to all $\alpha > 0$ (if $0 < \alpha < 1$ the sup *can be omitted).*

(c) An immediate consequence of the characterizations (1.5) and (1.11) as well as of the estimate (1.6) is the following Marchaud-type inequality for semigroups

$$\omega_\alpha(t, f) \lesssim t^\alpha \int_t^\infty u^{-\alpha-1} \omega_\gamma(u, f)\, du\,, \qquad t > 0\,,\ 0 < \alpha < \gamma\,. \tag{1.12}$$

This extends the inequality [15, (1.7)] from integer $\alpha, \gamma \in \mathbb{N}$, $\alpha < \gamma$, to all positive parameters α, γ with $\alpha < \gamma$.

(d) Let us point out that there exists a well developed quantitative approximation theory for commutative operators in Banach spaces in the case of nonoptimal approximation due to Butzer and Scherer [4] (see also [5, pp. 294]); this theory is based on interpolation spaces constructed with the aid of the K-functional. In Example 1.1 and in the following example, however, we argue directly with the K-functional (without using interpolation spaces).

Example 1.4 *Choose $X_{2\pi}$ as in Example 1.1 and consider the (holomorphic) Abel-Poisson semigroup $\{P_t\}$,*

$$P_t(f; x) := \sum_{k=-\infty}^{\infty} e^{-t|k|}\, \widehat{f}_k e^{ikx}\,, \qquad t > 0\,,\ f \in X_{2\pi},$$

with infinitesimal generator $A_P = -(d/dx)\mathcal{H}$, where $\mathcal{H}$ denotes the periodic Hilbert transform (cf. [3, p. 71]). Then $A_P^2 = -d^2/dx^2 = -A_T^2$, hence by (1.2) and its holomorphic counterpart (see Example 1.1)

$$\|P_{2t} f - 2P_t f + f\|_{X_{2\pi}} \approx \omega_2(t, f; X_{2\pi})$$

and, therefore, (1.6) and Theorem 1.2 give

$$\begin{aligned} \omega_2(t, f; X_{2\pi}) &\lesssim \|P_t(f) - f\|_{X_{2\pi}} \approx \sup_{0<s<t} s \,\Big\| \int_s^\infty u^{-2} [P_u - I]^2 f\, du \Big\|_{X_{2\pi}} \\ &\lesssim t \int_t^\infty u^{-2} \|[P_u - I]^2 f\|_{X_{2\pi}}\, du \approx t \int_t^\infty u^{-2} \omega_2(u, f; X_{2\pi})\, du\,, \end{aligned} \tag{1.13}$$

where the last inequality follows by Marchaud's inequality (1.12). This shows the well-known result that $O(t)$ is the optimal rate of convergence by which f can be approximated by its Poisson integral. Also note that the last integral can be estimated by $c_\gamma\,\omega_\gamma(t,f;X_{2\pi})$, $\gamma<1$ (but not by $\omega_1(t,f;X_{2\pi})$), when one uses (1.6) and (1.7). The above two-sided estimate immediately yields a nonoptimal approximation assertion in terms of the second modulus of continuity. Indeed, if $\omega_2(t,f;X_{2\pi})=O(\Omega(t))$ and Ω satisfies the averaging property $t\int_t^{1/2}u^{-2}\Omega(u)\,du\lesssim\Omega(t)$, then f is approximated by its Poisson integral with order $\Omega(t)$, and vice versa. The integral condition on Ω is satisfied, e.g., for $\Omega(t)=\log^\gamma(1/t)$, $\gamma<0$, or $t^\delta\log^\beta(1/t)$, $0<\delta<1$, $\beta\in\mathbb{R}$, but not when $\Omega(t)$ is "near" the optimal order t, e.g., if $\Omega(t)=t\log(1/t)$. For further averaging conditions on Ω see [4].

2 Proof of Theorem 1.2

The estimate of the truncated hypersingular integral by the K-functional is given in [15, Lemma 3.1]. To prove the converse we observe that for any $g_t\in D((-A)^\alpha)$ we trivially have

$$K(t^\alpha,f;X,D((-A)^\alpha))\le\|f-g_t\|+t^\alpha\|(-A)^\alpha g_t\|\,.$$

The main problem is to construct a suitable g_t which serves our purposes. To this end we need some notations and results developed in Westphal [17], [18] and which we recall for the convenience of the reader. The following two $L^1(0,\infty)$-functions play an essential role:

$$q_{\alpha,r}(u)=\frac{1}{\Gamma(\alpha+1)\,u}\sum_{j=0}^{r}(-1)^j\binom{r}{j}(u-j)_+^\alpha\,,\quad 0<\alpha<r,\tag{2.1}$$

$$p_{\alpha,r}(u)=\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{r}(-1)^j\binom{r}{j}(u-j)_+^{\alpha-1}\,,\quad 0<\alpha\le r\,;\tag{2.2}$$

here we use the standard notation ($a\in\mathbb{R}$ fixed)

$$(u-a)_+^\alpha=\begin{cases}(u-a)^\alpha & : \quad u>a\\ 0 & : \quad u\le a\,.\end{cases}$$

The functions $q_{\alpha,r}$ and $p_{\alpha,r}$ satisfy

$$\int_0^\infty q_{\alpha,r}(u)\,du = \int_0^\infty u^{-\alpha-1}(1-e^{-u})^r du = C_{\alpha,r}\,, \tag{2.3}$$

$$\int_0^\infty p_{\alpha,r}(u)\,du = \begin{cases} 0 & : \quad 0<\alpha<r \\ 1 & : \quad \alpha = r\,, \end{cases} \tag{2.4}$$

and have the Laplace transforms

$$\mathcal{L}[q_{\alpha,r}](\lambda) = \lambda^{-\alpha}\int_1^\infty u^{-\alpha-1}(1-e^{-\lambda u})^r du\,, \quad \mathcal{L}[p_{\alpha,r}](\lambda) = \lambda^{-\alpha}(1-e^{-\lambda})^r, \tag{2.5}$$

$\mathsf{Re}\,\lambda > 0$. The relevance of these two functions in our situation is indicated by the following two formulas. For each $f \in X$ and each $\varepsilon > 0$ there holds

$$(-A)^\alpha \int_0^\infty q_{\alpha,r}\Big(\frac{u}{\varepsilon}\Big)T(u)f\,\frac{du}{\varepsilon} = \int_\varepsilon^\infty u^{-\alpha-1}[I-T(u)]^r f\,du\,, \quad 0<\alpha<r, \tag{2.6}$$

and, for each $f \in D(A^r)$ and $t > 0$, one has

$$[I-T(t)]^r f = t^r \int_0^\infty p_{r,r}\Big(\frac{u}{t}\Big)T(u)(-A)^r f\,\frac{du}{t}\,. \tag{2.7}$$

For integer $\alpha = 1,\dots,r-1$, we essentially used in the proof of [15, Theorem 1.4] that $p_{\alpha,r}(u) = q_{\alpha,r}(u) = 0$ if $u > r$, a property we do not have at our disposal in the fractional case. The arising difficulties can be controlled with the aid of the following equality. For $f \in X$ there holds

$$\begin{aligned} &\sum_{k=1}^r (-1)^k \binom{r}{k} k^r \int_{k\varepsilon}^\infty u^{-r-1}[I-T(u)]^{r+1} f\,du \qquad (r\in\mathbb{N},\ \varepsilon>0) \\ &= \sum_{j=1}^r (-1)^j \binom{r+1}{j} j^r \int_{j\varepsilon}^{(r+1)\varepsilon} v^{-r-1}[I-T(v)]^r f\,dv\,, \end{aligned} \tag{2.8}$$

so that the infinite integration interval is reduced to a finite one. To verify (2.8) start with $u^{-r-1}[I-T(u)]^{r+1}f = \sum_{j=0}^{r+1}(-1)^j\binom{r+1}{j}u^{-r-1}T(ju)f$ and integrate this equality over $[k\varepsilon,\infty)$, the sum term by term. The substitution $v = ju/k$, each $j = 1,\dots,r+1$, $k = 1,\dots,r$, leads to

$$k^r\int_{k\varepsilon}^\infty [I-T(u)]^{r+1} f\,\frac{du}{u^{r+1}} = \sum_{j=1}^{r+1}(-1)^j\binom{r+1}{j}j^r\int_{j\varepsilon}^\infty T(kv)f\,\frac{dv}{v^{r+1}} + \frac{1}{r\varepsilon^r}f\,.$$

Insert this into the left side of (2.8) and interchange the order of summation to obtain nearly the right side of (2.8), nearly in so far as the integration runs over $[j\varepsilon,\infty)$. But this can be reduced to the final form since (cf. [3, p. 98]) $\sum_{j=1}^{r+1}(-1)^j\binom{r+1}{j}j^r=0$.

Now we set

$$g_t := F_r\sum_{k=1}^{r}(-1)^k\binom{r}{k}k^r\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}\int_0^\infty q_{r,r+1}(u)T(kj\frac{t}{r(r+1)}u)f\,du,$$

where the constant F_r is given by $(-1)^r r!\,C_{r,r+1}F_r=1$. In view of (2.6) it is clear that $g_t\in D((-A)^\alpha)$, $0<\alpha<r$. We observe that $\sum_{j=1}^r\ldots$ is needed for the estimate of $\|f-g_t\|$ and $\sum_{k=1}^r\ldots$ for the estimate of $t^\alpha\|(-A)^\alpha g_t\|$.

We start to dicuss $\|f-g_t\|$. Since $\sum_{k=1}^r(-1)^k\binom{r}{k}k^r=(-1)^r r!$, there holds

$$f-g_t=F_r\sum_{k=1}^{r}(-1)^k\binom{r}{k}k^r\int_0^\infty q_{r,r+1}(u)\Big[I-T(k\frac{t}{r(r+1)}u)\Big]^r f\,du\,;$$

thus, we have to estimate the integral for fixed k.

Lemma 2.1 *If $f\in X$, $\varepsilon>0$, $0<\alpha<r$, then*

$$\Big\|\int_0^\infty q_{r,r+1}\Big(\frac{u}{\varepsilon}\Big)[I-T(u)]^r f\,\frac{du}{\varepsilon}\Big\|\le c\sup_{0<\frac{s}{r+1}\le\varepsilon}s^\alpha\Big\|\int_s^\infty u^{-\alpha-1}[I-T(u)]^r f\,du\Big\|.$$

The constant c is given by $c=\alpha\,\|q_{r,r+1}\|_1+\|p_{r,r+1}\|_1$ and, therefore, is independent of f and $\varepsilon>0$. Also the standard notation $\|q\|_1=\int_0^\infty|q(u)|\,du$ is used.

Proof. Since $u\,q_{r,r+1}(u)=\int_0^u p_{r,r+1}(v)\,dv$, integration by parts leads to (cf. [15, (3.10)])

$$u^{\alpha+1}q_{r,r+1}\Big(\frac{u}{\varepsilon}\Big)=\int_0^u v^\alpha[\,\alpha\,q_{r,r+1}\Big(\frac{v}{\varepsilon}\Big)+p_{r,r+1}\Big(\frac{v}{\varepsilon}\Big)\,]\,dv\,.$$

Using this identity and Fubini's theorem, we obtain

$$\begin{aligned}&\int_0^\infty q_{r,r+1}\Big(\frac{u}{\varepsilon}\Big)[I-T(u)]^r f\,\frac{du}{\varepsilon}\\&=\int_0^\infty u^{-\alpha-1}[I-T(u)]^r f\left(\int_0^u v^\alpha[\,\alpha\,q_{r,r+1}\Big(\frac{v}{\varepsilon}\Big)+p_{r,r+1}\Big(\frac{v}{\varepsilon}\Big)\,]\,dv\right)\frac{du}{\varepsilon}\\&=\int_0^\infty[\,\alpha\,q_{r,r+1}\Big(\frac{v}{\varepsilon}\Big)+p_{r,r+1}\Big(\frac{v}{\varepsilon}\Big)\,]\left(v^\alpha\int_v^\infty u^{-\alpha-1}[I-T(u)]^r f\,du\right)\frac{dv}{\varepsilon}\,.\end{aligned}$$

The integration with respect to v extends, in fact, from 0 to $(r+1)\varepsilon$ since $q_{r,r+1}(u) = p_{r,r+1}(u) = 0$ for $u > r+1$ (as already remarked). Therefore, when taking norms on both sides, we obtain the assertion. □

Lemma 2.1 used in the case $\varepsilon = kt/r(r+1)$ enables us to get the "right" estimate for $\|f - g_t\|$:

$$\begin{aligned}\|f - g_t\| &\le |F_r| \sum_{k=1}^{r} \binom{r}{k} k^r c \sup_{0<v\le \frac{kt}{r}} v^\alpha \Big\| \int_v^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\| \\ &\le c_1 \sup_{0<v\le t} v^\alpha \Big\| \int_v^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\|, \end{aligned} \tag{2.9}$$

where $c_1 := |F_r| \sum_{k=1}^{r} \binom{r}{k} k^r (\alpha \, \|q_{r,r+1}\|_1 + \|p_{r,r+1}\|_1)$.

Now let us turn to the term $t^\alpha \|(-A)^\alpha g_t\|$. We write

$$g_t = F_r \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} h_{jt/r(r+1)} \,,$$

where, when $f \in X$ and $\varepsilon > 0$, we set

$$h_\varepsilon := \sum_{k=1}^{r} (-1)^k \binom{r}{k} k^r \int_0^\infty q_{r,r+1}\Big(\frac{u}{k\varepsilon}\Big) T(u) f \, \frac{du}{k\varepsilon} \,. \tag{2.10}$$

By (2.6) we have $h_\varepsilon \in D((-A)^r)$ implying $h_\varepsilon \in D((-A)^\alpha)$, $0 < \alpha < r$. We shall show that

$$\begin{aligned} t^\alpha \|(-A)^\alpha h_\varepsilon\| &\le c_2 t^\alpha \Big\| \int_t^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\| \\ &\quad + c_3 \Big(\frac{t}{\varepsilon}\Big)^r \sup_{\varepsilon \le s \le (r+1)\varepsilon} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\| \end{aligned} \tag{2.11}$$

with constants c_2, c_3 independent of f, ε, and t . Then the proof of Theorem 1.2 easily follows. Indeed, setting $\varepsilon = jt/r(r+1)$, $j = 1, \dots, r$, we get

$$\begin{aligned} t^\alpha \|(-A)^\alpha h_{jt/r(r+1)}\| &\le c_2 \, t^\alpha \Big\| \int_t^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\| \\ &\quad + c_3 \Big(\frac{r(r+1)}{j}\Big)^r \sup_{\frac{jt}{r(r+1)} \le s \le \frac{jt}{r}} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\| \end{aligned}$$

$$\leq \Big(c_2 + c_3\,(r(r+1)^r\Big) \sup_{\frac{t}{r(r+1)}\leq s\leq t} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I-T(u)]^r f\,du \Big\|\,.$$

In view of the definition of g_t this leads to

$$t^\alpha \|(-A)^\alpha g_t\| \lesssim \sup_{0<s<t} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I-T(u)]^r f\,du \Big\|$$

which, together with (2.9), gives the desired estimate of the K-functional.

Let us now establish (2.11). Having the definition (2.10) of h_ε in mind, we first consider only one term of the sum. By (1.10), for $t>0$, $0<\alpha<r$, we have

$$C_{\alpha,r}(-A)^\alpha \int_0^\infty q_{r,r+1}\Big(\frac{u}{k\varepsilon}\Big)T(u)f\,\frac{du}{k\varepsilon}$$

$$= \int_0^\infty v^{-\alpha-1}[I-T(v)]^r \int_0^\infty q_{r,r+1}\Big(\frac{u}{k\varepsilon}\Big)T(u)f\,\frac{du}{k\varepsilon}\,dv$$

$$= \int_t^\infty \ldots \int_0^\infty \ldots du\,dv + \int_0^t \ldots \int_0^\infty \ldots du\,dv =: I_{k,1}(f) + I_{k,2}(f)\,.$$

In a straightforward way one obtains

$$\|I_{k,1}(f)\| \leq M\,\|q_{r,r+1}\|_1 \Big\| \int_t^\infty v^{-\alpha-1}[I-T(v)]^r f\,dv \Big\|\,. \tag{2.12}$$

Summing over all k yields for the constant c_2 in (2.11)

$$c_2 := \frac{M}{C_{\alpha,r}}\,\|q_{r,r+1}\|_1 \sum_{k=1}^r \binom{r}{k} k^r\,.$$

Concerning the term $I_{k,2}(f)$ we first observe that by (2.7) and (2.6)

$$I_{k,2}(f) = \int_0^t w^{r-\alpha-1} \int_0^\infty p_{r,r}\Big(\frac{v}{w}\Big)T(v)(-A)^r \int_0^\infty q_{r,r+1}\Big(\frac{u}{k\varepsilon}\Big)T(u)f\,\frac{du}{k\varepsilon}\,\frac{dv}{w}\,dw$$

$$= \int_0^t w^{r-\alpha-1} \int_0^\infty p_{r,r}\Big(\frac{v}{w}\Big)T(v) \int_{k\varepsilon}^\infty u^{-r-1}[I-T(u)]^{r+1} f\,du\,\frac{dv}{w}\,dw\,.$$

Hence, by (2.8),

$$\sum_{k=1}^r (-1)^k \binom{r}{k} k^r I_{k,2}(f) \tag{2.13}$$

$$= \int_0^t \dots \int_0^\infty \dots \sum_{k=1}^r (-1)^k \binom{r}{k} k^r \int_{k\varepsilon}^\infty u^{-r-1}[I - T(u)]^{r+1} f \, du \, \frac{dv}{w} \, dw$$

$$= \int_0^t \dots \int_0^\infty \dots \sum_{k=1}^r (-1)^k \binom{r+1}{k} k^r \int_{k\varepsilon}^{(r+1)\varepsilon} u^{-r-1}[I - T(u)]^r f \, du \, \frac{dv}{w} \, dw \, .$$

There arises the problem to convert the innermost integrals into those of type $\int_u^\infty v^{-\alpha-1}[I - T(u)]^r f \, du$. This is taken care of by the next lemma.

Lemma 2.2 *If* $f \in X$, $\varepsilon > 0$, $0 < \alpha < r$, $0 < a < b < \infty$, *then*

$$\Big\| \int_{a\varepsilon}^{b\varepsilon} u^{-r-1}[I - T(u)]^r f \, du \Big\| \le 2(a\varepsilon)^{-r} \sup_{a\varepsilon \le s \le b\varepsilon} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I - T(u)]^r f \, du \Big\| \, .$$

Proof. Integration by parts yields for $0 < a < b < \infty$

$$\int_a^b u^{-r-1}[I - T(u)]^r f \, du = \int_a^b u^{\alpha-r}(u^{-\alpha-1}[I - T(u)]^r f) \, du$$

$$= -u^{\alpha-r} \int_u^\infty v^{-\alpha-1}[I - T(v)]^r f \, dv \Big|_{u=a}^b$$

$$-(r - \alpha) \int_a^b u^{-r-1} \Big(u^\alpha \int_u^\infty v^{-\alpha-1}[I - T(v)]^r f \, dv \Big) \, du \, .$$

Hence

$$\Big\| \int_a^b u^{-r-1}[I - T(u)]^r f \, du \Big\|$$

$$\le b^{-r} \Big\| b^\alpha \int_b^\infty v^{-\alpha-1}[I - T(v)]^r f \, dv \Big\| + a^{-r} \Big\| a^\alpha \int_a^\infty v^{-\alpha-1}[I - T(v)]^r f \, dv \Big\|$$

$$+ \sup_{a \le u \le b} \Big\| u^\alpha \int_u^\infty v^{-\alpha-1}[I - T(v)]^r f \, dv \Big\| \, \Big(\frac{r - \alpha}{-r} \Big) u^{-r} \Big|_{u=a}^b$$

$$\le [(2 - \frac{\alpha}{r})a^{-r} + \frac{\alpha}{r} b^{-r}] \sup_{a \le u \le b} \Big\| u^\alpha \int_u^\infty v^{-\alpha-1}[I - T(v)]^r f \, dv \Big\| \le 2a^{-r} \sup_{a \le u \le b} \dots$$

and the substitution $a \to a\varepsilon$, $b \to b\varepsilon$, gives the assertion. □

Now return to (2.13), take norms of this identity and apply Lemma 2.2 to obtain

$$t^\alpha \| \sum_{k=1}^r (-1)^k \binom{r}{k} k^r I_{k,2} f \|$$

$$\leq \frac{2M}{r-\alpha}\|p_{r,r}\|_1 \sum_{k=1}^{r}\binom{r+1}{k}\frac{t^r}{\varepsilon^r}\sup_{k\varepsilon\leq s\leq(r+1)\varepsilon} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I-T(u)]^r f\,du\Big\|$$

$$\lesssim \Big(\frac{t}{\varepsilon}\Big)^r \sup_{\varepsilon\leq s\leq(r+1)\varepsilon} s^\alpha \Big\| \int_s^\infty u^{-\alpha-1}[I-T(u)]^r f\,du\,\Big\|\,.$$

We draw out of these inequalities that the constant c_3 in (2.11) may be chosen as

$$c_3 = \frac{2M}{(r-\alpha)C_{\alpha,r}}\|p_{r,r}\|_1 \sum_{k=1}^{r}\binom{r+1}{k}\,.$$

The preceding estimate together with (2.12) establishes (2.11), thus the proof of Theorem 1.2 is complete.

