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del Circolo matematico di Palermo

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SUPPLEMENTO

AI

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO

DIRETTORE: B. PETTINEO

ATTI
DEL CONVEGNO CELEBRATIVO DEL I° CENTENARIO
DEL CIRCOLO MATEMATICO DI PALERMO
22-27 Ottobre 1984

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DISCORSO DEL PROF. BENEDETTO PETTINEO

PRESIDENTE DEL CIRCOLO MATEMATICO DI PALERMO

Signore, Signori

innanzi tutto Vi ringrazio. E non solamente ho l'onore di ringraziare i matematici, poichè accanto ai matematici, ai rappresentanti delle prestigiose Accademie e delle grandi Istituzioni culturali, sono qui presenti i rappresentanti del Parlamento Siciliano, della Città di Palermo e i Dirigenti dei Ministeri dell'Istruzione e dei Beni Culturali. E sono presenti i nostri allievi, anziani e giovani, e quanti ancora hanno sentito di partecipare con gioia alle celebrazioni del primo centenario del Circolo Matematico di Palermo. A tutti esprimo la mia gratitudine, anche agli amici lontani che hanno fatto giungere il loro messaggio. In maniera particolare a Giuseppe La Grutta, che ha voluto concludere l'opera di Rettore del nostro Ateneo partecipando a questa nostra festa.

Mi piace iniziare riportando le parole pronunziate da Michele Albeggiani, Presidente del Circolo Matematico nel 1914, in occasione del 30° anniversario della gloriosa Istituzione: " Nel febbraio del 1884 il Dr. G.B. Guccia, concepita l'idea della fondazione di un circolo matematico in Palermo, ne ragionò prima con amici e compagni di studi: adunati quindi i professori della facoltà matematica di questo Ateneo palermitano, insieme ad altri sia cultori sia matematici, sia comunque amici della scienza, nel dì 2 marzo 1884, per opera sua, venne approvato e sottoscritto da 27 soci lo Statuto provvisorio del Circolo Matematico di Palermo con lo scopo assai modesto dell'incremento degli studi matematici in Palermo: nè altro obiettivo per allora era dato presumere. E poichè lo scambio delle idee è il mezzo più efficace a rinvigorire le menti degli individui e a rendere gran-

di le istituzioni, si rese ben presto necessaria, per la più completa affermazione della nuova Società, la pubblicazione di un organo proprio a far conoscere, fuori dell'ambiente del Circolo, i risultati delle ricerche dei suoi soci, sino allora consegnati in un registro manoscritto,redatto dai segretarî. Quindi il Dr. Guccia, fondatore del Circolo, ne fondò pure il periodico dal titolo: Rendiconti del Circolo Matematico di Palermo. "

E mi sembra pure opportuno ricordare che in quell'epoca, solo da qualche anno era stato avvertito un segno di rinnovamento nella matematica palermitana: ad opera di Cesare Arzelà nel 1878 e di Alfredo Capelli nel 1880. Ernesto Cesàro, che finalmente avrebbe fatto capire cosa fosse la vera scienza, sarebbe arrivato nel 1886. Tutto questo, quando altrove uomini come Luigi Cremona, Francesco Brioschi, Enrico Betti si apprestavano già a concludere la loro opera di protagonisti.

E tuttavia, proprio Palermo, in condizioni di grave ritardo, nel giro di pochi anni avrebbe avuto l'eccezionale privilegio di diventare punto di riferimento della società matematica mondiale.

Non c'è di meglio che riportare le parole di Vito Volterra, il quale, a 30 anni dalla fondazione del Circolo Matematico, così rievoca gli inizi dell'opera del Guccia: " Pochi erano allora, sebbene zelanti e volenterosi, i soci del Circolo Matematico, non incominciato il primo fascicolo di pubblicazioni, modesta era la Biblioteca. Ma grande era l'entusiasmo che animava il Guccia. Pieno di speranza e di ardore, egli aveva sin da allora la visione che qualcosa di vitale e di utile dovesse uscire da quei modesti principi. La sua fede sincera rivelava una di quelle energie che creano le grandi istituzioni, che danno origine alle opere durature e feconde. Due anni dopo i professori italiani si riunivano a Milano in un congresso che rimarrà memorabile. Vi convennero matematici già celebri come Brioschi e Cremona, ed una larga schiera di giovani, all'inizio della loro carriera, pieni di slancio e di nobile idealità. Il Guccia era tra questi; ed il Circolo Matematico di Palermo, che iniziava allora i suoi " Rendiconti ", cominciò ad uscire dalla primitiva cerchia: giacchè i matematici italiani principiavano a conoscere la nuova istituzione. Poco tempo dopo il Guccia si recava a Parigi. I geometri francesi, ed in primo luogo il Bertrand, che era il segretario perpetuo dell'Accademia delle Scienze, ebbero fiducia nel giovane italiano, che con tanto schietto e giovanile entusiasmo parlava loro del Circolo fondato in Palermo. Gli incoraggiamenti ricevuti diedero nuova spinta. Già nel secondo volume dei " Rendiconti " apparivano Memorie e Note di Betti, Cerruti e Poincaré. Il Circolo cominciava ad orientarsi nella direzione che sempre ferma mantenne. Esso veniva poco a poco prendendo quell'indirizzo che ne fece la forza, che ne costituì la grandezza, che fu la fonte di futuri successi. I giornali scientifici di carattere regionale ed anche nazionale talvolta immiseriscono e muoiono; la scienza è universale, onde il Guccia dando al suo periodico una schietta impronta internazionale lo ha posto sopra basi sicure ed incrollabili. Egli ha fatto convergere verso i " Rendiconti ", mercè la stima e la simpatia acquistate, le memorie dei matematici di tutte le parti del mondo. Con opera assidua, perseverante e geniale, con fine discernimento, ha saputo scegliere fra gli autori, fra le scuole, fra gli indirizzi, tanto che il Circolo ha pubblicato dei lavori che hanno fatto epoca nel mondo scientifico e sono destinati a rimanere classici e fondamentali. "

" I matematici più insigni si ritengono onorati di pubblicarvi i loro scritti: i " Rendiconti " penetrano rapidamente dove esiste il culto della Matematica e vi diffondono rapidamente le più recenti scoperte ".

Le parole mirabili del grande Volterra non hanno bisogno di alcun commento.

Voglio solo ricordare che nel 1890 accanto al Guccia troviamo già Francesco Gerbaldi, maestro dei maestri, poichè furono suoi allievi a Palermo gli intramontabili Giuseppe Bagnera, Pasquale Calapso, Michele De Franchis, Michele Cipolla. A questi personaggi, nell'epoca felice della scuola palermitana, si aggiungeranno Corradino Mineo, Pia Nalli, Pietro Tortorici.

Alla morte del Guccia, nel 1914, Michele De Franchis assunse la direzione dei " Rendiconti ".

La sede del Circolo (sempre nel palazzo del Guccia), con

la sua Biblioteca, ricca di oltre diecimila volumi e delle riviste di ogni parte del mondo, con la sua prestigiosa Tipografia, venne poi devastata dai bombardamenti nell'immane flagello dell'ultima guerra mondiale. I locali, con la Tipografia al completo, ed i libri nella maggior parte, furono distrutti. Forse una delle più belle, assieme a tante grandiose istituzioni che civiltà ed onore ci avevano dato, era stata cancellata per sempre !

Ma così come ogni uomo, anche se umiliato per l'intera sua vita in una buia spelonca, riesce sempre a sperare sognando una libertà senza confini, con la stessa forza, con eguale fervore, il Circolo Matematico seppe risorgere. Mi piace ricordare come Eduardo Gugino (e solo Gugino!) ci conducesse tra le macerie per tentare di recuperare un bene tanto prezioso; purtroppo quel poco che era rimasto. E tuttavia i gloriosi " Rendiconti " riapparvero; seppure lentamente, il Circolo Matematico iniziò la sua faticosa risalita.

Quando nel 1965 Eduardo Gugino, ormai stanco dovette abbandonare, proprio io, immeritatamente, ricevetti la grande eredità! E a questo punto la mia relazione sui fatti deve avere termine.

Termino rivolgendomi ai giovani che sognano oggi un mondo più bello, ai giovani ancora una volta, e con immensa fiducia, ricordando che nel Circolo Matematico creato dal Guccia tutti quanti, e non i soli matematici, si ritrovavano amici e la loro patria era il mondo intero, senza violenza, senza trincee, senza confini.

Benedetto Pettineo

SALUTO DEL PROF. A.LICHNEROWICZ IN NOME DELLA COMUNITA' MATEMATICA INTERNAZIONALE

Quand le Comité d'Organisation a bien voulu me demander de porter témoignage sur le rôle et le rayonnement du Circolo Matematico di Palermo dans la communauté mathématique internationale, j'ai pensé à la fois que c'était un devoir d'accepter et une lourde tâche de remplir ce devoir. En ce dimanche 21 octobre, je n'avais pas sous la main les prestigieux volumes des Rendiconti et c'était un travail patient et profond d'historien des mathématiques qu'il aurait fallu faire. Je n'en avait ni le temps, ni la capacité, une capacité telle que celle de nôtre ami Taton, ici present.

Mais je puis porter aujourd'hui témoignage sur ce qu'était aux environs de 1934, à nos yeux de jeunes mathématiciens, le statut des " Rendiconti del Circolo Matematico di Palermo ". Il convient de rappeler qu'à cette époque c'est l'Europe qui menait le jeu mathématique, principalment l'Allemagne (dont le rôle avec Hitler allait se terminer), la France et l'Italie sans oublier ni la Pologne de Banach, ni la Suisse et les maîtres de l'Ecole Polytechnique Fédérale de Zürich. David Hilbert et ses élèves étaient le symbole de l'apport et de la contribution de la pensée allemande au développement mathématique. Les Francais d'Emile Borel à Henri Lebesgue et Paul Montel étaient principalment des analystes, mais avec Henri Poincaré et Elie Cartan ils avaient apporté, en particulair à tous les problèmes différentiels, quelque chose d'essentiel qui fait partie désormais de nôtre bien commun.

Les Italiens, parmi bien d'autres choses, avaient crée, avec Enriques et Severi, la géométrie algébrique et avait fait œuvre de grands analystes avec Vito Volterra. Mais surtout, à mes yeux, ils avaient été les créateurs de la géométrie différentielle et avaient prédit le rôle qu'elle pouvait jouer en Phisique; ce fut l'œuvre

de Ricci, Levi-Civita et Bompiani. Le célèbre mémoire de Ricci et Levi-Civita, paru en francais l'année 1900, dans une revue allemande les Mathematische Annalen, est comme un symbole de l'état de l'Europe scientifique en ces temps bénis. C'est ce mémoire, tout plein déjà d'applications à la Mécanique Analytique et à la théorie des milieux continus, qui, transmis par un truchement à Albert Einstein, va servir de base à ses réflections et être la source principale de la Relativité Générale, celle-ci, en retour, sera un des motives du développement de la géométrie différentielle. De 1923 à 1926, naît de mains d'Elie Cartan ce que nous appellerions, en langage contemporain, la théorie des connexions sur un espace fibré principal.

Vus de 1934, tous ces aspects des mathématiques sont ou ont été présents dans les Rendiconti del Circolo Matematico di Palermo, d'Emile Borel à Tibor Radò pour la théorie de la mesure; Henri Poincaré a donné au moins un mémoire célèbre dont nous reparlerons, les élèves de Hilbert sont présents. Rival des Annali di Matematica, le périodique que nous célébrons est, avec les Mathematische Annalen et Les Annales de l'Ecole Normale Supérieure de Paris, un des plus grands journaux mathématiques de l'Europe et par suite, en ce temps du monde.

Pour le jeune homme naïf que j'étais en 1934, je pensais ou rêvais que le Circolo Matematico avait une existence charnelle comme vrai cercle. Compte tenu de la qualité des articles et préstigeuse signatures qui apparaissaient dans les Rendiconti, je rêvai que, de temps à autre, certains des meilleurs mathématiciens d'Europe se réunissaient et discutaient sur les problèmes ouverts et le futur de nôtre science, un peu comme nous allons le faire cette semaine, mais de manière moins formelle, quelque chose pour les mathématiques, comme les congrés Solvay pour la Phisique au maximum de leur gloire vers 1930. Tel était mon rêve que partageaient d'autre très jeunes mathématiciens.

La réalité était evidemment autre, mais fort prestigeuse. Permettez moi une ou deux anectodes. Vers 1936, j'assistai à une conversation mathématique entre Paul Montel au sommet de sa réputation et Pierre Lelong qui est à peu près mon contemporain et qui obtenait déjà ses premiers résultats concernants la théorie des fonctions de

plusieures variables complexes, selon la terminologie du temps; Paul Montel concluait: " vos premiers résultats sont intéressants; rédigez un article qui soit clair et suffisamment larg pour être clair. S'il est digne de ce journal, nous le présenterons au " Circolo Matematico di Palermo " " et Pierre Lelong rougis presque de plaisir à cette perspective.

A quelque temp de là, Bompiani vint faire un cycle de conférences à Paris sur la géométrie différentielle. C'est là où j'eus le privilège de le connaître et il devins pour moi, au fil des ans, un véritable ami auquel j'ai porté une respecteuse et profonde affection. Je m'intéressais alors, sous l'influence d'Elie Cartan, à certains problèmes concernant ce qui devait devenir, nommées par André Weil pour une fois mal inspiré, les variétés kähleriennes. Ces variétés avaient été étudiées, quinze ans plus tôt, par le grand géométre hollandais Schouten dont les résultats allaient déjà beaucoup plus loin à l'époque que ceux bien postérieurs de Kähaler. Un des exemples de cette situation était fourni par la métrique dite de Bergmann par les domaines bornés de C^n qui intéressaient tant Elie Cartan. Je parlai de mes résultats très partiels avec Bompiani et il fut suffisamment intéressé pour me dire de les rédiger pour le " Circolo Matematico di Palermo " et moi aussi, je rougis de plaisir. Elie Cartan en décida autrement et le fit paraître dans les Annales de l'Ecole Normale. Voilà comment mon premier bon papier n'est pas paru aux Rendiconti et je le regrette profondément.

Cette trop longue évocation vous dira, je l'espère, quel fut pour les mathématiciens de ma génération le statut de ces Rendiconti del Circolo Matematico di Palermo, dont nous fêtons aujourd'hui le robuste centenaire. L'un des membres du Comité d'Organisation m'a dit qu'ils désiraient que cette semaine soit une fête , une fête entre amis. Ce sont certainment des amis, membres de ce Cercle idéal dont je rêvais autrefois, qui se trouvent réunis ici.

André Lichnerowicz

TECHNICS OF ANALYTIC CONTINUATIONS FOR THE LINEAR CAUCHY PROBLEM, AS IMPROVED BY Y. HAMADA AND A. TAKEUCHI,

Jean Leray.

Cauchy-Kowalewski theorem gives a local solution of the analytic Cauchy problem ; Y. Hamada, A. Takeuchi and myself are constructing analytic continuations of that solution [HLT]. Recently I gave several statements [L] resulting from that continuations ; here I quote only the simplest of them. More recently Y. Hamada and A. Takeuchi extended and clarified the technics of our analytic continuations ; I give here a survey of those improved and simplified technics.

1. Statement of a non-local existence theorem. - In the case of real variables, Cauchy problem is well settled for a hyperbolic equation in a variety given as the product of two others : space and time.

Similarly we study the analytic Cauchy problem in a space $X = \Sigma \times \Omega$ that is the product of a simply connected non-compact Riemannian surface Σ and a n-dimensional, complex analytic, connected and paracompact [B] variety Ω.

Recall that the assumption <<Ω is paracompact>> is here equivalent to <<Ω is metrisable >> and also to <<Ω can be provided with a Riemannian ds^2>>.

Denote a point of X by

$x = (\sigma,\omega)$, where $\sigma \in \Sigma$ and $\omega \in \Omega$.

Denote by

x_o , $(x_1, \ldots, x_n)$ and $(x_o, x_1, \ldots, x_n)$

local analytic coordinates of σ , ω and $x = (\sigma,\omega)$.

Recall that a germ u_E holomorph on a subset E of X is a class of complex numerical functions, each of them being holomorph on some neighborhood of E , any two of them being equal on some neighborhood of E . If $e \subset E$, then u_E has a restriction $u_e = u_E|_e$; it is a germ holomorph on e .

We consider here the Cauchy problem for only one equation. Its data are :

a point $\alpha \in \Sigma$;

a differential operator $a(x, \frac{\partial}{\partial x})$ of order m and a function $x \mapsto v(x)$, both of them being holomorph on a neighborhood of $\alpha \times \Omega = \{(\sigma,\omega) ; \sigma = \alpha , \omega \in \Omega\}$;

a germ w holomorph on $\alpha \times \Omega$.

The unknown u is a numerical holomorph function solution of the problem :

(1) $a\,u = v$; $u - w$ vanishes m-times on $\alpha \times \Omega$.

Our aim is a construction as simple as possible of an as large as possible neighborhood of $\alpha \times \Omega$ on which there exists a solution of problem (1).

Denote by ξ_o , ξ' and $(\xi_o , \xi') = \xi$ covectors of Σ at σ , of Ω at ω and of X at $x = (\sigma,\omega)$; namely :

$$(\sigma;\xi_o) \in T^*(\Sigma) \ , \ (\omega;\xi') \in T^*(\Omega) , (x;\xi) \in T^*(X),$$

where T^* denotes the cotangent bundle.

The characteristic polynomial of the operator a at x is a homogeneous polynomial of degree m

$$g : \xi \mapsto g(x,\xi) = \sum_{r=0}^{m} g_r(x_o,\omega;\xi') \, \xi_o^r \ ;$$

thus $g_r : \xi' \mapsto g_r(x_o,\omega;\xi')$ is a homogeneous polynomial of degree $m-r$; it depends on the choice of the local coordinate x_o of σ .

Assume that g is a function of x holomorph on the whole space X . Assume that

$$(\forall\, x_o,\omega) \ : \ g_m(x_o,\omega) \neq 0 \ ;$$

it means that $(\forall\, \sigma \in \Sigma)$ there is no point of the hypersurface $\sigma \times \Omega$ where that hypersurface is characteristic.

The propagation of the singular support of the solutions of the equation $au=v$ are ruled by the roots ξ_o of the equation

$$\sum_{r=0}^{m} g_r(x_o,\omega;\xi')\xi_o^r = 0 \ ,$$

which is the so called « characteristic equation ». Here we don't use those roots, but the estimate $\rho(x_o,\omega;\xi')$ of the maximum of their absolute values, defined as follows : $\rho(x_o,\omega;\xi')$ is the positive root of the equation

$$(2) \qquad \sum_{r=0}^{m-1} |g_r(x_o,\omega;\xi')|\rho^r = |g_m(x_o,\omega)|\rho^m \ .$$

Of course

$$(\forall\, \theta \in \mathbb{C}) \ : \ \rho(x_o,\omega;\theta\xi') = |\theta| \cdot \rho(x_o,\omega;\xi').$$

Moreover $\langle \forall\, (\omega;\xi') \in T^*(\Omega) \rangle$: the expression

(3) $\rho(x_o,\omega;\xi')|dx_o|$ is independent of the choice of the local coordinate x_o ;

(4) the function $x_o \mapsto \log\rho(x_o,\omega;\xi')$ is subharmonic (or $\rho \equiv -\infty$).

A closed subset ϕ^* of $T^*(\Omega)$ will be called admissible when the function

$$(5) \qquad x_o \mapsto \rho_{x_o} = \sup_{(\omega;\xi') \in \phi^*} \rho(x_o,\omega;\xi') \quad \text{is locally bounded.}$$

Then, by (3)

$$(6) \qquad ds = \rho_{x_o}|dx_o| \ .$$

is independent of the choice of the local coordinate x_o of $\sigma \in \Sigma$.

Define

$$(7) \qquad (\forall\, \sigma \in \Sigma) \ : \ \mathrm{dist}(\sigma,\alpha) = \inf. \int_{\alpha}^{\sigma} ds \ ;$$

the triangle inequality holds ; but $\mathrm{dist}(\sigma,\alpha) = 0$ does not necessarily imply $\sigma = \alpha$.

Moreover, by (4):

(8) the function $x_o \mapsto \log\rho_{x_o}$ is subharmonic (or $\rho \equiv -\infty$) ;

it means, when ds^2 is riemannian, that its curvature is $\leqslant 0$, which is essential.

We are now able to state a theorem, announced in [L] , whose complete proof will be given in [HLT] :

THEOREM. - 1°) Provide Ω with a riemannian metric ; it defines a lenght $|\xi'|$ of the covectors ξ' of Ω . By replacing that metric by a conformal one, decreasing sufficiently rapidly at infinity, one can fulfil the condition :

(9) $\phi^* = \{(\omega;\xi') \in T^*(\Omega) \ ; |\xi'| = 1\}$ is admissible.

2°) Define the distance on Ω by a metric such that (9) holds. Denote by $\mathrm{Dist}\,(\omega,\partial\Omega)$ the distance between $\omega \in \Omega$ and the point at infinity $\partial\Omega$, which compactifies Ω . Define dist on Σ by (5), (6) and (7).

The Cauchy problem (1) has a unique solution holomorph on the domain

(10) $\Delta = \{(\sigma,\omega) \in X \ ; \ \mathrm{dist}\,(\sigma,\alpha) < \mathrm{Dist}(\omega,\partial\Omega)\}$.

when the data a and v are holomorph on Δ.

EXAMPLE. - $\Delta = X$ when $(\forall\, \sigma \in \Sigma) \ : \ \mathrm{dist}(\sigma,\alpha) = 0$; i.e. when $a(x, \frac{\partial}{\partial x})$ does not contain derivations of order m others than $\frac{\partial^m}{\partial x_o^m}$

EXAMPLE. - $\Delta = X$ when $(\forall\, \omega \in \Omega) \ : \ \mathrm{Dist}(\omega,\partial\Omega) = \infty$. Particularly : $\Delta = X$ when $X = \mathbb{C}^n$ and $(\forall\, r) \ : \ \omega \mapsto g_r(x_o,\omega;\xi')$ is a polynomial of degree $\leqslant m-r$.

2. Analytic continuations of germs. - J.P. Serre observed that this section 2 relies on the following fact :

Proposition. - Any subspace E of X is paracompact.

Proof (J.P. Serre). - Since X is metrisable, E is also metrisable, hence paracompact, by a theorem due to A.H. Stone [St], and quoted by Kelley [K].

Actually, we only need the corollary :

Lemma 2. - Any open subset of X is paracompact.

There is a very simple proof of that lemma. Moreover the preceding proposition follows easily from that lemma.

Recall the well known

Definition. - A section (of the sheaf on X of the holomorph functions) holomorph on a subset E of X is a family $(u_x)_{x \in E}$, each u_x being a germ holomorph at x , the following condition of continuity being fulfilled : each point $x \in E$ has a neighborhood $N(x)$ in X on which there exists a holomorphic

function $u_{N(x)}$ such that

$$(\forall y \in E \cap N(x)) : \quad u_{N(x)}|_y = u_y .$$

Each germ u_E holomorph on $E \subset X$ generates as follows a section $(u_x)_{x \in E}$ holomorph on E :

$$u_x = u_E|_x .$$

Proposition 2.1.- The application

(Holomorph germ) $\mapsto$ (section generated by that germ) is bijective.

Proof. - R. Godement's theorem 3.3.1 in [G] p. 150, whose assumption results from lemma 2.

In our case Godement's proof can be simplified.

The following sections rely on the preceding proposition and on its two following consequences.

Proposition 2.2.- Let $e \subset E \subset X$. Assume a path $C(x)$ of E, from some point $y(x)$ of e to x, is associated to each $x \in E$. Assume that path belongs to the class C^0, depends continuously of x and reduces to x when $x \in e$. Assume that $u_e|_{y(x)}$ has an analytic continuation to $C(x)$. Then there is a unique analytic continuation u_E of u_e to E. (It means : there is a unique germ u_E holomorph on E such that $u_E|_e = u_e$).

Proposition 2.3.- Let $(E_i)_{i \in I}$ be a covering of $E \subset X$ by subsets E_i of X. Let $(\forall i \in I) : u_i$ be a germ holomorph on E_i such that $(\forall i,j \in I) : u_i$ and u_j have the same restriction to $E_i \cap E_j$. Then there is a unique germ holomorph on E such that $(\forall i \in I) : u$ and u_i have the same restriction to $E \cap E_i$, if the following condition is satisfied :

$$\begin{cases} (\forall x \in E) : \text{there is a finite subset } I(x) \text{ of } I \text{ such that :} \\ x \in \bigcap_{i \in I(x)} E_i \ ; \ \bigcup_{i \in I(x)} (E \cap E_i) \text{ is a neighborhood of } x \\ \text{in the subspace } E . \end{cases}$$

The preceding condition is obviously fulfilled in both cases :

1°) $(\forall i \in I) : E \cap E_i$ is an open subset of E.

2°) $(\forall i \in I) : E \cap E_i$ is a closed subset of E and $(E \cap E_i)_{i \in I}$ is a locally finite covering of E.

3. From Cauchy-Kowalewski theorem to a proposition suitable for analytic continuations of the solution of problem (1). - The singular support of any solution of an ordinary analytic linear differential equation belongs to the singular support of that equation. Hence the following version of

Cauchy-Kowalewski theorem. - Let $\zeta_0,\dots,\zeta_n,\tau \in \dot{\mathbb{R}}_+ =]0,\infty[$ and $\theta \in]0,1[$.

Assume

$$\Sigma = \{x_0 \in \mathbb{C} \ ; \ \zeta_0|x_0| < \tau\} \ ; \ \Omega = \{(x_1,\dots,x_n) \in \mathbb{C}^n \ ; \ \zeta_k|x_k| < \tau \ , \ k = 1,\dots,n\} \ ;$$

hence X is the polydisc

$$(11) \qquad \Sigma \times \Omega = X = \{(x_0,x_1,\dots,x_n) \in \mathbb{C}^{n+1} \ ; \ \zeta_j|x_j| < \tau \ , \ j = 0,\dots,n\} .$$

Assume the data a and v of problem (1) holomorph on X. Let :

$$(12) \qquad a(x,\frac{\partial}{\partial x}) = \frac{\partial^m}{\partial x_0^m} + \sum_{\{\lambda;|\lambda| \leq m\}} a_\lambda(x) (\frac{\partial}{\partial x})^\lambda ,$$

where $\lambda = (\lambda_0,\dots,\lambda_n)$ is a multi-index and $a_{(m,0,\dots,0)} \equiv 0$. Define the "spectral function" H_X by

$$(13) \qquad H_X(\zeta) = \sum_{\{\lambda;|\lambda| = m\}} \sup_{x \in X} |a_\lambda(x)| \ \zeta^\lambda .$$

Assume $\alpha = 0$. Do not assume the germ w holomorph in X on $\alpha \times \Omega$ but assume the germ w holomorph in X on

$$\alpha \times \Omega_\theta = \{(x_0,x_1,\dots,x_n) \in \mathbb{C}^{n+1} \ ; \ x_0 = 0 \ ; \ \zeta_k|x_k| < \theta\tau \ , \ k=1,\dots,n\} .$$

Then there exists a unique solution of problem (1) holomorph on the domain

$$(14) \qquad \Delta = \{(x_0,x_1,\dots,x_n) \in \mathbb{C}^{n+1} \ ; \ \sum_{j=0}^{n} \zeta_j|x_j| < \theta\tau\} ,$$

if

$$(15) \qquad H_X(\zeta) \leq (1-\theta) \ \zeta_0^m .$$

Note. - The preceding condition can be fulfilled by increasing ζ_0, since H_X is a polynomial in ζ_0 of degree $m - 1$.

We don't need the preceding theorem but only its obvious corollary :

Lemma 3. - Under the preceding assumptions there is a unique germ solution of problem (1), holomorph on the disc :

$$(16) \qquad \{(x_0,\dots,x_n) \in \mathbb{C}^{n+1} \ ; \ \zeta_0|x_0| < \theta\tau \ , \ x_1 = \dots = x_n = 0\}$$

From that corollary and from the propositions 2.1, 2.2 results the following proposition, which uses the

Notation 3. - Let $\overline{\Omega} = \Omega \cup \partial\Omega$ be the space Ω compactified by the adjunction of a point $\partial\Omega$, the so-called «point at infinity». Choose, what is possible since Ω is paracompact, a continuous function

$$f : \overline{\Omega} \to \mathbb{R}_+ = [0,\infty[,$$

of class C^1 on Ω, such that

$$f(\omega) > 0 \quad \text{for} \quad \omega \in \Omega \ , \ f(\partial\Omega) = 0 .$$

Denote by

$$\nabla f = (f_1,\dots,f_n)$$

the unique covector-field such that

$$df = \mathrm{Re}(f_1 \, dx_1 + \dots + f_n \, dx_n) \quad \text{on} \quad \Omega .$$

Assume

$$(17) \qquad \Phi^* = \{(\omega;\nabla f(\omega)) \ ; \ \omega \in \Omega\} \text{ admissible } [\text{see } (9)] .$$

Denote

$$(18) \qquad \rho(x_0,\omega) = \rho(x_0, \omega \ ; \ \nabla f(\omega)) ,$$

where $\rho(.,.;.)$ is defined by (2)

Proposition 3. - Let Γ be a compact subset of Σ and $s : \Gamma \to \dot{\mathbb{R}}_+ =]0,\infty[$ be a continuous function. Then the subset of X

(19) $\{(\gamma,\omega) \in X \; ; \; \gamma \in \Gamma \; , \; s(\gamma) \leqslant f(\omega)\}$

is compact. Assume a and v holomorph on that subset and the germ w holomorph in X on

(20) $\alpha \times \{\omega \in \Omega \; ; \; s(\alpha) < f(\omega)\}$ where $\alpha \in \Gamma$.

Then there is a unique solution of the problem (1) holomorph on the open subset of X

$$\{(\sigma,\omega) \in X \; ; \; |\sigma-\alpha| < \varepsilon \; , \; [\rho(\alpha,\omega) + \eta(\varepsilon)].|\sigma-\alpha| < f(\omega) - s(\alpha)\} \; .$$

Note 3. - It is assumed that Σ is $\mathbb{C}$ or its unitary disc (see H. Poincaré) and that ε is any point of $]0,\bar{\varepsilon}[$. The number $\bar{\varepsilon} \in \dot{\mathbb{R}}_+$ and the function, vanishing with ε , $\eta \; : [0,\bar{\varepsilon}] \ni \varepsilon \to \eta(\varepsilon) \in \mathbb{R}_+ = [0,\infty[$ are defined by the data Γ , s, a and v , but are independent of α and w .

Proof. - Let us only outline how the spectral function H_X used in lemma 3 disappears by introducing the function ρ : if we neglect functions of ε vanishing with ε , then the rôle of the spectral function reduces to the definition of the root $\zeta_o \geqslant 0$ of the equation $H_x(\zeta) = \zeta_o^m$, for $\zeta_1 = |\nabla f|$, $\zeta_2 = \ldots = \zeta_n = 0$, at a point $x = (\alpha,\omega)$ where $f_2 = \ldots = f_n = 0$; now that root is $\rho(\alpha,\omega)$.

4. Analytic continuation of the solution of (1) above an arc Γ of Σ .- Let us complete notation 3 by the following one and substitute for proposition 3 a more adequate one.

Notation 4.1.- Let σ be a point of Σ and x_o its local coordinate ; denote

$$\rho[x_o,t] = \max_{\{\omega;f(\omega)=t\}} \rho(x_o,\omega) \quad \text{if } 0 < t \leqslant \bar{f} = \max_{\Omega} f \; ,$$

$$\rho[x_o,t] = \rho[x_o,\bar{f}] \quad \text{if } \bar{f} < t \; ;$$

obviously the function $\rho[.,.]$ is continuous and the expression

$$\rho|x_o,t|\;|dx_o|$$

is independent of the choice of the local coordinate x_o .

Preliminary proposition. - Assume a and v holomorph on (19) and the germ w holomorph in X on (20). Then for any $t > s(\alpha)$ there is a unique germ solution of problem (1) holomorph on the compact subset of X :

$$\{\sigma \in \Sigma \; ; \; |\sigma-\alpha| \leqslant \varepsilon\} \times \{\omega \in \Omega \; ; \; t \leqslant f(\omega)\} \; ,$$

if

$$[\rho[\alpha,t] + \eta(\varepsilon)]\cdot\varepsilon < t - s(\alpha) \; .$$

Note 3 has still to be taken into account.

Let us now substitute for the definitions of Γ and s the following ones.

Notation 4.2.- Let Γ be a compact orientated polygon of Σ , with origin at α . That polygon Γ has no multiple point ; its sides are curves belonging to class C^1 .

Let $s : \Gamma \to \mathbb{R}_+$ be a strictly increasing function of class C^1; assume $ds > 0$.

Denote by

$$s(\gamma) = s[x_o]$$

the value of s at the point $\gamma \in \Gamma$ having x_o as local coordinate in Σ .

Iterating the preliminary proposition, letting ε tend to zero, what removes ε and η , and using the proposition 2.3 one gets :

Proposition 4. - Use notations 4.1 and 4.2. Assume

(21) $\rho[x_o, s[x_o]]\;|dx_o| \leqslant ds[x_o]$ on Γ .

Then there is a unique germ solution of problem (1) holomorph on the subset of X :

(22) $\{(\gamma,\omega) \; ; \; \gamma \in \Gamma \; , \; s(\gamma) < f(\omega)\}$,

if the data a and v are holomorph on that subset and if the germ w is holomorph in X on $\alpha \times \{\omega \in \Omega; \; s(\alpha) < f(\omega)\}$.

Comment. - That proposition leads to the following ones, but is not contained in them.

5. Analytic continuation of the solution of (1) above a domain of Σ.- Preliminary proposition. - Assume (17) ; define ρ_{x_o} by (5). But, instead of defining ds by (6), let ds^2 be riemannian on Σ , of class C^∞ , of curvature $\leqslant 0$ and such that

$$\rho_{x_o}|dx_o| \leqslant ds \; .$$

Denote by $\overset{o}{\Phi}$ any simply connected domain of Σ containing α and having the following properties : its closure Φ is compact ; its boundary $\partial\Phi$ is a closed polygon, without multiple points, whose sides are geodesic arcs. Denote by dist and dist_Φ the distances defined by ds respectively in Σ and in Φ . Then there is a unique germ solution of problem (1) holomorph on the subset of X

(23) $\{(\sigma,\omega) \in X \; ; \; \sigma \in \Phi, \; \text{dist}_\Phi \; (\sigma,\alpha) < f(\omega)\}$

if the data a and v are holomorph on the subset

$$\{(\sigma,\omega) \in X \; ; \; \text{dist}(\sigma,\alpha) < f(\omega)\} \; ,$$

which contains (23).

Proof relies on propositions 4, 2.1 and 2.2 and the following geometrical property, which furnishes the pathes C(x) required by proposition 2.2 :

A geometrical property. - There is in Φ a unique shortest path $\Gamma(\sigma,\alpha)$ from α to $\sigma \in \Phi$; it belongs to the class C^o and depends continuously on σ .

Note. - $\Gamma(\sigma,\alpha)$ is a polygon, whose sides are geodesic arcs, and whose vertices other than σ and α are re-entrant vertices of $\partial\Phi$; $\Gamma(\sigma,\alpha)$ is "subtended" by $\partial\Phi$. Those properties characterize $\Gamma(\sigma,\alpha)$; they are nothing else than the properties of the strings stretched on Φ .

An open problem. - What precedes gives in a simple case a positive answer to the question, which to my knoledge is open :

Let Σ be a surface provided with a riemannian metric, whose curvature is $\leqslant 0$; let $\bar{\Sigma}$ be the completion of Σ for that metric ; is there always a unique shortest path on $\bar{\Sigma}$ between two of its points ?

By trivial approximations and trivial applications of proposition 2.3, the preliminary proposition gives :

Proposition 5. - Assume that f satisfies (17) ; define ρ_{x_o}, ds and dist. by (5), (6) and (7). Then there is a unique solution of problem (1) holomorph on the domain of X

(24) $\Delta = \{(\sigma,\omega) \in X ; \operatorname{dist}(\sigma,\alpha) < f(\omega)\}$,

if the data a and v are holomorph on Δ.

6. Conclusion. - The preceding proposition extends easily as follows :

Proposition 6.1. - Use the notations defined by section 1 and the following ones :

Let $(\Omega_N)_{N \in \dot{\mathbb{R}}_+}$ be a family of domains of Ω such that :

(25) $\overline{\Omega}_N$ is compact ; Ω_N increases with N ; $\lim_{N\to\infty} \Omega_N = \Omega$.

Denote by $\partial\Omega_N = \overline{\Omega}_N \setminus \Omega_N$ the boundary of Ω_N .

Let $(f_N)_{N \in \dot{\mathbb{R}}_+}$ be a family of continuous functions

$f_N : \overline{\Omega}_N \to \mathbb{R}_+$,

of class C^1 on Ω_N , such that :

(26) $f_N(\omega) > 0$ for $\omega \in \Omega_N$, $f_N(\omega) = 0$ for $\omega \in \partial\Omega_N$, $f_N(\omega)$ increases with N.

Let

$F(\omega) = \lim_{N\to\infty} f_N(\omega) \leqslant \infty$.

Assume there is an admissible [see : (5)] subset Φ^* of $T^*(\Omega)$ and a family $(\theta_N)_{N \in \dot{\mathbb{R}}_+}$ of functions $\theta_N : \Omega_N \to C$ such that

(27) $(\forall N \in \dot{\mathbb{R}}_+ , \omega \in \Omega_N) : |\theta_N(\omega)| \leqslant 1 , (\omega , \frac{1}{\theta_N(\omega)} \nabla f_N(\omega)) \in \Phi^*$.

Recall that ∇ is defined by notation 3, and moreover ρ_{x_o}, ds and dist by (5), (6) and (7) . Then the problem (1) has a unique solution holomorph on the domain

(28) $\Delta = \{(\sigma,\omega) \in \Sigma \times \Omega ; \operatorname{dist}(\sigma,\alpha) < F(\omega)\}$

if the data a and v of the problem (1) are holomorph on that domain Δ.

Comment. - F is a lower semi-continuous function, which does not necessarily vanish at the "point at infinity" $\partial\Omega$ of Ω .

The preceding proposition makes use of an admissible subset Φ^* of $T^*(\Omega)$. Such subsets can be obtained as follows.

Definition 6. - Denote by Π^* any closed subset of $T^*(\Omega)$ whose natural mapping in Ω is proper ; i.e. : the largest subset of Π^* that naturally maps onto a compact subset of Ω is compact.

The construction of such subsets Π^* is easy ; the generate admissible subsets Φ^* as follows :

Proposition 6.2. - Denote by R any function $\Omega \to \dot{\mathbb{R}}_+$ such that $\lim_{\omega\to\partial\Omega} R(\omega) = \infty$

Such functions exist since Ω is paracompact.

Given Π^* and R , there exist continuous functions

$B[.] : \dot{\mathbb{R}}_+ \to [1,\infty[$

such that the sets

(29) $\Phi^* = \{(\omega;\xi') / B[R(\omega)]) \in T^*(\Omega) ; (\omega;\xi') \in \Pi^*\}$

are admissible.

The preceding proposition is deduced from the

Lemma. - Use definition 6. Let $Q : \Sigma \to \dot{\mathbb{R}}_+$ and $R : \Omega \to \dot{\mathbb{R}}_+$ be continuous functions such that

$$\lim_{\sigma \to \partial\Sigma} Q(\sigma) = \lim_{\omega \to \partial\Omega} R(\omega) = \infty .$$

Let $\psi : \Sigma \times \Pi^* \to \mathbb{R}_+$ be a continuous function. Then there exists an increasing function of class C^1

$$A[.] : \dot{\mathbb{R}}_+ \to \dot{\mathbb{R}}_+$$

such that

$$\psi(\sigma,\omega;\xi') \leqslant \max\{A[Q(\sigma)], A[R(\omega)]\} .$$

The theorem stated in section 1 and the two others theorems established in [HLT] and quoted by [L] follow easily from propositions 6.1 and 6.2.

Thus we have completed our review of the technics that [HLT] uses for its analytic continuation of the solution of Cauchy problem, in the case of a unique differential equation or of a differential system.

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IL TEOREMA DI H.WEYL SULLA DISTRIBUZIONE ASINTOTICA DEGLI AUTOVALORI NELLA ELASTICITA'

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I Rendiconti del Circolo Matematico di Palermo e, con essi, gli Atti dell'Accademia dei Lincei e gli Annali di Matematica sono, indubbiamente, le riviste matematiche italiane di più antica e nobile tradizione. E' a tutti noto che i "Rendiconti" di Palermo ospitarono, nel primo trentennio della loro esistenza, i risultati di ricerche che ebbero importanza determinante nei successivi sviluppi di fondamentali rami della Matematica. Fra i lavori apparsi in quell'epoca, uno dei più notevoli è certamente la Memoria [1] di Hermann Weyl. Essa è dedicata ai teoremi di esistenza per i problemi al contorno per il classico sistema delle equazioni tridimensionali dell'Elasticità per un corpo isotropo ed omogeneo, nonchè alla dimostrazione della legge di distribuzione asintotica degli autovalori nei relativi problemi di vibrazioni libere.

Il lavoro di H.Weyl che, per quanto riguarda i teoremi di esistenza, non supera in interesse i risultati di analoghe ricerche dedicate, in quel lasso di tempo, da altri eminenti Analisti a questa tematica, acquista, invece, uno straordinario valore per il teorema sulla legge di distribuzione asintotica degli autovalori. In effetti, H.Weyl, per la prima volta, dimostra un siffatto teorema in connessione ad un sistema alle derivate parziali: quello dell'Elasticità tridimensionale; sistema da considerarsi di notevole complessità matematica, specie se riferito a qual'erano le possibilità dell'Analisi nell'epoca in cui il lavoro venne pubblicato.

Ω indichi un campo (cioè un aperto connesso) limitato di R^3, Δ_2 sia l'operatore di Laplace tridimensionale e vettoriale, σ un parametro reale assoggettato alla condizione: $\sigma > -1$, u un vettore a tre com

ponenti.

Si consideri il problema di autovalori

$$(1)\qquad \begin{cases} \Delta_2 u + \sigma \operatorname{grad}\operatorname{div} u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

Questo problema ammette esclusivamente una successione crescente di autovalori reali positivi, ciascuno di molteplicità (geometrica) finita:

$$(2)\qquad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

e tali che

$$\lim_{n\to\infty} \lambda_n = +\infty .$$

Nella successione (2) ciascun autovalore compare un numero di volte pari alla rispettiva molteplicità. Indichiamo, per ogni $\lambda > 0$, con $N(\lambda)$ il numero dei termini della successione (2) che non superano λ, poniamo cioè:

$$N(\lambda) = \sum_{\lambda_n \leq \lambda} 1 .$$

La legge asintotica di Weyl si esprime al modo seguente:

$$(3)\qquad N(\lambda) \simeq \frac{\operatorname{vol}\Omega}{6\pi^2}\left[\frac{1}{(\sigma+1)^{3/2}} + 2\right]\lambda^{3/2}.$$

Il simbolo $\simeq$ indica che il rapporto fra $N(\lambda)$ ed il secondo membro tende ad 1 per $\lambda \to +\infty$. Dalla (3) si deduce

$$(4)\qquad \lambda_n \simeq \left(\frac{6\pi^2}{\operatorname{vol}\Omega}\right)^{2/3} \frac{\sigma+1}{\left[1+2(\sigma+1)^{3/2}\right]^{2/3}}\, n^{2/3}.$$

E' del tutto inutile soffermarsi a commentare l'interesse, sia teorico che applicativo, di relazioni quali la (3) e la (4). Esse sono state il punto di partenza per un ampio e fiorente capitolo dell'Analisi matematica.

Lo scopo cui è dedicata la presente trattazione costituisce una ulteriore esegesi della profondità dei risultati forniti da formule asintotiche quali la (3) e la (4). In effetti, volendo far luce sul problema della affidabilità numerica della (4), ci si accorge che gli strumenti di Analisi matematica e di Algebra, ai quali bisogna far ricorso, sono particolarmente elevati e sofisticati. Il presente lavoro è, appunto, dedicato ad un'esposizione sintetica delle teorie matematiche che permettono di forgiare siffatti strumenti.