3 Equibounded groups of operators

Let us now consider n-parameter equibounded $(\mathcal{C}_0)$-groups $\{G(t): t\in\mathbb{R}^n\}$ of linear operators defined via the properties

$$\begin{aligned} G(s)G(t) &= G(s+t) \quad \text{for } s,t\in\mathbb{R}^n, \qquad G(0)=I\\ \|G(t)\| &\leq M \quad \text{for } t\in\mathbb{R}^n, \quad M \text{ being a constant}\\ \lim_{t\to 0}\|G(t)f-f\| &= 0 \quad \text{for each } f\in X \quad ((\mathcal{C}_0)\text{-property}). \end{aligned}$$

Such a group can be composed by n commutative one-parameter groups

$$\{G_k(\tau):\tau\in\mathbb{R}\}\,, \qquad G_k(\tau):=G(\tau e^k), \quad 1\leq k\leq n,$$

where $e^k\in\mathbb{R}^n$ is the standard k-th unit vector; this leads only to partial infinitesimal generators $A_k=\lim_{\tau\to 0}\tau^{-1}(G_k(\tau)-I)$. For the definition of more general (fractional) derivatives it is convenient to make use of a functional calculus in the sense of L. Schwartz, which is based on the (n-dimensional) Fourier transform on $\mathcal{D}'_{L^1}(\mathbb{R}^n)$. Here $U\in\mathcal{D}'_{L^1}(\mathbb{R}^n)$ if U can be represented as a finite sum of (distributional, partial) derivatives of finite Borel measures, i.e., if $U=\sum_{0\leq|\beta|\leq m} D^\beta\mu_\beta$, $\mu_\beta\in M(\mathbb{R}^n)$, $\beta\in\mathbb{N}_0^n$. This calculus is described for the one-dimensional situation in Westphal and Neubauer [20], for the n-dimensional one in Neubauer [10]. Let us indicate it and consider a finite Borel measure on $\mathbb{R}^n$ with Fourier transform

$$\mathcal{F}[\mu](\xi)=\widehat{\mu}(\xi)=\int_{\mathbb{R}^n} e^{-i\xi\cdot t}\,d\mu(t)\,, \qquad \xi\in\mathbb{R}^n, \quad \xi\cdot t=\sum_{k=1}^{n}\xi_k t_k\,.$$

Then, in view of the formal correspondence $e^{-i\xi\cdot t} \sim G(t) = e^{t\cdot(A_1,\cdots,A_n)}$, a bounded linear operator $H(\mu)$ is defined by

$$H(\mu)\, f := \int_{\mathbb{R}^n} G(t) f \, d\mu(t)\,, \qquad f \in X.$$

The mapping H, which assigns to each μ the operator $H(\mu)$, is an algebra homomorphism of the algebra $M(\mathbb{R}^n)$ of finite Borel measures on $\mathbb{R}^n$ – with convolution as multiplication – into the Banach algebra of endomorphisms of X. Now the crucial point is that this mapping can be extended to the class $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ by a limiting process. Thus, to each $U \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ one can associate a closed linear operator $H(U)$ (unbounded in general) with a domain dense in X. Inspired by the classical Laplacian Δ with Fourier symbol $-|\xi|^2$ we now define an abstract Riesz derivative R^α, $\alpha > 0$, by

$$R^\alpha f := H(V^\alpha) f\,, \qquad f \in D(H(V^\alpha)\,), \quad \text{where } \mathcal{F}[V^\alpha](\xi) = |\xi|^\alpha.$$

Thus, if $G(t)$ is the translation group on $L^p(\mathbb{R}^n)$, the operator R^α may be interpreted as the fractional power $(-\Delta)^{\alpha/2}$. We now give a characterization of $K(\tau^\alpha, f; X, D(R^\alpha))$ via abstract approximation processes, generated by convolution with certain $L^1(\mathbb{R}^n)$-kernels.

Theorem 3.1 *Let $\{G(t) : t \in \mathbb{R}^n\}$ be an n-parameter equibounded group of operators of class $(\mathcal{C}_0)$ on a Banach space $(X, \|\cdot\|)$. Let R^α, $\alpha > 0$, be the abstract Riesz derivative as above. Let $\psi \in L^1(\mathbb{R}^n)$ have the following properties:*

(i) $\quad 1 - \mathcal{F}[\psi](\xi) \neq 0 \quad$ *for all* $\xi \neq 0\,,$

(ii) $\quad \lim\limits_{\xi\to 0} \dfrac{1 - \mathcal{F}[\psi](\xi)}{|\xi|^\alpha} \neq 0\,,$

(iii) $\quad \dfrac{1 - \mathcal{F}[\psi](\xi)}{|\xi|^\alpha} \in \mathcal{F}[M(\mathbb{R}^n)]\,.$

Then there holds for all $f \in X$ and $\tau > 0$

$$K(\tau^\alpha, f; X, D(R^\alpha)) \approx \Big\| \int_{\mathbb{R}^n} \tau^{-n} \psi\Big(\frac{u}{\tau}\Big)\, G(u) f \, du - f \Big\|\,.$$

Theorem 3.1 is the abstract version of [16, Theorem 5] proved there on $L^1(\mathbb{R}^n)$. Hence, on account of the functional calculus for the group case (based on the Fourier transform), we can take over the arguments given there. As an application we obtain the following corollary.

Corollary 3.2 *Let* $\{G(t); t \in \mathbb{R}^n\}$ *be an* n*-parameter equibounded* $\mathcal{C}_0$*-group of operators; let* R^α, $\alpha > 0$, *be the abstract Riesz derivative. Then, if* $0 < \alpha < 2m$, $m \in \mathbb{N}$, *we have for all* $f \in X$ *and* $\tau > 0$

$$K(\tau^\alpha, f; X, D(R^\alpha)) \approx \tau^\alpha \Big\| \int_{|u| \geq \tau} |u|^{-\alpha-n} [G(u/2) - G(-u/2)]^{2m} f \, du \Big\| .$$

It is shown in [8] that the kernel, leading to the truncated hypersingular integral on the right hand side of the above equivalence, satisfies the properties (i) – (iii) of Theorem 3.1.

Remark 3.3 *(a) If* $\{G(t); t \in \mathbb{R}\}$ *is a one-parameter group with infinitesimal generator* A, *Corollary 3.2 improves a result in [15], since in this case* $R^\alpha = (-A^2)^{\alpha/2}$ *and Corollary 3.2 shows that we can omit the supremum occurring in [15, Theorem 1.6].*

(b) Also in the one-parameter situation one can vary the Riesz derivative in the sense that the symbols involved are essentially of the type

$$|\xi|^\alpha (1 + |\xi|)^\beta, \quad |\xi|^\alpha \log^\gamma (1 + |\xi|) , \qquad \alpha > 0, \, \beta > -\alpha, \, \gamma \geq 1 ,$$

– for the precise result see [1, Theorem 3.1]. Interpreted in the group setting this means that one can generalize the fractional powers R^α, *e.g., to power-logarithmic ones in imitation of the idea of power-logarithmic kernels in [12, § 21].*

(c) In connection with Theorem 3.1 we should mention a result due to Neubauer [10] in the sense of the Grünwald-Letnikow approach. Define a central difference operator of order $\alpha > 0$ *by*

$$\overline{\Delta}_\tau^\alpha := (2n)^{\alpha/2} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha/2}{m} \Big\{ \frac{1}{2n} \sum_{k=1}^{n} [G_k(\tau) + G_k(-\tau)] \Big\}^m , \quad \tau \neq 0,$$

(recall $G_k(\tau) = G(\tau e^k)$*) which, for* $\alpha = 2$, *reduces to the difference operator*

$$\overline{\Delta}_\tau^2 = - \sum_{k=1}^{n} [G_k(\tau) - 2\, I + G_k(-\tau)] .$$

This is well known in the case of the classical translation on $L^p(\mathbb{R}^n)$, $p \geq 1$, *for the approximation of the Laplacian. With the notation* $\widetilde{\omega}_\alpha(\tau, f) := \sup_{0<\sigma<\tau} \|\overline{\Delta}_\sigma^\alpha f\|$ *Satz 3.8 in [10] reads*

$$K(\tau^\alpha, f; X, D(R^\alpha)) \approx \widetilde{\omega}_\alpha(\tau, f) , \qquad 0 < \alpha \leq 2.$$

Example 3.4 *Choose* $X = L^p(\mathbb{R}^n)$, $p \geq 1$, *with the standard norm and consider the generalized Weierstrass means* W_τ^α, $\tau > 0$, $\alpha > 0$,

$$W_\tau^\alpha f(x) := \int_{\mathbb{R}^n} w_\tau^\alpha(y) f(x-y)\, dy\,, \qquad \mathcal{F}[w_\tau^\alpha](\xi) = e^{-\tau|\xi|^\alpha}\,, \quad \xi \in \mathbb{R}^n,$$

or the Riesz means $R_{\alpha,\lambda;\tau}$, $\tau > 0$, $\alpha > 0$, $\lambda > (n-1)/2$,

$$R_{\alpha,\lambda;\tau} f(x) := \int_{\mathbb{R}^n} r_{\alpha,\lambda;\tau}(y) f(x-y)\, dy\,, \quad \mathcal{F}[r_{\alpha,\lambda;\tau}](\xi) = (1-\tau|\xi|^\alpha)_+^\lambda\,,\ \xi \in \mathbb{R}^n.$$

Then the kernels w_1^α *and* $r_{\alpha,\lambda;1}$ *satisfy the requirements of Theorem 3.1 (or one may also argue for the first equivalence to follow that* $\{W_\tau^\alpha : \tau > 0\}$ *is an equibounded holomorphic one-parameter semigroup of class* $(\mathcal{C}_0)$ *and apply the Ditzian-Ivanov result of Sec. 1) and thus, for* $0 < \alpha < 2m$,

$$\|W_\tau^\alpha f - f\|_{L^p(\mathbb{R}^n)} \approx K(\tau, f; L^p(\mathbb{R}^n), D(R^\alpha)) \approx \|R_{\alpha,\lambda;\tau} f - f\|_{L^p(\mathbb{R}^n)} \quad (3.1)$$

$$\approx \tau \Big\| \int_{|t|\geq\tau^{1/\alpha}} |t|^{-\alpha-n} \sum_{j=-m}^{m} (-1)^j \binom{2m}{j} f(\cdot + jt)\, dt \Big\|_{L^p(\mathbb{R}^n)}\,.$$

The last equivalence is just Trigub's result [9]. Further, making use of Neubauer's characterization one obtains for $0 < \alpha \leq 2$

$$\|R_{\alpha,\lambda;\tau^\alpha} - f\|_{L^p(\mathbb{R}^n)} \approx \widetilde{\omega}_\alpha(\tau, f; L^p(\mathbb{R}^n))\,,$$

which supplements Theorem 4 in [14] in a natural way; concerning the $\widetilde{\omega}_2(\tau, f)$*-modulus of smoothness, see [14], [6], and [16].*

Next we want to give a slight modification of Theorem 3.1 and again illustrate the advantage of using the K-functional as an interface.

Theorem 3.5 *Let* $\{G(t) : t \in \mathbb{R}^n\}$ *be an* n*-parameter equibounded group of operators of class* $(\mathcal{C}_0)$ *on a Banach space* $(X, \|\cdot\|)$. *Let* $\Psi : \mathbb{R}_+ \to \mathbb{C}$ *be a sufficiently smooth function; set* $g_1 := \Psi$, $g_2(\tau) := (1 - \Psi(\tau))/\tau$ *and assume that* g_2 *is continuous,* $g_2(\tau) \neq 0$ *for all* $\tau \geq 0$, *and*

$$\int_0^\infty \tau^{k-1} |g_i^{(k)}(\tau)|\, d\tau < \infty\,, \qquad 1 \leq k \leq [(n+3)/2]\,,\ \ i = 1, 2.$$

Further, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be strictly increasing with* $\lim\limits_{\tau\to 0+} \phi(\tau) = 0$, $\lim\limits_{\tau\to\infty} \phi(\tau) = \infty$, *and* $\tau^{k-\ell} |\phi^{(k)}(\tau)| \lesssim \phi^{(\ell)}(\tau)$ *for* $\ell = 0, 1$ *and* $1 \leq k \leq [(n+3)/2]$. *Set* $\widetilde{V} := \mathcal{F}^{-1}[\phi(|\xi|)]$. *Then*

$$K(\tau, f; X, D(H(\widetilde{V}))) \approx \Big\| \int_{\mathbb{R}^n} \mathcal{F}^{-1}[\Psi(\tau\phi(|\xi|))](u)\, G(u) f\, du - f \Big\|\,.$$

The conditions on g_i and ϕ are such that the $\mathcal{F}[M(\mathbb{R}^n)]$-multiplier criterion [13, Theorem 5.10] can be used and the proof of Theorem 3.1 carries over immediately.

As application consider Weierstrass-type means (with ϕ from above) in $L^p(\mathbb{R}^n)$

$$V_\tau^\phi f(x) := \int_{\mathbb{R}^n} v_\tau^\phi(y) f(x-y)\, dy\,, \quad \mathcal{F}[v_\tau^\phi](\xi) = e^{-\tau\phi(|\xi|)}\,, \quad \xi \in \mathbb{R}^n\,.$$

If also $\phi(|\xi|)/|\xi|^\alpha$, $|\xi|^\alpha/\phi(|\xi|) \in \mathcal{F}[M(\mathbb{R}^n)]$ holds, then $D(A_{W^\alpha}) = D(A_{V^\phi})$ and $\|A_{V^\phi} f\|_{L^p(\mathbb{R}^n)} \approx \|A_{W^\alpha} f\|_{L^p(\mathbb{R}^n)}$. Hence

$$\|W_\tau^\alpha f - f\|_{L^p(\mathbb{R}^n)} \approx K(\tau, f; L^p(\mathbb{R}^n), D(A_{V^\phi})) \approx \|V_\tau^\phi f - f\|_{L^p(\mathbb{R}^n)}\,, \quad \tau > 0.$$

Observe that the proof of this comparison (in the sense of (3.1)) between the generalized Weierstrass means and some perturbed ones is reduced to a discussion of the involved infinitesimal generators. Examples for admissible ϕ's are:
$\phi_1 := \tau^\alpha + \log^\gamma(1+\tau)$, $0 < \alpha < \gamma$, or $\phi_2 := \tau^\alpha + \tau^\gamma\, b(\tau)/(1+\tau)^\gamma$, $0 < \alpha \le \gamma$, $b(\tau) \lesssim \tau^{\alpha-\varepsilon}$, $\tau \ge 1$, $0 < \varepsilon < \alpha$, and b like ϕ ($\lim_{\tau\to 0+} b(\tau) = 0$ is not needed). We mention that in the case ϕ_2 we need a factor with b which behaves like $O(\tau^\gamma)$, $\tau \to 0$, since otherwise $\|A_{V^\phi} f\|_{L^p(\mathbb{R}^n)}$ can in general not be dominated by $\|A_{W^\alpha} f\|_{L^p(\mathbb{R}^n)}$ (e.g., if $b(\tau) \ge c > 0$ and $\gamma = 0$ then the seminorm property of the K-functional with respect to the pair $(L^p(\mathbb{R}^n), D(A_{V^\phi}))$ would get lost).

Example 3.6 *Choose X to be the Lorentz-Zygmund spaces $L^{p,q}(Log\, L)^\beta(\mathbb{T})$ or $L^{p,q}(Log\, L)^\beta(\mathbb{R}^n)$ with norms*

$$\|\tau^{1/p-1/q}(1+|\log\tau|)^\beta f^{**}(\tau)\|_{q,(0,1)}$$

and

$$\|\tau^{1/p-1/q}(1+|\log\tau|)^\beta f^{**}(\tau)\|_{q,(0,\infty)}\,,$$

resp.; here $\|\cdot\|_{q,(a,b)}^q = \int_a^b |\cdot|^q\, dx$ denotes the standard one-dimensional Lebesgue integral; these spaces are Banach spaces if $1 < p \le \infty$, $1 \le q \le \infty$, $\beta \in \mathbb{R}$ – see Bennett and Rudnick [2]. Thus there remains to check if the involved n-parameter groups are equibounded. Up to the case $p = \infty$ this follows immediately, e.g., by [11, Lemma 4.3] which we write down for the convenience of the reader:

Let T be a linear operator such that $T : L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ is bounded for all $r \in (1,\infty)$. Then $T : L^{p,q}(Log\, L)^\beta(\mathbb{R}^n) \to L^{p,q}(Log\, L)^\beta(\mathbb{R}^n)$ is bounded if $1 < p < \infty$, $1 \le q \le \infty$ and $\beta \in \mathbb{R}$.

Hence one can state results analogous to those in Example 3.4 and in the application to Theorem 3.5.

References

[1] E. Belinsky and W. Trebels, *Generalized Liouville differentiation, truncated hypersingular integrals and K-functionals,* Math. Z. **246** (2004), 339 – 357.

[2] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces,* Dissert.Math. **175** (1980), 1 – 72.

[3] P. L. Butzer and H. Berens, Semi-Groups of Operators and Approximation, *Springer, Berlin, 1967.*

[4] P. L. Butzer and K. Scherer, *Jackson and Bernstein-type inequalities for families of commutative operators in Banach spaces,* J. Approx. Theory **5** (1972), 308 – 342.

[5] R. A. DeVore and G. G. Lorentz, Constructive Approximation, *Springer, Berlin, 1993.*

[6] Z. Ditzian, *A measure of smoothness related to the Laplacian,* Trans. Amer. Math. Soc. **326** (1991), 407 – 422.

[7] Z. Ditzian and K. G. Ivanov, *Strong converse inequalities,* J. d'Analyse Math. **61** (1993), 61 – 111.

[8] M. Kojima and G. Sunouchi, *Characterization of saturation classes in $L(\mathbf{R}^n)$,* Studia Math. **48** (1973), 71 – 81.

[9] O. I. Kuznecova and R. M. Trigub, *Two-sided estimates of the approximation of functions by Riesz and Marcinkiewicz means,* Soviet Math. Dokl. **21** (1980), 374 – 377.

[10] M. Neubauer, *($\mathcal{C}_0$)-Gruppen von Operatoren: zentrale Differenzen und gebrochene Potenzen,* Dissertation, Fb Mathematik und Informatik, U Hannover, 2002.

[11] B. Opic and W. Trebels, *Sharp embeddings of Bessel potential spaces with logarithmic smoothness,* Math. Proc. Camb. Phil. Soc. **134** (2003), 347 – 384.

[12] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives, *Gordon and Breach, Langhorne, PA, 1993.*

[13] W. Trebels, Multipliers for (C, α)-bounded Fourier Expansions in Banach Spaces and Approximation, *Lecture Notes in Math.* **329**, *Springer, Berlin, 1973.*

[14] W. Trebels, *On the approximation behavior of the Riesz means in $L^p(\mathbb{R}^n)$*, Lect. Notes in Math. **556**, pp. 428 – 438, Springer, Berlin, 1976.

[15] W. Trebels and U. Westphal, *Characterizations of K-functionals built from fractional powers of infinitesimal generators of semigroups,* Constr. Approx. **19** (2003), 355 – 371.

[16] W. Trebels and U. Westphal, *K-functionals on $L^1(\mathbf{R}^n)$ related to the Laplacian.* Contemporary Mathematics Vol. **320**, Amer. Math. Soc. 2003, pp. 459 – 465.

[17] U. Westphal, *Ein Kalkül für gebrochene Potenzen infinitesimaler Erzeuger von Halbgruppen und Gruppen von Operatoren. I: Halbgruppenerzeuger,* Compositio Math. **22** (1970), 67 – 103; *II: Gruppenerzeuger,* ibid. **22**, (1970), 104 – 136.

[18] U. Westphal, *An approach to fractional powers of operators via fractional differences,* Proc. London Math. Soc. (3. Ser.) **29** (1974), 557 – 576.

[19] U. Westphal, *Fractional powers of infinitesimal generators of semigroups,* In: Applications of Fractional Calculus in Physics (R. Hilfer, Ed.), World Scientific Publishing Co., Singapore, 2000, 131 – 170.