Al fine di ottenere risultati del tutto concreti nell'indagine sull'efficacia numerica della (4), è imprescindibile la preventiva risoluzione di un problema particolarmente difficile, sia sul piano teorico che su quello numerico: l'approssimazione rigorosa di ogni autovalore λ_n. Con ciò s'intende la costruzione esplicita, per ogni assegnato n, di due successioni $\{\tau_n^{(s)}\}$ e $\{\lambda_n^{(s)}\}$ tali che

$$(5)\qquad \tau_n^{(s)} \leq \lambda_n \leq \lambda_n^{(s)},$$

$$(6)\qquad \tau_n^{(s)} \leq \tau_n^{(s+1)}, \qquad \lambda_n^{(s)} \geq \lambda_n^{(s+1)},$$

$$(7)\qquad \lim_{s\to\infty} \tau_n^{(s)} = \lim_{s\to\infty} \lambda_n^{(s)} = \lambda_n .$$

Allo studio teorico di questo problema ed alle teorie algebriche, che ne facilitano l'effettiva soluzione nei campi con un gruppo di simmetrie, sono dedicati i paragrafi che seguono.

1. Il metodo di Rayleigh-Ritz per la costruzione della successione $\{\lambda_n^{(s)}\}$.

Siano u^1, u^2, u^3 le componenti del vettore u, che supponiamo complesse. Si ponga

$$(1.1)\qquad (u,v) = \sum_{i=1}^{3} \int_\Omega u^i \bar{v}^i \, dx \qquad (dx = dx^1 dx^2 dx^3).$$

Per le funzioni (vettoriali) u(x) appartenenti a $\mathring{H}_1(\Omega)$, cioè tali che $u \in H_1(\Omega)$, $u|_{\partial\Omega} = 0$ [1], porremo

$$(1.2)\qquad ((u,v)) = \int_\Omega \left(\frac{\partial u^i}{\partial x^k} \frac{\partial \bar{v}^i}{\partial x^k} + \sigma \frac{\partial u^i}{\partial x^i} \frac{\partial \bar{v}^k}{\partial x^k} \right) dx .$$

L'ipotesi $\sigma > -1$ assicura che $((u,v))$ è un prodotto scalare.

Sia $\{w_k\}$ una successione di funzioni (vettoriali) linearmente indipendenti e costituenti un sistema completo nello spazio $\mathring{H}_1(\Omega)$. Per ogni intero $s \geq 1$, si consideri l'equazione di tipo secolare:

$$(1.3)\qquad \det\left\{ ((w_h, w_k)) - \lambda (w_h, w_k) \right\} = 0 \qquad (h,k = 1,\ldots,s).$$

[1] Per la definizione degli spazi $H_m(\Omega)$, $\mathring{H}_m(\Omega)$, cfr. [15], pag. 349.

Siano

$$0 < \lambda_1^{(s)} \leq \cdots \leq \lambda_s^{(s)}$$

le sue s radici. Si ha

$$\lambda_n^{(s)} \geq \lambda_n^{(s+1)} \geq \lambda_n \ , \quad \lim_{s\to\infty} \lambda_n^{(s)} = \lambda_n \ .$$

E' questo il classico metodo di Rayleigh-Ritz che consente il calcolo delle successioni $\{\lambda_n^{(s)}\}$. Per la giustificazione teorica di tale metodo si veda [13].

Esamineremo in seguito il problema consistente nel calcolo dei valori per eccesso $\lambda_n^{(s)}$, sfruttando le eventuali simmetrie di Ω al fine di migliorare le approssimazioni.

2. Il metodo degli invarianti ortogonali per il calcolo della successione $\{\tau_n^{(s)}\}$.

Per esporre i metodi per il calcolo dei "valori per difetto" $\{\tau_n^{(s)}\}$, conviene porre il problema in un ambiente astratto. Sia, allora, S uno spazio di Hilbert astratto separabile, che supporremo complesso. Sia (u,v) il prodotto scalare di due vettori di S. Sia L un operatore lineare avente come dominio la varietà lineare $\mathcal{D}_L$ di S e codominio contenuto in S. Sia U una fissata varietà lineare contenuta in $\mathcal{D}_L$.

Si consideri il seguente problema di autovalori:

$$Lu - \lambda u = 0 \ , \quad u \in U. \tag{2.1}$$

Si assuma la seguente ipotesi:

Hp) Esiste un operatore lineare G tale che: 1) G è un operatore compatto e positivo di S; 2) $G(S) \subset U$; 3) quali si siano u e v, rispettivamente in U e in S, si ha: $GLu = u$, $LGv = v$.

Posto $\lambda = \mu^{-1}$, il problema di autovalori

$$Gv - \mu v = 0 \ , \quad v \in S \tag{2.2}$$

è equivalente al problema (2.1). Il problema (2.2) ha una successione di autovalori

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots > 0$$

tutti positivi, ciascuno di molteplicità finita, tali che

$$\lim_{n\to\infty} \mu_n = 0.$$

Quindi la successione $\{\lambda_n\}$, con $\lambda_n = \mu_n^{-1}$, gode delle proprietà sopra enunciate per la successione di autovalori (2).

Aggiungeremo all'ipotesi Hp) l'ulteriore istanza:

4) esiste l'intero $q_0 > 0$ tale che G^{q_0} abbia traccia di Hilbert-Schmidt finita.

E' ovvio che il problema di autovalori (1) rientra, con evidente scelta di S, L, $\mathcal{D}_L$, U, nel problema (2.1). Inoltre, dalla teoria dei sistemi di equazioni alle derivate parziali di tipo ellittico, segue che per esso sono verificate tutte le istanze dell'ipotesi Hp) (cfr. [15], p.369).

Per $q \geq q_0$ il prodotto infinito

$$\Delta_q(z) = \prod_{k=1}^{\infty} (1 - z\mu_k^q)$$

rappresenta una trascendente intera nella variabile complessa z. Poniamo, per m=0,1,2,...,

$$\mathfrak{J}_m^q(G) = \frac{(-1)^m}{2\pi i} \int_{+C} \frac{\Delta_q(z)}{z^{m+1}} \, dz,$$

essendo C una curva rettificabile, semplice e chiusa attorno al punto z=0.

Le funzioni di G: $\mathfrak{J}_m^q(G)$ sono invarianti ortogonali dell'operatore G, cioè se X è un qualsiasi operatore unitario di S,

$$\mathfrak{J}_m^q(G) = \mathfrak{J}_m^q(XGX^{-1}).$$

Si dimostrano facilmente le seguenti proprietà degli invarianti ortogonali $\mathfrak{J}_m^q(G)$ (cfr.[13]):

$$\mathfrak{J}_0^q(G) = 1 \ , \qquad \mathfrak{J}_m^q(G) > 0 \ , \qquad \mathfrak{J}_m^q(G^p) = \mathfrak{J}_m^{qp}(G).$$

Inoltre, $G_1 \leq G_2$ implica $\mathfrak{J}_m^q(G_1) \leq \mathfrak{J}_m^q(G_2)$. Se P è il proiettore ortogonale di S sul sottospazio W, si ha: $\mathfrak{J}_m^q(PGP) \leq \mathfrak{J}_m^q(G)$. Se la successione $\{G_s\}$ converge uniformemente a G e se per ogni s si ha $G_s \leq G_0$, riesce

$$\lim_{s\to\infty} \mathfrak{J}_m^q(G_s) = \mathfrak{J}_m^q(G).$$

La successione $\{\mathfrak{J}_m^q(G)\}$ (m=0,1,...) caratterizza G a meno di una equivalenza unitaria, cioè si ha $\mathfrak{J}_m^q(G') = \mathfrak{J}_m^q(G)$ (m=0,1,...) se e solo se esiste l'operatore unitario X di S tale che $G' = XGX^{-1}$.

L'idea del metodo degli invarianti ortogonali per il calcolo per difetto di ogni λ_n consiste nell'osservare che λ_n è esso stesso un invariante ortogonale dell'operatore G e che, quindi, deve potersi rappresentare mediante invarianti quali gli $\mathfrak{J}_m^q(G)$. Si può, allora, tentare di modificare tale rappresentazione in modo che essa fornisca valori per difetto di λ_n. Tale programma è attuabile (cfr. [13], [16]). Si consideri, infatti, un qualsiasi sistema completo in S costituito da vettori linearmente indipendenti. Sia esso $\{w_h\}$. Sia W_s la varietà s-dimensionale determinata da $w_1, \dots, w_s$, sia $s \geq n$ e sia P_s il proiettore ortogonale di S su W_s. Diciamo $\mu_1^{(s)} \geq \dots \geq \mu_s^{(s)}$ gli autovalori di $P_s G P_s$ e $w_1^{(s)}, \dots, w_s^{(s)}$ un sistema di corrispondenti autovettori. Sia $W_s = W_{n,s} \oplus [w_n^{(s)}]$, avendo indicato con $[w_n^{(s)}]$ la varietà 1-dimensionale determinata da $w_n^{(s)}$ e con $W_{n,s}$ il complemento ortogonale di questa rispetto a W_s. Sia $P_{n,s}$ il proiettore ortogonale di S su $W_{n,s}$. Ponendo

$$(2.3) \qquad \tau_n^{(s)} = \left\{ \frac{\mathfrak{J}_m^q(G) - \mathfrak{J}_m^q(P_s G P_s)}{\mathfrak{J}_{m-1}^q(P_{n,s} G P_{n,s})} + [\mu_n^{(s)}]^q \right\}^{-\frac{1}{q}}$$

la successione $\tau_n^{(s)}$ verifica le condizioni (5), (6), (7).

Per il calcolo esplicito degli invarianti ortogonali che compaiono al secondo membro di (2.3), si osservi che le radici dell'equazione secolare

$$\det \left\{ (Gw_h, w_k) - \mu (w_h, w_k) \right\}_{(h,k = 1,\dots,s)} = 0$$

sono $\mu_1^{(s)}, \dots, \mu_s^{(s)}$ e si ha

$$\mathfrak{J}_m^q(P_s\ GP_s) = \sum_{i_1 < \dots < i_m}^{1,s} \left[\mu_{i_1}^{(s)} \cdots \mu_{i_m}^{(s)} \right]^q$$

$$\mathfrak{J}_{m-1}^q(P_{n,s}\ GP_{n,s}) = \sum_{i_1 < \dots < i_{m-1}}^{1,s\ (n)} \left[\mu_{i_1}^{(s)} \cdots \mu_{i_{m-1}}^{(s)} \right]^q .$$

Il simbolo $\sum_{i_1 < \dots < i_{m-1}}^{1,s\ (n)}$ esprime che la somma è estesa unicamente alle disposizioni (crescenti) $i_1, \dots, i_{m-1}$ degli indici $1, \dots, n-1, n+1, \dots, s$.

Rimane solo da calcolare (per eccesso!) l'invariante $\mathfrak{J}_m^q(G)$. A tal fine, se Ω è un insieme di un qualsiasi spazio nel quale è introdotta una misura α non negativa, tale che possa considerarsi la

realizzazione di S sullo spazio $L^2(\Omega; \alpha)$, si dimostra che l'operatore G^q ha la seguente rappresentazione

$$G^q u = \int_\Omega K(x,y) u(y) d\alpha_y$$

nella quale il nucleo $K(x,y)$ è dato dalla seguente convoluzione

$$K(x,y) = \int_\Omega H(x,t) H(t,y) d\alpha_t ,$$

ove $H(x,y)$ è un "nucleo hermitiano", cioè è tale che

$$H(x,y) = \overline{H(y,x)} \qquad , \qquad H \in L^2(\Omega \times \Omega\ ;\ \alpha \times \alpha).$$

Si ha la seguente rappresentazione integrale di $\mathfrak{J}_m^q(G)$:

$$\mathfrak{J}_m^q(G) = \frac{1}{m!} \int_\Omega \cdots \int_\Omega \det \left\{ K(x_i, x_j) \right\} d\alpha_{x_1} \cdots d\alpha_{x_m} \qquad (i,j = 1, \dots, m).$$

Nell'applicazione di questa teoria ai problemi al contorno per le equazioni ed i sistemi alle derivate parziali si urta contro la difficoltà rappresentata dalla mancanza della conoscenza esplicita del nucleo $K(x,y)$, che, in ultima analisi, equivale alla conoscenza della funzione (o matrice) di Green, o di una sua iterata, relativa al problema in istudio. Con tecniche piuttosto sofisticate si riesce a superare questa difficoltà in ipotesi di larghissima generalità. Rimandiamo, per ciò, ai lavori specifici dedicati a questo argomento: [13], [14], [16]. L'idea, che ispira i procedimenti contenuti in questi lavori, è quella consistente nel sostituire G con l'elemento G_s, esplicitamente calcolabile, di una successione $\{G_s\}$, la quale converge uniformemente a G decrescendo. Si può allora, nella (2.3), sostituire $\mathfrak{J}_m^q(G)$ con $\mathfrak{J}_m^q(G_s)$, con che si seguita ad ottenere una successione $\{\tau_n^{(s)}\}$ di valori per difetto di λ_n che converge a questo autovalore, ma tale che $\tau_n^{(s)}$ riesce esplicitamente calcolabile. Ci limitiamo qui ad indicare i risultati che si ottengono nel caso del problema (1). Per le dimostrazioni si veda [13].

Poniamo

$$F_{ij}(x) = \frac{\sigma}{8\pi(1+\sigma)} \frac{\partial^2 |x|}{\partial x^i \partial x^j} - \frac{\delta_{ij}}{4\pi} \frac{1}{|x|} \qquad [x = (x^1, x^2, x^3)],$$

$$\gamma_{ij}(x,y) = -\sum_{h,k}^{1,3} \int_\Omega \left[F_{ik/h}(x-t) F_{jk/h}(t-y) + \sigma F_{ik/k}(x-t) F_{jh/h}(t-y) \right] dt.$$

Sia $\{\omega^h\}$ una successione completa di vettori soluzioni dell'equa-

zione: $\Delta_2 u + \sigma \operatorname{grad} \operatorname{div} u = 0$ (cfr.[12]).

Per pura semplicità espositiva supponiamo che

$$(2.4) \qquad (\!(\omega^h, \omega^k)\!) = \delta_{hk} .$$

Ciò consente di scrivere formule più compatte, ma l'ortonormalizzazione (2.4) non è richiesta nell'applicazione della teoria. Poniamo

$$\varphi_i^s(x) = \sum_{h,k}^{1,3} \int_\Omega \left[F_{ik/h}(x-t)\omega^s_{k/h}(t) + \sigma F_{ik/k}(x-t)\omega^s_{h/h}(t) \right] dt.$$

Dalla (2.3), scritta per m=1, q=2, si trae

$$\tau_n^{(s)} = \left\{ \sum_{i,j}^{1,3} \left[\iint_{\Omega\times\Omega} |\gamma_{ij}(x,y)|^2 dxdy + \sum_{h,k}^{1,s} \int_\Omega \varphi_i^h(x)\varphi_i^k(x)dx \int_\Omega \varphi_j^h(x)\varphi_j^k(x)dx \right.\right.$$
$$\left.\left. - 2\sum_{h}^{1,s} \iint_{\Omega\times\Omega} \gamma_{ij}(x,y)\varphi_i^h(x)\varphi_j^h(y)dxdy \right] - \sum_i^{1,s} (m) \left[\lambda_i^{(s)}\right]^{-2} \right\}^{-1/2} .$$

Tale formula esplicita per il calcolo di $\tau_n^{(s)}$ è fra le più semplici per raggiungere questo scopo. Non è, però, detto che essa sia fra le più efficaci nel contesto delle infinite altre che il metodo degli invarianti ortogonali fornisce. Per l'effettivo calcolo numerico conviene, in effetti, servirsi di altre rappresentazioni di $\tau_n^{(s)}$ le quali, ancorchè meno eleganti, consentono una più rapida approssimazione per difetto di λ_n (cfr.[8],[10]).

3. Problemi di autovalori in campi dotati di un gruppo di simmetrie.

A scopo esemplificativo consideriamo il semplicissimo problema di autovalori

$$(3.1) \qquad \begin{cases} \dfrac{d}{dx}\left[a(x)\dfrac{du}{dx} \right] + c(x)u = \lambda u , & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

nel quale $a(x) \in C^1[-1,1]$, $a(x) > 0$, $c(x) \in C^0[-1,1]$. Le funzioni $a(x)$ e $c(x)$ sono funzioni assegnate, che supponiamo funzioni pari della variabile x. Se poniamo

$$Lu = \frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + c(x)u,$$

è evidente che l'operatore L trasforma ogni funzione pari (dispari) di x in una funzione pari (dispari) . Segue da ciò l'ovvia convenienza

di spezzare il problema (3.1) in due problemi distinti: il primo che ricerca la u fra le funzioni pari, il secondo fra le dispari. E' immediato dimostrare che in tal modo si ottengono tutti gli autovalori per il problema (3.1) e che un sistema completo di autofunzioni del primo problema, unito ad un sistema completo di autofunzioni del secondo, costituiscono un sistema completo per il problema (3.1).

L'utilità pratica di tale spezzamento è del tutto ovvia. Infatti, l'efficacia del calcolo per eccesso e di quello per difetto di ogni λ_n tanto più migliora quanto più alta è l'approssimazione s cui si può spingere il metodo di Rayleigh-Ritz.

Ora, se i mezzi di calcolo a disposizione permettono di spingere l'approssimazione fino ad un certo s_0, cioè se in pratica si riescono a calcolare gli zeri di equazioni matriciali di "tipo secolare" quali la (1.3), fino all'ordine s_0, l'indicato spezzamento permette una approssimazione fino all'ordine $2s_0$. Lo spazio vettoriale U delle funzioni ammissibili da usare per le approssimazioni viene decomposto in somma diretta di due sottospazi invarianti: U^0, spazio delle funzioni pari, e U^1, spazio delle funzioni dispari, e si ha per ogni $u(x) \in U$: $u = P^0u + P^1u$, con $P^0u = 2^{-1}[u(x)+u(-x)]$, $P^1 = 2^{-1}[u(x)-u(-x)]$. P^0 e P^1 sono i due proiettori che proiettano U, rispettivamente, su U^0 e su U^1 e, detto I l'operatore identità, si ha: $I = P^0 + P^1$, $P^0P^0 = P^0$, $P^1P^1 = P^1$, $P^0P^1 = P^1P^0 = 0$. Ciascuno degli spazi U^0 e U^1 è un sottospazio invariante per l'operatore L.

Il problema della determinazione dei sottospazi invarianti di un operatore che, considerato in un campo dotato di particolari proprietà di simmetria, goda di corrispondenti proprietà di invarianza, diventa assai più complicato quando si passa da una a due dimensioni.

Come esempio, si consideri il caso in cui Ω è il campo quadrato del piano delle variabili x e y: $\Omega = (-a,a)\times(a,a)$. Si voglia studiare in Ω il problema di autovalori

$$(3.2) \qquad \begin{cases} \Delta_2 u + \lambda u = 0 & \text{in } \Omega , \\ u = 0 & \text{su } \partial\Omega , \end{cases}$$

oppure il problema

$$(3.3) \qquad \begin{cases} \Delta_2\Delta_2 u - \lambda u = 0 & \text{in } \Omega , \\ u = u_x = u_y = 0 & \text{su } \partial\Omega . \end{cases}$$

In ciascuno di questi due casi lo spazio U delle funzioni ammissibili si decompone in somma diretta di sei sottospazi invarianti. I proiettori che proiettano U su ognuno di tali sottospazi risultano dalla decomposizione

$$u(x,y) = P^{000}u + P^{001}u + P^{110}u + P^{111}u + P^{01}u + P^{10}u, \tag{3.4}$$

ove

$$v^{\alpha\beta} = P^{\alpha\beta}u$$

$$= \frac{1}{4}\left[u(x,y) + (-1)^{\alpha}u(-x,y) + (-1)^{\beta}u(x,-y) + (-1)^{\alpha+\beta}u(-x,-y)\right]$$

$$(\alpha, \beta = 0,1)$$

e

$$P^{\alpha\alpha\gamma}u = \frac{1}{2}\left[v^{\alpha\alpha}(x,y) + (-1)^{\gamma}v^{\alpha\alpha}(y,x)\right] \qquad (\gamma = 0,1) \quad (2).$$

Indicando, brevemente, con $P_1, \ldots, P_6$ i sei proiettori che intervengono nella (3.4), si ha

$$I = \sum_{k=1}^{6} P_k \quad , \quad P_h P_k = \delta_{hk} P_h .$$

Se il numero n delle dimensioni è > 2, il problema diventa straordinariamente complesso e non si può più sperare di risolverlo con mezzi "artigianali". Bisogna, invece, approntare gli adeguati strumenti matematici che ne permettano una razionale soluzione. A tal fine occorre, prima, enunciare con chiarezza in che cosa consiste il problema da risolvere. Nel far questo considereremo il caso generale di uno spazio $\mathbb{R}^n$ e , in vista delle applicazioni all'Elasticità, considereremo spazi funzionali U, costituiti da vettori ad n componenti (che conviene considerare complesse) ed operatori L che trasformano ogni vettore di U in un vettore di U.

I dati del problema sono i seguenti:

1) Un campo Ω di $\mathbb{R}^n$ avente un assegnato gruppo $\mathcal{G}$ di simmetrie. (3)

(2) Cfr. [11]. E' significativo il fatto che N.Aronszajn, considerando il problema (3.3) in $(-a,a)\times(-a,a)$, decompose U in solo quattro spazi invarianti, non accorgendosi che era possibile una decomposizione più fine ([2], cfr. anche [17], p.175).

(3) Il significato da dare alla frase: Ω ha $\mathcal{G}$ come gruppo di simmetrie è classicamente noto e, comunque, verrà ricordato fra breve.

2) Uno spazio vettoriale di vettori ad n componenti, ciascuna funzione complessa del punto $x = (x^1, \ldots, x^n)$ di Ω . Sia U tale spazio.