[20] U. Westphal and M. Neubauer, *Fractional powers of group-generators via fractional central differences,* Rend. Circ. Mat. Palermo (2) Suppl. 68 (2002), 897 – 909.

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Computing Orthogonal Rational Functions on a Halfline

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Abstract. We present some transformation formulas which relate orthogonal rational functions (ORF) on $[0, \infty)$ to ORF on $[-1, 1]$. From a computational point of view, we can then limit our attention to the interval case, which has been well studied. We also use these formulas to obtain quadrature rules and asymptotic behaviour for ORF on $[0, \infty)$. Some examples serve as illustration.

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1 Introduction

Orthogonal rational functions (ORF) are widely used in applications such as signal processing and system identification. To approximate the (unknown) rational transfer function in system theory, researchers first used rational functions with one pole [17] or a pair of complex conjugate poles [18], then with a finite number of poles that are cyclically repeated [3, 8, 10] and finally with an arbitrary pole sequence [5, 6, 7]. These rational functions are orthogonal with respect to the Lebesgue measure on the complex unit circle and are special cases of the so-called Malmquist basis. Later on this was generalized to arbitrary orthogonality measures on the unit circle and (subsets of) the real line [1].

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The accurate computation of orthogonal rational functions for arbitrary measures is a far from trivial problem. For the case of orthogonality on a finite real interval, several results were obtained in [13, 14]. It is the purpose of the present paper to show that the case of orthogonality on a halfline can be completely reduced to the case of the interval, using a suitable transformation. We will start by reviewing the basic theory of orthogonal rational functions on subsets of the real line, including the three term recurrence relation. Next we derive a relation between functions orthogonal on a halfline and on an interval. Some examples are given to illustrate the theory. As an application, we study the asymptotic behaviour of ORF on a halfline, using known results from the interval case, and we look at quadrature formulas.

2 Preliminaries

The real line is denoted by $\mathbb{R}$ and the extended real line by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

By a measure μ we will mean a positive bounded Borel measure whose support $\mathrm{supp}(\mu) \subset \overline{\mathbb{R}}$ is an infinite set and normalized such that $\mu(\overline{\mathbb{R}}) = 1$. The inner product in the metric space $L_2(\mu)$ is then defined as

$$\langle f\,, g\rangle = \int f\overline{g}d\mu. \tag{2.1}$$

Next we will introduce the spaces of rational functions with real poles. Let a sequence $A = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{R}}\backslash\{0\}$ be given such that $A \cap \mathrm{supp}(\mu) = \emptyset$. As a consequence we cannot have $\mathrm{supp}(\mu) = \overline{\mathbb{R}}$. Define factors

$$Z_n(z) = \frac{z}{1 - z/\alpha_n}$$

and basis functions

$$b_0(z) \equiv 1, \quad b_n(z) = b_{n-1}(z)Z_n(z), \quad n = 1, 2, \ldots$$

Then the space of rational functions with poles in A is defined as

$$\mathcal{L}_n = \mathrm{span}\{b_0, \ldots, b_n\}.$$

Note that we recover the polynomial case if we take all poles equal to infinity.

Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n and define

$$\pi_n(z) = \prod_{k=1}^{n}(1 - z/\alpha_k),$$

then we may write equivalently

$$\mathcal{L}_n = \{p_n/\pi_n, p_n \in \mathcal{P}_n\}.$$

Orthonormalizing the basis $\{b_0, \ldots, b_n\}$ with respect to μ we obtain orthogonal rational functions $\{\varphi_0, \ldots, \varphi_n\}$ where we choose the leading coefficient κ_n in the expansion $\varphi_n(z) = \kappa_n b_n(z) + \ldots$ to be real. The φ_n will be uniquely determined once the sign of κ_n is fixed. We will get back to this later on. The following lemma from [1] will be useful.

Lemma 2.1. *The orthonormal functions φ_n have real coefficients with respect to the basis $\{b_k\}$.*

It follows in particular that $\varphi_n(z)$ is real for real z.

The orthogonal rational function φ_n is called *regular* if its numerator polynomial satisfies $p_n(\alpha_{n-1}) \neq 0$. The system $\{\varphi_n\}$ is regular if φ_n is regular for every n. We now mention the most important theorem for the computation of orthogonal rational functions on the real line, which states that they satisfy a three term recurrence relation, analogous to the one for the polynomial case. For the proof of the theorem we refer to [1].

Theorem 2.2. *Put by convention $\alpha_{-1} = \alpha_0 = \infty$. Then for $n = 1, 2, \ldots$ the orthonormal rational functions φ_n satisfy the following three term recurrence relation if and only if φ_n and φ_{n-1} are regular,*

$$\varphi_n(z) = \left(E_n Z_n(z) + F_n \frac{Z_n(z)}{Z_{n-1}(z)}\right)\varphi_{n-1}(z) - \frac{E_n}{E_{n-1}}\frac{Z_n(z)}{Z_{n-2}(z)}\varphi_{n-2}(z). \quad (2.2)$$

The initial conditions are $\varphi_{-1}(z) \equiv 0$, $\varphi_0(z) \equiv 1$ and the coefficients E_n are nonzero.

Note that the coefficient E_0 is never used and can be arbitrarily chosen. We take it equal to $E_0 = 1$. If we take the coefficient E_n to be positive, then the functions φ_n will be uniquely determined. This amounts to fixing the sign of κ_n. We will use this convention in the sequel.

If we take all poles outside the convex hull of $\mathrm{supp}(\mu)$, then the system $\{\varphi_n\}$ will be regular and thus the recurrence relation will hold for every n. This follows from the fact that in this case the zeros of φ_n are inside the convex hull of $\mathrm{supp}(\mu)$. Therefore, if $\mathrm{supp}(\mu)$ is connected then $\{\varphi_n\}$ will be regular (because of the assumptions we made on the location of the poles).

3 Mapping the halfline to the interval

In what follows, we only consider two different cases: orthogonality on the interval $[-1,1]$ and orthogonality on the halfline $[0,\infty)$. To avoid the introduction of more notation, we will denote the case of the halfline merely by adding a tilde to all quantities involved. So suppose we are given a measure $\tilde{\mu}$ on the halfline $[0,\infty]$ and a set of poles $\tilde{A} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\} \subset (-\infty, 0)$. With this measure and poles we associate a set of orthonormal rational functions $\{\tilde{\varphi}_n\}$. Since $\mathrm{supp}(\tilde{\mu})$ is connected, the recurrence relation holds for every n. According to the conventions from section 2, we cannot have poles at infinity, because this is in the support of the measure.

We map the halfline to the interval $[-1,1]$ using the transformation

$$\tau(x) = \frac{1-x}{1+x}, \quad x \in [0,\infty]. \tag{3.1}$$

Note that $y = \tau(x)$ implies $x = \tau(y)$. Then associate to $\tilde{\mu}$ and $\tilde{A}$ a measure μ on $[-1,1]$ and a set of poles $A = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{R}} \backslash [-1,1]$ in the following way. For every Borel measurable set $E \subset [-1,1]$ set

$$\mu(E) = \tilde{\mu}(\{\tau(y), y \in E\}) \tag{3.2}$$

and for the poles A set

$$\alpha_n = \tau(\tilde{\alpha}_n), \quad n = 1, 2, \ldots \tag{3.3}$$

From (3.2) we have $\mu' = |\tau'|(\tilde{\mu}' \circ \tau)$, where the prime means derivative (in case of a measure this is of course the Radon-Nikodym derivative with respect to the Lebesgue measure). More explicitly this yields

$$\mu'(y) = \frac{2}{(1+y)^2} \tilde{\mu}' \left(\frac{1-y}{1+y} \right) .$$

We will need several simple lemmas before we can prove the main theorem. Let $\tilde{\mathcal{L}}_n$ denote the space of rational functions of degree n with poles in $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ and $\tilde{b}_n(z) = \prod_{k=1}^n \tilde{Z}_k(z)$ the corresponding basis functions. As before, the symbols without a tilde refer to the interval. The proof of the following lemma is a matter of straightforward computation and we omit it.

Lemma 3.1. *Put $y = \tau(x)$ then we have*

$$\tilde{Z}_n(x) = -\frac{1}{2} \left(1 - \frac{1}{\alpha_n} \right) \left[\left(1 - \frac{1}{\alpha_n} \right) Z_n(y) - 1 \right], \quad n \geq 1$$

$$\frac{\tilde{Z}_n(x)}{\tilde{Z}_m(x)} = \frac{1 - 1/\alpha_n}{1 - 1/\alpha_m} \frac{Z_n(y)}{Z_m(y)}, \quad n, m \geq 1$$

and for every pair of complex numbers (a, b) *we have*

$$aZ_n(y) + b = \left(a - \frac{b}{Z_{n-1}(\alpha_n)}\right) Z_n(y) + b\frac{Z_n(y)}{Z_{n-1}(y)}, \quad n \geq 1$$

Next we derive a relation between the rational functions in $\mathcal{L}_n$ orthonormal with respect to μ and those in $\tilde{\mathcal{L}}_n$ orthonormal with respect to $\tilde{\mu}$.

Lemma 3.2. *Let* $\{\tilde{\varphi}_n\}_{n=1}^{\infty}$ *denote the set of orthonormal rational functions associated with* $(\tilde{A}, \tilde{\mu})$ *and* $\{\varphi_n\}_{n=1}^{\infty}$ *those associated with* (A, μ)*. Then we have*

$$\tilde{\varphi}_n \circ \tau = \pm\varphi_n, \quad n \geq 1$$

where the sign is determined by the normalization $\tilde{E}_n > 0$ *for* $\tilde{\varphi}_n$ *and* $E_n > 0$ *for* φ_n*.*

Proof. It is clear that $\tilde{\varphi}_n \circ \tau \in \mathcal{L}_n$. Because $\tilde{\alpha}_k \neq 0$ for all k we have $\alpha_k \neq 1$ and then it follows from lemma 3.1 that the basis functions b_k and $\tilde{b}_k$ satisfy the following relation,

$$\tilde{b}_k = \sum_{j=0}^{k} c_j^{(k)} (b_j \circ \tau), \quad c_k^{(k)} \neq 0$$

for some constants $\{c_j^{(k)}\}$. Next expand $\tilde{\varphi}_n$ in the basis $\{\tilde{b}_0, \ldots \tilde{b}_n\}$,

$$\tilde{\varphi}_n = \sum_{k=0}^{n} d_k^{(n)} \tilde{b}_k$$

where $d_n^{(n)} \neq 0$ because $\tilde{\varphi}_n \in \tilde{\mathcal{L}}_n$ and $\tilde{\varphi}_n \perp \tilde{\mathcal{L}}_{n-1}$. Then with the previous relation and the fact that $\tilde{b}_k \circ \tau \circ \tau = \tilde{b}_k$ it follows that

$$\begin{aligned}
\tilde{\varphi}_n \circ \tau &= \sum_{k=0}^{n} d_k^{(n)} (\tilde{b}_k \circ \tau) \\
&= \sum_{k=0}^{n} d_k^{(n)} \sum_{j=0}^{k} c_j^{(k)} b_j \\
&= \sum_{k=0}^{n} \hat{d}_k^{(n)} b_k
\end{aligned}$$

and $\hat{d}_n^{(n)} = d_n^{(n)} c_n^{(n)} \neq 0$ (note that the coefficients $\hat{d}_k^{(n)}$ are implicitly defined by the last equation). This shows that $\tilde{\varphi}_n \circ \tau \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and thus $\{\tilde{\varphi}_0 \circ \tau, \ldots, \tilde{\varphi}_n \circ \tau\}$ forms a basis for $\mathcal{L}_n$.

Because of orthogonality we have

$$\int \tilde{\varphi}_n \tilde{\varphi}_m d\tilde{\mu} = \delta_{nm}, \quad n, m \geq 0$$

where δ_{nm} is the Kronecker symbol. Using the definition of τ this becomes

$$\int (\tilde{\varphi}_n \circ \tau)(\tilde{\varphi}_m \circ \tau) d\mu = \delta_{nm}, \quad n, m \geq 0.$$

This means that $\tilde{\varphi}_n \circ \tau \perp \mathcal{L}_{n-1}$ and $\|\tilde{\varphi}_n \circ \tau\| = 1$. But we also have $\varphi_n \perp \mathcal{L}_{n-1}$ and $\|\varphi_n\| = 1$ so it follows that $\tilde{\varphi}_n \circ \tau = \gamma_n \varphi_n$ with $|\gamma_n| = 1$. Using lemma 2.1 we then get

$$\tilde{\varphi}_n \circ \tau = \pm \varphi_n$$

which proves the lemma. □

Using the previous two lemmas we can express the recurrence coefficients $\{\tilde{E}_n, \tilde{F}_n\}$ for $\{\tilde{\varphi}_n\}$ in terms of the recurrence coefficients $\{E_n, F_n\}$ for $\{\varphi_n\}$. We will need one more lemma before we can prove our main theorem.

Lemma 3.3. *If $n = 1$ then the recurrence coefficients for φ_1 satisfy*

$$E_1 > F_1.$$

Proof. From the recurrence relation, taking the inner product with $\varphi_0 = 1$ on both sides, we obtain,

$$-F_1 = E_1 \frac{\int_{-1}^{1} Z_1(x) d\mu(x)}{\int_{-1}^{1} \frac{Z_1(x)}{x} d\mu(x)}$$

or writing Z_1 explicitly

$$E_1 - F_1 = E_1 \frac{\int_{-1}^{1} \frac{x+1}{1-x/\alpha_1} d\mu(x)}{\int_{-1}^{1} \frac{1}{1-x/\alpha_1} d\mu(x)} .$$

Both integrands are positive on $[-1, 1]$ and so is E_1. Therefore also $E_1 - F_1 > 0$, proving the lemma. □

We now state and prove our main theorem.

Theorem 3.4. *With the definitions of this section we obtain the following relations between* $\{E_n, F_n\}$ *and* $\{\tilde{E}_n, \tilde{F}_n\}$ *for* $n \geq 1$,

$$\tilde{E}_n = [2E_n - \delta_{n1}(E_1 + F_1)] \left(1 - \frac{1}{\alpha_{n-1}}\right)^{-1} \left(1 - \frac{1}{\alpha_n}\right)^{-1},$$
$$\tilde{F}_n = -\left(1 - \frac{1}{\alpha_n}\right)^{-1} \left[F_n \left(1 - \frac{1}{\alpha_{n-1}}\right) + E_n - \delta_{n2} \frac{E_2}{E_1} \frac{1 - 1/\alpha_1}{F_1 - E_1}\right].$$

Note that the formulas simplify for $n > 2$ *because of the Kronecker symbols.*

Proof. First we will prove the theorem for the cases $n = 1$ and $n = 2$ and then for general $n > 2$. The first two cases are special because by convention we have $\alpha_{-1} = \alpha_0 = \infty$ but also $\tilde{\alpha}_{-1} = \tilde{\alpha}_0 = \infty$ which shows that $\alpha_k \neq \tau(\tilde{\alpha}_k)$ for $k = -1, 0$.

For $n = 1$ use lemma 3.2 to write $(\tilde{\varphi}_1 \circ \tau)(x) = c\varphi_1(x)$ where $c = \pm 1$. Then write down the recurrence relations for $\tilde{\varphi}_1$ and φ_1, use the definition of τ and equate the coefficients of like powers of x to obtain (recall that $\tilde{\varphi}_0 = \varphi_0 = 1$)

$$\left(1 - \frac{1}{\alpha_1}\right) \tilde{E}_1 = c(F_1 - E_1),$$
$$\left(1 - \frac{1}{\alpha_1}\right) \tilde{F}_1 = c(F_1 + E_1),$$

Now it follows from $|\alpha_1| > 1$, $\tilde{E}_1 > 0$ and lemma 3.3 that $c = -1$, proving the theorem for $n = 1$.

For $n = 2$ we proceed in the same way. Write down the recurrence relation for $\tilde{\varphi}_2 \circ \tau$, using the fact that we already know that $\tilde{\varphi}_1 \circ \tau = -\varphi_1$ and of course $\tilde{\varphi}_0 \circ \tau = \varphi_0$. Then use the relation

$$E_2 = \left.\frac{\varphi_2(x)}{\varphi_1(x) Z_2(x)}\right|_{x=\alpha_1}$$

and $E_2 > 0$ to find

$$E_2 = \frac{\tilde{E}_2}{2} \left(1 - \frac{1}{\alpha_2}\right) \left(1 - \frac{1}{\alpha_1}\right).$$

To find the relation for $\tilde{F}_2$ use $\varphi_2(0) = F_2\varphi_1(0) - E_2/E_1$ and $\varphi_1(0) = F_1$, again comparing the recurrence relations for $\tilde{\varphi}_2 \circ \tau$ and φ_2 and using all the previous results.

The general case $n > 2$ is the easiest to prove. Write down the recurrence relation for φ_n and for $\tilde{\varphi}_n \circ \tau$ using $\tilde{\varphi}_k \circ \tau = c_k \varphi_k$ for $k = n, n-1, n-2$ and $c_k = \pm 1$ and the formulas from lemma 3.1. Comparing the factors in front of φ_{n-1} and φ_{n-2} and using $E_n > 0$ immediately yields the result, thus proving the theorem. □

Remark 3.5. *If we take all poles $\tilde{\alpha}_k = -1$ then transforming to the interval we obtain the orthogonal polynomials on $[-1, 1]$ with respect to μ. For the case of Legendre polynomials we would have to take $\tilde{\mu}$ absolutely continuous with weight $\tilde{\mu}'(x) = (1+x)^{-2}$. See [2] for an application of this special case.*

We could also transform orthogonal polynomials on $[0, \infty)$ to get orthogonal rational functions on $[-1, 1]$. In this case the poles $\tilde{\alpha}_n$ are in the support of the measure (polynomials have poles at infinity) and thus far we have always excluded this case. However, as long as all the integrals

$$\mu_{kl} = \int \tilde{b}_k(x) \tilde{b}_l(x) d\tilde{\mu}(x)$$

exist and are finite, the recurrence relation remains valid whenever the $\tilde{\varphi}_n$ are regular, and so do the transformation formulas from theorem 3.4. Next we look at an example of this case.

Example 3.6. *The Laguerre polynomials $L_n(x)$ are orthonormal on $[0, \infty)$ with respect to the weight function e^{-x} and usually normalized such that the coefficient of x^n has sign $(-1)^n$, see e.g. [9, p. 100]. In that case they satisfy the recurrence formula*

$$nL_n(x) = (-x + 2n - 1)L_{n-1}(x) - (n-1)L_{n-2}(x).$$

with initial values $L_{-1}(x) = 0$ and $L_0(x) = 1$. If we regard them as orthogonal rational functions on the halfline and normalize them according to our convention $\tilde{E}_n > 0$, then it follows from the above recurrence relation that

$$\tilde{E}_n = \frac{1}{n}, \quad \tilde{F}_n = \frac{1}{n} - 2, \quad n = 1, 2, \ldots$$

Because of the remark following theorem 3.4, we may still use the transformation formulas from that theorem to obtain ORF on $[-1, 1]$. Of course now we have to use them in the opposite direction to obtain $\{E_n, F_n\}$ from $\{\tilde{E}_n, \tilde{F}_n\}$. We then get rational functions $\varphi_n(x)$ on $[-1, 1]$ with all poles in -1 and orthogonal with respect to the absolutely continuous measure

$$d\mu(x) = \frac{2}{(1+x)^2} \exp\left\{\frac{x-1}{x+1}\right\}.$$

n	E_n	F_n
1	$\frac{10}{3}$	$-\frac{2}{3}$
2	1	$\frac{37}{40}$
$n > 2$	$\frac{2}{n}$	$2\left(1-\frac{1}{n}\right)$

Table 1: Recurrence coefficients for φ_n

Some computations then yield the recurrence coefficients as given in table 1.

Example 3.7. *Among the very few examples of explicitly known orthogonal rational functions (and in fact the only known example on $[-1,1]$ with arbitrary poles), are ORF with respect to the Chebyshev weight functions $(1-x)^\gamma(1+x)^\delta$ on $[-1,1]$, with γ and δ belonging to $\{\pm 1/2\}$ and real poles outside $[-1,1]$. These ORF are studied in detail in [16]. For $\gamma = \delta = -1/2$ they are related to the Malmquist basis on the complex unit circle, which we mentioned in the introduction.*

For $n > 2$ the recurrence coefficients are independent of the specific choice of γ and δ. Just as in the polynomial case, they are the same for the different weight functions. The general expressions are quite complicated, but if we limit our attention to the case of one multiple pole $A = \{\alpha, \alpha, \alpha, \ldots\}$ then we have for $n > 2$

$$E_n = 2\left(1 - \frac{1}{\alpha^2}\right),$$
$$F_n = -\frac{2}{\alpha}.$$

Note that we obtain the well-known recurrence coefficients for Chebyshev polynomials if we take $\alpha = \infty$.

Using theorem 3.4, some computations yield the recurrence coefficients for ORF on the halfline $[0,\infty)$ with respect to the weight functions

$$\tilde{w}(x) = \frac{2}{(1+x)^2}\left(\frac{2x}{1+x}\right)^\gamma\left(\frac{2}{1+x}\right)^\delta$$

and with a multiple pole in $\tilde{\alpha}$. They are given by

$$\tilde{E}_n = -\frac{4}{\tilde{\alpha}},$$
$$\tilde{F}_n = -2,$$

for $n > 2$.

4 Quadrature formulas

We can use orthogonal rational functions to construct Gaussian quadrature formulas, analogous to the polynomial case. The nodes are the zeros of φ_n and the weights can be expressed in terms of these nodes, see [1, 11]. These quadrature formulas are exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$.

Of course, using lemma 3.2 and the formulas from section 3, we can compute quadrature formulas on the halfline, given the nodes and weights for quadrature on the interval, as given in the following corollary.

Corollary 4.1. *Suppose we are given the zeros $\{x_{nk}\}_{k=1}^n$ of φ_n and weights $\{\lambda_{nk}\}_{k=1}^n$ defined by*

$$\lambda_{nk} = \left[\sum_{k=0}^{n-1} (\varphi_k(x_{nk}))^2\right]^{-1},$$

such that the quadrature formula

$$\int_{-1}^{1} f(x) d\mu(x) \approx \sum_{k=1}^{n} \lambda_{nk} f(x_{nk})$$

is exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$. Then putting

$$\Lambda_{nk} = \lambda_{nk},$$
$$\xi_{nk} = \tau(x_{nk}),$$

the formula

$$\int_{0}^{\infty} f(x) d\tilde{\mu}(x) \approx \sum_{k=1}^{n} \Lambda_{nk} f(\xi_{nk})$$

is exact in $\tilde{\mathcal{L}}_n \cdot \tilde{\mathcal{L}}_{n-1}$.

5 Asymptotic behaviour

In this section we look at several types of asymptotic behaviour for orthogonal rational functions on a halfline. This is another very straightforward application of lemma 3.2, since we can simply translate results about the convergence of ORF on an interval to the halfline case. So far, we have studied ratio asymptotics, convergence of the recurrence coefficients [12] and some weak-star convergence results [15] for orthogonal rational functions on $[-1,1]$. From these two references we can get the following theorem, which we give without proof.

Theorem 5.1. *Let $x = J(z)$ denote the Joukowski transform $x = \frac{1}{2}(z + z^{-1})$. The inverse mapping is denoted by $z = J^{-1}(x)$ and is chosen so that $|z| < 1$ if $x \notin [-1,1]$. Assume the sequence $A = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{R}} \setminus [-1,1]$ is bounded away from $[-1,1]$ and let μ satisfy the condition $\mu' > 0$ a.e. Then uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$ we have*

$$\lim_{n\to\infty} \frac{z-\beta_{n+1}}{1-\beta_n z}\sqrt{\frac{1-\beta_n^2}{1-\beta_{n+1}^2}}\frac{\varphi_{n+1}(x)}{\varphi_n(x)} = 1$$

where $z = J^{-1}(x)$ and $\beta_k = J^{-1}(\alpha_k)$ for $k = n, n+1$. Under the same assumptions, the following relations hold in the sense that the ratio of the left hand side and the right hand side tends to 1 as n tends to infinity,

$$\begin{aligned} E_n &\sim 2\frac{\sqrt{(1-\beta_{n-1}^2)(1-\beta_n^2)}(1-\beta_{n-1}\beta_n)}{(1+\beta_{n-1}^2)(1+\beta_n^2)}, \\ F_n &\sim -\sqrt{\frac{1-\beta_n^2}{1-\beta_{n-1}^2}}\frac{(1-\beta_{n-1}^2)(\beta_n+\beta_{n-2})+2\beta_{n-1}(1-\beta_n\beta_{n-2})}{(1+\beta_n^2)(1-\beta_{n-1}\beta_{n-2})}, \end{aligned}$$

Furthermore, still under the same assumptions, we have

$$\frac{1-x/\alpha_n}{\sqrt{1-1/\alpha_n^2}}\varphi_n^2(x)d\mu(x) \xrightarrow{*} \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}.$$

where $\xrightarrow{*}$ means convergence in the weak-star topology for measures.

Using lemma 3.2 and theorem 3.4 it is very easy to obtain convergence results for orthogonal rational functions on the halfline $[0,\infty)$ when the poles stay away from infinity (this excludes the polynomial case). In fact, because of lemma 3.2 and the remark following (3.1) we know that $\tilde{\varphi}_n =$

$\pm\varphi_n \circ \tau$, so we can rephrase the previous theorem, putting $\tau(x)$ in place of x, writing $\tau(\tilde{\alpha}_n)$ for α_n, etc. Instead of repeating all the formulas in their general form, we feel it is more informative to look at the special case where all poles are equal to each other, because then we can rewrite the expressions to obtain explicit representations for the limit functions. The computations are very straightforward and we only give the result in the following corollary.

Corollary 5.2. *Let $\tilde{\mu}$ be a measure on $[0,\infty)$ satisfying the condition $\tilde{\mu}' > 0$ a.e. and assume that all poles are equal to each other, $\tilde{A} = \{\tilde{\alpha}, \tilde{\alpha}, \ldots\}$ where $\tilde{\alpha} \in (-\infty, 0]$ (so the pole is finite). Then locally uniformly in $\mathbb{C} \setminus [0,\infty)$ we have*

$$\lim_{n\to\infty} \frac{\tilde{\varphi}_n(x)}{\tilde{\varphi}_{n-1}(x)} = \frac{(\sqrt{-x} + \sqrt{-\tilde{\alpha}})^2}{x - \tilde{\alpha}}$$

where the branch of the square root is such that the limit is real for $x < 0$. For the recurrence coefficients we have the limits

$$\lim_{n\to\infty} \tilde{E}_n = -\frac{4}{\tilde{\alpha}},$$
$$\lim_{n\to\infty} \tilde{F}_n = -2.$$

We also have the weak-star convergence result

$$(x - \tilde{\alpha})\tilde{\varphi}_n^2(x) d\tilde{\mu}(x) \xrightarrow{*} \frac{\sqrt{-\tilde{\alpha}}}{\pi} \frac{dx}{(1+x)\sqrt{x}}.$$

Remark 5.3. *Note that the asymptotic values for the recurrence coefficients are exactly the values from example 3.7. This is not surprising; also in the polynomial case it is well known that the recurrence coefficients behave asymptotically like the coefficients of the Chebyshev polynomials if the weight is strictly positive almost everywhere.*

6 Conclusion

In this paper we presented some formulas to transform orthogonal rational functions on $[0,\infty)$ to $[-1,1]$ and applied these formulas to obtain quadrature formulas and asymptotics for ORF on a halfline. The main reason for studying these formulas is that they allow us to limit our attention to the interval case. Especially from a computational point of view, there no longer exists a fundamental difference between orthogonal rational functions on a halfline and on an interval. However, the asymptotic results obtained

through this transformation suffer from the limitation that the poles cannot tend to infinity, which excludes the polynomial case. As is well known, the study of convergence for polynomials orthogonal on a halfline (or on the whole real line), is much more complicated than the case of the interval, see e.g. [4] and the references therein.

References

[1] A. Bultheel, P. González Vera, E. Hendriksen, and O. Njåstad. *Orthogonal Rational Functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[2] B. Guo, J. Shen, and Z. Wang. A rational approximation and its applications to differential equations on the half line. *J. Sci. Comput.*, 15(2):117–147, 2000.

[3] P.S.C. Heuberger, P.M.J. Van den Hof, and O. Bosgra. A generalized orthogonal basis for linear dynamical systems. *IEEE Trans. Automat. Control*, 40:451–465, 1995.

[4] E. Levin and D.S. Lubinsky. *Orthogonal polynomials for exponential weights*. CMS Books in Mathematics. Springer-Verlag, New York, 2001.

[5] B. Ninness. Frequency domain estimation using orthonormal bases. In *Proceedings of the 13th IFAC World Congress*, volume 8, pages 381–386, San Francisco, 1996.

[6] B. Ninness and H. Akçay. Rational basis functions for robust identification from frequency and time domain measurements. *Automatica*, vol. 34, no. 9:1101–1117, 1998.

[7] B. Ninness and F. Gustafsson. A unified construction of orthogonal bases for system identification. *IEEE Trans. Automat. Control*, 42:515–522, 1997.

[8] Z. Szabó, J. Bokor, and F. Schipp. Identification of rational approximate models in H_∞ using generalized orthogonal bases. *IEEE Trans. Automat. Control*, vol. 44, no. 1:153–158, 1999.

[9] G. Szegő. *Orthogonal Polynomials*, volume 23 of *Am. Math. Soc. Colloq. Publ.* Am. Math. Soc., Providence, RI, 1967.

[10] P.M.J. Van den Hof, P.S.C. Heuberger, and J. Bokor. System identification with generalized orthogonal basis functions. *Automatica*, 31:1821–1834, 1995.

[11] J. Van Deun and A. Bultheel. Orthogonal rational functions and quadrature on an interval. *J. Comput. Appl. Math.*, 153(1-2):487–495, 2003.

[12] J. Van Deun and A. Bultheel. Ratio asymptotics for orthogonal rational functions on an interval. *J. Approx. Theory*, 123:162–172, 2003.

[13] J. Van Deun and A. Bultheel. The computation of orthogonal rational functions on an interval. *J. Comput. Appl. Math.*, 2004. Accepted.

[14] J. Van Deun and A. Bultheel. An interpolation algorithm for orthogonal rational functions. *J. Comput. Appl. Math.*, 164-165:749–762, 2004.

[15] J. Van Deun and A. Bultheel. A weak-star convergence result for orthogonal rational functions. *J. Comput. Appl. Math.*, 2004. Accepted.

[16] J. Van Deun, A. Bultheel, and P. González Vera. On computing rational Gauss-Chebyshev quadrature formulas. *Math. Comp.*, 2004. Accepted.

[17] B. Wahlberg. System identification using Laguerre models. *IEEE Trans. Automat. Control*, vol. AC-36, no. 5:551–562, 1991.

[18] B. Wahlberg. System identification using Kautz models. *IEEE Trans. Automat. Control*, vol. AC-39, no. 6:1276–1282, 1994.

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One-point Location Problems from a Very General Point of View

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Abstract. Given a nonempty set A in a metric space (X, d), we consider the problem of minimizing a function $\varphi\colon X \to \mathbb{R}$ of the form

$$\varphi(x) = f(\Delta_x)\,,$$

where f is a monotone functional on the set $\mathrm{Lip}_1^+(A)$ of all nonnegative 1-Lipschitz functions on A, and $\Delta_x\colon A \to \mathbb{R}_+$ is the (1-Lipschitz) function $\Delta_x(\cdot) = d(x, \cdot)$. The minimizers (if any) of φ over X are called *f-centers* of the set A.

We present an existence theorem for f-centers, based on compactness in the so called *ball*-topology and on the Fatou property of f. We discuss sufficient conditions for the assumptions being satisfied. The last section is devoted to significant particular cases: generalized integral medians, generalized Chebyshev centers, and "generalized centers with neglect".

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Introduction

In Approximation Theory and Mathematical Economy, one often looks for a point that would approximate (in a certain given sense) a given set. Such problems, called *one-point location problems*, consist in minimizing a function depending only on the distances from the elements of the set.

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Let us give a list of several examples of one-point location (with some relative references), among which the most known and studied is the case of Chebyshev centers.

Let X be a Banach space.

(a) *Chebyshev centers* of a bounded set $A \subset X$ are the minimizers (if any) over X of the function

$$\varphi(x) = \sup_{a \in A} \|x - a\|.$$

(See [4], [5], [12], [3], [13], [20].)

(b) *Medians* (or Fermat points) of a finite set $A \subset X$ are the minimizers (if any) over X of the function

$$\varphi(x) = \sum_{a \in A} \|x - a\|.$$

(See [6], [7], [8].)

(c) According to [16], if $A = \{a_1, \dots, a_n\} \subset X$ and $1 \leq k \leq n$, *k-centers* of A are the minimizers (if any) over X of the function

$$\varphi(x) = \sum_{i=1}^{k} \|x - a_{\pi_x(i)}\|,$$

where π_x is a permutation of $\{1, \dots, n\}$ such that $\|x - a_{\pi_x(1)}\| \geq \|x - a_{\pi_x(2)}\| \geq \cdots \geq \|x - a_{\pi_x(n)}\|$. (Observe that $k = 1$ gives Chebyshev centers, and $k = n$ gives medians of A.)

(d) According to [19], if $A = \{a_1, \dots, a_n\} \subset X$ and $f\colon [0, \infty)^n \to \mathbb{R}$ is a monotone (w.r.t. the coordinate-wise partial ordering) function, *f-centers* of A are the minimizers (if any) over X of the function

$$\varphi(x) = f\big(\|x - a_1\|, \dots, \|x - a_n\|\big).$$

(See [19], [9], [10], [18].)

The only known tool for proving existence of such minimizers in infinite-dimensional spaces is *weak* or *weak** compactness. (Only for Chebyshev centers, another sufficient condition, so called quasi-uniform rotundity, is known – cf. [5]. Note that Chebyshev centers always exist in ultrametric spaces [15].) One usually proves that the function φ is coercive, hence it is sufficient to minimize it over a sufficiently large closed ball B; then, if X is

reflexive or dual, and φ is *weak* or *weak** (respectively) lower semicontinuous, the existence of minimizers on B follows.

The aim of the present paper is to give a very general point of view to existence of one-point location problems using compactness. We formulate such problems in the context of metric spaces and we show how the so-called *ball*-topology (the coarsest topology in which the closed balls remain closed) can replace the weak topologies of Banach spaces (Theorem 1.2). After a basic information about the *ball*-topology and its compactness, we study the Fatou property of monotone functionals on $\mathrm{Lip}_1^+(A)$ (the set of all nonnegative 1-Lipschitz functions on A), since it is related to *ball*-lower semicontinuity of our function φ. The last section is devoted to significant particular cases: generalized integral medians, generalized Chebyshev centers, and "f-centers with neglect" (a generalization of (c) and (d) above). Our unified approach covers all existence results for one-point location, known to us, based on compactness.

1 General existence theorem

If (X,d) is a metric space, we denote by $\mathrm{Lip}_1^+(X)$ the set of all 1-Lipschitz nonnegative real-valued functions on X. Then $e \in \mathrm{Lip}_1^+(X)$ denotes the constant function $e(\cdot) \equiv 1$. Let ∞e be the function (which does not belong to $\mathrm{Lip}_1^+(X)$) constantly equal to ∞ on X. Observe that the set $\mathrm{Lip}_1^+(X) \cup \{\infty e\}$ is closed under making arbitrary suprema and infima.

A functional $f\colon \mathrm{Lip}_1^+(X) \to \mathbb{R}$ is said to be *monotone* if $f(u) \le f(v)$ whenever $u, v \in \mathrm{Lip}_1^+(A)$, $u \le v$. We shall say that f satisfies the *Fatou property for nets* if

$$f(\liminf u_\gamma) \le \liminf f(u_\gamma) \tag{1.1}$$

whenever (u_γ) is a net in $\mathrm{Lip}_1^+(X)$; if necessary, we put $f(\infty e) = \sup f$ in (1.1). (Of course, "lim inf" is intended pointwise).

The *ball-topology* on X is the coarsest topology τ such that each closed ball is τ-closed.

Definition 1.1 *Let (X,d) be a metric space. Let $A \subset X$ be a nonempty set and $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$ be a monotone functional. We shall say that a point $x_0 \in X$ is an* f-center *for A if x_0 is a (global) minimizer for the function $\varphi\colon X \to \mathbb{R}$, given by*

$$\varphi(x) = f(\Delta_x)\,,$$

where $\Delta_x(\cdot) = d(x,\cdot) \in \mathrm{Lip}_1^+(A)$.

Let us state the following simple theorem — a general existence theorem for f-centers.

Theorem 1.2 *Let A be a nonempty subset of a metric space (X,d), and let $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$ be monotone. If X is* ball*-compact and f satisfies the Fatou property for nets, then A admits at least one f-center.*

Proof. It suffices to show that φ is *ball*-l.s.c. on X. Since all closed balls are *ball*-closed, for each $a \in A$ the function $x \mapsto d(x,a)$ is *ball*-l.s.c. on X. Hence, if $x_\gamma \overset{ball}{\to} x_0$, we have $\Delta_{x_0} \le \liminf \Delta_{x_\gamma}$. Then, by monotonicity and Fatou property, we obtain

$$\varphi(x_0) = f(\Delta_{x_0}) \le f(\liminf \Delta_{x_\gamma}) \le \liminf f(\Delta_{x_\gamma}) = \liminf \varphi(x_\gamma),$$

which completes the proof. □

The above theorem admits the following –apparently more general– corollary.

Corollary 1.3 *Let A be a nonempty subset of a metric space (X,d). Let τ be a topology on X such that each closed ball in X is τ-closed and τ-compact. If $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$ is monotone and satisfies the Fatou property for nets, then A admits at least one f-center.*

Proof. If the closed balls are τ-closed, then τ is finer than (or equal to) the *ball*-topology. Hence τ-compactness of all closed balls implies their *ball*-compactness which, in turn, implies *ball*-compactness of X (cf. Proposition 2.2). Apply Theorem 1.2. □

To use the above theorem, one needs to know sufficient conditions for compactness in the *ball*-topology, and sufficient conditions for Fatou property of monotone functionals on $\mathrm{Lip}_1^+(A)$. The next two sections are devoted to these two questions.

2 Some facts about the *ball*-topology

The *ball*-topology in Banach spaces was widely and deeply studied in a paper by Godefroy and Kalton [11], especially in connection with uniqueness of preduals. In this section, we present some basic properties of the *ball*-topology which are related to compactness. Most of the presented results are contained in [11].

Let (X,d) be a metric space. Recall that the *ball*-topology is the coarsest topology τ on X such that all closed balls are τ-closed. Thus, given $x \in X$, a basis of *ball*-neighborhoods of x consists of all sets of the form

$$X \setminus \bigcup_{i=1}^{n} B_i\,,$$

where B_i's are closed balls not containing x.

If X is a normed linear space, the *ball*-topology on X is the coarsest topology τ on X satisfying the following two properties:

- B_X is τ-closed, and
- for each $a \in X$ and each $\lambda \in \mathbb{R} \setminus \{0\}$, the mappings $x \mapsto x + a$ and $x \mapsto \lambda x$ are τ-continuous (and hence they are homeomorphisms of (X,τ)).

Observation 2.1 *A net (x_γ) converges to x_0 in the* ball-*topology of (X,d) if and only if $d(x_0,y) \le \liminf d(x_\gamma,y)$ for each $y \in X$.* Consequently, if $d(x_\gamma,a) \to \infty$ for some ($\Leftrightarrow$ for each) $a \in X$, then (x_γ) *ball*-converges to any element of X. (Thus, the *ball*-topology on X is not Hausdorff whenever X is unbounded.)

Proof. Consider the following chain of equivalent statements:

$$\begin{aligned}
&x_\gamma \to x_0\\
\Leftrightarrow\ &\forall y \in X\ \forall r \ge 0 \text{ s.t. } x_0 \notin B(y,r) : x_\gamma \notin B(y,r) \text{ for each sufficiently large } \gamma\\
\Leftrightarrow\ &\forall y \in X\ \forall 0 \le r < d(x_0,y)\ \exists \gamma_0 : d(x_\gamma,y) > r \text{ whenever } \gamma \ge \gamma_0\\
\Leftrightarrow\ &\forall y \in X : d(x_0,y) \le \liminf d(x_\gamma,y).
\end{aligned}$$

□

The following proposition is an easy consequence of the Aleksander subbase theorem.