3) Lo spazio vettoriale complesso $\mathcal{L}$ degli operatori lineari L tali che: a) il dominio di L sia U; b) il codominio L(U) di L sia contenuto in U ; c) L sia invariante rispetto al gruppo $\mathcal{G}$. (4)

Si pone il seguente

PROBLEMA

I) Trovare una decomposizione massimale di U in somma diretta di sottospazi

$$U = U_1 \oplus U_2 \oplus U_3 \oplus \ldots \oplus U_\mu ,$$

ciascuno di essi invariante rispetto ad ogni $L \in \mathcal{L}$, cioè tali che

$$L(U_k) \subset U_k \qquad (k=1,\ldots,\mu) \ , \ \forall\, L \in \mathcal{L}.$$

II) Costruire l'operatore di proiezione P_k che proietta U su U_k $(k=1,\ldots,\mu)$:

$$I = \sum_{k=1}^{\mu} P_k \ , \ P_h P_k = \delta_{hk} P_h \ , \ P_k(U) = U_k .$$

Aspetti di questo problema furono da me proposti, a partire dalla fine degli anni '60, a Lucilla Bassotti ed a Gerald F.Smith, i quali, in lavori apparsi fra il 1967 ed il 1980 (cfr.[3],[4],[5],[6]) , hanno, come risultato delle ricerche da loro, indipendentemente, condotte, elaborato una teoria di indubbio valore teorico, ma, soprattutto, di grande interesse applicativo: una teoria che, a quanto mi consta, non è conosciuta dai matematici e , in particolare, dagli analisti numerici, come essa meriterebbe, visti i grandi servigi che è in grado di rendere.

Nei prossimi paragrafi vorrò tratteggiare i risultati salienti dovuti ai due valorosi ricercatori testè citati, e indicarne poscia le applicazioni al controllo della reliability, dal punto di vista numerico, della legge asintotica di H.Weyl fornita dalla (4), nel caso che Ω sia un cubo di $\mathbb{R}^3$. Questo problema, proposto a L.Bassotti, fu da lei brillantemente risolto nel 1978 [7]. Ne riporteremo i risultati alla fine del § 6.

(4) Il significato della frase: L è invariante rispetto a $\mathcal{G}$ sarà ben specificato fra breve.

4. Operatori invarianti rispetto ad un gruppo di matrici ortogonali. Fondamenti della teoria delle rappresentazioni lineari dei gruppi.

Sia Ω un campo di $\mathbb{R}^n$, il cui punto generico indichiamo con $x = (x^1,\dots,x^n)$. Sia $\mathcal{G} \equiv \{\gamma^1,\dots,\gamma^q\}$ un gruppo finito di matrici reali di ordine n ortogonali; sia γ^1 la matrice unità.

Dicendo che Ω ha $\mathcal{G}$ come gruppo di simmetrie, intendiamo che, per ogni γ^h $(h=1,\dots,q)$ di $\mathcal{G}$, dall'essere $x \in \Omega$ segue $\gamma^h x \in \Omega$ $(\forall\, x \in \Omega)$.

Come spazio vettoriale U assumiamo quello costituito da tutte le funzioni $u(x)$ [5] appartenenti a $C^\infty(\Omega)$ oppure a $C^\infty(\bar\Omega)$ o, ancora, quello delle funzioni di $C_c^\infty(\Omega)$ (funzioni di $C^\infty(\Omega)$ con supporto compatto contenuto in Ω). Volendo, potremmo proporre altre definizioni per U. Ma su ciò non riteniamo di dover insistere.

Per ogni $u(x) \in U$, detta γ una matrice $n \times n$ ortogonale, consideriamo la funzione

$$\Gamma u = \gamma u(\overset{*}{\gamma} x) \qquad (\gamma \in \mathcal{G}),$$

avendo indicato con $\overset{*}{\gamma}$ la matrice trasposta di γ e, quindi, $\overset{*}{\gamma} = \gamma^{-1}$. Quale si sia, fra quelle proposte, la definizione di U, accade che Γ è una trasformazione lineare di U su U. Orbene, se Γ^h è la trasformazione lineare che corrisponde a $\gamma^h \in \mathcal{G}$, sussiste il seguente lemma ([4],[5]).

I. Il gruppo $\{\Gamma^1,\dots,\Gamma^q\}$ è isomorfo al gruppo $\mathcal{G} \equiv \{\gamma^1,\dots,\gamma^q\}$.

Sia ora L un qualsiasi operatore lineare definito in U ed avente codominio contenuto in U, tale, cioè, che $L(U) \subset U$. Si dirà che L è un operatore invariante rispetto al gruppo $\mathcal{G}$ se $L\,\Gamma^h u = \Gamma^h L u$, $\forall\, u \in U$. Sia $\mathcal{L}$ lo spazio vettoriale di tutti gli operatori L ciascuno dei quali è invariante rispetto a $\mathcal{G}$. Supponiamo che l'operatore L sia, in particolare, un operatore differenziale lineare matriciale. Sia $Lu \equiv (L^1u,\dots,L^nu)$ e (adottando la convenzione sommatoria) si abbia:

$$L_j^i u^j = \sum_{\nu=0}^{m} a_j^{i\,s_1\dots s_\nu}(x)\, \frac{\partial^\nu u^j}{\partial x^{s_1}\dots\partial x^{s_\nu}}\,.$$

[5] D'ora in avanti, parlando di funzione (senza ulteriori specifiche) intenderemo sempre una funzione vettoriale ad n componenti complesse.

I coefficienti $a_j^{i\,s_1\dots s_\nu}(x)$ siano funzioni scalari appartenenti a $C^\infty(\mathbb{R}^n)$.

Sussiste il seguente teorema [5]:

II. L'operatore L appartiene a $\mathcal{L}$, cioè è invariante rispetto al gruppo $\mathcal{G}$, se e solo se per ogni matrice $\gamma \equiv \{\gamma_i^h\}$ di $\mathcal{G}$ si ha in Ω

$$a_k^{h\,r_1\dots r_\nu}(\gamma x) = a_j^{i\,s_1\dots s_\nu}(x)\,\gamma_i^h\,\gamma_{s_1}^{r_1}\dots\gamma_{s_\nu}^{r_\nu}\,\overset{*}{\gamma}{}_k^j\,.$$

Si vede quindi che, per l'invarianza di L rispetto a $\mathcal{G}$, i coefficienti $a_j^{i\,s_1\dots s_\nu}$ devono comportarsi come le componenti di un tensore con un solo indice di covarianza e $\nu+1$ indici di controvarianza, quando i cambiamenti di coordinate ammesse siano solo quelli determinati dalle matrici ortogonali di $\mathcal{G}$.

E' opportuno ora richiamare alcuni risultati basilari della teoria della rappresentazione lineare dei gruppi finiti, risultati sui quali poggiano i più significativi sviluppi della teoria di L.Bassotti e G.F.Smith. [6] Occorre dire che la rappresentazione lineare dei gruppi è stata ampiamente usata nello studio di certe proprietà di simmetria che s'incontrano nella Fisica teorica (cfr.ad es.[2]).Tuttavia, le applicazioni fattene al problema che ci interessa, hanno carattere di originalità, se non nei procedimenti, certo nei risultati.

Se con $\mathbb{C}^m$ indichiamo lo spazio vettoriale complesso ad m dimensioni (complesse), denoteremo con T una trasformazione lineare di $\mathbb{C}^m$ in se stesso e supporremo sempre che T sia non degenere, con ciò intendendo che $Tz=0$, $z \in \mathbb{C}^m$ implica $z=0$.

Sia $\mathcal{G} \equiv \{\gamma^1,\dots,\gamma^q\}$ un qualsiasi gruppo astratto finito e γ^1 l'elemento unità del gruppo. Sia data una corrispondenza la quale associa ad ogni $\gamma^h \in \mathcal{G}$ una trasformazione lineare T^h del tipo anzidetto: $\gamma^h \longrightarrow T^h$. Se: $\gamma^h\gamma^k \longrightarrow T^hT^k$, diremo che il gruppo $\{T^1,\dots,T^q\}$, omomorfo a $\mathcal{G}$, è una rappresentazione lineare di $\mathcal{G}$ di ordine m. E' evidente che dall'essere $\gamma^1 \longrightarrow T^1$ segue che T^1 è la trasformazione identica. Introducendo in $\mathbb{C}^m$ un sistema di coordinate cartesiane (complesse), ciascuna T^h si rappresenterà univocamente mediante una

[6] Per esposizioni sistematiche della teoria delle rappresentazioni lineari dei gruppi, possono consultarsi le Monografie: [22],[20],[19],[18],[9].

matrice $m \times m$ A^h non degenere ad elementi complessi; se A^k è la matrice che rappresenta T^k, avremo che $T^h T^k$ viene rappresentata da $A^h A^k$. E' allora spontaneo chiamare anche $\{A^1,\dots,A^q\}$ una rappresentazione lineare di $\mathcal{G}$ (mediante un gruppo di matrici non degeneri) di ordine m. Se X è una qualsiasi matrice non degenere di ordine m, le due rappresentazioni lineari di $\mathcal{G}$: $\{A^1,\dots,A^q\}$ e $\{XA^1X^{-1},\dots,XA^qX^{-1}\}$ saranno dette equivalenti, dato che A^h e XA^hX^{-1} sono rappresentazioni matriciali della stessa trasformazione lineare T^h e dall'una si passa all'altra mediante il cambiamento di coordinate in $\mathbb{C}^m$ determinato da X.

Si dimostra che in ogni classe di equivalenza di rappresentazioni del gruppo finito $\mathcal{G}$ mediante un gruppo di matrici di ordine m, esiste sempre una rappresentazione costituita da matrici unitarie. Una siffatta rappresentazione viene detta rappresentazione unitaria di ordine m. E' ovvio che se $\mathcal{G}$ è il gruppo di matrici ortogonali, da noi inizialmente considerato, esso stesso costituisce una rappresentazione unitaria di $\mathcal{G}$ di ordine n.

Tornando ad un gruppo finito qualsiasi $\mathcal{G}$, si dice che la rappresentazione $\{T^1,\dots,T^q\}$ di $\mathcal{G}$ di ordine m è riducibile se esiste in $\mathbb{C}^m$ un sottospazio S di dimensione complessa r, $0 < r < m$, il quale sia sottospazio invariante per ogni T^h, cioè $T^h(S) \subset S$ $(h=1,\dots,q)$. E' allora possibile scegliere in $\mathbb{C}^m$ le coordinate cartesiane complesse in guisa tale che, se $\tilde{A}^h$ è la matrice $m \times m$ che rappresenta T^h in tale sistema, si ha

$$\tilde{A}^h = \begin{pmatrix} B^h & 0 \\ 0 & C^h \end{pmatrix} \qquad (h=1,\dots,q), \tag{4.1}$$

cioè $\tilde{A}^h$ è una matrice a blocchi, con B^h matrice $r \times r$ e C^h matrice $s \times s$ $(s=m-r)$. E' facile dimostrare che $\{B^1,\dots,B^q\}$ e $\{C^1,\dots,C^q\}$ sono, esse stesse, rappresentazioni lineari di $\mathcal{G}$ degli ordini r ed s, rispettivamente. Se $\{A^1,\dots,A^q\}$ è una rappresentazione lineare di $\mathcal{G}$ di ordine m, essa è riducibile se e solo se esiste una matrice $m \times m$ X non degenere, tale che $\tilde{A}^h = XA^hX^{-1}$ sia data da (4.1). Se una siffatta X non esiste, se cioè non esiste un sottospazio S di $\mathbb{C}^m$ che sia simultaneamente invariante per ogni T^h $(h=1,\dots,q)$, la rappresentazione $\{A^1,\dots,A^q\}$ di ordine m di $\mathcal{G}$ dicesi irriducibile.

Si pone il problema di stabilire quante sono le rappresentazioni

irriducibili di $\mathcal{G}$. A tal fine bisogna suddividere $\mathcal{G}$ in classi di equivalenza, dicendo due elementi g^h e g^k di $\mathcal{G}$ equivalenti se esiste qualche $g^l \in \mathcal{G}$ tale che $g^h = g^l g^k (g^l)^{-1}$. In tal modo $\mathcal{G}$ viene suddiviso in ν $(1 \le \nu \le q)$ classi di equivalenza. Si dimostra che esistono ν interi positivi $m_1, m_2, \dots, m_\nu$ tali che per ogni m_r esiste una rappresentazione lineare irriducibile di ordine m_r di $\mathcal{G}$; sia essa

$$\{A^{(r)}(g^1),\dots,A^{(r)}(g^q)\} \qquad (r=1,\dots,\nu), \tag{4.2}$$

avendo indicato con $A^{(r)}(g^h)$ la matrice $m_r \times m_r$ che corrisponde a g^h. Ogni altra rappresentazione lineare di $\mathcal{G}$ o è equivalente a qualcuna delle rappresentazioni (4.2), oppure è riducibile. Inoltre se per $i \neq j$ riesce $m_i = m_j$, le due rappresentazioni (4.2) di $\mathcal{G}$ corrispondenti a i ed a j non sono equivalenti.

Pertanto, può dirsi che le (4.2) sono tutte le rappresentazioni irriducibili di $\mathcal{G}$. Mediante esse può costruirsi qualsiasi rappresentazione lineare di $\mathcal{G}$. Poichè ogni rappresentazione (4.2) è determinata a meno di un'equivalenza del tipo $\{XA^{(r)}(g^1)X^{-1},\dots,XA^{(r)}(g^q)X^{-1}\}$, con X arbitraria matrice $m_r \times m_r$ non degenere, può ben supporsi, per un risultato sopra ricordato, che la rappresentazione (4.2) sia unitaria, cioè che le matrici che intervengono in essa siano unitarie. Assumeremo, pertanto, d'ora in avanti, che la matrice

$$A^{(r)}(g^h) = \{a_{ij}^{(r)}(g^h)\} \qquad (h = 1,\dots,q \; ; \; r = 1,\dots,\nu)$$
$$(i,j = 1,\dots,m_r)$$

sia unitaria.

Si dimostra che gli elementi delle matrici unitarie $A^{(r)}(g^h)$ sono legati dalle seguenti relazioni

$$\sum_{h=1}^{q} a_{ij}^{(r)}(g^h)\,\overline{a_{k\ell}^{(s)}(g^h)} = 0 \qquad \text{per } r \neq s, \tag{4.3}$$

$$\sum_{h=1}^{q} a_{ij}^{(r)}(g^h)\,\overline{a_{k\ell}^{(r)}(g^h)} = \frac{q}{m_r}\,\delta_{ik}\delta_{j\ell}\,, \tag{4.4}$$

$$\sum_{r=1}^{\nu} m_r \sum_{i=1}^{m_r} a_{ii}^{(r)}(g^h) \begin{cases} = q & \text{per } h=1 \\ = 0 & \text{per } h>1. \end{cases} \tag{4.5}$$

Si noti che la (4.5), per h=1, diventa

$$\sum_{r=1}^{\nu} m_r^2 = q.$$

5. I teoremi di G.F.Smith e di L.Bassotti.

Indicheremo con $\mathcal{A}(\mathcal{G})$ l'algebra (sui complessi) di tutte le combinazioni lineari (a coefficienti complessi)

$$a_1\Gamma_1 + \ldots + a_q\Gamma^q$$

degli operatori Γ^h, di U su U, dianzi introdotti. Sussiste il seguente importante teorema dovuto a G.F.Smith [21].

III. Posto $\mu = m_1 + m_2 + \ldots + m_\nu$, si considerino i seguenti operatori di $\mathcal{A}(\mathcal{G})$

$$P_{ri} = \frac{m_r}{q}\sum_{h=1}^{q} a_{ii}^{(r)}(\gamma^h)\,\Gamma^h \qquad (r=1,..,\nu\,;\ i=1,..,m_r). \tag{5.1}$$

Detto I l'operatore identità, si ha

$$P_{ri}\,P_{sj} = \delta_{rs}\,\delta_{ij}\,P_{ri}\,, \tag{5.2}$$

$$I = \sum_{r=1}^{\nu}\sum_{i=1}^{m_r} P_{ri}\,. \tag{5.3}$$

Pertanto, gli operatori (5.1) sono proiettori che decompongono U in somma diretta di sottospazi, ciascuno dei quali è invariante per ogni $L \in \mathcal{L}$. Questa decomposizione è massimale.

Dicendo che la decomposizioni di U, come somma diretta dei codomini dei μ proiettori P_{ri}, è massimale, s'intende che, qualunque sia P_{ri}, se si ha: $P_{ri} = P'_{ri} + P''_{ri}$, $P'_{ri}P'_{ri} = P'_{ri}$, $P''_{ri}P''_{ri} = P''_{ri}$, $P'_{ri}P''_{ri} = 0$, $P'_{ri} \in \mathcal{A}(\mathcal{G})$, $P''_{ri} \in \mathcal{A}(\mathcal{G})$, deve essere: $P'_{rv} = 0$, $P''_{ri} = P_{ri}$, oppure: $P'_{ri} = P_{ri}$, $P''_{ri} = 0$.

In altri termini, non è possibile, adoperando proiettori di $\mathcal{A}(\mathcal{G})$, raffinare ulteriormente la decomposizione ottenuta di U in somma diretta di sottospazi invarianti per gli operatori di $\mathcal{L}$.

L'interesse del teorema di G.F.Smith risiede, specie dal punto di vista applicativo, nella costruzione esplicita di tutti i proiettori P_{ri} mediante le (5.1). Data l'importanza del teorema di G.F.Smith, vogliamo qui darne una dimostrazione.

Se γ è un qualsiasi elemento di $\mathcal{G}$, indicheremo con $\overset{*}{\Gamma}$ l'operatore di $\mathcal{A}(\mathcal{G})$ che corrisponde a $\overset{*}{\gamma}$. Essendo $\gamma^{-1} = \overset{*}{\gamma}$, avremo $\Gamma^{-1} = \overset{*}{\Gamma}$. Sia $\gamma^h\gamma^k = \gamma^l$ e quindi $\gamma^h = \gamma^l\,\overset{*}{\gamma}{}^k$. Si ha, ricordando (4.3),(4.4):

$$P_{ri}\,P_{sj} = \frac{m_r m_s}{q^2}\sum_{h=1}^{q}\sum_{k=1}^{q} a_{ii}^{(r)}(\gamma^h)\,a_{jj}^{(s)}(\gamma^k)\,\Gamma^h\Gamma^k$$

$$= \frac{m_r m_s}{q^2}\sum_{k=1}^{q} a_{jj}^{(s)}(\gamma^k)\sum_{l=1}^{q} a_{ii}^{(r)}(\gamma^l\,\overset{*}{\gamma}{}^k)\,\Gamma^l$$

$$= \frac{m_r m_s}{q^2}\sum_{k=1}^{q} a_{jj}^{(s)}(\gamma^k)\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\,a_{mi}^{(r)}(\overset{*}{\gamma}{}^k)\,\Gamma^l$$

$$= \frac{m_r m_s}{q^2}\sum_{k=1}^{q} a_{jj}^{(s)}(\gamma^k)\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\,\overline{a_{im}^{(r)}(\gamma^k)}\,\Gamma^l$$

$$= \frac{m_r m_s}{q^2}\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\sum_{k=1}^{q} a_{jj}^{(s)}(\gamma^k)\,\overline{a_{im}^{(r)}(\gamma^k)}\,\Gamma^l$$

$$= \frac{m_r m_s}{q^2}\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\,\delta_{rs}\,\frac{q}{m_r}\,\delta_{ji}\,\delta_{jm}\,\Gamma^l$$

$$= \delta_{rs}\,\delta_{ij}\,\frac{m_r}{q}\sum_{l=1}^{q} a_{ii}^{(r)}(\gamma^l)\,\Gamma^l.$$

E' così provata la (5.2).

Si ha, tenendo presente la (4.5),

$$\sum_{r=1}^{\nu}\sum_{i=1}^{m_r} P_{ri} = \sum_{r=1}^{\nu}\sum_{i=1}^{m_r}\frac{m_r}{q}\sum_{h=1}^{q} a_{ii}^{(r)}(\gamma^h)\,\Gamma^h$$

$$= \frac{1}{q}\sum_{h=1}^{q}\Big[m_1\sum_{i=1}^{m_1} a_{ii}^{(1)}(\gamma^h) + \ldots + m_\nu\sum_{i=1}^{m_\nu} a_{ii}^{(\nu)}(\gamma^h)\Big]\Gamma^h = I,$$

quindi la (5.3). Occorre, per completare la dimostrazione, provare la massimalità della (5.3).

Si ha, ricordando la (4.4),

$$P_{ri}\,\Gamma^k\,P_{ri} = \frac{m_r^2}{q^2}\sum_{h=1}^{q} a_{ii}^{(r)}(\gamma^h)\sum_{j=1}^{q} a_{ii}^{(r)}(\gamma^j)\,\Gamma^h\Gamma^k\Gamma^j$$

$$= \frac{m_r^2}{q^2}\sum_{j=1}^{q} a_{ii}^{(r)}(\gamma^j)\sum_{l=1}^{q} a_{ii}^{(r)}(\gamma^l\,\overset{*}{\gamma}{}^j\,\overset{*}{\gamma}{}^k)\,\Gamma^l$$

$$= \frac{m_r^2}{q^2}\sum_{j=1}^{q} a_{ii}^{(r)}(\gamma^j)\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\,a_{mi}^{(r)}(\overset{*}{\gamma}{}^j\,\overset{*}{\gamma}{}^k)\,\Gamma^l$$

$$= \frac{m_r^2}{q^2}\sum_{j=1}^{q} a_{ii}^{(r)}(\gamma^j)\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\sum_{t=1}^{m_r} a_{mt}^{(r)}(\overset{*}{\gamma}{}^j)\,a_{ti}^{(r)}(\overset{*}{\gamma}{}^k)\,\Gamma^l$$

$$= \frac{m_r^2}{q^2}\sum_{l=1}^{q}\sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l)\sum_{t=1}^{m_r}\overline{a_{it}^{(r)}(\gamma^k)}\sum_{j=1}^{q} a_{ii}^{(r)}(\gamma^j)\,\overline{a_{tm}^{(r)}(\gamma^j)}\,\Gamma^l$$

$$= \frac{m_r^2}{q^2} \sum_{l=1}^{q} \sum_{m=1}^{m_r} a_{im}^{(r)}(\gamma^l) \sum_{t=1}^{m_r} \overline{a_{it}^{(r)}(\gamma^k)} \frac{q}{m_r} \delta_{it}\delta_{im} \Gamma^l$$

$$= \frac{m_r}{q} \sum_{l=1}^{q} a_{ii}^{(r)}(\gamma^l) \overline{a_{ii}^{(r)}(\gamma^k)} \Gamma^l = \overline{a_{ii}^{(r)}(\gamma^k)} P_{ri} .$$

Poichè è $P'_{ri} \in \mathcal{A}(\mathcal{G})$, si ha, detta σ una costante, $P_{ri} P'_{ri} P_{ri} = \sigma P_{ri}$. Si ha d'altra parte: $P_{ri} P'_{ri} P_{ri} = (P'_{ri} + P''_{ri}) P'_{ri} (P'_{ri} + P''_{ri}) = P'_{ri}$. Quindi $P'_{ri} = \sigma P_{ri}$. Ne viene $P'_{ri} = P'_{ri} P'_{ri} = \sigma P_{ri} P'_{ri} = \sigma P_{ri}$. Deve, quindi, essere $\sigma = 1$, oppure $P'_{ri} = 0$, $\sigma = 0$.

Supponiamo di introdurre in U un prodotto scalare, che indichiamo con $((u,v))$. Si dirà che è questo un prodotto scalare invariante rispetto al gruppo $\mathcal{G}$ se: $((\Gamma^h u, \Gamma^h v)) = ((u,v))$ $(h=1,\dots,q; {}^{*}\forall u,v \in U)$. E' assai utile, per gli ulteriori sviluppi della teoria, il seguente teorema [6].

IV. Se $\mathcal{H}$ è lo spazio di Hilbert ottenuto per completamento funzionale tramite il prodotto scalare $((u,v))$, invariante rispetto a $\mathcal{G}$, ciascuno degli operatori (5.1) è un proiettore ortogonale di $\mathcal{H}$.

Occorre solo far vedere che P_{ri} è un operatore hermitiano di $\mathcal{H}$. Sia $w \in U$ e, detto Γ un qualsiasi operatore di $\mathcal{G}$, sia $v = \overset{*}{\Gamma} w$, cioè $\Gamma v = w$. Si ha: $((\Gamma u, w)) = ((\Gamma u, \Gamma v)) = ((u,v)) = ((u, \overset{*}{\Gamma} w))$. Pertanto l'operatore Γ, ovviamente continuo in $\mathcal{H}$, ha come operatore aggiunto $\overset{*}{\Gamma}$. Tenendo presente che, al variare di h da 1 a q, la matrice $\overset{*}{\gamma}{}^h$ descrive tutto il gruppo $\mathcal{G}$, si ha, detto $\overset{*}{P}_{ri}$ l'operatore aggiunto di P_{ri},

$$\overset{*}{P}_{ri} = \frac{m_r}{q} \sum_{h=1}^{q} \overline{a_{ii}^{(r)}(\gamma^h)}\, \overset{*}{\Gamma}{}^h = \frac{m_r}{q} \sum_{h=1}^{q} a_{ii}^{(r)}(\overset{*}{\gamma}{}^h)\, \overset{*}{\Gamma}{}^h = P_{ri} .$$

6. Il caso in cui Ω è un cubo di $\mathbb{R}^3$. Verifica dell'attendibilità numerica della (4).

Sia Ω il cubo di $\mathbb{R}^3$: $|x^h| < \pi/2$ $(h=1,2,3)$. E' noto che il gruppo $\mathcal{G}$ delle simmetrie di Ω è, in questo caso, composto da 48 matrici ortogonali $\mathcal{G} = \{ \gamma^1, \dots, \gamma^{48} \}$. Esse sono le seguenti:

$$\gamma^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \gamma^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad \gamma^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\gamma^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad \gamma^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad \gamma^6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^7 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \gamma^8 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad \gamma^9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^{17} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \gamma^{25} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \gamma^{33} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^{41} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ;$$

$$(6.1) \quad \begin{cases} \gamma^{8+i} = \gamma^i \gamma^9 , \quad \gamma^{16+i} = \gamma^i \gamma^{17} , \quad \gamma^{24+i} = \gamma^i \gamma^{25} , \\ \gamma^{32+i} = \gamma^i \gamma^{33} , \quad \gamma^{40+i} = \gamma^i \gamma^{41} \\ (i = 1,\dots,8) . \end{cases}$$

Nel caso particolare che stiamo ora considerando, si ha: $\nu = 10$ e $m_1 = m_2 = m_3 = m_4 = 1$, $m_5 = m_6 = 2$, $m_7 = m_8 = m_9 = m_{10} = 3$. Riesce, quindi, $\mu = 20$. Dalla tabella seguente si deduce qual'è la matrice unitaria $A^{(r)}$ $(r = 1,\dots,10)$ (guardare la corrispondente colonna) che corrisponde alla matrice γ^h di $\mathcal{G}$ (guardare la riga, o servirsi delle (6.1)).

	$A^{(1)}$	$A^{(2)}$	$A^{(3)}$	$A^{(4)}$	$A^{(5)}$	$A^{(6)}$	$A^{(7)}$	$A^{(8)}$	$A^{(9)}$	$A^{(10)}$
g^1	(1)	(1)	(1)	(1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
g^2	(1)	(-1)	(1)	(-1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
g^3	(1)	(-1)	(1)	(-1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
g^4	(1)	(-1)	(1)	(-1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$
g^5	(1)	(1)	(1)	(1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
g^6	(1)	(1)	(1)	(1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$
g^7	(1)	(-1)	(1)	(1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
g^8	(1)	(1)	(1)	(-1)	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
g^9	(1)	(1)	(-1)	(-1)	$\begin{pmatrix}\frac12&\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}\frac12&\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$
g^{17}	(1)	(1)	(-1)	(-1)	$\begin{pmatrix}\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}$
g^{25}	(1)	(1)	(-1)	(-1)	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}$
g^{33}	(1)	(1)	(1)	(1)	$\begin{pmatrix}-\frac12&\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}-\frac12&\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
g^{41}	(1)	(1)	(1)	(1)	$\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$

I 20 proiettori che danno la decomposizione dello spazio funzionale U in sottospazi invarianti per gli operatori di $\mathcal{L}$ sono i seguenti:

$$P_k = \frac{1}{48}\sum_{i=1}^{48} a_{11}^{(k)}(g^i)\,\Gamma^i \qquad , k = 1,2,3,4$$

$$P_k = \frac{1}{24}\sum_{i=1}^{48} a_{11}^{(k)}(g^i)\,\Gamma^i \qquad , k = 5,6$$

$$P_{k+2} = \frac{1}{24}\sum_{i=1}^{48} a_{22}^{(k)}(g^i)\,\Gamma^i \qquad , k = 5,6$$

$$P_{k+2} = \frac{1}{16}\sum_{i=1}^{48} a_{11}^{(k)}(g^i)\,\Gamma^i \qquad , k = 7,8,9,10$$

$$P_{k+6} = \frac{1}{16}\sum_{i=1}^{48} a_{22}^{(k)}(g^i)\,\Gamma^i \qquad , k = 7,8,9,10$$

$$P_{k+10} = \frac{1}{16}\sum_{i=1}^{48} a_{33}^{(k)}(g^i)\,\Gamma^i \qquad , k = 7,8,9,10$$

Agli operatori di $\mathcal{L}$ appartiene sicuramente l'operatore classico dell'Elasticità tridimensionale $L = \Delta_2 + \sigma$ grad div, il quale è invariante rispetto al gruppo ortogonale.

La teoria può applicarsi al problema (1), considerato nel cubo Ω. E' facile, infatti, verificare che, sia il prodotto scalare (1.1), che il prodotto scalare (1.2) sono invarianti rispetto a $\mathcal{G}$.

Il potere spezzare il problema (1) in 20 problemi rende possibile spingere la limitazione rigorosa degli autovalori del problema iniziale, in modo da dare senso ad un confronto numerico fra i valori approssimati (per difetto e per eccesso) degli autovalori, ottenuti con i metodi prima descritti, e quelli forniti dalla (4). Indicheremo ora con δ_n il valore per difetto di λ_n, con ε_n quello per eccesso e con α_n quello fornito dalla (4), che, nel caso attuale, diventa:

$$\alpha_n = \left(\frac{6}{\pi}\right)^{2/3} \frac{\sigma+1}{\left[1+2(\sigma+1)^{3/2}\right]^{2/3}}\, n^{2/3}. \tag{7}$$

Ecco i risultati numerici ottenuti nel caso $\sigma = 1$ (cfr. [8], [7]).

(7) Il lettore non sia sorpreso nel constatare che, al crescere dell'indice n dell'autovalore λ_n, lo scarto fra δ_n e ε_n diviene, in genere, sempre maggiore. In effetti, il problema del calcolo di λ_n è un problema

./.

$\sigma = 1$

n	δ_n	ε_n	α_n
1	3.868	3.872	0.870
5	6.132	6.146	2.873
10	8.720	8.822	4.034
20	10.774	10.985	6.410
40	16.60	17.44	10.18
60	18.83	20.30	13.33
80	21	24.09	16.15
100	22	26.99	18.74
130	26	30.85	22.32

Per meglio valutare la rapidità di convergenza di α_n verso λ_n, conviene normalizzare i risultati numerici confrontando δ_n / ε_n con α_n / ε_n.

$\sigma = 1$

n	δ_n / ε_n	α_n / ε_n
1	0.9989	0.2247
5	0.9977	0.4139
10	0.9884	0.4572
20	0.9808	0.5835
40	0.9518	0.5837
60	0.9275	0.6566
80	0.8717	0.6704
100	0.8427	0.7235

---

che diviene sempre meno stabile al crescere di n. Ciò viene posto in luce dalla seguente diseguaglianza di Aronszajn (cfr.[13],p.115 e [17] p.159)

$$\left| 1 - \frac{\lambda_n}{\tilde{\lambda}_n} \right| \leq \lambda_n \| G - \tilde{G} \| ,$$

dove G e $\tilde{G}$ sono due operatori compatti e positivi, con λ_n^{-1}, $\tilde{\lambda}_n^{-1}$ i rispettivi autovalori di indice n.

Questi risultati numerici mettono bene in evidenza la notevole lentezza della convergenza della formula asintotica. Ciò seguita a verificarsi anche per altri valori di σ, come mostrano le tabelle seguenti.

$\sigma = 2$

n	δ_n / ε_n	α_n / ε_n
1	0.9954	0.1979
5	0.9948	0.4296
10	0.9859	0.4601
20	0.9644	0.5390
40	0.9101	0.5849
60	0.8535	0.6205
80	0.8145	0.6567
100	0.7555	0.6747

$\sigma = 3$

n	δ_n / ε_n	α_n / ε_n
1	0.9916	0.1771
5	0.9934	0.4363
10	0.9717	0.4472
20	0.9399	0.5011
40	0.8773	0.5755
60	0.8207	0.6090
80	0.7803	0.6425
100	0.7279	0.6637

$\sigma = 4$

n	δ_n / ε_n	α_n / ε_n
1	0.9838	0.1617
5	0.9918	0.4395
10	0.9599	0.4173
20	0.9389	0.4870
40	0.8526	0.5660
60	0.8049	0.6165
80	0.7478	0.6228

$\sigma = 5$

n	δ_n / ε_n	α_n / ε_n
1	0.9790	0.1492
5	0.9795	0.4358
10	0.9570	0.3999
20	0.9463	0.4874
40	0.8325	0.5528
60	0.7585	0.5731
80	0.7274	0.6100

L.Bassotti [7] ha osservato che, detto $\lambda_n^{(\sigma)}$ lo n-esimo autovalore del problema (1), corrispondente al valore σ del parametro dell'Elasticità, si ha, qualunque sia $\sigma > 0$, e per ogni n, $\lambda_n^{(0)} \leq \lambda_n^{(\sigma)}$.

Il numero $\lambda_n^{(0)}$, specie se è σ abbastanza grande, costituisce una limitazione inferiore piuttosto grossolana di $\lambda_n^{(\sigma)}$. Eppure, se è $n \leq 170$, si constata (cfr.[7]) che riesce $\alpha_n < \lambda_n^{(0)}$, qualunque sia $\sigma > 0$.

Si può, quindi, affermare che la formula asintotica (4) di Weyl, di straordinario valore teorico, appare, invece, assai poco affidabi-

le dal punto di vista numerico.(8) Analoga circostanza era stata messa in luce nel problema piano (3.3),considerato in un quadrato (cfr. [16], pp.74-89).

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(8) Il Sig. Hersch (Zurich), recensendo la Nota [7], fa la seguente osservazione: Reviewer's remark: Since Weyl's asymptotic formula remains valid with a factor $1+o(1)$,it could presumably be refined in view of numerical applications (Math.Reviews 82d:49041).
Il perfezionamento delle formule asintotiche per gli autovalori , aggiungendo termini di ordine inferiore, è, tranne che in qualche caso particolare, problema ancora aperto e, come ben noto, di assai severa difficoltà.

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PERIODIC SOLUTIONS OF A WAVE EQUATION WITH STRONG NONLINEARITY

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In the present paper we consider the problem

$$(0)\qquad \begin{aligned} &u_{tt} - u_{xx} = \pm g(u) + f(t,x), \qquad 0 < x < \pi,\ -\infty < t < +\infty, \\ &u(t,0) = u(t,\pi) = 0, \quad u(t,x) = u(t + 2\pi, x), \end{aligned}$$

for $f(t,x)$ also 2π-periodic in t, and where $g: \mathbb{R} \to \mathbb{R}$ is monotone nondecreasing in u. We treat this problem as written in the form $Eu = Nu$, where E is the linear wave operator with the boundary conditions in (0), and $Nu = \pm g(u) + f(t,x)$. The problem has been treated by others recently (e.g. Brezis [3]) on the basis of much information on the partial differential operator E.

Our approach is based solely on functional analysis, and particularly on remarks we have made in [5-12]. Concerning g we assume that, for some $p > 2$ and constants k, $K > 0$ we have

$$(h_1)\quad (g(u) - g(v))(u-v) \geq k\,|u-v|^p, \qquad u,v \in \mathbb{R},$$

$$(h_2)\quad |g(u) - g(v)| \leq K(|u|^{p-2} + |v|^{p-2}), \qquad u,v \in \mathbb{R},$$

$$(h_3)\quad g(-u) = -g(u), \quad u \in \mathbb{R}.$$

Thus, g is continuous and locally Lipschitzian, odd and strictly increasing in $\mathbb{R}$. Note that $g(0) = 0$ and that (h_1), (h_2) for $v=0$ yield

$$(h_4)\quad k|u|^{p-1} \leq |g(u)| \leq K|u|^{p-1}, \quad u \in \mathbb{R}.$$

The case $p = 2$ will be treated in a subsequent paper.

Though g is monotone here, we have not used in the proofs theorems on monotone operators. Instead, we have used the monotonicity property of g to prove the relation $\langle QNu, u_o \rangle \geq 0$, and other analogous ones, which we have showed in previous papers [9-12] to imply the solvability of the problem through alternative considerations.

For a treatment of problem (0) for g not monotone we refer to [10-12] and particularly to [11,12].

In the past, problem (0) and analogous ones had been studied only as perturbation problems, that is, say, with g replaced by εg where ε is a "small" parameter. We just mention the work of O. Vejvoda and School (see, e.g. [27] also for further references), L. Cesari [5], L. DeSimon and G. Torelli [13], J. K. Hale [14] W. S. Hall [15], J. Kurzweil [16], H. Lovicarova [18], M. Nakao [20], P. Rabinowitz [22,23], M. Sova [24].

1. The eigenvalues of the wave operator E.

The eigenvalues and eigenelements of the linear operator $Eu = u_{tt}-u_{xx}$ with the boundary conditions stated in (0) are

$$\lambda_k = k^2-\ell^2,\quad e_{k\ell} = \langle \sin kx \cos\ell t,\ \sin kx \sin\ell t\rangle,\quad k=1,2,\cdots,\ \ell=0,1,2,\cdots.$$

In particular, the eigenvalues $\lambda_{ko} = k^2$ have eigenelements $e_{ko} = \langle \sin kx\rangle$, $k=1,2,\cdots$.

Let us consider the sets

$S = \langle \sin kx,\ k=1,2,\cdots;\ \sin kx \cos\ell t,\ \sin kx \sin\ell t,\ k,\ell=1,2,\cdots\rangle$

$S_0 = \langle \sin kx \cos kt,\ \sin kx \sin kt,\ k=1,2,\cdots\rangle$

$S_1 = \langle \sin kx,\ k=1,2,\cdots;\ \sin kx \cos\ell t,\ \sin kx \sin\ell t,\ k\neq\ell, k,\ell=1,2,\cdots\rangle$

Below we shall consider certain Banach spaces, say X and Y, and decompositions $X = X_0 + X_1$, $Y = Y_0 + Y_1$, where X, X_0, X are the completions of S, S_0, S_1 in the topology of X, and Y, Y_0, Y_1 are the completions of S, S_0, S_1 in the topology of Y. As it is well known, the functions $u_0 \in X_0$ [$\in Y_0$] are of the form $u_0(x,t) = p(x+t) - p(-x+t)$, where $p(\xi)$ is 2π-periodic in ξ.

2. Remarks on the linear wave equation

Let S denote the space of all real valued functions $\psi(t,x)$, $(t,x) \in \mathbb{R}^2$, which are odd in x, which are doubly 2π-periodic in t,x, and which are of class C^2 in $\mathbb{R}^2$. Then the operator Q defined by

$$(1)\qquad \psi(t,x) = Q\phi = \frac{1}{2\pi}\int_0^{2\pi}\phi(s,x+t-s)ds - \frac{1}{2\pi}\int_0^{2\pi}\phi(s,-x+t-s)ds$$

is a projection, (i.e., $Q^2 = Q$), and ψ is odd in x, is doubly 2π-periodic in t,x, is of class C^2 in $\mathbb{R}^2$, and ψ is of the form $\psi(t,x) = p(x+t) - p(-x+t)$, where $p(\xi)$ is 2π-periodic, and of class C^2 in $\mathbb{R}$. Since $\psi_{tt}-\psi_{xx} = 0$, we briefly say that

Q maps S into S_0, the kernel of E. We note that $Q(\psi-Q\psi) = Q\psi-Q\psi = 0$. Let $S_1 = (I-Q)S$. We know that the functions $\lambda \in S_1$, or $\lambda = \psi - Q\psi$, $\psi \in S$, are orthogonal to all elements of ker E, in other words,

$$\iint_G \lambda(t,x)\psi(t,x)\,dt\,dx = 0$$

for every $\lambda \in S_1 = (I-Q)S$, and all $\psi \in \ker E$ (Vejvoda [26]), Hale [15]). Now let S* denote the class of all functions u(t,x) of the form

$$(2)\qquad u(t,x) = \mathcal{K}\lambda = \frac{1}{2}\int_0^t\int_{x-t+\theta}^{x+t-\theta}\lambda(\theta,\xi)\,d\xi\,d\theta,\quad \lambda \in S_1.$$

Again from Vejvoda [26] we know that u(t,x) is odd in x, is doubly 2π-periodic in t,x, is of class C^2 in $\mathbb{R}$, with $u(t,0) = u(t,\pi) = 0$ and, everywhere in $\mathbb{R}^2$, $Eu = u_{tt}-u_{xx} = (t,x)$, $(t,x) \in \mathbb{R}^2$. The elements $u \in S^*$ may not be orthogonal to ker E, but the functions $u_1 \in S_1^*$ $(I-Q)S^*$, or $u_1 = u-Qu$, certainly have this property. Moreover (Vejvoda [26]),

$$(3)\qquad \begin{aligned} u_t &= \frac{1}{2}\int_0^t [\lambda(\theta,x+t-\theta) + \lambda(\theta,x-t+\theta)]d\theta,\\ u_x &= \frac{1}{2}\int_0^t [\lambda(\theta,x+t-\theta) - \lambda(\theta,x-t+\theta)]d\theta,\\ u_{tt}-u_{xx} &= \lambda, \end{aligned}$$

and these functions are of class C^2 in $\mathbb{R}^2$.

We shall now extend the sets S, S_1, S*, S_1^*. Let $1/p + 1/q = 1$, $p > 2$, $1 \leq q < 2$.

Let Y denote the space of all real valued functions $\phi(t,x)$, $(t,x) \in \mathbb{R}^2$, which are odd in x, which are doubly 2π-periodic in t,x, and which are of class $L_q(G)$ on $G = [0,\pi] \times [0,2\pi]$. Then Q still applies to the elements $\phi \in Y$ and the functions $\psi = Q\phi$, $\phi \in Y$, are defined almost everywhere in $\mathbb{R}^2$, are odd in x, are doubly 2π-periodic in t,x, are of class $L_q(G)$, and are of the form $\psi(t,x) = p(x+t)-p(-x+t)$, where now $p(\xi)$ is odd and 2π-periodic in $\mathbb{R}$ and of class $L_q[0,2\pi]$.

We say briefly that Q maps Y into ker E.

Let $Y_1 = (I-Q)Y$. The functions $\lambda \in Y_1$, or $\lambda = \psi - Q\psi$, $\psi \in Y$, have the same properties above, and moreover they are orthogonal to ker E in the sense that

$$\iint_G \lambda(t,x)\psi(t,x)dt\,dx = 0$$

for all $\psi \in \ker E \cap L_p(G)$, $1/p + 1/q = 1$, and all $\lambda \in Y_1$. All these statements are a consequence of the fact that we can approximate λ by elements $\lambda_s \in S_1$ of

class C^2 in the topology of L_q and the elements $\psi \in \ker E \cap L_p(G)$ by elements $\lambda_s \in \ker E \cap C^2$ in the topology of L_p, and then $0 = (\lambda_s,\psi_s) \to (\lambda,\psi)$ as $s \to \infty$, hence $(\lambda,\psi) = 0$.

For any element $\lambda \in Y_1$ take $u(t,x) = \mathfrak{K}\lambda$ by the same operation $\mathfrak{K}$ above. Again $u(t,x)$ is odd in x, doubly 2π-periodic in t,x, continuous in $\mathbb{R}^2$, and satisfies $Eu = \lambda$ in the weak sense, that is

$$\iint_G (uE\phi - \lambda\phi)\, dt\, dx = 0$$

for all $\phi \in Y$. Again, this can be seen by approximating, say, ϕ by elements $\phi_s \in S$ of class C^2 in the topology of $L_q(G)$, by approximating u by elements $u_s \in S^*$ of class C^2 in the topology of $L_p(G)$, and noticing that, by duality

$$\iint_G u_s(E\phi_s)\, dt\, dx = \iint_G (Eu_s)\phi_s\, dt\, dx = \iint_G \lambda_s\phi_s\, dt\, dx,$$

and $\iint_G \lambda_s\phi_s\, dt\, dx \to \iint_G \lambda\phi dt\, dx$ as $s \to \infty$. Let $Y^* = \mathfrak{K}Y_1$. The elements $u \in Y^*$ may not be orthogonal to ker E but the elements $u_1 \in Y_1 = (I-Q)Y^*$ have this property. Let $H = (I-Q)\mathfrak{K}$.