Proposition 2.2 *For a metric space (X,d), the following assertions are equivalent:*

(i) *X is* ball-*compact;*

(ii) *each closed ball of X is* ball-*compact;*

(iii) *X satisfies the following "finite-infinite intersection property" ($IP_{f,\infty}$): every family of closed balls, whose each finite subfamily has a nonempty intersection, has a nonempty intersection.*

Moreover, if X is a normed linear space, any of the above conditions is equivalent to

(iv) B_X (the closed unit ball of X) is ball-*compact.*

Corollary 2.3 *Every* ball-*compact metric space is complete.*

Proof. If (X, d) is not complete, it contains a sequence of closed balls $B_1 \supset B_2 \supset \cdots$ whose radii tend to 0, such that $\bigcap B_n = \emptyset$. It follows that X is not *ball*-compact. □

Proposition 2.4 *Let X be a Banach space. Then the implications*

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longleftrightarrow (d)$$

hold for the following properties:

(a) X is (isometric to) a dual space;

(b) X is norm-one complemented in its bidual;

(c) there exists a (non necessarily linear) mapping $P\colon X^{**} \to X$ such that $\|y - P(x^{**})\| \leq \|\hat{y} - x^{**}\|$ whenever $x^{**} \in X^{**}$, $y \in X$;

(d) X is ball-*compact.*

(For $x \in X$, $\hat{x}$ denotes the canonical image of x in X^{**}.)

Proof. It suffices to show the equivalence of (c) and (d) since the other implications are well known.

If $\{B(x_\alpha, r_\alpha)\}_\alpha$ is a family of closed balls, whose any finite subfamily has a nonempty intersection, then (by *weak**-compactness) the intersection of the family $\{B(\hat{x}_\alpha, r_\alpha)\}_\alpha$ contains a point $z^{**} \in X^{**}$. If (c) holds, then $P(z^{**})$ is contained in each ball $B(x_\alpha, r_\alpha)$. By Proposition 2.2, (d) holds.

Now, suppose (d) holds. Given $x^{**} \in X^{**}$, we want to show that the family of balls

$$\left\{ B(y, \|\hat{y} - x^{**}\| + \varepsilon) : y \in X,\, \varepsilon > 0 \right\}$$

has a nonempty intersection. By (d), it suffices to show that every finite subfamily of these balls intersect. And this follows from an easy application of the principle of local reflexivity (see e.g. [14]). □

We conclude this section with properties of the *ball*-topology in some classical Banach spaces. The proofs are omitted, though some of them are not completely trivial. They can be found in Chapter 8 of [17].

Examples 2.5 *Let X be a Banach space with closed unit ball B_X.*

1. *$(B_X, ball)$ is compact (and hence X is ball-compact) whenever X is either dual or $L_1[0,1]$. It is not compact if X is one of the spaces c_0, c, $C[0,1]$.*
2. *$(B_X, ball)$ is Hausdorff whenever X is either reflexive or one of the spaces c_0, c, ℓ_∞. It is not Hausdorff if X is one of the spaces $C[0,1]$, ℓ_1, $L_1[0,1]$, $L_\infty[0,1]$.*
3. *The* ball- *and the weak-topology coincide on B_X whenever either X is reflexive or $X = c_0$. They do not coincide if X is one of the spaces c, $C[0,1]$, ℓ_1, $L_1[0,1]$, ℓ_∞, $L_\infty[0,1]$.*
4. *The* ball- *and the weak*-topology coincide on B_X for $X = \ell_\infty$. They do not coincide if X is one of the spaces ℓ_1, $L_\infty[0,1]$.*

3 Fatou property for nets

3.1 General properties

Let (A,d) be a metric space. We shall say that $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$ satisfies the *monotone convergence property for nets* if $f(v_\gamma) \to f(v)$ whenever $(v_\gamma) \subset \mathrm{Lip}_1^+(A)$ is a nondecreasing net such that $v_\gamma \nearrow v \in \mathrm{Lip}_1^+(A) \cup \{\infty e\}$. (Again we put $f(\infty e) = \sup f$, if necessary.)

Let τ_p be the topology of the *pointwise* convergence on $\mathrm{Lip}_1^+(A)$.

Lemma 3.1 *For a monotone $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$, the following are equivalent.*

(i) f satisfies the Fatou property for nets.

(i') f satisfies the Fatou property for nets whose lim inf *is finite.*

(ii) f satisfies the monotone convergence property for nets.

(ii') f satisfies the monotone convergence property for nets whose supremum is finite.

(iii) f is τ_p-lower semicontinuous on $\mathrm{Lip}_1^+(A)$.

Proof. $(i) \Rightarrow (i')$ is obvious.
$(i') \Rightarrow (iii) \Rightarrow (ii')$ are easy.
$(ii') \Rightarrow (ii)$. We have to prove that $f(v_\gamma) \to \sup f$ whenever (v_γ) is a net in $\mathrm{Lip}_1^+(A)$ such that $v_\gamma \nearrow \infty e$. Fix $\alpha < \sup f$. Let $u \in \mathrm{Lip}_1^+(A)$ be such that $f(u) > \alpha$. Put $u_\gamma := \min\{u, v_\gamma\} \in \mathrm{Lip}_1^+(A)$. Then $u_\gamma \nearrow u$, hence

$f(u_\gamma) \to f(u)$. Thus, for each sufficiently large γ, we have $f(v_\gamma) \geq f(u_\gamma) > \alpha$. Hence $f(v_\gamma) \to \sup f$.
$(ii) \Rightarrow (i)$. Let $(v_\gamma)_{\gamma\in\Gamma}$ be a net in $\mathrm{Lip}_1^+(A)$. For $\beta \in \Gamma$ denote $u_\beta = \inf_{\gamma\geq\beta} v_\gamma$. Then $u_\beta \nearrow \liminf v_\gamma$.

Thus, by (ii) and monotonicity of f,

$$f(\liminf_\gamma v_\gamma) = \lim_\beta f(u_\beta) = \sup_\beta f(u_\beta) \leq \sup_\beta \inf_{\lambda\geq\beta} f(v_\lambda) = \liminf_\lambda f(v_\lambda). \quad \square$$

Remark 3.2 *The properties from Lemma 3.1 remain equivalent if we write "sequences" instead of "nets", and "sequentially τ_p-lower semicontinuous" instead of "τ_p-lower semicontinuous".*

As an immediate consequence of well known stability results for lower semicontinuous functions, we get the following

Corollary 3.3 (Stability) *Let $\mathcal{F}$ be the family of all functionals on $\mathrm{Lip}_1^+(A)$ that are monotone and satisfy the Fatou property for nets. Then the family $\mathcal{F}$ is closed under multiplication by nonnegative real numbers, under making sums and minima of finite subfamilies, and under making pointwise finite suprema of arbitrary subfamilies.*

Observation 3.4 *Because of unique extensibility of Lipschitz functions to the closure, we can identify $\mathrm{Lip}_1^+(A)$ either with $\mathrm{Lip}_1^+(D)$, where D is a dense subset of A, or with $\mathrm{Lip}_1^+(\hat{A})$ where $\hat{A}$ is the completion of A.*

3.2 Case of A totally bounded

If A is bounded, every Lipschitz function on A is bounded. Thus the *sup*-norm $\|\cdot\|_\infty$ is well defined for such functions.

Let $\mathrm{Lip}_1(A)$ be the set of all 1-Lipschitz real functions on A. For $w \in \mathrm{Lip}_1(A)$, the function $w^+ := \max\{w, 0\}$ belongs to $\mathrm{Lip}_1^+(A)$.

Theorem 3.5 *Let (A, d) be totally bounded and $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$ be monotone. Then f has the Fatou property for nets provided* at least one *of the following three properties is satisfied:*

(a) f is convex;

(b) f is $\|\cdot\|_\infty$-lower semicontinuous;

(c) for each $u \in \mathrm{Lip}_1^+(A)$, the function $t \mapsto f\big((u + te)^+\big)$ $(t \in \mathbb{R})$ is lower semicontinuous at $t = 0$.

Proof. First, let us prove the implications $(a) \Rightarrow (c) \Rightarrow (b)$. If f is convex, then the functional $\tilde{f}\colon \mathrm{Lip}_1(A) \to \mathbb{R}$, given by $\tilde{f}(w) = f(w^+)$, is convex, too. Thus (c) follows, since every convex function on $\mathbb{R}$ is continuous.
Let f satisfy (c). Given $u \in \mathrm{Lip}_1^+(A)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $f\big((u + te)^+\big) > f(u) - \varepsilon$ whenever $|t| \le \delta$. Then, for every $v \in \mathrm{Lip}_1^+(A)$ with $\|v - u\|_\infty \le \delta$, we have $v \ge (u - \delta e)^+$, and hence $f(v) \ge f\big((u - \delta e)^+\big) > f(u) - \varepsilon$.

Now, it remains to show that (b) implies the Fatou property for nets. By Lemma 3.1, it suffices to prove that f has the monotone convergence property for nets with finite suprema. Let (v_γ) be a nondecreasing net in $\mathrm{Lip}_1^+(A)$ such that $v_\gamma \nearrow v \in \mathrm{Lip}_1^+(A)$. Then the set $K = \{u \in \mathrm{Lip}_1^+(A) : u \le v\}$ is $\|\cdot\|_\infty$-compact by the Ascoli-Arzelà theorem. Consequently, the topology τ_p and the $\|\cdot\|_\infty$-topology coincide on K. Thus $\|v_\gamma - v\|_\infty \to 0$. By (b), we have $f(v) \le \lim f(v_\gamma)$, which implies $f(v_\gamma) \to f(v)$ (by monotonicity). □

3.3 Case of A separable

Theorem 3.6 *Let (A, d) be separable and $f\colon \mathrm{Lip}_1^+(A) \to \mathbb{R}$ be monotone. Then f has the Fatou property for nets whenever it has the Fatou property for sequences.*

Proof. By Observation 3.4, we can suppose that A is at most countable. Then $\big(\mathrm{Lip}_1^+(A), \tau_p\big)$ is (homeomorphic to) a subspace of the topological product $\mathbb{R}^{\mathbb{N}}$ which is well-known to be metrizable by the metric

$$\rho(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{|u(n) - v(n)|}{1 + |u(n) - v(n)|}\,.$$

Consequently, τ_p-lower semicontinuity coincides with sequential τ_p-lower semicontinuity on $\mathrm{Lip}_1^+(A)$. The rest follows by Remark 3.2. □

One of important cases in which Theorem 3.6 can be applied is that of integral functionals because they satisfy —by the classical Fatou's lemma— the Fatou property for sequences. From this point of view, measures which are concentrated on a separable set are important. The following Proposition 3.8 gives a characterization of such measures and shows that separability of the support of a measure is, in some sense, equivalent to the Fatou property for nets of integral functionals.

Let μ be a nonnegative Borel measure on A. We say that μ is concentrated on a Borel set $C \subset A$ if $\mu(E) = \mu(E \cap C)$ for each Borel set $E \subset A$.

We also say that μ *admits a support* if there exists a minimal closed set C such that μ is concentrated on C.

Remark 3.7 *It is easy to see that a Borel measure μ admits a support if and only if the union of any family of open μ-null sets is μ-null, too.*

Proposition 3.8 *Let μ be a nonnegative σ-finite Borel measure on (A, d). Then the following assertions are equivalent:*

(i) μ is concentrated on a separable set;

(ii) μ is separably regular in the following sense:

$$\mu(E) = \sup\{\mu(S) : \ S \subset E \ \textit{separable}\}$$

for each Borel set $E \subset A$;

(iii) each open set of positive μ-measure contains a separable subset of positive μ-measure;

(iv) μ admits a support.

Moreover, if $\int_A v \, d\mu < \infty$ for each $v \in \mathrm{Lip}_1^+(A)$, then the above conditions are equivalent to

(v) the functional $f(v) = \int_A v \, d\mu$ satisfies the Fatou property for nets.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (iv)$. Let $\{G_\alpha\}_\alpha$ be a family of open sets in A such that $\mu(G_\alpha) = 0$ for each α. Denote $G = \bigcup_\alpha G_\alpha$. By Lindelöf theorem, for each separable set $S \subset G$, there exists a countable subfamily of $\{G_\alpha\}_\alpha$ that covers S, and hence $\mu(S) = 0$. Then $\mu(G) = 0$ by (iii). By Remark 3.7, μ admits a support.

$(iv) \Rightarrow (i)$. Let C be a support for μ. We shall prove that C is separable. By minimality of C, we have

$$\mu\big(B^0(c, r)\big) > 0 \qquad \text{whenever } c \in C, \ r > 0.$$

By σ-finiteness, $C = \bigcup_{k=1}^\infty C_k$ where $\mu(C_k) < \infty$ for each k. Suppose C is not separable. Then C contains an uncountable dispersed set, i.e. a set whose any two distinct points have a distance greater than a certain fixed positive number. Thus, there exists an uncountable family $\mathcal{B}$ of pairwise disjoint open balls centered at points of C.

Given a $B \in \mathcal{B}$, we have $\mu(B \cap C) = \mu(B) > 0$ and $B \cap C = \bigcup_{k=1}^{\infty}(B \cap C_k)$; hence there exists a positive integer $k(B)$ such that $\mu(B \cap C_{k(B)}) > 0$. Then there exists a positive integer k_0 such that the family

$$\mathcal{B}_{k_0} = \{B \in \mathcal{B} : \ k(B) = k_0\}$$

is uncountable. There exists $\varepsilon > 0$ and an infinite (even uncountable!) subfamily $\mathcal{D}$ of $\mathcal{B}_{k_0}$ such that

$$\mu(B \cap C_{k_0}) \geq \varepsilon \qquad \text{for each } B \in \mathcal{D}.$$

It follows that $\mu(C_{k_0}) = \infty$, a contradiction.

Let us prove the second part concerning the case when $f(v) = \int_A v \, d\mu$ defines a real-valued functional on $\mathrm{Lip}_1^+(A)$. Obviously, μ is finite, in this case.

$(i) \Rightarrow (v)$. If μ is concentrated on a separable set, we can suppose that A is separable. By Theorem 3.6, f has the Fatou property for nets.

$(v) \Rightarrow (iv)$. It suffices to prove the following

Claim. *If $W \subset \mathrm{Lip}_1^+(A)$ is such that $z := \sup W \in \mathrm{Lip}_1^+(A)$ and $f(w) = 0$ for each $w \in W$, then $f(z) = 0$.*

Suppose that our Claim holds true. Let $\mathcal{G}$ be a family of open sets such that $\mu(G) = 0$ for all $G \in \mathcal{G}$. Define $u_G(x) = \min\{1, dist(x, A \setminus G)\}$ ($G \in \mathcal{G}$, $x \in A$). Then $u_G \in \mathrm{Lip}_1^+(A)$ and $f(u_G) = 0$ for each $G \in \mathcal{G}$. The function $u := \sup\{u_G : \ G \in \mathcal{G}\}$ belongs to $\mathrm{Lip}_1^+(A)$, is positive on $\bigcup \mathcal{G}$, and $\int_{\bigcup \mathcal{G}} u \, d\mu \leq f(u) = 0$ by Claim. Consequently, $\mu(\bigcup \mathcal{G}) = 0$. Then (iv) follows by Remark 3.7.

Proof of Claim. Let us proceed by transfinite induction on $\mathrm{card}(W)$. If W is finite, then

$$0 \leq f(\max W) \leq f(\sum W) = \sum_{w \in W} f(w) = 0.$$

Let W be infinite, and suppose that Claim holds for any family whose cardinality is less than $\mathrm{card}(W)$. Let α be the smallest ordinal number such that $\mathrm{card}(\alpha) = \mathrm{card}(W)$. Obviously, α is a limit ordinal and we can index the elements of W so that

$$W = \{w_\beta : \ \beta < \alpha\}.$$

For $\gamma < \alpha$, put $v_\gamma = \sup_{\beta < \gamma} w_\beta$. Then $f(v_\gamma) = 0$ by our assumption, since $\mathrm{card}(\gamma) < \mathrm{card}(\alpha)$. Moreover, $(v_\gamma)_{\gamma < \alpha}$ is a non decreasing net in $\mathrm{Lip}_1^+(A)$

such that $v_\gamma \nearrow z$. By (v) and Lemma 3.1, we get $f(z) = \lim f(v_\gamma) = 0$, which proves our Claim.
The proof of Proposition 3.8 is complete. □

4 Significant particular cases

4.1 Generalized integral medians

Let (X, d) be a metric space, μ be a nonnegative Borel measure on a set $A \subset X$. Let us consider the functionals of type

$$f(v) = \int_A K(a, v(a))\, d\mu(a)\,, \tag{4.1}$$

where K is a real function on $A \times [0, \infty)$.
We shall write $\mathbb{R}_+ = [0, \infty)$ for brevity. Let us consider the following three properties of K:

(K1) for each $a \in A$, the function $K(a, \cdot)$ is continuous and non decreasing on $\mathbb{R}_+$;

(K2) for each $t \in \mathbb{R}_+$, the function $K(\cdot, t)$ is μ-integrable on A (in particular, it is Borel measurable);

(K3) if A is unbounded, there exist $\theta > 1$ and $a_0 \in A$ such that the function $a \mapsto K\big(a, \theta d(a, a_0)\big)$ is μ-integrable on A.

Remark 4.1 *The condition (K3) is satisfied whenever there exists $k \in L_1(\mu)$ such that $|K(a, t)| \le k(a)$ for all $a \in A$, $t \in \mathbb{R}_+$.*

Theorem 4.2 *Let $K\colon A \times \mathbb{R}_+ \to \mathbb{R}$ satisfy the conditions (K1), (K2) and (K3). Then (4.1) defines a monotone functional on $\mathrm{Lip}_1^+(A)$. Moreover, if μ is concentrated on a separable set (e.g., if A is separable) and X is* ball-*compact, then A admits at least one f-center.*

Proof. Let $v \in \mathrm{Lip}_1^+(A)$. The function $K(\cdot, v(\cdot))$ is Borel measurable on A since K is jointly Borel measurable (cf. [1]). If A is bounded, v is bounded on A and hence

$$K(a, 0) \le K(a, v(a)) \le K(a, \|v\|_\infty) \qquad (a \in A),$$

which implies that $K(\cdot, v(\cdot))$ is μ-integrable on A.
If A is unbounded, put $r = \frac{v(a_0)}{\theta - 1}$ (where a_0, θ are as in (K3)). We already

know that $K(\cdot, v(\cdot))$ is μ-integrable on $B(a_0, r)$. If $a \in A$ is such that $d(a, a_0) > r$, then $v(a_0) + d(a, a_0) < \theta d(a, a_0)$, and hence

$$K(a, 0) \le K(a, v(a)) \le K(a, v(a_0) + d(a, a_0)) \le K(a, \theta d(a, a_0)).$$

Thus $K(\cdot, v(\cdot))$ is μ-integrable also on $A \setminus B(a_0, r)$ by (K3). The monotonicity of f is obvious.
Existence of an f-center will follow if we show that f has the Fatou property for nets (Theorem 1.2). By Theorem 3.6 and Lemma 3.2, it suffices to show the monotone convergence property for sequences with finite suprema. Let (u_n) be a sequence in $\mathrm{Lip}_1^+(A)$ such that $u_n \nearrow u \in \mathrm{Lip}_1^+(A)$. Then, by the classical Monotone Convergence Theorem,

$$\begin{aligned} f(u) &= \int_A K(a, u(a))\, d\mu(a) = \int_A \lim_{n\to\infty} K(a, u_n(a))\, d\mu(a) \\ &= \lim_{n\to\infty} \int_A K(a, u_n(a))\, d\mu(a) = \lim_{n\to\infty} f(u_n). \end{aligned}$$ □

Example 4.3 (integral weighted p-medians) *Let X be a dual Banach space (or, more generally, a Banach space which is norm-one complemented in its bidual), $A \subset X$ be a nonempty bounded separable Borel set, μ be a nonnegative finite Borel measure on A and $\rho \in L_\infty(\mu)$ be a nonnegative function. By Theorem 4.2 and Proposition 2.4, the functional*

$$\varphi(x) = \int_A \rho(a)\|x - a\|^p\, d\mu(a)$$

attains its infimum over X. (Observe that if A is finite and $p = 1$, we obtain existence of [weighted] medians of A.)

4.2 Generalized Chebyshev centers

Let A be a nonempty set in a metric space (X, d). Let us consider the functionals of type

$$f(v) = \sup_{a\in A} G(a, v(a)), \tag{4.2}$$

where G is a real function on $A \times [0, \infty)$.
Let us consider the following three properties of G:

(G1) for each $a \in A$, the function $G(a, \cdot)$ is continuous and non decreasing on $\mathbb{R}_+$;

(G2) for each $t \in \mathbb{R}_+$, the function $G(\cdot, t)$ is bounded on A;

(G3) if A is unbounded, there exist $\theta > 1$ and $a_0 \in A$ such that the function $a \mapsto G\big(a, \theta d(a, a_0)\big)$ is bounded on A.