First we note that

$$\psi(t,x) = Q\phi = \frac{1}{2\pi}\int_0^{2\pi} \phi(s,x+t-s)ds - \frac{1}{2\pi}\int_0^{2\pi} \phi(s,-x+t-s)ds = \psi_1 - \psi_2$$

with

$$\int_0^T |\psi_1(t,x)|^q\, dx = \int_0^\pi dx \left|\frac{1}{2\pi}\int_0^{2\pi} \phi(s,x+t-s)ds\right|^q$$

$$\le \int_0^{2\pi} dx\, (2\pi)^{-q}\, ((2\pi)^{1/p}(\int_0^{2\pi} |\phi(s,x+t-s)|^q ds)^{1/q})^q$$

$$= (2\pi)^{-q}\, (2\pi)^{q/p} \iint_\Pi |\phi(s,z)|^q\, ds\, dz,$$

where Π is the parallelogram of vertices $(0, x+t)$, $0, x+\pi+t)$, $(2\pi, x+t-2\pi)$, $(2\pi, x+t-\pi)$. By periodicity then

$$\int_0^\pi |\psi_1(t,x)|^q\, dx \le (2\pi)^{-q}\, (2\pi)^{q/p} \iint_G |\phi(s,z)|^q\, ds\, dz$$

$$= (2\pi)^q\, (2\pi)^{q/p}\, \|\phi\|_q^q$$

$$\int_0^{2\pi}\int_0^\pi |\psi_1(t,x)|^q\, dx\, dt \le (2\pi)^{1-q}\, (2\pi)^{q/p}\, \|\phi\|_q^q$$

where $q = \frac{p}{p-1}$, $1-q = -\frac{1}{p-1}$ and $-\frac{1}{p-1} + \frac{q}{p} = -\frac{1}{p-1} + \frac{1}{p-1} = 0$

Thus

$$\int_0^{2\pi}\int_0^\pi |\psi_1(t,x)|^q\, dt\, dx \le \|\phi\|_q^q,$$

and $\|\psi_1\|_q \le \|\phi\|_q$. The same holds for ψ_2 so that $\|\psi\|_q \le 2\|\phi\|_q$, or $\|Q\phi\|_q \le 2\|\phi\|_q$.

If ϕ is bounded, we also have

$$\|\psi\|_\infty = \left\|\frac{1}{2\pi}\int_0^{2\pi} \phi(s,x+t-s)\, ds - \frac{1}{2\pi}\int_0^{2\pi} \phi(s,-x+t-s)ds\right\|_\infty$$

$$\le \left\|\frac{1}{2\pi}\int_0^{2\pi} |\phi(s,x+t-s)\, ds + \frac{1}{2\pi}\int_0^{2\pi} |\phi(s,-x+t-s)|ds\right\|_\infty$$

$$\le 2\|\phi\|_\infty .$$

or $\|Q\phi\|_\infty \le 2\|\phi\|_\infty$. Finally $\|(I-Q)\phi\|_q \le 3\|\phi\|_q$, $\|(I-Q)\phi\|_\infty \le 3\|\phi\|_\infty$.

We have now $u(t,x) = \mathfrak{K}\lambda = \frac{1}{2}\int_0^t \int_{x-t+\theta}^{x+t-\theta} \lambda(\theta,\xi)d\theta\, d\xi = \frac{1}{2}\iint_T \lambda(\theta,\xi)\, d\theta\, d\xi$, where T is the triangle of vertices $(0,x-t)$,$(0,x+t)$, (t,x) and area $|T| = |t|^2 \le 4\pi^2$. Thus, if λ is bounded we have $\|u\|_\infty \le (2\pi^2)\|\lambda\|_\infty$, and if $\lambda \in L_q$ also $|u(t,x)| = |\frac{1}{2}\iint_T \lambda(\theta,\xi)\, d\theta\, d\xi| \le \frac{1}{2}\, |T|^{1/p}(\iint_T |\lambda|^q\, d\theta\, d\xi)^{1/q}$, where the integral on T is $\le$ the integral on a rectangle twice the size of G. Thus $|u(t,x)| \le \frac{1}{2}\, (4\pi^2)^{1/p} \cdot 2^{1/q}\|\lambda\|_q = 2^{-1/p}(4\pi^2)^{1/p}\, \|\lambda\|_q$, $\|u\|_\infty \le (2\pi^2)^{1/p}\, \|\lambda\|_q$, or $\|\mathfrak{K}\lambda\|_\infty \le (2\pi^2)^{1/p}\, \|\lambda\|_q$.

We also have $\|u\|_p = (\iint_G |u|^p\, dt\, dx)^{1/p} \le (2\pi^2)^{1/p}\|u\|_\infty$, $\|u\|_p \le (2\pi^2)^{2/p}\, \|\lambda\|_q$, that is, $\|\mathfrak{K}\lambda\| \le (2\pi^2)^{2/p}\, \|\lambda\|_q$.

Since $H = (I-Q)\,\mathfrak{K}$, and $\|(I-Q)\mathfrak{K}\|_q \le 3\|\phi\|_q$, $\|(I-Q)\phi\|_\infty \le 3\|\phi\|_\infty$, we have

$$\|H\lambda\|_p \le 2(2\pi^2)^{2/p}\, \|\lambda\|_q,\quad \|H\lambda\|_\infty \le 3(2\pi^2)^{1/p}\, \|\lambda\|_q,$$

$$\|H(I-Q)\phi\|_p \le 3^2(2\pi^2)^{2/p}\, \|\phi\|_q, \tag{4}$$

$$\|H(I-Q)\phi\|_\infty \le 3^2(2\pi^2)^{1/p}\, \|\phi\|_q .$$

Relations (3) still hold, hence for $u = \mathfrak{K}\lambda$, we have $u_t, u_x \in L_q$, $u \in W^{1,q}(G)$, and

$$\|u_t\|_q,\ \|u_x\|_q \le 2\pi\|\lambda\|_q. \tag{5}$$

Moreover, $Eu = u_{tt} - u_x$ exists in the sense of distributions, and equals the function λ a.e. in G; hence,

$$\|Eu\|_q = \|u_{tt} - u_{xx}\|_q = \|\lambda\|_q.$$

3. Reduction of problem (0) to an alternative problem

Let X denote the Banach space of all functions $u(t,x)$, $(t,x) \in \mathbb{R}^2$, odd in x, doubly 2π-periodic in t,x, of class $L_p(G)$, $G = [0,\pi] \times [0, 2\pi]$, and norm $\|u\|_X = \|u\|_p$ in G.

Let Y denote the Banach space of all functions $\lambda(t,x)$, $(t,x) \in \mathbb{R}^2$, odd in x, doubly 2π-periodic in t,x, of class $L_q(G)$, and norm $\|\lambda\|_Y = \|\lambda\|_q$ in G.

Let $\langle\lambda,u\rangle$ denote the usual linear operation $\langle\lambda,u\rangle = \iint_G \lambda u \, dt\, dx$, $\lambda \in Y$, $u \in X$, so that $|\langle\lambda,u\rangle| \le \|\lambda\|_q \|u\|_p = \|\lambda\|_Y \|u\|_X$.

Let X_0 and X_1 be the closures of $S_0 = \ker E$ and S_1 in L_p, respectively, so that $X = X_0 + X_1$, and let $P: X \to X$ be the projection operator we denoted by Q in Section 1, and which now is thought of as applied to $X = L_p$, so that $PX = X_0$, $(I-P)X = X_1$.

Let Y_0 and Y_1 be the closures of S_0 and S_1 in L_q, respectively, so that $Y = Y_0 + Y_1$, and let $Q: Y \to Y$ be the same projection operator thought of as applied to $Y = L_q$, so that $QY = Y_0$, $(I-Q)Y = Y_1$.

For $u \in X$, hence L_p-integrable in G, we have, by (h_4)

$$
\begin{aligned}
&|g(u)| \le K|u|^{p-1} = K|u|^{p/q}, \\
&\|g(u)\|_q \le K\|u\|_p^{p/q} = K\|u\|_p^{p-1} .
\end{aligned} \tag{6}
$$

In this section the sign (+), or (-), in (0) is not relevant. We shall choose the sign (+). Let f be an element of Y; hence $f \in L_q(G)$. Let N denote the Nemitsky operator defined by $Nu = g(u) + f$ for $u \in Y$. Then, $N: X \to Y$, and in view of (4), (6) also $Nu \in Y$, $(I-Q)Nu \in Y_1$, $QNu \in Y_0$, and

$$
\begin{aligned}
&\|Nu\|_q \le \|g(U)\|_q + \|f\|_q \le K\|u\|_p^{p-1} + \|f\|_q \\
&\|H(I-Q)N(u)\|_p \le 3^2(2\pi^2)^{2/p}\,(K\|u\|_p^{p-1} + \|f\|_q) \\
&\|QN(u)\|_q \le 3(K\|u\|_p^{p-1} + \|f\|_q).
\end{aligned} \tag{7}
$$

In view of Section 2 we have reduced problem (0), or $Eu = Nu$, to the alternative form

$$
\begin{aligned}
&u_1 = H(I-Q)N(u_0+u_1), \\
&0 = QN(u_0+u_1),\ u = u_0+u_1,\ \ u_0 \in X_0,\ u_1 \in X_1.
\end{aligned}
$$

Let us prove that $N: X \to Y$ is a continuous operator. Indeed, for $u,v \in X$ we have, as before,

$$
\begin{aligned}
\|Nu-Nv\|_Y = \|Nu-Nv\|_q &= [\iint_G |g(u)-g(v)|^q \, dt\, dx]^{1/q} \\
&\le K\,[\iint_G |u-v|^q\,(|u|^{p-2} + |v|^{p-2})^q \, dt\, dx]^{1/q} \\
&\le K\,[(\iint_G |u-v|^{q\frac{p}{q}} \, dt\, dx)^{\frac{q}{p}} \cdot \iint_G (|u|^{p-2} + |v|^{p-2})^{\frac{qp}{p-q}} \, dt\, dx)^{\frac{p-q}{p}}]^{1/q} \\
&\le 2^{p-2} K \|u-v\|_p \,(\iint_G (|u|^p + |v|^p) dt\, dx)^{\frac{p-2}{p}} \\
&\le 2^{(p-2)(p+1)/p} K(\|u\|_p^{p-2} + \|v\|_p^{p-2}) \|u-v\|_p .
\end{aligned} \tag{8}
$$

As a consequence, the operator $H(I-Q)N$ is a continuous operator $X \to X_1$.

Moreover, for $u = u_0 + u_1$, $u_0 \in \bar{S}_0$, $u_1 \in S_1$, we have $u \in X = L_p(G)$, $g(u) \in L_q(G)$, and $Nu = g+f \in L_q(G)$, and

$$
\begin{aligned}
\langle QNu,\, u_0\rangle &= \langle Nu, u_0\rangle = \iint_G [g(u_0+u_1)+f]u_0 \, dt\, dx \\
&= \iint_G [g(u_0+u_1)-g(u_1)]u_0 \, dt\, dx + \iint_G g(u)_1)u_0 \, dt\, dx + \iint_G f u_0 \, dt\, dx \\
&\ge k\|u_0\|_p^p - \|u_0\|_p[\|g(u_1)\|_q + \|f\|_q] \\
&\ge k\|u_0\|_p^p - \|u_0\|_p[K\|u_1\|_p^{p-1} + \|f\|_q]
\end{aligned}
$$

Thus, for $\|u_0\|_p \ge \mu_0$, $\|u_1\|_p \le \mu$, and $\|f\|_q \le c_0$, we have

$$
\langle QNu, u_0\rangle \ge k\,\mu_0^p - \mu_0\,[K\mu^{p-1} + c_0]
$$

and finally

$$
\langle QNu, u_0\rangle \ge 0 \text{ for } \|u_0\|_p \ge R, \|u_1\| \le r, \text{ and } R \ge k^{-\frac{1}{p-1}} (Kr^{p-1} + c_o)^{\frac{1}{p-1}}. \tag{9}
$$

Note that, for $\bar{u}_1 = H(I-Q)Nu$, we have $\bar{u}_1 \in L_\infty$, $\|\bar{u}_1\|_\infty \le$ $\le 3^2(2\pi^2)^{1/p}[K\|u\|_p^{p-1} + \|f\|_q]$, and moreover $\bar{u}_1 \in W^{1/q}(G)$ with

$$
\begin{aligned}
&\|\bar{u}_1\|_p \le 3^2(2\pi^2)^{1/p}\, K[\|u_0\|_p^{p-1} + \|f\|_q],\ \|\bar{u}_1\|_\infty \le 3^2(2\pi^2)^{1/p}[K\|u\|_p^{p-1} + \|f\|_q], \\
&\bar{u}_{1t},\ \bar{u}_{1q} \in L_q,\ \|\bar{u}_{1t}\|_q,\ \|\bar{u}_{1x}\|_q \le 2\pi K\|u\|_p^{p-1} ,
\end{aligned} \tag{10}
$$

and $E\bar{u}_1 = u_{1tt} - u_{1xx}$ in the sense of distributions is a function in L_q with

$$
\|E\bar{u}_1\|_q = \|\bar{u}_{1tt} - \bar{u}_{1xx}\|_q \le \|(I-Q)Nu\|_q \le 3[K\|u\|_p^{p-1} + \|f\|_q]. \tag{11}
$$

We now denote by $\tilde{X}$ the Banach space of all functions $u(t,x)$, $(t,x) \in \mathbb{R}^2$, odd in x, doubly 2π-periodic in t and x, of class $L_p(G)$, whose derivatives

$u_t, u_x, u_{tt}-u_{xx}$ in the sense of distributions are functions of class $L_q(G)$. We can take in $\tilde{X}$ any of the norms

$$|||u||| = \alpha\|u\|_p + \beta(\|u_t\|_q + \|u_x\|_q) + \|u_{tt}-u_{xx}\|_q, \text{ or}$$

$$|||u||| = \beta(\|u_t\|_q + \|u_x\|_q) + \gamma\|u_{tt}-u_{xx}\|_q.$$

where α,β,γ are positive constants. Note that $\tilde{X} \subset X$ and that the injection operation $j: \tilde{X} \to X$ is compact. Also note that, by the usual decomposition $\tilde{X} = \tilde{X}_0 + \tilde{X}_1$, $\tilde{X}_0 \subset X_0$, $\tilde{X}_1 \subset X_1$, we see that $Eu = u_{tt}-u_{xx} = 0$ for every element $u \in \tilde{X}_0$, and that we can take in $\tilde{X}_0$ the simpler norm

$$\|u\|^* = 2^{-1}(\|u_t\|_q + \|u_x\|_q) \text{ for } u \in X_0.$$

Note that for $\bar{u}_1 = H(I-Q)Nu$, $u \in X$, we have $\bar{u}_1 \in \tilde{X}$.

Let us study further properties of the space $\tilde{X}$.

First, we may think of $\tilde{X}$ as the completion of the class S of all functions $u(t,x)$, $(t,x) \in \mathbb{R}^2$, odd in x, doubly 2π-periodic in t,x, of class C^2 in $\mathbb{R}^2$, with respect to the norm $|||\ |||$.

Note that for any function $u \in \tilde{X}$, then $u(\bar{t},x)$ is an AC function of x for a.a. $\bar{t}$, and $u(t,\bar{x})$ is an AC function of t for a.a. $\bar{x}$, and

$$(12) \qquad u(\bar{t},x_2)-u(\bar{t},x_1) = \int_{x_1}^{x_2} u_x(\bar{t},\eta)d\eta, \quad u(t_2,\bar{x})-u(t_1,\bar{x}) = \int_{t_1}^{t_2} u_t(\xi,\bar{x})d\xi.$$

We need another property of the functions $u \in \tilde{X}$.

We consider the transformation $\xi = x+t$, $\eta = -x+t$, $x = 2^{-1}(\xi-\eta)$, $t = 2^{-1}(\xi+t)$, and the functions $\bar{u}(\xi,\eta) = u(2^{-1}(\xi-\eta),2^{-1}(\xi+\eta))$ for $u \in X$. Then the functions $\bar{u}(\xi,\eta)$ are 4π-periodic in ξ,η, are of class $L_p(G')$ with $G' = [0,4\pi] \times [0,4\pi]$, and the partial derivatives, in the sense of distributions $\bar{u}_\xi$, $\bar{u}_\eta$, $\bar{u}_{\xi\eta}$ exist a.e. in G', and are of class $L_q(G)$. Moreover, as before, $\bar{u}(\xi,\bar{\eta})$ is AC in ξ for a.a. $\bar{\eta}$, and $\bar{u}(\bar{\xi},\eta)$ is AC in η for a.a. $\bar{\xi}$, with the relations analogous to (12), and the partial derivatives $\bar{u}_\xi$, $\bar{u}_\eta$, u_x, u_t are related by the usual relations a.e. in G.

Finally, we can prove that any function $u \in \tilde{X}$ is continuous. We shall indeed prove that $\bar{u}(\xi,\eta)$ is continuous. First we note that there must be a set Σ of positive measure of points $\eta \in [0,4\pi]$ where

$$(13) \qquad \int_0^{4\pi} |u_\xi(\alpha,\eta)|^q d\alpha \le \frac{1}{4\pi} \int_0^{4\pi}\int_0^{4\pi} |u_\xi(\alpha,\beta)|^q d\alpha d\beta = (4\pi)^{-1}\|u_q\|_q^q.$$

Indeed, in the opposite case, we would have the relation $>$ for a.a. η, and then by integration with respect to η, also

$$\int_0^{4\pi}\int_0^{4\pi} |u_\xi(\alpha,\eta)|^q \, d\alpha d\eta > \int_0^{4\pi}\int_0^{4\pi} |u_\xi(\alpha,\beta)|^q \, d\alpha d\beta$$

a contradiction. Now we can take from Σ a point $\bar{\eta}$ such that (13) holds, and such that $u(\xi,\bar{\eta})$ is an AC function of ξ with derivative $u_\xi(\xi,\bar{\eta})$ for a.a. ξ. We have now for any two ξ, $\xi+h \in [0,4\pi]$, and $\eta \in [0,4\pi]$:

$$|\bar{u}(\xi,\eta) - \bar{u}(\xi+h,\eta)| =$$

$$|[\bar{u}(\xi,\bar{\eta}) - \bar{u}(\xi+h,\bar{\eta})] + [\bar{u}(\xi,\eta) - \bar{u}(\xi+h,\eta) - \bar{u}(\xi,\bar{\eta}) + \bar{u}(\xi+h,\bar{\eta})]|$$

$$= \left|\int_\xi^{\xi+h} \bar{u}_\xi(\alpha,\bar{\eta})d\alpha + \int_\xi^{\xi+h}\int_{\bar{\eta}}^{\eta} \bar{u}_{\xi\eta}(\alpha,\beta)d\alpha d\beta\right|$$

$$\le |h|^{1/p}\left(\int_\xi^{\xi+h} |\bar{u}_\xi(\alpha,\bar{\eta})|^q d\alpha\right)^{1/q} + (|h|\,|\eta-\bar{\eta}|)^{1/p}\left(\int_\xi^{\xi+h}\int_{\bar{\eta}}^{\eta} |\bar{u}_{\xi\eta}|^q d\alpha d\beta\right)^{1/q}$$

$$\le \left(\frac{1}{4\pi}\|\bar{u}_\xi\|_q + (4\pi)^{1/p}\|\bar{u}_{\xi\eta}\|_q\right)|h|^{1/p}$$

Analogously we prove that

$$|\bar{u}(\xi,\eta) - \bar{u}(\xi,\eta+k)| \le \left((4\pi)^{-1}\|\bar{u}_\eta\|_q + (4\pi)^{1/p}\|\bar{u}_{\xi\eta}\|_q\right)|k|^{1/q}$$

We have proved that the functions $\bar{u} \in \tilde{X}$ not only are continuous, but they also have a modulus of continuity which depends only on $\|u\|_{\tilde{X}}$.

As a consequence, the injection $j: \tilde{X} \to X$ is a compact operation. Indeed, for any sequence $[u_n]$ of elements $u_n \in \tilde{X}$ with $|||u_n||| \le M$, the same functions u_n have a common modulus of continuity, hence they are equicontinuous, and certainly equibounded. By Ascoli's theorem there is a subsequence $[u_{n_k}]$ which converges uniformly, and then certainly strongly in $L_p(G)$.

Let D denote the class of all functions $y(t,x)$, odd in t, 2π-periodic in t,x, of class $L_q(G)$, with partial derivatives y_t, y_x in the sense of distributions of class $L_p(G)$. Let us prove that for every element $u_0 \in \tilde{X}_0$ and $y \in D$, we have

$$(16) \qquad \langle u_{0t}, y_t\rangle = \langle u_{0x}, y_x\rangle.$$

Indeed, for functions u, y of class C^2 we have, because of the boundary conditions and by integration by parts,

$$< u_{0t}, y_t > = < u_{0tt}, y > = < u_{0xx}, y > = < u_{0x}, y_x > .$$

Thus, (16) is true for u,y of class C^2. By a passage to the limit in L_p and L_q respectively we obtain the same relation (16) under the stated conditions.

We now define an operator $J: Y_0 \to X_0$ as follows. First, we note that any element $\lambda_0(x,t) \in Y_0$ is of the form $\lambda_0(x,t) = v(x+t)-v(-x+t)$, with λ_0 odd in x, doubly 2π-periodic in t,x, of class $L_q(G)$, and $v(\xi)$ 2π-periodic, of class $L_q[x,2\pi]$. Let us prove that $\int_0^{2\pi} v(\xi)d\xi = 0$.