Theorem 4.4 *Let $G\colon A \times \mathbb{R}_+ \to \mathbb{R}$ satisfy the conditions (G1), (G2) and (G3). Then (4.2) defines a monotone functional on* $\mathrm{Lip}_1^+(A)$*, which satisfies the Fatou property for nets. Thus, if X is* ball-*compact, then A admits at least one f-center.*

Proof. An easy modification of the first part of the proof of Theorem 4.2 shows that (4.2) defines a real-valued monotone functional on $\mathrm{Lip}_1^+(A)$. For each fixed $a \in A$ the functional $v \mapsto G(a, v(a))$ is τ_p-continuous on $\mathrm{Lip}_1^+(A)$, hence it satisfies the Fatou property for nets. By Corollary 3.3, f has the Fatou property for nets, too. □

Example 4.5 (weighted Chebyshev centers) *Let X be a dual Banach space (or, more generally, a Banach space which is norm-one complemented in its bidual), $A \subset X$ be a nonempty bounded set, ρ be a nonnegative bounded function on A. By Theorem 4.4 and Proposition 2.4, the functional*

$$\varphi(x) = \sup_{a \in A} \rho(a)\|x - a\|$$

attains its infimum over X. (Observe that for $\rho \equiv 1$ we obtain existence of Chebyshev centers of A.)

4.3 f-centers for finite sets. Rearrangement and neglect

Let (X, d) be a metric space. Let $A = \{a_1, \dots, a_n\}$ be a finite subset of X. For each $x \in X$, the vector

$$\Delta_x = \big(d(x, a_1), \dots, d(x, a_n)\big),$$

as well as any other element of $\mathrm{Lip}_1^+(A)$, belongs to $\mathbb{R}_+^n := [0, \infty)^n$.

Thus, given a monotone function $f\colon \mathbb{R}_+^n \to \mathbb{R}$, we can speak about f-centers of A, i.e. the minimizers (if any) over X of the function

$$\varphi(x) = f(\Delta_x) = f\big(d(x, a_1), \dots, d(x, a_n)\big).$$

The following theorem holds by Theorem 1.2 and Theorem 3.5.

Theorem 4.6 *Let $A = \{a_1, \dots, a_n\}$ be a finite set. Let $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ be monotone and continuous (e.g., monotone and convex). If X is* ball-*compact, then A admits at least one f-center.*

Definition 4.7 *Denote $D_n = \{\lambda \in \mathbb{R}^n_+ :\ \lambda_1 \geq \cdots \geq \lambda_n\}$. For each $\lambda \in \mathbb{R}^n_+$ there exists a unique $\lambda^* \in D_n$ which is a rearrangement of λ: there exists a (not necessarily unique) permutation π of $\{1, \dots, n\}$ such that*

$$\lambda^* = \lambda \circ \pi := (\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}).$$

Although it is likely that the following lemma is known, we give a short proof of it since we have not been able to find any proper reference.

Lemma 4.8 *The non increasing rearrangement mapping $\lambda \mapsto \lambda^*$ is monotone and 1-Lipschitz on $\mathbb{R}^n_+$.*

Proof. Monotonicity. By symmetry, it suffices to prove:

$$\lambda \in D_n\,,\ \lambda \leq \mu \quad \Longrightarrow \quad \lambda \leq \mu^*\,.$$

Let π be a permutation such that $\mu^* = \mu \circ \pi$. Suppose $\lambda \not\leq \mu^*$. Let k be the minimal integer in $[1, n]$ with $\lambda_k > \mu^*_k = \mu_{\pi(k)}$. We must have $k > 1$ since $\mu_{\pi(1)} \geq \mu_1 \geq \lambda_1$. We have $\mu_{\pi(k)} < \lambda_k \leq \mu_k$ which implies

$$k \in \{\pi(i) :\ 1 \leq i \leq k-1\}.$$

On the other hand, for $1 \leq j \leq k-1$, we have $\mu_j \geq \lambda_j \geq \lambda_k > \mu_{\pi(k)}$ which implies

$$\{\pi(i) :\ 1 \leq i \leq k-1\} = \{1, \dots, k-1\},$$

a contradiction.
1-Lipschitz property. For $\lambda \in \mathbb{R}^n_+$, put

$$\Pi(\lambda) = \{\pi :\ \pi \text{ permutation},\ \lambda^* = \lambda \circ \pi\}.$$

($\Pi(\lambda)$ contains only one element if and only if the coordinates of λ are pairwise distinct.) Fix a $\lambda \in \mathbb{R}^n_+$.

Claim: $\Pi(\mu) \subset \Pi(\lambda)$ *for each μ sufficiently near to λ.*

To prove this, put $\eta = \min\{|\lambda_i - \lambda_j| :\ \lambda_i \neq \lambda_j\}$. Suppose $\|\mu - \lambda\|_\infty < \eta/2$ and $\pi \in \Pi(\mu) \setminus \Pi(\lambda)$. Then $\lambda_{\pi(i)} < \lambda_{\pi(i+1)}$ for some i. But this implies

$$\eta \leq \lambda_{\pi(i+1)} - \lambda_{\pi(i)} < \mu_{\pi(i+1)} - \mu_{\pi(i)} + \eta \leq \eta\,,$$

a contradiction. Our Claim is proved.

Since, for any permutation π, the affine mapping $\lambda \mapsto \lambda \circ \pi$ is an isometry, our Claim easily implies that

$$\forall \lambda \in \mathbb{R}^n_+ \ \exists \delta_\lambda > 0 : \quad \|\mu^* - \lambda^*\| = \|\mu - \lambda\| \text{ whenever } \mu \in \mathbb{R}^n_+,\ \|\mu - \lambda\| < \delta_\lambda. \tag{4.3}$$

Now, given two distinct points $\lambda, \mu \in \mathbb{R}^n_+$, (4.3) and the compactness of the segment $[\lambda, \mu]$ easily imply that $\|\mu^* - \lambda^*\| \leq \|\mu - \lambda\|$. (We leave this exercise to the reader.) □

Corollary 4.9 *If $g\colon D_n \to \mathbb{R}$ is monotone and continuous, then also the function $f\colon \mathbb{R}^n_+ \to \mathbb{R}$, given by*

$$f(\lambda) = g(\lambda^*)\,,$$

is monotone and continuous.

The idea of "generalized centers with neglect" is the following. Given $A = \{a_1, \ldots, a_n\}$ and $x \in X$, consider the distances $d(x, a_i)$ $(i = 1, \ldots, n)$, neglect the smallest p and the largest q of them, and then apply a function depending only on the remaining ones. In this way, we obtain a function on X to minimize. This is the meaning of the following definition.

Definition 4.10 *Let $A = \{a_1, \ldots, a_n\}$. Let $p, q \geq 0$ be integers such that $\nu := n - p - q > 0$. Let $h\colon D_\nu \to \mathbb{R}$ be a monotone function. For each $x \in X$, let π_x be a permutation of $\{1, \ldots, n\}$ such that*

$$d(x, a_{\pi_x(1)}) \geq d(x, a_{\pi_x(2)}) \geq \cdots \geq d(x, a_{\pi_x(n)}).$$

We shall say that A admits h-centers with (p, q)-neglect *if the function $\varphi\colon X \to \mathbb{R}$, given by*

$$\varphi(x) = h\big(d(x, a_{\pi_x(p+1)}), \ldots, d(x, a_{\pi_x(n-q)})\big)\,,$$

attains its global minimum over X.

Observation 4.11 *The function φ from Definition 4.10 can be written in the form*

$$\varphi(x) = h\big(P_{p,q}\big((\Delta_x)^*\big)\big)\,,$$

where $P_{p,q}\colon \mathbb{R}^n_+ \to \mathbb{R}^\nu_+$ is the projection

$$P_{p,q}(\lambda) = (\lambda_{p+1}, \ldots, \lambda_{n-q})\,.$$

Now, we are ready to state an existence result for "generalized centers with neglect".

Corollary 4.12 *Let (X,d) be a metric space. Let A, p, q, ν be as in Definition 4.10. Let $h\colon \mathbb{R}^\nu_+ \to \mathbb{R}$ be continuous and monotone (e.g. convex and monotone). If X is* ball-*compact, then A admits h-centers with (p,q)-neglect.*

Proof. By Observation 4.11, we have to prove that A admits f-centers, where $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ is given by

$$f(\lambda) = h(P_{p,q}(\lambda^*)).$$

Use Corollary 4.9 and Theorem 4.6. □

Example 4.13 *Let X be a dual Banach space (or, more generally, a Banach space which is norm-one complemented in its bidual). For every finite set $A = \{a_1, \dots, a_n\} \subset X$, every $I \subset \{1, \dots, n\}$ and every $p, r_1, \dots, r_n > 0$, the functional*

$$\varphi(x) = \sum_{i \in I} r_i \|x - a_{\pi_x(i)}\|^p$$

(where π_x is as in (c) in Introduction) attains its infimum over X. This includes existence of k-centers of A from [16]*, as well as of "k-media" of A, studied in* [2].

References

[1] Aliprantis C. D. and Border K. C., *Infinite Dimensional Analysis, A Hitchhiker's Guide,* 2nd edition, Springer, 1999.

[2] Alvoni E., On k nearest points of a finite set in a normed linear space, *Discrete Appl. Math.* **143** (2004), 23–30.

[3] Amir D., Chebyshev centers and uniform convexity, *Pacific J. Math.* **77** (1978), 1–6.

[4] Amir D., Best simultaneous approximation (Chebyshev centers), in: *Parametric Optimization and Approximation (Oberwolfach 1983),* International Series of Numer. Math. **72** (B. Brosowski and F. Deutsch, eds.), Birkhäuser Verlag, Basel, 1985, pp. 19–35.

[5] Amir D., Mach J. and Saatkamp K., Existence of Chebyshev centers, best n-nets and best compact approximants, *Trans. Amer. Math. Soc.* **271** (1982), 513–524.

[6] Baronti M., Casini E. and Papini P. L., Equilateral sets and their central points, *Rend. Mat. Appl.* **13** (1993), 133–147.

[7] Baronti M., Casini E. and Papini P. L., Central points of finite sets, *Rend. Circ. Mat. Palermo (2) Suppl.* **40** (1996), 63–72.

[8] Benítez C., Fernández M. and Soriano M. L., Location of the Fermat-Torricelli medians of three points, *Trans. Amer. Math. Soc.* **354** (2002), 5027–5038.

[9] Durier R., The general one center location problem, *Math. Oper. Res.* **20** (1995), 400–414.

[10] Durier R., The Fermat-Weber problem and inner product spaces, *J. Approx. Theory* **78** (1994), 161–173.

[11] Godefroy G. and Kalton N. J., The ball topology and its applications, *Contemporary Math.* **85** (1989), 195–237.

[12] Holmes R. B., *A Course in Optimization and Best Approximation,* Lecture Notes in Mathematics **257,** Springer-Verlag, 1972.

[13] Konyagin S. V., A remark on renormings of nonreflexive spaces and the existence of a Chebyshev center, *Moscow Univ. Math. Bull.* **43** (1988), no. 2, 55–56.

[14] Lindenstrauss J. and Tzafriri L., *Classical Banach Spaces I, Sequence spaces,* Springer-Verlag, 1977.

[15] Martínez-Maurica J. and Pellón M. T., Non-archimedean Chebyshev centers, *Indagationes Math.* **49** (1987), 417–421.

[16] Papini P. L. and Puerto J., Averaging the k-largest distances among n: k-centra in Banach spaces, *J. Math. Anal. Appl.* **291** (2004), 477–487.

[17] Togni C., *Topologia ball in spazi di Banach,* Diploma Thesis, Facoltà di Scienze Mat. Fis. Nat., Università degli Studi di Milano, 2004 (see also *http://users.mat.unimi.it/users/libor/Varie/c-togni.htm*).

[18] Veselý L., A characterization of reflexivity in the terms of the existence of generalized centers, *Extracta Math.* **8** (1993), 125–131.

[19] Veselý L., Generalized centers of finite sets in Banach spaces, *Acta Math. Univ. Comenianae* **66** (1997), 83–115.

[20] Veselý L., A Banach space in which all compact sets, but not all bounded sets, admit Chebyshev centers, *Arch. Math.* **79** (2002), 499–506.

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Pseudo-Resolvents on Locally Convex Spaces

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Dedicated to my parents

Abstract. The notion of generated pseudo-resolvent is introduced and sufficient conditions under which a pseudo-resolvent is gene-rated are also given.

We have additionally introduced a more general concept of L_∞-type pseudo-resolvent, as a generalization of the similar concept introduced by F.Hirsch [4] on Banach spaces.

The class of L_∞-type pseudo-resolvents is characterized in different terms and a theorem of characterization for their generators is proved.

A relation between L_∞-type pseudo-resolvents and C_o-equicontinuous semigroups is pointed out.

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1 Introduction

Throughout this paper X is a complex Hausdorff locally convex space, C the complex field, $C_s(X)$ the continuous seminorms on X and $L(X)$ the algebra of the continuous linear operators.

This paper is in final form and no version of it will be submitted for publication elsewhere.

First of all let us recall a few elements of spectral theory for linear operators acting on locally convex spaces. In this sense we refer to [3] and [17].

Definition 1.1 *Let $V : D \to X$ be a linear operator and $\lambda \in C$. We say that $\lambda \in \rho(V)$ if it satisfies the following conditions:*
a) $\lambda I - V : D \to X$ is one to one
b) $\overline{(\lambda I - V)(D)} = X$ and $(\lambda I - V)^{-1} : (\lambda I - V)(D) \to X$ is conti-nuous. In this case $\rho(V)$ is called the resolvent set of V.

Definition 1.2 *Let $V : D \to X$ be a linear operator and $\rho(V) \neq \phi$. The mapping defined on $\rho(V)$ by $\lambda \to (\lambda I - V)^{-1} = R(\lambda, V)$ is called the resolvent of V. The concept of pseudo-resolvent was introduced by Hille on Banach spaces in [3] and investigated by Kato in [6] and Yosida in [16].*

Definition 1.3 *A mapping $R : \Delta \subset C \to L(X)$ is called a pseudo-resolvent if*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \text{ for all } \lambda, \mu \in \Delta. \tag{1}$$

According to ([17], p.299) one can state the following result:

Proposition 1.4 *$R : \Delta \to L(X)$ is a pseudo-resolvent then: $Ker\ R(\lambda)$, $Ker\ (I - \lambda R(\lambda))$, $R(\lambda)(X)$ and $(I - \lambda R(\lambda))(X)$ are independent of $\lambda \in \Delta$.*

Proposition 1.5 *([17], p.293) Let $V : D \to X$ be linear, $\lambda, \mu \in \rho(V)$ such that $R(\lambda, V)$ and $R(\mu, V) \in L(X)$. Then we have*

$$R(\lambda, V) - R(\mu, V) = (\mu - \lambda)R(\lambda, V)R(\mu, V).$$

Remark 1.6 *1.6. By Definitions 1.2 and 1.3 it follows that a resolvent is not necessarily a pseudo-resolvent, because in general on locally convex spaces $R(\lambda, V) \notin L(X)$.*

We introduce in this paper the class of generated pseudo-resolvents. After a short characterization of this class we prove sufficient conditions under which a pseudo-resolvent is generated. We also point out a practical connection between a generated pseudo-resolvent and its generator.

In 1972 F. Hirsch [4] defined the class of L_∞ pseudo-resolvents on Banach spaces, whose generators included the class of infinitesimal ge-nerators of the C_o-semigroups.

In the present paper we introduce a more general concept of L_∞-type pseudo-resolvent. We give here necessary and sufficient conditions in spectral terms under which a linear operator is the generator of a L_∞-type pseudo-resolvent.

Finally, we have shown that the class of generators of L_∞-type pseudo-resolvents generalizes the class of infinitesimal generators of C_o-equicontinuous semigroups acting on sequentially complete locally convex spaces.

2 Generated Pseudo-Resolvents

Definition 2.1 *A pseudo-resolvent $R : \Delta \to L(X)$ is called generated if there exists a linear operator $V : D \to X$ provided that: $\Delta \subset \rho(V)$, $(\lambda I - V)(D) = X$ and $R(\lambda) = (\lambda I - V)^{-1} = R(\lambda, V)$ for all $\lambda \in \Delta$. In this case V is unique and it is called the generator of R.*

Proposition 2.2 *$R : \Delta \to L(X)$ be a pseudo-resolvent. Then the following assertions are equivalent:*

1) R is generated

2) $Ker\ R(\lambda) = \{0\}$, $\lambda \in \Delta$.

In this case the operator $\lambda I - R(\lambda)^{-1}$ is independent of λ and the generator of R is $V = \lambda I - R(\lambda)^{-1}$.

Proof: Let $\lambda, \mu \in \Delta$ and $D = R(\lambda)(X)$. Then by the resolvent equation (1) it follows that $\lambda I - R(\lambda)^{-1} = \mu I - R(\mu)^{-1}$. Hence the operator $V : D \to X$ defined by $V = \lambda I - R(\lambda)^{-1}$ is the generator of R. For details see ([8], p.36).

Example 2.3 *Let E be a locally compact space, $\mathcal{K}(E)$ the family of compact subsets of E and $X = \{x : E \to C : x$ is bounded on the compact sets of $E\}$. For each $K \in \mathcal{K}(E)$ we consider the functional defined by $\|x\|_K = \sup_{t \in K} |x(t)|$. Then $(X, \|, \|_K)_{K \in \mathcal{K}(E)}$ is a Hausdorff locally convex space.*

Let also $K_0 \in \mathcal{K}(E)$ and $T : X \to X$ defined by

$$T(x)(t) = \begin{cases} x(t), & t \in Ko \\ 0, & t \notin Ko. \end{cases}$$

It is clear that T is linear and for any $K \in \mathcal{K}(E)$ we have $\|T(x)\|_K \leq \|x\|_K$. Moreover it follows that $T^2 = T$ (i.e. T is a projection).

Let $\Delta = C \setminus \{0\}$ and $R : \Delta \to L(X)$ defined by $R(\lambda) = \frac{1}{\lambda}T$. Let $\lambda, \mu \in \Delta$. Then we have $R(\lambda) - R(\mu) = \frac{\mu - \lambda}{\lambda\mu}T$ and $(\mu - \lambda)R(\lambda)R(\mu) = (\mu - \lambda)R(\lambda)\frac{1}{\mu}T = \frac{\mu - \lambda}{\lambda\mu}T^2 = \frac{\mu - \lambda}{\lambda\mu}T$. Therefore R is a pseudo-resolvent. We remark on the other hand that R is not generated, because T is not one to one.

Next we give more practical sufficient conditions, under which a pseudo-resolvent be generated.

Theorem 2.4 *Let $R : \Delta \to L(X)$ be a pseudo-resolvent and $(\lambda_n)_{n\in\mathbf{N}} \subset \Delta$ verifying that $\lim_n \lambda_n R(\lambda_n)(x) = x$ for all $x \in X$.*

Then the following statements hold:

1) R is generated and its generator is densely defined.

2) If $V : D \to X$ is its generator, then

$$D = \left\{x \in X : \exists \lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x)\right\}$$

and $V(x) = \lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x)$, $x \in D$.

Proof. Let $\mu \in \Delta$ and $x \in Ker\ R(\mu)$. Then by Proposition 1.4 it follows that $x \in Ker\ R(\lambda_n)$ for all $n \in \mathbf{N}$.

Since $\lim_n \lambda_n R(\lambda_n)(x) = x$ we deduce that $x = 0$ and $\overline{R(\mu)(X)} = X$. By taking into account that $D = R(\mu)(X)$ the first point is proved.

2) We have to prove that

$$R(\mu)(X) = \left\{x \in X : \exists \lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x)\right\}.$$

Let $x \in X$. According to (1) one can write the following equalities: $R(\lambda_n)(x) - R(\mu)(x) = (\mu - \lambda_n)R(\lambda_n)R(\mu)(x)$ and

$$(2)\quad \lambda_n\left(\lambda_n R(\lambda_n)R(\mu)(x) - R(\mu)(x)\right) = \mu\lambda_n R(\lambda_n)R(\mu)(x) - \lambda_n R(\lambda_n)(x).$$

By passing to the limit as $n \to \infty$ in (2) we get

$$\lim_n \lambda_n\left(\lambda_n R(\lambda_n)(x) - R(\mu)(x)\right) = \mu R(\mu)(x) - x = VR(\mu)(x)$$

for all $x \in X$. Hence it follows that $D \subset \left\{x \in X : \exists \lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x)\right\}$ and $VR(\mu)(x) = \mu R(\mu)(x) - x$. Let now $x \in X$ such that $\mu \in \Delta$ and $\lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x) = z$. Then it results that

$$\lim_n \lambda_n \left(\lambda_n R(\lambda_n) R(\mu)(x) - R(\mu)(x)\right) = R(\mu)(z).$$

Hence it follows that $R(\mu)(z) = \mu R(\mu)(x) - x$, or equivalently $x = R(\mu)(\mu x - z)$. In conclusion $x \in D$ and the proof is complete.