It is enough to verify this point for the particular functions below. Indeed, for $\lambda_0 = \sin kx \cos kt = 2^{-1}(\sin k(x+t)-\sin k(-x+t))$, then $v(\xi) = 2^{-1}\sin k\xi$, $\int_0^{2\pi} v\, d\xi = 0$; for $\lambda_0 = \sin kx \sin kt = -2^{-1}(\cos k(x+t)-\cos k(-x+t))$, then $v(\xi) = -2^{-1}\cos k\xi$, $\int_0^{2\pi} v\, d\xi = 0$, and this holds for all $k=1,2,\cdots$. Thus, $\int_0^{2\pi} v\, d\xi = 0$ for every element v in the completion in L_q of the same functions. We take now

$$u_0 = J\lambda_0 = V(x+t)-V(-x+t), \text{ for } \lambda_0 = v(x+t)-v(-x+t),$$

$$V(\xi) = \int_0^{\xi} v(\eta)d\eta - c, \quad c = (2\pi)^{-1}\int_0^{2\pi} d\xi \int_0^{\xi} v(\eta)d\eta.$$

Now V is 2π-periodic, with $\int_0^{2\pi} V(\xi)d\xi = 0$, V is continuous in $(-\infty,+\infty)$, AC in $[0, 2\pi]$, and u_0 is bounded, continuous, odd in x, doubly 2π-periodic in t,x, and with (usual) partial derivatives $u_{0t} = v(x+t)-v(-x+t)$, $u_{0x} = v(x+t)+v(-x+t)$, $u_{0t}, u_{0x} \in L_q(G)$. Of course $Eu_0 = 0$ in the sense of distributions. We note that $u_{0t} = \lambda_0$ a.e. in G.

Note that J is a linear operator with the property that $J^{-1}(0) = 0$. Indeed, if $u_0 \equiv 0$, hence, for $\xi = x+t$, $\eta = -x+t$, also $V(\xi) = V(\eta)$ for all ξ and η, thus $V \equiv$ constant, and since $\int_0^{2\xi} V d\xi = 0$, also $V \equiv 0$ and $v \equiv 0$.

We shall actually need the iterate operation J^2.

To estimate the operator $\lambda_0 \to J^2\lambda_0$ note that, for $x+t = \xi$, $-x+t = \eta$, $x = 2^{-1}(\xi-\eta)$, $t = 2^{-1}(\xi+\eta)$, we have

$$v(\xi) - v(\eta) = \lambda_0(2^{-1}(\xi-\eta), 2^{-1}(\xi+\eta))$$

and by integration with respect to η over $[0,2\pi]$ also

$$v(\xi) = (2\pi)^{-1}\int_0^{2\pi}\lambda_0(2^{-1}(\xi-\eta), 2^{-1}(\xi+\eta))\, d\eta ,$$

$$|v(\xi)| \le (2\pi)^{-1}\int_0^{2\pi}|\lambda_0(2^{-1}(\xi-\eta), 2^{-1}(\xi+\eta))|d\eta ,$$

$$\int_0^{2\pi}|v(\xi)|d\xi \le (2\pi)^{-1}\int_0^{2\pi}\int_0^{2\pi}|\lambda_0(2^{-1}(\xi-\eta), 2^{-1}(\xi+\eta))|d\eta\, d\xi$$

$$= 2^{-1}(2\pi)^{-1}\iint_{\Pi}|\lambda_0(x,t)|dx\, dt,$$

where Π is the parallelogram of vertices (0,0), (π, π), $(-\pi,\pi)$, $(0,2\pi)$ in the xt-plane. Hence, v is certainly L_1-integrable with

$$\int_0^{2\pi}|v(\xi)|dt \le (2\pi)^{-1}\iint_G|\lambda_0(x,t)|dx\, dt \le (2\pi)^{-1}(2\pi^2)^{1/p}\|\lambda_0\|_q.$$

Thus, $V(\xi) = \int_0^{\xi} v(\eta)d\eta - c$ is AC, with

$$|c| \le (2\pi)^{-1}\int_0^{2\pi} d\alpha \int_0^{\alpha}|v(\xi)|d\xi \le \int_0^{2\pi}|v(\xi)|d\xi \le (2\pi)^{-1}(2\pi^2)^{1/p}\|\lambda_0\|_q,$$

$$\|V(\xi)\|_\infty \le \int_0^{2\pi}|v(\xi)|dt + |c| \le 2(2\pi)^{-1}(2\pi^2)^{1/p}\|\lambda_0\|_q = 2^{\frac{1}{p}}\pi^{-1+\frac{2}{p}}\|\lambda_0\|,$$

Actually, v is also of class L_q with

$$\int_0^{2\pi}|v(\xi)|^q d\xi \le (2\pi)^{-1}\int_0^{2\pi}[\int_0^{2\pi}|\lambda_0(2^{-1}(\xi-\eta), 2^{-1}(\xi+\eta))|d\eta]^q d\xi$$

$$\le (2\pi)^{-1}(2\pi)^{q/p}\int_0^{2\pi}\int_0^{2\pi}|\lambda_0(2^{-1}(\xi-\eta), 2^{-1}(\xi+\eta))|^q d\xi\, d\eta$$

$$= (2\pi)^{-1}(2\pi)^{q/p}\iint_{\Pi}|\lambda_0(x,t)|^q dx\, dt \le 2^{-1}(2\pi)^{-1+q/p}\iint_G|\lambda_0(x,t)|^q dx\, dt,$$

or

$$\|v\|_q \le 2^{-1/q}(2\pi)^{-1/q+1/p}\|\lambda_0\|_q.$$

Hence, the L_q norm of V can be estimated as follows:

$$\|V\|_q \le \|\int_0^{\xi} v(\eta)d\eta\|+\|c\|_q \le \|\int_0^{2\pi}|v(\eta)|d\eta\|_q + \|c\|_q$$

$$= (2\pi)^{1/q}[\int_0^{2\pi}|v(\eta)|d\eta + |c|]$$

$$\le 2(2\pi)^{1/q}[(2\pi)^{-1}(2\pi^2)^{1/p}\|\lambda_0\|_q]$$

$$= 2\pi^{1/p}\|\lambda_0\|_q$$

On the other hand, from (8), we have

$$\|H(I-Q)Nu-H(I-Q)Nv\|_p \le 3^2(2\pi^2)^{2/p}\|Nu-Nv\|_q \tag{17}$$

$$\le 3^2(2\pi^2)^{2/p}\,2^{(p-2)(p+1)/p}K(\|u\|_p^{p-2}+\|v\|_p^{p-2})\|u-v\|_p.$$

In particular, for $u = u_0+u_1$, $v = u_0+u_1'$, $u_0 \in X_0$, $u_1, u_1' \in X_1$, $\|u_0\|_p \le R$, $\|u_1\|_p$, $\|u_1'\|_p \le r$, we have

$$\|H(I-Q)N(u_0+u_1) - H(I-Q)N(u_0+u_1')\|_p \le 3^2\, 2^{(p-2)(p+1)/p}(2\pi^2)^{2/p} K(R+r)^{p-2}\|u_1-u_1'\|_p.$$

Thus, for

(18) $$3^2 \cdot 2^{1+(p-2)(p+1)/p}(2\pi^2)^{2/p}(R+r)^{p-2}K < 1$$

the transformation $H(I-Q)N(u_0+u_1)$ is a contraction map with respect to u_1 in the topology of L_p. But we shall not need this restriction.

From (11) we have

$$\|H(I-Q)N(u_0+u_1)\|_p \le 3^2(2\pi^2)^{2/p}(K\|u_0+u_1\|_p^{p-1}+\|f\|_q) \le$$
$$\le 3^2(2\pi^2)^{2/p}(K(R+r)^{p-1} + c_o)$$

and this is $< r$ provided

(19) $$3^2(2\pi^2)^{2/p}(K(R+r)^{p-1} + c_o) < r$$

For $W_0(\xi) = \int_0^\xi V(\eta)d\eta$ we have then

$$|W_0(\xi)| \le \int_0^{2\pi}|V(\eta)|d\eta = 2(2\pi)^{-1}(2\pi^2)^{1/p}\ \|\lambda_0\|_q.$$

Finally, $U_0 = J^2\lambda_0 = W_0(x+t) - W_0(-x+t)$ is a continuously differentiable function, $U_0 \in \tilde{X}_0$, and

$$|U_0| = |J^2\lambda_0| = |W_0(x+t) - W_0(x-t)| \le 4(2\pi)^{-1}(2\pi^2)^{1/p}\ \|\lambda_0\|_q$$

or

$$\|J^2\lambda_0\|_\infty \le 2^{1+1/p}\pi^{-1+2/p}\ \|\lambda_0\|_q,$$

$$\|(J_0\lambda_0)_t\|_\infty = \|(W_0(x+t) - W_0(-x+t))_t\|_\infty = \|V(x+t)-V(x+t)\|_\infty \le 2^{1+1/p}\pi^{-1+2/p}\|\lambda_0\|_q$$

$$\|(J_0\lambda_0)_x\|_\infty = \|(W_0(x+t) - W_0(-x+t))_t\|_\infty = \|V(x+t)+V(-x+t)\|_\infty \le 2^{1+1/p}\pi^{-1+2/p}\|\lambda_0\|_q.$$

Finally, in the sense of distributions, $E(J^2\lambda_0) = 0$.

A bilinear operation on $\tilde{X}_0$ is the following one:

$$\{u_0,v_0\} = \langle u_{0t}, v_{0t}\rangle + \langle u_{0x}, v_{0x}\rangle$$

so that, $\{u_0,u_0\} \ge 0$ and $\{u_0,u_0\} = 0$ if and only if $u_0 = 0$.

Also we note that, for $\|u_0\|_p \le R$, $\|u_1\|_p \le r$, $\|f\|_q \le c_0$, we have from (10)

$$\|H(I-Q)N(u_0+u_1)\|_p \le 3^2\,(2\pi^2)^{2/p}\,[K(R+r)^{p-1} + c_0],$$

$$\|J^2QN(u_0+u_1)\|^* = \|(J^2QN(u_0+u_1)_t\| + \|(J^2QN(u_0+u_1)_x\| \le 2^{2+1/p}\pi^{-1+2/p}[K(\mu_0+\mu)^{p-1}+c_0]$$

$$\{J^2QN(u_0+u_1),u_0\} = \langle (J^2QN(u_0+u_1))_t, u_{0t}\rangle + \langle (J^2QN(u_0+u_1))_x, u_{0x}\rangle$$
$$= \langle JQN(u_0+u_1), u_{0t}\rangle + \langle JQN(u_0+u_1), u_{0x}\rangle$$
$$= 2\langle QN(u_0+u_1), u_0\rangle$$

and finally, from (15) and for $\|u_0\| = R$, $\|u_1\|_p \le r$,

we also have

$$\{J^2QN(u_0+u_1),u_0\} \ge 2(kR^p - R[Kr^{p-1}+c_0]) \ge 0$$

provided

(20) $$R \ge k^{-\frac{1}{p-1}}\,(Kr^{p-1}+c_0)^{\frac{1}{p-1}}$$

We assume now that the constants R, r are chosen so that relations (19), (20) hold. Let

$$S_0 = [u_0 \in X_0 \mid \|u_0\|_p \le R],\quad S_1 = [u_1 \in X_1 \mid \|u_1\| \le r],\ \Omega = S_0 + S_1,$$

and let $T: \Omega \to X$ denote the map

$$Tu = Pu + H(I-Q)Nu - J^2QNu,\ u = u_0+u_1,\ u_0 \in S_0,\ u_1 \in S_1.$$

Now problem (10), (11) is reduced to the fixed point problem $u = Tu$.

By the Leray-Schauder theory we shall probe for solutions of the equation $u = \lambda Tu$, or $(I-\lambda T)$ for $u \in \partial\Omega$, where $0 < \lambda < 1$, and

$$(I-\lambda T)u = u - \lambda Pu - \lambda H(I-Q)Nu + \lambda J^2QNu$$
$$= (1-\lambda)u_0 + u_1 - \lambda H(I-Q)Nu + \lambda J^2QNu\ .$$

For solutions of $(I-\lambda T)\ u = 0$ we need that

(21) $$\begin{cases} u_1 = \lambda H(I-Q)N(u_0+u_1), \\ (1-\lambda)u_0 + \lambda J^2QN(u_0+u_1) = 0\ . \end{cases}$$

We have seen that for fixed u_0, $\|u_0\|_p \le R$, the map $u_1 \to \lambda H(I-Q)N(u_0+u_1)$ maps any u_1 with $\|u_1\|_p = r$ into an $\bar{u}_0 = \lambda H(I-Q)N(u_0+u_1)$ with $\|\bar{u}_0\|_p < r$. Thus, system (21) has no solution for $\|u_1\|_p = r$, $\|u_0\|_p \le R$. For $\|u_1\| \le r$, $\|u_0\| = R$, we may well solve the first equation (21) by Banach's fixed point theorem obtaining some $u_1 = \psi(u_0)$, $\|u_0\| = R$, $\|u_1\| \le r$. The second equation (21) becomes

$$(1-\lambda)u_0 + \lambda J^2QN(u_0+\psi(u_0)) = 0,$$

where $1 - \lambda > 0$, $\lambda > 0$. We first note that if this equation had a solution u_0 with $\|u_0\| = R$; then u_0 should be of class X_0, so that by application on u_0 we would have

$$(1-\lambda)\{u_0,u_0\} + \lambda\{J^2QN(u_0+u(u_0)),\ u_0\} = 0$$

where $\{u_0,u_0\} \geq 0$ and $\{J^2QN(u_0+\psi(u_0)),u_0\} \geq 0$, a contradiction. We have proved that $(I-\lambda T)u \neq 0$ on $\partial\Omega$ for all $0 < \lambda < 1$; hence the equation $u = Tu$ has at least one solution $u = u_0+u_1 \in \Omega$, and necessarily $u_0 \in X$, $u_1 \in X$, hence $u \in \tilde{X}$.

Let us prove that conditions (18), (19), (20) are compatible. First we note that $(\alpha+\beta)^\gamma \leq 2^\gamma(\alpha^\gamma+\beta^\gamma)$ for all α, β, $\gamma > 0$. Therefore, it is enough we satisfy the stronger inequalities

$$3^2(2\pi^2)^{2/p}\ (2^{p-1}KR^{p-1}+2^{p-1}Kr^{p-1}+c_0) \leq r,$$

$$R \geq 2^{\frac{1}{p-1}}(k^{-1}K)^{\frac{1}{p-1}} r + 2^{\frac{1}{p-1}} k^{-\frac{1}{p-1}} c_0 . \tag{22}$$

First, we may require

$$r \leq B \text{ with } B = (3^{-3}(2\pi^2)^{-2/p}K^{-1}2^{-(p-1)})^{\frac{1}{p-2}}, \tag{23}$$

$$R \leq C r^{\frac{1}{p-1}} \text{ with } C = (3^{-3}(2\pi^2)^{-2/p}K^{-1}2^{-(p-1)})^{\frac{1}{p-1}}, \tag{24}$$

$$c_0 \leq Dr \text{ with } D = 3^{-3}(2\pi^2)^{-2/p}, \tag{25}$$

so that first relation (22) is certainly satisfied. On the other hand, we write the second relation (22) in the form

$$R \geq Er + Fc_0 \text{ with } E = 2^{\frac{1}{p-1}}(k^{-1}K)^{\frac{1}{p-1}},\ F = 2^{\frac{1}{p-1}} k^{-\frac{1}{p-1}}. \tag{26}$$

Note that the two curves $R = Cr^{\frac{1}{p-1}}$ and $R = Er$ in the Rr-plane meet at the point $r_0 = (CE^{-1})^{(p-1)/(p-2)}$, $R_0 = C(CE^{-1})^{\frac{1}{p-2}}$. Thus, if we take $r < r_0$, then $Cr^{\frac{1}{p-2}} > Er$. Thus, for

$$r = \min [B,(CE^{-1})^{(p-1)/p-2)}],\ R = Cr^{\frac{1}{p-1}} \tag{27}$$

both relations (23), (24) are satisfied, with $R > Er$, and we satisfy both (25) and (26) by taking

$$c_0 = \min [Dr,\ F^{-1}(R-Er)].$$

We have proved the statement:

Theorem. Under conditions (h_{123}), $p > 2$, $1/p + 1/q = 1$, f odd in x, doubly 2π-periodic in t,x, $\|f\|_q \leq c_0$ for c_0 sufficiently small, problem (0) (either sign) has always at least one weak solution $u \in \tilde{X}$, continuous in $\mathbb{R}^2$, odd in x, doubly 2π-periodic in t, x, with $u(\bar{t},.)$ AC for a.a. $\bar{t}$, $u(\cdot,\bar{x})$ AC for a.a. $\bar{x}$, with usual derivatives u_t, u_x a.e., $u_t,u_x \in L_q(G)$, and distributional $Eu = g(u) + f \in L_q(G)$.

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In questi ultimi anni sono stati studiati vari problemi di Calcolo delle Variazioni del tipo seguente: fissata una successione $\{F_h\}_{h\in N}$ di funzionali, trovare un funzionale F_∞ tale che per ogni perturbazione G appartenente ad una famiglia $\mathcal{G}$ abbastanza ampia risulti

$$(1) \qquad \lim_{h\to\infty} \min\,(F_h(u) + G(u)) = \min\,(F_\infty(u) + G(u)).$$

Per esempio, dati $a,b \in R$ con $a<b$ ed una funzione reale g continua in $[a,b] \times R$ e verificante la condizione

esiste $c>0$ tale che $g(x,u) \geq |u|-c$,

risulta

$$\lim \min \int_a^b (2 + \frac{\operatorname{sen}(hx)}{|\operatorname{sen}(hx)|})(\frac{du}{dx})^2 + g(x,u)\,dx = \min\, 3/2 \int_a^b (\frac{du}{dx})^2 + g(x,u)\,dx.$$

In questo esempio è interessante notare che $F_\infty(u) = 3/2 \int_a^b (\frac{du}{dx})^2\,dx$ non coincide con il limite puntuale:

$$\lim_{h\to\infty} F_h(u) = \lim \int_a^b (2 + \frac{\operatorname{sen}(hx)}{\operatorname{sen}(hx)})(\frac{du}{dx})^2 dx = 2 \int_a^b (\frac{du}{dx})^2 dx.$$

Questa situazione in cui il funzionale F non coincide con il limite ordinario dei funzionali F_h è tipica dei problemi di omogeneizzazione, problemi che oltre all'interesse teorico hanno anche un notevole interesse applicativo essendo collegati allo studio delle proprietà dei materiali composti. Esistono anzi molti casi in cui la differenza tra il limite dei funzionali F_h e il funzionale F_∞ non consiste solo in un fattore numerico ma coinvolge in modo molto più complesso tutta la struttura del funzionale.

Se per esempio consideriamo un aperto limitato Ω contenuto in R^n e poniamo

$$(2) \qquad F_h(u) = \int_\Omega \tau(hx)|Du|^2 dx$$

ove Du indica il gradiente di u, essendo τ una funzione reale definita in R^n misurabile, periodica e compresa tra due costanti positive, si può dimostrare che esiste una forma quadratica

$\Sigma a_{hk}\,\lambda_h\,\lambda_k$ definita positiva tale che posto

$$F_\infty(u) = \int_\Omega \Sigma\, a_{hk}\, \frac{\partial u}{\partial x_h}\, \frac{\partial u}{\partial x_k}\, dx \tag{3}$$

vale la (1) per ogni G del tipo: $\int_\Omega g(x,u)\,dx$ con g funzione reale continua definita in $\overline{\Omega} \times \mathbb{R}$ e verificante la condizione: $g(x,u) \geq |u|-c$.

E' stato dimostrato che data comunque una forma quadratica definita positiva $\Sigma a_{hk}\,\lambda_h\,\lambda_k$ si può trovare una funzione τ per cui dalla (2) segua la (3) (v.[275]).

Questo risultato ha un'interessante interpretazione fisica: precisamente corrisponde al fatto che un materiale con caratteristiche anisotrope può essere ottenuto attraverso una composizione di diversi materiali con caratteristiche isotrope.

Per una più ampia discussione dei problemi fisici e tecnici connessi alla teoria della omogeneizzazione rinviamo ad esempio a [356].

Da un punto di vista teorico notiamo che i funzionali F_∞ considerati negli esempi precedenti possono essere ottenuti come Γ-limite dei funzionali F_h ed anzi che dalla riflessione sui problemi di omogeneizzazione è nata la teoria generale dei Γ-limiti.

Di questa teoria daremo soltanto qualche definizione e qualche risultato particolarmente interessante per le applicazioni al Calcolo delle Variazioni.

Consideriamo perciò uno spazio topologico S, una successione di insiemi $\{E_h\}$ contenuti in S e densi in S, una successione $\{F_h\}$ di funzioni reali estese definite in tali insiemi. Porremo, per ogni $x \in S$:

$$\Gamma(S^-)\ \operatorname*{minlim}_{h\to\infty} F_h(x) = \sup_{A \in I_x}\ \operatorname*{minlim}_{h\to\infty}\ \inf_{y \in A} F_h(y)$$

$$\Gamma(S^-)\ \operatorname*{maxlim}_{h\to\infty} F_h(x) = \sup_{A \in I_x}\ \operatorname*{maxlim}_{h\to\infty}\ \inf_{y \in A} F_h(y)$$

ove I_x denota la famiglia degli intorni di x in S.

Se il $\Gamma(S^-)\ \operatorname*{minlim}_{h\to+\infty} F_h(x)$ coincide con $\Gamma(S^-)\ \operatorname*{maxlim}_{h\to+\infty} F_h(x)$ allora il valore comune viene indicato con il simbolo $\Gamma(S^-)\ \lim_{h\to\infty} F_h(x)$.

E' da notare che nei due esempi precedenti si ha rispettivamente che:

$$F_\infty(u) = \Gamma(L^2(a,b)^-)\ \lim_{h\to\infty} F_h(u)$$

$$F_\infty(u) = \Gamma(L^2(\Omega)^-)\ \lim_{h\to\infty} F_h(u)$$

ove L^2 indica l'usuale spazio delle funzioni misurabili secondo Lebesgue con quadrato sommabile.

Le proprietà del Γ-limite che hanno maggior interesse per il Calcolo delle Variazioni sono le seguenti:

PROPOSIZIONE 1: Se A_h è una successione crescente di aperti di S ed

$$F_\infty(x) = \Gamma(S^-)\ \lim_{h\to\infty} F_h(x)$$

allora vale

$$\inf_{x \in A_\infty} F(x) \geq \operatorname*{maxlim}_{h\to\infty}\ \inf_{u \in A_h \cap E_h} F_h(u)$$

ove $A_\infty = \bigcup_{h=1}^{\infty} A_h$.