Corollary 2.5 *Let $\Delta \subset C$ unbounded and $(\lambda_n)_{n\in\mathbf{N}} \subset \Delta$ such that $|\lambda_n| \to \infty$. Let also $R : \Delta \to L(X)$ be a pseudo-resolvent and*

$$M = \left\{x \in X : \exists \lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x)\right\}.$$

Then the following statements are equivalent:

1) $\lim_n \lambda_n R(\lambda_n)(x) = x$ *for all* $x \in X$.

2) $\begin{cases} a)\ \overline{M} = X \\ b)\ \forall p \in C_s(X), \exists q \in C_s(X) \text{ s.t. } \limsup_n p\left(\lambda_n R(\lambda_n)(x)\right) \le q(x), \\ \forall x \in X. \end{cases}$

Proof. $1 \Rightarrow 2$ is a consequence of Theorem 2.4.

$2 \Rightarrow 1$. Let $x \in M$. Since the sequence $(\lambda_n(\lambda_n R(\lambda_n)(x) - x))_{n\in\mathbf{N}}$ is bounded and $|\lambda_n| \to \infty$ it follows that $\lim_n \lambda_n R(\lambda_n)(x) = x$. Let now $x \in X$ and $(x_j)_{j\in J} \subset M$ verifying that $\lim_{j\in J} x_j = x$. For any $j \in J$ and $n \in \mathbf{N}$ we have

$$\lambda_n R(\lambda_n)(x) - x = \lambda_n R(\lambda_n)(x - x_j) + \lambda_n R(\lambda_n)(x_j) - x_j + x_j - x. \quad (3)$$

Let $p, q \in C_s(X)$ verifying 2 b). By (3) and 2 b) it follows that

$\limsup_n p\left(\lambda_n R(\lambda_n)(x) - x\right) \le q(x - x_j) + \limsup_n p\left(\lambda_n R(\lambda_n)(x_j) - x_j\right) + p(x_j - x)$.

Therefore $\lim_n \lambda_n R(\lambda_n)(x) = x$.

Corollary 2.6 *Let $\Delta \subset C$ unbounded and $(\lambda_n)_{n\in\mathbf{N}} \subset \Delta$ such that $|\lambda_n| \to \infty$. Let also $R : \Delta \to L(X)$ be a pseudo-resolvent. The following assertions are equivalent:*

1) $\lim_n \lambda_n R(\lambda_n)(x) = x$ *for all* $x \in X$

2) $\begin{cases} a)\ \overline{R(\lambda)(X)} = X \\ b)\ \forall p \in C_s(X), \exists q \in C_s(X) \text{ s.t. } \limsup_n p\left(\lambda_n R(\lambda_n)(x)\right) \le q(x), \\ \forall x \in X. \end{cases}$

Proof. $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 1$. Let $x \in X,\ \mu \in \Delta,\ \mu \neq \lambda_n$ for any $n \in \mathbf{N}$. From (1) it results the following equality

(4) $\lambda_n R(\lambda_n)R(\mu)(x) = \frac{1}{\mu-\lambda_n}\lambda_n R(\lambda_n)(x) + \frac{\lambda_n}{\lambda_n-\mu}R(\mu)(x),\ \forall n \in \mathbf{N}$.

By passing to the limit in (4) it follows that $\lim_n \lambda_n R(\lambda_n)R(\mu)(x) = R(\mu)(x)$. By the density of $R(\mu)(X)$, one can prove as in Corollary 2.5 that $\lim_n \lambda_n R(\lambda_n)(x) = x$ for all $x \in X$.

3 L_∞-Type Pseudo-Resolvents

Definition 3.1 *Let $\Delta \subset C$ be unbounded. A pseudo-resolvent $R : \Delta \to L(X)$ is of L_∞-type if there exists a sequence $(\lambda_n)_{n\in\mathbf{N}} \subset \Delta$ with the property $|\lambda_n| \to \infty$, such that $\lim_n \lambda_n R(\lambda_n)(x) = x$ for all $x \in X$.*

Remark 3.2 *According to Theorem 2.4 it follows that any L_∞-type pseudo-resolvent is a generated one.*

By taking into account Theorem 2.4 and its consequenees, one can state a theorem of characterization for L_∞-type pseudo-resolvents as follows:

Theorem 3.3 *Let $\Delta \subset C$ unbounded and $R : \Delta \to L(X)$ a pseudo-resolvent. The following statements are equivalent:*

1) R is a L_∞-type pseudo-resolvent

2)
- *a)* $\overline{R(\lambda)(X)} = X$
- *b)* $\exists(\lambda_n)_{n\in\mathbf{N}} \subset \Delta,\ |\lambda_n| \to \infty$ *verifying that:*
  $\forall p \in C_s(X), \exists q \in C_s(X)$ *s.t.* $\limsup_n p\left(\lambda_n R(\lambda_n)(x)\right) \leq q(x)$,
  $\forall x \in X$.

3)
- *c)* $\exists(\lambda_n)_{n\in\mathbf{N}} \subset \Delta,\ |\lambda_n| \to \infty$ *verifying that:*
  $\left\{x \in X : \exists \lim_n \lambda_n(\lambda_n R(\lambda_n)(x) - x)\right\}$ *is dense in* X
- *d)* $\forall p \in C_s(X), \exists q \in C_s(X)$ *s.t.* $\limsup_n p\left(\lambda_n R(\lambda_n)(x)\right) \leq q(x)$,
  $\forall x \in X$.

The proof is an immediate consequence of *Definition* 3.1 and coro-llaries 2.5 and 2.6.

Example 3.4 *Let $X = C([0,1])$ the set of all continuous real functions defined on [0,1]. It is well known that X is a Banach space with respect*

to the norm given by $\|f\| = \sup_{t\in[0,1]} |f(t)|$, $f \in X$. *Consider also the subspace of* X, $D = \left\{u \in X \cap C^1([0,1]) : u(0) = 0\right\}$ *and the linear operator* $V : D \to X$ *defined by* $V(u) = -u'$. *Let* $\lambda \in \mathbf{R}$ *and* $f \in X$. *The first order linear differential equation* $\lambda u + u' = f$ *has a unique solution* $u \in D$ *whose expression is the following*

$$u(x) = \int_0^x e^{\lambda(t-x)} f(t)dt. \tag{5}$$

This is equivalent to say that for each $\lambda \in \mathbf{R}$, *the linear operator* $\lambda I - V : D \to X$ *is a bijection and* $(\lambda I - V)(u) = f$. *For any* $x \in [0,1]$ *the following inequalities are valid:*

$\left|(\lambda I - V)^{-1}(f)(x)\right| = |u(x)| \leq \int_0^x e^{\lambda(t-x)} |f(t)|\, dt \leq \frac{1}{\lambda}(1 - e^{-\lambda x}) \|f\| \leq \frac{1}{\lambda} \|f\|$ *for all* $\lambda > 0$;

$\left|V^{-1}(f)(x)\right| \leq \|f\|$ *if* $\lambda = 0$;

$\left|(\lambda I - V)^{-1}(f)(x)\right| \leq \frac{1}{|\lambda|} e^{-\lambda} \|f\|$ *for all* $\lambda < 0$.

Hence it follows that $\mathbf{R} = \rho(V)$ *and according to Definition 2.1 the mapping* $R : \mathbf{R} \to L(X)$ *defined by* $R(\lambda) = R(\lambda, V) = (\lambda I - V)^{-1}$ *is a generated pseudo-resolvent. On the other hand one can remark that* R *is not a* L_∞*-type pseudo-resolvent, because* D *is not dense in* X.

Remark 3.5 *The class of generated pseudo-resolvents is a non trivial extension of the class of* L_∞*-type pseudo-resolvents.*

The next result gives necessary and sufficient conditions under which a linear operator is the generator of a L_∞-type pseudo-resolvent.

Theorem 3.6 *A linear operator* $V : D \to X$ *is the generator of a* L_∞*-type pseudo-resolvent if and only if it satisfies the following conditions:*

1) $\overline{D} = X$

2) $\exists \Delta \subset C$ *unbounded verifying that:*

$\Delta \subset \rho(V)$ *and* $(\lambda I - V)(D) = X$ *for all* $\lambda \in \Delta$

3) $\exists (\lambda_n)_{n\in\mathbf{N}} \subset \Delta$, $|\lambda_n| \to \infty$ *such that:*

$\forall p \in C_s(X), \exists q \in C_s(X)$ *s.t.* $\limsup_n p\left(\lambda_n(\lambda_n I - V)^{-1}(x)\right) \leq q(x)$,

$\forall x \in X$.

Proof. Let us suppose that V is the generator of a L_∞-type pseudo-resolvent $R : \Delta \to L(X)$ and $(\lambda_n)_{n\in\mathbf{N}} \subset \Delta$, $|\lambda_n| \to \infty$ such that $\lim_n \lambda_n R(\lambda_n)(x) = x$ for all $x \in X$. Since $R(\lambda) = (\lambda I - V)^{-1}$ it is clear that V fulfils the three conditions.

Assume now that V satisfies the above conditions and define the mapping $R : \Delta \to L(X)$ by $R(\lambda) = (\lambda I - V)^{-1}$. Since $D = R(\lambda)(X)$, by Theorem 3.3 it follows that R is a L_∞-type pseudo-resolvent and V is its generator.

Corollary 3.7 *Let X be a Hausdorff and complete locally convex space. A linear operator $V : D \to X$ is the generator of a L_∞-type pseudo-resolvent if and only if it fulfils the following conditions:*

1) V is closed and $\overline{D} = X$

2) $\exists \Delta \subset C$ unbounded such that $\Delta \subset \rho(V)$

3) $\exists (\lambda_n)_{n\in\mathbf{N}} \subset \Delta$ unbounded, $|\lambda_n| \to \infty$ such that: $\forall p \in C_s(X)$, $\exists q \in C_s(X)$ s.t. $\limsup_n p\left(\lambda_n(\lambda_n I - V)^{-1}(x)\right) \leq q(x)$, $\forall x \in X$.

Proof. It is sufficient to prove that $(\lambda I - V)(D) = X$ for any $\lambda \in \Delta$ and to take into account Theorem 3.6. Let $\lambda \in \Delta$ and $x \in X$. According to *Definition* 1.1 there exists $(x_j)_{j\in J} \subset (\lambda I - V)(D)$ such that $x_j \to x$. Let also $(y_j)_{j\in J} \subset D$ such that $x_j = (\lambda I - V)(y_j)$, $\forall j \in J$. Since $(\lambda I - V)^{-1} : (\lambda I - V)(D) \to X$ is continuous it results that $(\lambda I - V)^{-1}(x_j) = y_j$ is Cauchy and consequently it is convergent. Let $y = \lim_{j\in J} y_j$. Thus we have $(y_j, (\lambda I - V)(y_j)) \to (y, x)$. Since V is closed it follows that $y \in D$ and $(\lambda I - V)(y) = x$. Therefore $(\lambda I - V)(D) = X$ for all $\lambda \in \Delta$ and the proof is complete.

Corollary 3.8 *Let X be a Banach space. A linear operator $V : D \to X$ is the generator of a L_∞-type pseudo-resolvent if and only if it satisfies the following conditions:*

1) V is closed and $\overline{D} = X$

2) $\exists \Delta \subset C$ unbounded such that $\Delta \subset \rho(V)$

3) $\exists (\lambda_n)_{n\in\mathbf{N}} \subset \Delta$, $|\lambda_n| \to \infty$ such that, the sequence $\left(\lambda_n(\lambda_n I - V)^{-1}\right)_{n\in\mathbf{N}}$ is equicontinuous.

Proof. Necessity: In this context the third condition of Corollary 3.7 becomes the following: $\exists (\lambda_n)_{n\in\mathbf{N}} \subset \Delta$, $|\lambda_n| \to \infty$ such that: there exists $M > 0$ such that $\limsup_n \left\|\lambda_n(\lambda_n I - V)^{-1}(x)\right\| \leq M \|x\|$ for all $x \in X$.

Therefore the sequence of operators $\left(\lambda_n(\lambda_n I - V)^{-1}\right)_{n\in\mathbf{N}}$ is pointwise bounded and consequently equicontinuous.

The sufficiency is obvious and its proof is omitted.

Example 3.9 *Let $X = C([0,\infty))$ the set of all complex continuous functions defined on $[0,\infty)$. For each $n \in \mathbf{N}$ we consider the seminorm $p_n : X \to \mathbf{R}$ defined by $p_n(u) = \sup_{x\in[0,n]} |u(x)|$. Thus X is a Frèchet space whose topology is given by the family of seminorms $(p_n)_{n\in\mathbf{N}}$. We denote by $i \in X$ the identical function (i.e. $i(x) = x$, $\forall x \in [0,\infty)$) and consider the linear operator $V : X \to X$ defined by $V(u) = -iu$. Let $\lambda \in C$ with the property that $\mathrm{Re}(\lambda) > 0$ and $u \in X$ such that $(\lambda I - V)(u) = 0_X$. This means that $(\lambda + x)u(x) = 0$ for all $x \in [0,\infty)$. Hence it follows that $u = 0_X$ and in consequence the linear operator $\lambda I - V$ is one to one. Let now $w \in X$ arbitrary and $u = \dfrac{w}{\lambda + i}$ $\left(i.e.\ u(x) = \dfrac{w(x)}{\lambda + x}\right)$. Then $(\lambda I - V)(u) = w$, which shows that $(\lambda I - V)(X) = X$. Let now $u \in X$, $x \in [0,n]$, and $n \in \mathbf{N}$. Then we have $((\lambda I - V)(u))(x) = (\lambda + x)u(x)$ and $|(\lambda I - V)(u)(x)| \geq |\lambda|\,|u(x)|$. Hence it follows that $p_n((\lambda I - V)(u)) \geq |\lambda|\, p_n(u)$, which is equivalent to $p_n\left(\lambda(\lambda I - V)^{-1}(w)\right) \leq p_n(w)$ for all $w \in X$. Therefore V fulfils the conditions required by Theorem 3.6 and consequently it is the generator of a L_∞-type pseudo-resolvent $R : \{\lambda \in C : \mathrm{Re}(\lambda) > 0\} \to L(X)$ whose expresion is the following: $R(\lambda)(w) = (\lambda I - V)^{-1}(w) = u = \dfrac{w}{\lambda + i}$ for all $w \in X$.*

Remark 3.10 *3.10. Theorem 3.6 and its consequences suggest a possible connection between the class of L_∞-type pseudo-resolvents and the class of C_o-equicontinuous semigroups.*

Theorem 3.11 *Let X be a sequentially complete locally convex space, $\Delta \subset C$ such that $(0,\infty) \subset \Delta$ and $V : D \to X$ a linear operator. The following statements are equivalent:*

1) V is the generator of a C_o-equicontinuous semigroup

2) V is the generator of a L_∞-type pseudo-resolvent $R : \Delta \to L(X)$ provided that the family $(nR(n))^k_{\substack{n\in\mathbf{N}^* \\ k\in\mathbf{N}}}$ is equicontinuous.

Proof. 1 $\Rightarrow$ 2. By Hille - Yosida theorem ([17], p.332) it follows that the infinitesimal generator of a C_o-equicontinuous semigroup fulfils obviously the conditions of Theorem 3.6 and in conclusion it is the generator of a L_∞-type pseudo-resolvent, which in this case is the Laplace transform of the semigroup.

It is also known that the family $(\lambda R(\lambda))^k_{\substack{\lambda>0 \\ k\in\mathbf{N}}}$ is equicontinuous.

$2 \Rightarrow 1$. In this case it follows that $\Delta \subset \rho(V)$, $R(\lambda) = (\lambda I - V)^{-1}$ for all $\lambda \in \Delta$ and $\overline{D} = X$. In addition the family $\left(n(nI - V)^{-1}\right)^k_{\substack{n\in\mathbf{N}^* \\ k\in\mathbf{N}}}$ is equicontinuous.

Under these conditions, V is the generator of a C_o-equicontinuous semigroup ([17], p.339).

Remark 3.12 *The class of the generators of L_∞-type pseudo-resolvents generalizes the class of the infinitezimal generators of C_o-equicontinuous semigroups acting on sequentially complete locally convex spaces.*

Example 3.13 *Let $X = C([0,\infty))$ the locally convex space given in Example 3.9 and $V : X \to X$ defined by $V(u) = -iu$. We have proved that V was the generator of the L_∞ type pseudo-resolvent $R : \{\lambda \in C : \mathrm{Re}(\lambda) > 0\} \to L(X)$ whose expression was $R(\lambda)(u) = \dfrac{u}{\lambda + i}$. For any $n, m \in \mathbf{N}^*$ and $x \in [0, n]$ we have: $|(mR(m)(u))(x)| = m\left|\dfrac{u(x)}{m + x}\right| \leq |u(x)|$. Hence it follows that $p_n(mR(m)(u)) \leq p_n(u)$, which implies that the family $(mR(m))^k_{\substack{m\in\mathbf{N}^* \\ k\in\mathbf{N}}}$ is equicontinuous. By Theorem 3.11 it follows that V is the generator of a C_o-equicontinuous semigroup $T : [0,\infty) \to L(X)$. From the classical theory of semigroups [17] it results that $T(t)(u) = \lim_n e^{tn(nR(n)-I)}(u) = e^{-ti}u$ (i.e. $(T(t)(u)(x) = e^{-tx}u(x)$).*

References

[1] Arendt W., *Resolvent positive operators*, Proc. London Mat. Soc. **54** (1987), 321-349.

[2] Davies E.B., *One Parameter Semigroups*, Academie Press, London, 1980.

[3] Hille E., *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ., 31, 1948.

[4] Hirsch F., *Familles resolventes, generateurs, cogenerateurs, potentieles*, Ann. Inst. Fourier **22** (1972), 89-210.

[5] Hirsch F., *Sur les familles resolvantes et les generateurs associés*, C. R. Acad. Sc. Paris **271** (1970), Serie A, 714-717.

[6] Kato T., *Remarks on pseudo-resolvents and infinitesimal generators of semi-groups*, Proc. Japan Acad. **35** (1959), 467-468.

[7] Nagel R. Ed., *One Parameter semigroups of positive operators*, Springer, Lecture Notes 1184, 1986.

[8] Pazy A., *Semigroups of linear operators and Applications to Partial Differential Equations*, Springer, 1983.

[9] Voicu M., *Resolvent families on locally convex spaces*, Stud. Cerc. Mat., 29, 2 (1977) (in romanian).

[10] Voicu M., *Limit operators of resolvents on locally convex spaces*, Stud. Cerc. Mat., 29, 5 (1977) (in romanian).

[11] Voicu M., *Positive resolvents and the property of dispersiveness in locally convex spaces*, Semesterbericht Functionalanalysis Sommersemester, (1985), 201-210.

[12] Voicu M., *Resolvent positive operators*, Ann. Univ. Bucharest, XXXVIII, 3 (1989), 83-90.

[13] Voicu M., *Dissipative operators and resolvents*, Romanian J. of Pure and Appl. Math., XLII, 5-6, (1997), 449-459.

[14] Voicu M., *Dissipative and accretive operators on locally convex spaces*, Rend. Circ. Matem., Palermo, Serie II, **52** (1998), 805-815.

[15] Voicu M., *Resolvents on locally convex spaces*, Order structures in functional analysis, Printing house of Romanian Academy **4** (2001), 204-258.

[16] Yosida K., *Ergodic theorems for pseudo-resolvents*, Proc. Japan Acad. **37** (1961), 422-425.

[17] Yosida K., *Functional Analysis*, Springer, 1965.

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Integral Frames and Riesz Bases

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Abstract. In this work some facts of the theory of frames (namely the theorem giving necessary and sufficient conditions for system to be a frame) are generalized to a continuous case. That is, to get "discrete" Riesz basis we employ linear bounded invertible operator to the orthonormal basis in Hilbert space and in the definition of integral Riesz system we employ linear bounded invertible operator to the system, similar to orthogonal. Their definition will be given below. Such systems possess most of the properties of the orthogonal systems, such as the Parseval formula, Riesz-Fisher theorem and others.

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1 Basic information

Let H be a Hilbert space over the field $\mathbb{R}$ or $\mathbb{C}$. In 1946 N.K. Bary ([1]) introduced the following definition of Riesz basis (system):

Definition 1.1. *A sequence* $\{\psi_n\}_{n\in\mathbb{N}} \subset H$ *is called Riesz basis if it is complete and there exist constants* $0 < A \leqslant B < \infty$ *so that for each sequence* $\{\gamma_n\}_{n\in\mathbb{N}} \in l_2(\mathbb{N})$

$$A\sum_{n=1}^{\infty}|\gamma_n|^2 \leqslant \left\|\sum_{n=1}^{\infty}\gamma_n\psi_n\right\|_H^2 \leqslant B\sum_{n=1}^{\infty}|\gamma_n|^2.$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Today, Riesz systems are commonly used in the theory of decomposition and functional analysis (for more information see [2, 3]). Particularly, D. Han and D. R. Larson ([4]) and (independently) B.S. Kashin and T. Yu. Koulikova ([5]) used these systems for describing frames of general form.

Definition 1.2. *A sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset H$ is called frame if there exist constants $0 < A \leqslant B < \infty$ so that*

$$A \parallel y \parallel_H^2 \leqslant \sum_{n=1}^{\infty} |(y, \varphi_n)|^2 \leqslant B \parallel y \parallel_H^2$$

for all $y \in H$. The numbers A, B are called the lower and upper frame bounds respectively.

These necessary and sufficient conditions for system to be a frame are proved in [4] and in [5]:

Theorem 1.3. *The sequence $\Phi = \{\varphi_j : j \in \mathbb{N}\}$ is frame in Hilbert space H with constants A, B if and only if there exist a larger Hilbert space $H' \supset H$ and a Riesz basis for H' $\Psi = \{\psi_j : j \in \mathbb{N}\}$ with the same constants A, B so that*

$$\varphi_n = \pi_{H'\to H}(\psi_n), \quad n \in \mathbb{N},$$

where $\pi_{H'\to H}$ is the orthogonal projection of the space H' onto H.

Interesting generalization of this theorem to the case of (Ω, μ)-frames could be found in [6].

2 Integral Riesz bases

Let Ω be a space with countably additive measure μ.

Definition 2.1. *The system $\{e^w\}_{w\in\Omega} \subset H$ is called system, similar to orthogonal, if all elements $y \in H$ could be represented as*

$$y = \int_{\Omega} \widehat{y}_\omega e^\omega d\mu(\omega),$$

where $\widehat{y}_\omega = (y, e^\omega)$. The integral is regarded as proper or improper Lesbegue integral of a function with values from H, in the latter case there exist an exhaustion $\{\Omega_k\}_{k=1}^{\infty}$ of the space Ω (all the subspaces Ω_k are measurable,

$\Omega_k \subset \Omega_{k+1}$ *for all* $k \in \mathbb{N}$ *and* $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$*), which may depend on* y *(it is called suitable for* y*) so that the function* $\hat{y}_\omega e^\omega$ *is integrable on* Ω_k *and*

$$y = \int_\Omega \hat{y}_\omega e^\omega \, d\mu(\omega) = \lim_{k\to\infty} (L) \int_{\Omega_k} \hat{y}_\omega e^\omega \, d\mu(\omega).$$

This definition was proposed by T.P. Lukashenko in 1997 as a continuous analogue for orthogonal systems.

Remark 2.2. *If* $\Omega = \mathbb{N}$ *and for all* $k \in \mathbb{N}$ $\mu(k) = 1 (\mu(k) > 0)$ *and* $\Omega_k = \{1, 2, ..., k\}$*, we get the definition of countable non-negative systems similar to orthogonal with common convergence, including countable complete orthonormal (orthogonal) systems as a special case.*

Remark 2.3. *Unlike the case of orthonormal systems, generally the coefficients in the decomposition formula in the definition of the systems similar to orthogonal are not unique.*

For more details, see [7, 8].

Now we will put in a notion, which is analogous to the notion of Riesz basis.

Definition 2.4. *Let* A *be a linear bounded operator with bounded inverse operator, which depicts a system similar to orthogonal* $\{e^w\}_{w\in\Omega}$ *to some system* $\{\psi^w\}_{w\in\Omega}$*. Then we call the system* $\{\psi^w\}_{w\in\Omega}$ *an integral Riesz system.*

Here, according to the definition of the system similar to orthogonal, for all $y \in H$

$$A^{-1}y = \int_\Omega (A^{-1}y, e^w) e^w d\mu(w) = \int_\Omega (y, A^{-1^*} e^w) e^w d\mu(w).$$

Hence, according to [9],

$$y = \int_\Omega (y, A^{-1^*} e^w) A e^w d\mu(w) = \int_\Omega (y, A^{-1^*} e^w) \psi^w d\mu(w).$$

Definition 2.5. *The system* $\{\psi^w\}_{w\in\Omega}$ *is called integral Riesz basis, if it becomes system similar to orthogonal in space* H *after the corresponding replacement of the inner product* (f, g) *with some new one, topologically equivalent to the first,* $(f, g)_1$*, i.e. there exist* $c_1, c_2 > 0$, $c_1(y, y) \leqslant (y, y)_1 \leqslant c_2(y, y)$ *for all* $y \in H$*.*

Definition 2.6. *The system $\Psi = \{\psi^w\}_{w\in\Omega} \subset H$ is called integral Riesz system, if there are constants $0 < A \leqslant B < \infty$, so that for either $y \in H$ we take $\gamma_w = (y, \psi^w)$ and*

$$A\int_\Omega |\gamma_w|^2 d\mu(w) \leqslant \left|\left|\int_\Omega \gamma_w \psi^w d\mu(w)\right|\right|_H^2 \leqslant B\int_\Omega |\gamma_w|^2 d\mu(w).$$

Definition 2.7. *Suppose that for the system $\{\psi^w\}_{w\in\Omega}$ in H and for all $y \in H$*

$$\int_\Omega |(y, \psi^w)|^2 d\mu < \infty.$$

Let us define a system $\{\chi^w\}_{w\in\Omega}$ so that $(\psi^{w_1}, \chi^{w_2}) = \delta^{w_2}_{w_1}$. If for all $y \in H$

$$\int_\Omega |(y, \chi^w)|^2 d\mu < \infty,$$

then $\{\psi^w\}_{w\in\Omega}$ is integral Riesz system.

Definition 2.8. *Let $\{(\psi^{w_1}, \psi^{w_2})\}_{w_1,w_2\in\Omega}$ generate in $L_2(\Omega)$ a linear bounded operator V, so that*

$$V\alpha^w = \int_\Omega (\psi^\gamma, \psi^w)\alpha^w d\mu(\gamma)$$

for all $\{\alpha^w\}_{w\in\Omega} \in L^2(\Omega)$. Then a system $\{\psi^w\}_{w\in\Omega}$ is integral Riesz system.

Theorem 2.9. *The definitions* 2.4 — 2.8 *of integral Riesz system are equivalent.*

Proof. Let us prove that definition 2.5 follows from definition 2.4. We define new inner product by formula $(f, g)_1 = (A^{-1}f, A^{-1}g)$. It is equivalent to initial one, because by virtue of the boundedness of the operator A we have $\| A^{-1}f \|^2 \leqslant \| A^{-1} \|^2 \| f \|^2$, where $0 < \| A^{-1} \|^2 < \infty$, and $(f, f) = (AA^{-1}f, f) = (A^{-1}f, A^*f) \leqslant \| A^{-1}f \| \| A^*f \|$. If we define $c_1 = \| A \|^{-2}$, $c_2 = \| A^{-1} \|^2$, we get $c_1(f, f) \leqslant (f, f)_1 \leqslant c_2(f, f)$ for all $f \in H$. As $(f, \psi^w)_1 = (A^{-1}f, A^{-1}\psi^w) = (A^{-1}f, e^w)$, and $A^{-1}f = \int_\Omega (A^{-1}f, e^w)e^w d\mu(w)$ according to the definition of the system similar to the orthogonal, then

$$\| f \|_1^2 = \| A^{-1}f \|^2 = \int_\Omega |(A^{-1}f, e^w)|^2 d\mu(w) = \int_\Omega |(A^{-1}f, A^{-1}\psi^w)|^2 d\mu(w),$$

in accord to the integral analogue of the Parseval formula ([7]). Using the criteria for system similar for orthogonal ([7]), we see that after the replacement of the inner product the system $\{\psi^w\}_{w\in\Omega}$ becomes system similar to the orthogonal.

We derive now definition 2.6 from definition 2.5. Let us choose $c_1, c_2 > 0$, so that for all $f \in H$ the inequalities $c_1(f,f) \leqslant (f,f)_1 \leqslant c_2(f,f)$ take place. We rewrite them in the form: $c_2^{-1} \parallel f \parallel_1^2 \leqslant \parallel f \parallel^2 \leqslant c_1^{-1} \parallel f \parallel_1^2$. As

$$\parallel f \parallel^2 = \left|\left| \int_\Omega \gamma_w \psi^w d\mu(w) \right|\right|^2,$$

where $\gamma_w = (f, \psi^w)_1$, and

$$f = \int_\Omega (f, \psi^w)_1 \psi^w d\mu(w) = \int_\Omega (A^{-1}f, A^{-1}\psi^w)\psi^w d\mu(w),$$

then according to integral Parseval formula,

$$\parallel f \parallel_1^2 = \int_\Omega \left|(A^{-1}f, A^{-1}\psi^w)\right|^2 d\mu(w).$$

After taking $a = c_2^{-1}, \quad b = c_1^{-1}$, we get the required inequality.

Now we will deduce definition 2.4 from definition 2.6. $\{e^w\}_{w\in\Omega}$ is any given system similar to orthogonal, $y \in H$. Let us define operators

$$A_1 \left(\int_\Omega \gamma_w e^w d\mu(w) \right) = \int_\Omega \gamma_w \psi^w d\mu(w),$$
$$A_2 \left(\int_\Omega \gamma_w \psi^w d\mu(w) \right) = \int_\Omega \gamma_w e^w d\mu(w)$$

for $\gamma_w = (y, e^w)$. Then

$$\left|\left| A_1 \left(\int_\Omega \gamma_w e^w d\mu(w) \right) \right|\right| \leqslant b^{1/2} \int_\Omega |\gamma_w|^2 d\mu = b^{1/2} \left|\left| \int_\Omega \gamma_w e^w d\mu(w) \right|\right|.$$

Similarly,

$$\left|\left| A_2 \left(\int_\Omega \gamma_w \psi^w d\mu(w) \right) \right|\right| = \int_\Omega |\gamma_w|^2 d\mu(w) \leqslant a^{-1/2} \left|\left| \int_\Omega \gamma_w \psi^w d\mu(w) \right|\right|.$$

Thus, the operators A_1 and A_2 are bounded. Now we can expand A_1 and A_2 to the space H to linear bounded operators. Since $A_1 A_2 = A_2 A_1 = I$, the operator A_1 is invertible.

To bring out definition 2.4 from definition 2.7, we consider a bulged functional $p(y) = \left(\int_\Omega |(y, \psi^w)|^2 d\mu(w)\right)^{1/2}$. It is evident that

$$p(y) = \sup_k \left(\int_{\Omega_k} |(y, \psi^w)|^2 d\mu(w)\right)^{1/2},$$

where $\{\Omega_k\}_{k\in\mathbb{N}}$ are chosen in such a way that $\Omega_k \subset \Omega$ for all $k \in \mathbb{N}$ and $\cup_{k=1}^{\infty}\Omega_k = \Omega$. Then functional $p(y)$ is continuous by virtue of the Gelfund's lemma ([10]). It means, that there exists a constant $d > 0$, such that $p(y) \leqslant d \parallel y \parallel$. Consider the operators R and S:

$$R\left(\int_\Omega \gamma_w \psi^w d\mu(w)\right) = \int_\Omega \gamma_w e^w d\mu(w),$$
$$S\left(\int_\Omega \gamma_w \chi^w d\mu(w)\right) = \int_\Omega \gamma_w e^w d\mu(w)$$

for $\gamma_w = (y, e^w)$, where $\{e^w\}_{w\in\Omega}$ is system similar to orthogonal, the integrals are proper or improper Lesbegue integrals. Then

$$\left(R\left(\int_\Omega \gamma'_w \psi^w d\mu(w)\right), S\left(\int_\Omega \gamma''_w \chi^w d\mu(w)\right)\right) =$$
$$= \left(\int_\Omega \gamma'_w e^w d\mu(w), \int_\Omega \gamma''_w e^w d\mu(w)\right) = \int_\Omega \gamma'_w \gamma''_w d\mu(w)$$

and $(Rf, Sg) = (f, g)$, $\quad (R^*Sf, g) = (f, g)$ for all $f, g \in H$, i.e., $R^*S = I$. So, the operator R is invertible (because the set of its values is dense in H) and we get the definition 2.4.

Taking $\gamma_w = (y, \psi^w)$ in definition 2.6, we have

$$\int_\Omega |\gamma_w|^2 d\mu(w) \leqslant A^{-1} \left|\left|\int_\Omega (y, \psi^w)\psi^w d\mu(w)\right|\right|^2.$$

This means that $\int_\Omega |\gamma_w|^2 d\mu(w) < \infty$. Analogously, $\int_\Omega |(y, \chi_w)|^2 d\mu(w) < \infty$.

Finally, we will take out definition 2.8. According to definition 2.2, $Ae^w = \psi^w$, then $(A^*Ae^w, e^\gamma) = (\psi^w, \psi^\gamma), w, \gamma \in \Omega$; thus, $\{(\psi^w, \psi^\gamma)\}_{w,\gamma\in\Omega}$ corresponds to the linear and bounded operator A^*A.

Conversely, let us take a system similar to orthogonal $\{e^w\}_{w\in\Omega}$. Let us consider the operator V, defined by the following formula:

$$V\left(\int_\Omega \gamma_w e^w d\mu(w)\right) = \int_\Omega \varphi^w \left(\int_\Omega (\psi^{w_1}, \psi^w)\gamma_{w_1} d\mu(w_1)\right) d\mu(w),$$

where $y \in H$, $\gamma_w = (y, e^w)$ and $\int_\Omega |\gamma_w|^2 d\mu(w) < \infty$. As

$$\left(V\left(\int_\Omega \gamma_w e^w d\mu(w)\right), y\right) = \left(\int_\Omega \int_\Omega e^w(\psi^{w_1}, \psi^w)\gamma_{w_1} d\mu(w_1) d\mu(w), y\right) =$$
$$\int_\Omega \int_\Omega \gamma_{w_1}\overline{\gamma_w}(\psi^{w_1}, \psi^w) d\mu(w_1) d\mu(w) = \int_\Omega \int_\Omega (\gamma_{w_1}\psi^{w_1}, \gamma_w\psi^w) d\mu(w_1) d\mu(w)$$
$$= \left(\int_\Omega \gamma_w\psi^w d\mu(w), \int_\Omega \gamma_w\psi^w d\mu(w)\right),$$

then

$$\| V^{-1} \|^{-1} \int_\Omega |\gamma_w|^2 d\mu(w) \leqslant \left\| \int_\Omega \gamma_w\psi^w d\mu(w) \right\|^2 \leqslant \| V \| \int_\Omega |\gamma_w|^2 d\mu(w),$$

hence, definition 2.6 takes place. The proof is complete. □

3 Integral frames

Definition 3.1. *The system $\Phi = \{\varphi^w\}_{w\in\Omega} \subset H$ is called integral frame, if there exist constants $0 < A \leqslant B < \infty$ and*

$$A \| y \|_H^2 \leqslant \int_\Omega |(y, \varphi^w)|^2 d\mu(w) \leqslant B \| y \|_H^2$$

for all $y \in H$.

Now we will generalize the theorem (necessary and sufficient conditions for system to be a frame) mentioned above to the case of integral frames.

Theorem 3.2. *$\Phi = \{\varphi^w\}_{w\in\Omega} \subset H$ is integral frame with constants a, b if and only if there exists Hilbert space H', so that $H \subset H'$ and integral Riesz basis $\Psi = \{\psi^w\}_{w\in\Omega} \subset H'$ with constants a, b, so that*

$$\varphi^w = \pi_{H'\to H}(\psi^w), \quad w \in \Omega,$$

where $\pi_{H'\to H}$ is the orthogonal projection of the space H' onto H.

Proof. **Sufficiency.** Let us see, that system $\chi^w = (A^{-1})^* e^w$ is integral Riesz system in H^* with constants a^{-1}, b^{-1}. It is obvious that $\{\chi^w\}_{w\in\Omega}$ is integral Riesz system (according to one of the definitions). Then, if Riesz constants are a', b' respectively,

$$a' = \| ((A^{-1})^*)^* \|^{-2} = b^{-1}, b' = \| A^{-1} \|^2 = a^{-1}.$$

For all $y \in H^*$

$$b^{-1}\int_\Omega |\gamma_w|^2 d\mu(w) \leqslant \left|\left|\int_\Omega \gamma_w \chi^w d\mu(w)\right|\right|_H^2 \leqslant a^{-1}\int_\Omega |\gamma_w|^2 d\mu(w)$$

and

$$a \parallel y \parallel_H^2 \leqslant \int_\Omega |(y,\psi^w)|^2 d\mu(w) \leqslant b \parallel y \parallel_H^2 .$$

As H and H' are isometric we pass on to H. It is evident that

$$(f,\psi^w) = (f,\varphi^w),$$

because

$$\varphi^w = \pi_{H'\to H}(\psi^w), \quad w \in \Omega.$$

Then, as for all $y \in H$ we have

$$a \parallel y \parallel_H^2 \leqslant \int_\Omega |(y,\varphi^w)|^2 d\mu(w) \leqslant b \parallel y \parallel_H^2,$$

$\{\varphi^w\}_{w\in\Omega}$ is integral frame.

Necessity. Let $H = L^2(\Omega_1)$ without loss of generality. Define $v^\gamma = \{\varphi^w(x_\gamma)\}_{w\in\Omega_1}$, where $x_\gamma \in H$. Then from

$$a \parallel y \parallel_H^2 \leqslant \int_\Omega |(y,\varphi^w)|^2 d\mu(w) \leqslant b \parallel y \parallel_H^2$$

for all $y \in H$ we have

$$a \parallel \alpha \parallel_H^2 \leqslant \left|\left|\int_\Omega \alpha_\gamma v^\gamma d\mu(\gamma)\right|\right|_H^2 \leqslant b \parallel \alpha \parallel_H^2$$

for all $\alpha \in L^2(\Omega)$ so $\{v^\gamma\}_{w\in\Omega_1}$ is integral Riesz system in H. Consider in H a system, similar to orthogonal $\{e^w\}_{w\in\Omega_2}$. Define $\Gamma = \{\gamma^w\}_{w\in\Omega_1\bigcup\Omega_2}$ as

$$\gamma^w = \begin{cases} v^w, w \in \Omega_1, \\ e^w, w \in \Omega_2. \end{cases}$$

Γ is integral Riesz system for $L^2(\Omega)$. If

$$r^{w,\lambda} = \begin{cases} (v^w)_\lambda, w \in \Omega_1, \\ (e^w)_\lambda, w \in \Omega_2, \end{cases}$$

and

$$\psi^w = \{r^{w,\lambda} : w \in \Omega\}.$$

Then,

$$\varphi^w = \pi_{L^2(\Omega)\to H}(\psi^w), \quad w \in \Omega,$$

and $\{\psi^w\}_{w\in\Omega}$ is integral Riesz system in $L^2(\Omega)$, because Γ is integral Riesz system and

$$\left|\left|\int_\Omega \beta_w \psi^w d\mu(w)\right|\right|^2_{L^2(\Omega)} = \int_\Omega |(\beta, \gamma^w)|^2 d\mu(w).$$

So we have

$$a\int_\Omega |\beta_w|^2 d\mu(w) \leqslant \left|\left|\int_\Omega \beta_w \psi^w d\mu(w)\right|\right|^2_{L^2(\Omega)} \leqslant b\int_\Omega |\beta_w|^2 d\mu(w). \qquad \square$$

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References

[1] Bary N. K., Biorthogonal systems, *Sc. Proc. of MSU* **148** (1951), 4, 68–107.

[2] Daubechies I., *Ten lectures on wavelets,* SIAM, Philadelphia, PA, 1992.

[3] Chui C. K., *An introduction to wavelets,* Academic Press, New York, 1992.

[4] Han D. and Larson D. R., Frames, bases and group representations, *Mem. Amer. Math. Soc.* **147** (2000), 697, x+94.

[5] Kashin B. S. and Koulikova T. Yu., A remark on the description of frames of general form, *Math. Notes* **72** (2002), 5-6, 863-867.

[6] Gabardo J.-P. and Han D. Frames associated with measurable spaces, *Adv. Comp. Math.* **18** (2003), 2-4, 127-147.

[7] Lukashenko T. P., Coefficients with respect to orthogonal-like decomposition systems, *Sb. Math.* **188** (1997), 12, 1783-1798.

[8] Lukashenko T. P., Properties of expansion systems similar to orthogonal, *Izv. Math.* **62** (1998), 5, 1035-1054.

[9] Dunford N. and Schwartz J. T., *Linear operators,* New York, Interscience, 1957.

[10] Akhiezer N. I. and Glazman I. M., *Theory of linear operators in Hilbert space,* Boston, Pitman, 1981.

[11] Casazza P. G., The art of frame theory, *Taiwanese J. Math.* **4** (2002), 2, 129-201.

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