PROPOSIZIONE 2: Se $K_h \subseteq E_h$ è una successione decrescente di compatti non vuoti e $K = \bigcap_{h=1}^{\infty} K_h$ e $F_\infty = \Gamma(S^-)\ \lim F_h$ allora vale

$$\min_K F \leq \operatorname*{minlim}_{h\to\infty}\ \inf_{K_h} F_h$$

Dalle due proposizioni discende il seguente

COROLLARIO: Se $x_h \to x_\infty$ in S, $F_\infty = \Gamma(S^-)\ \lim_{h\to\infty} F_h$

$$\lim_{h\to\infty} F_h(x_h) = \lim_{h\to\infty}\ \inf F_h$$

allora

$$\min_S F_\infty = F_\infty(x_\infty) = \lim_{h\to\infty} F_h(x_h) \tag{4}$$

In particolare la (4) ci dice che i limiti di punti di minimo sono punti di minimo della funzione limite nella Γ-convergenza.

Questo fatto è ovviamente molto importante in tutte le applicazioni della Γ-convergenza al Calcolo delle Variazioni, e spiega i motivi per cui i Γ-limiti appaiono particolarmente utili quando si voglia studiare il problema limite di una successione di problemi variazionali. E' anche da notare che il Γ-limite commuta con le perturbazioni continue: precisamente se $G \in C(S)$ e se $F_\infty = \Gamma(S^-) \lim_{h\to\infty} F_h$ allora

$$F_\infty + G = \Gamma(S^-) \lim_{h\to\infty} (F_h + G).$$

Esiste ormai una letteratura molto ampia sulla teoria astratta della Γ-convergenza e sulle sue applicazioni, ma restano ancora moltissimi problemi aperti.

In questa conferenza mi limiterò a segnalare qualche ulteriore risultato e qualche problema aperto che mi sembra più interessante.

Se $F_\varepsilon(u) = \int_\Omega f(x/\varepsilon, Du)dx$ con $f(x,p)$ una funzione periodica in x e verificante la

$$|p|^\alpha \leq f(x,p) \leq C(1 + |p|^\alpha) \qquad \text{con} \quad \alpha \geq 1$$

e u è una funzione da Ω in $\mathbf{R}^N$, Du la matrice delle sue derivate, allora esiste il $\Gamma(L^\alpha(\Omega)^-) \lim_{\varepsilon\to 0} F_\varepsilon(u)$, (v. [93]), e ammette una rappresentazione del tipo $\int_\Omega \varphi(Du)\,dx$ con φ funzione quasi-convessa nel senso di Morrey.

Rimane come problema aperto lo studio del Γ-limite per i funzionali $F_\varepsilon(u) + \int_\Omega J(\operatorname{div} u)dx$ dove J è la funzione indicatrice definita dalla relazione

$$\begin{cases} J(0) = 0 \\ J(t) = +\infty \quad \text{per} \quad t \neq 0. \end{cases}$$

Notiamo che mentre lo studio qualitativo dei problemi di omogeneizzazione è stato realizzato in ipotesi molto generali, anche in situazioni abbastanza semplici dal punto di vista qualitativo sono ancora scarsi i casi in cui si conosce una espressione esplicita del Γ-limite in funzione di alcuni parametri che intervengono nei funzionali F_h. Un caso particolarmente interessante della situazione espressa dalla (2) è quello in cui $\Omega \subseteq \mathbf{R}^2$, τ è periodica e il suo intervallo di periodicità è il quadrato unitario, e questo può essere diviso in quattro quadrati uguali su ciascuno dei quali τ è costante. Relativamente a questa struttura alcuni risultati sperimentali tramite calcolatore e alcune considerazioni teoriche hanno indotto a formulare una congettura sulla espressione del funzionale Γ-limite in termini dei quattro valori (diciamo a b c d) che la funzione τ assume. Precisamente la matrice (a_{ij}) associata al funzionale Γ-limite espresso in (2) sarebbe la seguente: (v. [298])

$$a_{12} = a_{21} = 0; \quad a_{11} = \sqrt{\frac{abc+abd+acd+bcd}{a+b+c+d} \frac{(a+c)(b+d)}{(a+b)(c+d)}}$$

$$a_{22} = \sqrt{\frac{abc+abd+acd+bcd}{a+b+c+d} \frac{(a+b)(c+d)}{(a+c)(b+d)}}$$

Per quanto riguarda problemi di Γ-convergenza diversi da problemi di omogeneizzazione, segnaliamo un risultato (v. [293], [294]) in cui la successione dei funzionali studiati è:

$$F_h(u) = \int_\Omega \left|\frac{Du}{h}\right|^2 + h(1-u^2)^2 dx. \tag{5}$$

In questo caso si è potuto dimostrare che il Γ-limite è un funzionale legato alla teoria delle superfici minimali, precisamente:

$F_\infty(u) = 8/15 \cdot \int_\Omega |Du|$ se $u^2=1$ quasi ovunque, $F_\infty(u) = 0$ altrimenti,

ove $\int_\Omega |Du|$ indica $\sup \left(\int_\Omega \sum_1^n u(x)\, D_i g_i(x)dx : g_i \ldots g_n \in C_0^1(\Omega);\ \sum_{i=1}^n g_i^2 \leq 1\right)$ (v. [286]).

Questo risultato costituisce un collegamento tra funzionali del tipo (5) e funzionali del tipo considerato nello studio delle superfici di area minima che dovrebbe essere approfondito. In particolare dovrebbero esistere delle notevoli analogie di comportamento tra le superfici di area minima e le linee di livello di soluzioni dell'equazione di Eulero dei funzionali del tipo (5).

Per esempio sarebbe interessante stabilire se è verificata la seguente congettura: se $u \in C^2(R^n)$ e soddisfa l'equazione $\Delta u = u^3 - u$ in tutto R^n, con $\frac{\partial u}{\partial x_n} > 0$, è vero che per ogni $\lambda \in R$ gli insiemi $\{u=\lambda\}$ sono iperpiani, per lo meno se $n<8$?

Ricordo a questo proposito che per $n<8$ gli unici coni di area minima sono gli iperpiani (v. [223]).

Un altro caso interessante di cambiamento della natura dei funzionali è stato trovato in [126]. Precisamente si trova che la successione di funzionali del tipo

$$F_h(u) = \int_\Omega |Du|^2 + J(u-g_h)^+ dx$$

ove J è la funzione indicatrice già definita, $(u(x)-g_h(x))^+$ indica la parte positiva di $u-g_h$ e le funzioni g_h sono opportune funzioni definite in Ω. Il funzionale $F_\infty = \Gamma(L^2(\Omega)^-)F_h$ è in tal caso del tipo

$$\int_\Omega |Du|^2 + ((u(x))^+)^2 dx.$$

Più in generale, per Ω sufficientemente regolare si può dimostrare (v. [186], [164]) che mediante opportune scelte della successione g_h si possono ottenere tutti e soli i funzionali del tipo

$\int_\Omega |Du|^2 dx + \int_\Omega \varphi(x,u)d\mu + C$ ove μ è una misura nulla

sugli insiemi di capacità nulla, C è una costante non negativa, φ una funzione a valori reali estesi non negativa, di Borel nella variabile x e crescente, convessa, semicontinua inferiormente nella variabile u, verificante inoltre la condizione seguente: se $\varphi(x,u)$ è finita per $u<a$ allora anche $\frac{\partial\varphi}{\partial u}$ è convessa per $u<a$; notiamo che nel caso in cui $\varphi(x,u)$ è finita ed ammette derivata terza su tutto l'asse reale le condizioni precedenti si possono riassumere nella formula $\frac{\partial^h\varphi}{\partial u^h} \geq 0$, $h=0,1,2,3$.

Più in generale sono stati studiati largamente funzionali del tipo

$$F_h = \int_\Omega f_h(x,u,Du) + J(u-g_h)^+ + J(u+\varphi_h)^- dx \quad (\text{v. [163]}),$$

cioè, sostanzialmente sono stati studiati i limiti di ostacoli unilateri e bilateri per funzioni u scalari.

Resta invece aperto il problema dei Γ-limiti dei funzionali del tipo

$\int_\Omega f(x,u,Du) + J(\varphi_h(x,u(x))dx$ dipendente da una funzione vettoriale u,

cioè in sostanza la ricerca dei limiti dei problemi variazionali che si ottengono imponendo alla funzione incognita il vincolo variabile $\varphi_h(x,u(x)) \leq 0$.

Notiamo infine che le Proposizioni 1 e 2 e il relativo Corollario descrivono abbastanza bene il comportamento dei punti di minimo ma non danno direttamente informazioni sul comportamento di eventuali punti critici che non sono punti di minimo. Non potendo in questa conferenza affrontare il problema dei limiti dei punti critici (problema non meno vasto di quello dei limiti di punti di minimo) mi limiterò a segnalare un gruppo di lavori dedicati a questo argomento (v. [191], [192], [195]).

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THE RADON TRANSFORM - SOME RECENT RESULTS

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0. The definition of the Radon transform

The Radon transform

$$(1)\qquad Rf(x,p) = \int_{x\cdot y=p} f(y)dy$$

associates with a function f, defined on $\mathbb{R}^n (u \geq 2)$, a function Rf, defined on the space $\mathbb{P}^n$ of hyperplanes $x\cdot y = p$ of $\mathbb{R}^n$, where $(x,p) \in S^{n-1} \times \mathbb{R}$. So, $(x,p) \in S^{n-1} \times \mathbb{R}$ and $(-x,-p)$ respresent the same hyperplane ($S^{n-1} \times \mathbb{R}$ is a double foliation of of $\mathbb{P}^n$). In (1), dy denotes the measure, induced by the n-dim. Lebesgue measure $d\lambda^n$ on the hyperplane, in question. The Radon transform acts as a linear continuous operator on the spaces $\mathcal{D}(\mathbb{R}^n)$, $S(\mathbb{R}^n)$, and $L^1(\mathbb{R}^n)$, respectively.

1. The projection lemma

The projection lemma, due to BRACEWELL [1], tells us that for suitable functions g on $\mathbb{R}$ and h on $\mathbb{R}^n$, and for $x \in S^{n-1}$,

$$(2)\qquad \int_{\mathbb{R}} g(t)Rh(x,t)dt = \int_{\mathbb{R}^n} g_x(y)h(y)d\lambda^n(y)$$

holds, where $g_x(y) = g(x\cdot y)$ and $x\cdot y$ denotes the Euclidean inner product of x and y. From (2) follows, for $g(t) = e^{itp}$, the connection with Fourier transforms:

$$(3)\qquad F_1 Rh(x,\cdot)(p) = F_n h(p\cdot x),$$

where F_n denotes the n-dimensional Fourier transform. From (3) fol-

lows the injectivity of the Radon transform on $\mathcal{D}(\mathbb{R}^n)$, $S(\mathbb{R}^n)$, and $L^1(\mathbb{R}^n)$. From (2) follows further, for $g(t) = t^m$, that, for a function k on $\mathbb{P}^n$, a necessary condition to be in the range of the Radon transform on $\mathcal{D}(\mathbb{R}^n)$, or $S(\mathbb{R}^n)$, is:

$$(4)\quad \begin{cases} k \text{ is an even function, and} \\ \forall\, m \in \mathbb{Z}^+ : \int_{\mathbb{R}} t^m \cdot k(x,t)\,dt \text{ is} \\ \text{a polynomial of degree} \le m. \end{cases}$$

HELGASON [2] and GELFAND et al. [1] proved the (much harder) converse that (4) is also sufficient for k to be in the range of R, known as the range theorem for the Radon transform. An open problem is to find a characterization of the range of the Radon transform on $L^1(\mathbb{R}^n)$.

Recently, MICCHELLI [1], initiated an application of the projection Lemma to multivariate splines and interpolation. First, he stated the following projection lemma for B-splines M_k on $\mathbb{R}^n$ with k nodes $(x_1^o,\ldots,x_k^o) = x^o \in \mathbb{R}^{n\cdot k}$:

$$RM_k(\cdot\,|\,x^o)((x,t) = M_1(t\,|\,x\cdot x^o) \qquad \forall\, x \in S^{n-1} \qquad \forall\, m \in \mathbb{N},$$

which follows easily from (2).

Later, CAVARETTA et al. [1] made the following abstraction of Kergin's multivariate Hermite interpolation (KERGIN [1]): Starting out from the observation that any univariate interpolation $T : \mathcal{D}(\mathbb{R}) \times \mathbb{R}^n \to E(\mathbb{R})$ on $\mathbb{R}^n$ obeys the projection law:

$$T(g_x\,|\,x^o)(x) = T(g\,|\,x\cdot x^o)(x\cdot x^o),$$

a k-variate extension T_k of an abstract univariate interpolation $T : E'(\mathbb{R}) \times \mathbb{R} \to E'(\mathbb{R})$ on $\mathbb{R}^n$ is defined as a map $T_k : E'(\mathbb{R}^k) \times \mathbb{R}^n \to E'(\mathbb{R}^k)$ such that

$$T_k(g_x\,|\,x^o) = T(g\,|\,x\cdot x^o)$$
$$\forall\, g \in E(\mathbb{R}) \qquad \forall\, x \in S^{n\cdot k} \qquad \forall\, x^o \in \mathbb{R}^{n\cdot k}.$$

Next, the inverse Radon transform $R^{-1}T$ of an abstract univariate interpolation T on $\mathbb{R}^n$ is defined by $R^{-1}T \in E'(\mathbb{R}^n)$ such that

$$R^{-1}T(g_{x^o}) = T(g\,|\,x^o) \quad \forall\, g \in E(\mathbb{R}) \quad \forall\, x^o \in \mathbb{R}^n.$$

The projection lemma yields the consistency with the Radon transform of a usual univariate interpolation on $\mathbb{R}^n$. CAVARETTA et al. [1] proved the following multivariate extension criterion for interpolations:

An abstract univariate interpolation T on $\mathbb{R}^n$ has a k-variate multivariate extension T_k, $\forall\, k \in \mathbb{N}$, iff R^{-1} exists; T_K can be found as follows:

$$T_k(g\,|\,x^o) = R^{-1}T(L_k^{x^o} g),$$

where $L_k^{x^o} E(\mathbb{R}^k) \to E(\mathbb{R}^n)$ is defined by

$$L_k^{x^o} g(S_1,\ldots,S_n) = g(S_1 x_1^o + \ldots + S_n x_n^o).$$

2. The adjoint Radon transform

The adjoint Radon transform

$$(5)\quad R^*g(x) = \int_{S^{n-1}} g(y, x\cdot y)\,d\sigma^{n-1}(y),$$

where σ^{n-1} denotes the spherical measure on S^{n-1}, associates with every function g, defined on $\mathbb{P}^n$, a function R*g, defined on $\mathbb{R}^n$. It was introduced already by RADON [1], who gave credit to Blaschke, for its suggestion. As well known, JOHN [1] and [2] used the adjoint Radon transform to derive his inversion formula

$$(6)\quad R^{-1} = R^* \circ K,$$

where $K : E(P^n) \to E(P^n)$ is defined by

$$Kg(\omega,p) = \begin{cases} c_n^0 \cdot D_p^{n-1} g(\omega,p), & \text{for n odd,} \\ c_n^1 \cdot H_p D_p^{n-1} g(\omega,p), & \text{for n even,} \end{cases}$$

where D_p denotes differentiation and H_p the Hilbert transform, both

with respect to p. Later, the adjoint Radon transform was used by LUDWIG [1] to define the Radon transform on the space $\mathcal{E}'(\mathbb{R}^n)$ of distributions with compact support, as common in distribution theory, by duality. Analogously, one can define the Radon transform on the space $M(\mathbb{R}^n)$ of finite signed Borel measures by

$$(7) \quad R\mu(g) = \mu(R^*g) \qquad \forall\, g \in S(\mathbb{R}^n) \qquad \forall\, \mu \in M(\mathbb{R}^n).$$

HERTLE [1] proved that this definition gives a linear, continuous and injective map $R : M(\mathbb{R}^n) \to M(P^n)$, which is norm-norm and weakly-weakly continuous, but vaguely-vaguely discontinuous, and consisted with the Radon transform for L^1-functions, i.e.:

$$R(f\cdot\lambda^n) = R(f)\cdot\sigma^{n-1} \otimes \lambda^1 \qquad \forall\, f \in L^1(\mathbb{R}^n).$$

The inverse of the Radon transform on $M(\mathbb{R}^n)$ is given by

$$(8) \quad R^{-1}\nu(f) = \nu(KRf) \qquad \forall\, \nu \in M(\mathbb{R}^n) \qquad \forall\, G \in S(\mathbb{R}^n).$$

As a complement to the half space representation of the Radon transform for functions

$$(9) \quad Rf(x,p) = \frac{\partial}{\partial p} \int_{x\cdot y<p} f(y)dy \qquad \forall\, f \in S(\mathbb{R}^n),$$

given by GELFAND et al. [1], HERTLE [1] proved that

$$R\mu(g) = -\int_{S^{n-1}} \int_{\mathbb{R}} \frac{\partial}{\partial p}\, g(x,p) \int_{x\cdot y<p} d\mu(y)dp\,d\sigma^{n-1}(x)$$

$$\forall\, g \in S(\mathbb{R}^n), \quad \forall\, \mu \in M(\mathbb{R}^n).$$

This yields the uniqueness theorem of WOLD-CRAMER [1], and as a supplement, a reconstruction formula for measures $\mu \in M(\mathbb{R}^n)$ by half spaces.

3. Well/ill-posedness of the Radon transform

The question of well-posedness of the Radon transform R depends very much on the domain of R: LUDWIG [1] proved that $R^{-1} : R(S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous. HERTLE [1] and [2] proved that $R^{-1}: R(L^1(\mathbb{R}^n)) \to L^1(\mathbb{R}^n)$ is norm-norm and even norm-weakly discontinuous and that

$R^{-1} : R(M(\mathbb{R}^n)) \to M(\mathbb{R}^n)$ is norm-weakly discontinuous, but vaguely-vaguely sequentially continuous, if n is odd. In [3], HERTLE proved the sequential continuity of $R^{-1} : \mathcal{D}'_{L^1}(\mathbb{R}^n) \to \mathcal{D}'_{L^1}(\mathbb{R}^n,\gamma)$, where $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ denotes the space of integrable distributions and $\mathcal{D}'_{L^1}(\mathbb{R},\gamma)$ is the strong dual of $C^\infty_o(\mathbb{R}^n,\gamma)$, where γ stands for some polynomial weight. Recent contributions concerning numerical problems in connection with the ill-posedness of the Radon-transform are LOUIS [1] and NATTERER [1] and [2].

4. The Radon transform on $L^2(\mathbb{R}^n)$

The Radon transform on $L^2(\mathbb{R}^n)$ was established by LUDWIG [1] as a unitary map $R : L^2(\mathbb{R}^n) \to L^2_K(\mathbb{P}^n)$, where $L^2_K(\mathbb{P}^n)$ denotes $L^2(\mathbb{P}^n)$ with a new inner product, according to the following Plancherel theorem for the Radon transform:

$$\int_{\mathbb{R}^n} fgd^n(x) = \int_{\mathbb{P}^n} (\sqrt{K}\,Rf)(\sqrt{K}\,Rg)d\sigma^{n-1}d\lambda^1.$$

Hertle [1] started an operational calculus for the Radon transform by collecting pairs of operatos $A \in \mathcal{L}(L^2(\mathbb{R}^n))$ and $B \in \mathcal{L}(L^2_K(\mathbb{P}^n))$ such that $RAR^{-1} = B$. He proved that for every such a pair A,B a Heisenberg uncertainty relation for the Radon transform of the following form holds:

$$< \Delta^2 A(F_n f) > < \Delta^2 B > (Rf) \geq \frac{\|f\|^2}{4} \qquad \forall\, f \in L^2(\mathbb{R}^n),$$

where $< \Delta^2 A >$ denotes the variation of A. If D_x denotes the derivative in direction $x \in S^{n-1}$, one can define a multiplication operator by $x\cdot = R^{-1}D_xR$. The corresponding Heisenberg uncertainty relation is related to the more quantitative answer, given by LOGAN [1], on the question how many directions are needed to reconstruct a function from the projection of its Fourier and its Radon transforms, in these directions.

An application of the Radon transform to group representation was made by TAKENAKA [1] who first showed that by

$$U : g \to \left(\frac{d\mu^{g^{-1}}}{d\mu}\right)^{1/2}(x)\cdot f(x\cdot g),$$

where μ denotes a measure on $\mathbb{R}^n$, invariant under the Euclidean motion group, a unitary representation of $SL(n+1,\mathbb{R})$ on $L^2_K(\mathbb{P}^n)$ is defined (which is not necessary continuous in g). He shows further that, for n odd, this representation U and the classical unitary representation T of $SL(n+1,\mathbb{R})$ on $L^2(\mathbb{R}^n)$, defined via the Lie algebra, generated by the shifts, the dilations and their canonical conjugations on $L^2(\mathbb{R})$, are related by

$$U_{\hat{g}} = RT_gR^{-1}, \text{ where } \hat{g} = [{}^tg]^{-1}.$$

For the proof, Takenaka used the white noise W defined by μ, and the corresponding $\mathbb{R}^n$-parameter Brownian motion, in the sense of LEVY [1]. To the ref.'s knowledge, no analytic proofs are available. Those might be provided by the duality theory of HELGASON [1] and [2].
- For an application of the Radon transform to linear filtering see TAKENAKA [2].

5. An application to Lévy's multiparameter Brownian motion

NODA [1] applied the Radon transform to a problem, raised by McKEAN [1], on the Markov property of Lévy's $\mathbb{R}^n$-parameter Brownian motion X. For this, Noda started out from the white noise (half space) representation of X, due to Čencov [1]:

$$X(x) = c_nW(V_x) = c_n\int_{V_x} W(dy,dp) \quad \forall\ x\in\mathbb{R}^n,$$

where $V_x = \{(\eta,p)\in S^{n-1}\times]0,\infty[\ : x\cdot\eta \geq p\}$.

Noda considered the corresponding integral transform:

$$Vf(x) = \int_{V_x} f(\eta,p)d\eta dp$$

on $L^2(S^{n-1}\times]0,\infty[)$ and proved for $\theta\in S^{n-1}$ and $n = \frac{p}{\theta\cdot\eta}$, the following relation to the Radon transform:

$$Vf(r\cdot\theta) = \int_0^r R^*(m(f))(\eta\cdot\theta)\frac{d\eta}{u},$$

where $m(f)(\eta,p) = |p|f(\eta,p)$.

The expansion of the even $f\in L^2(S^{n-1}\times\mathbb{R})$ by spherical harmonics $S_{m,e}$:

$$f(\eta,p) = \sum_{m,e} f_{m,e}(p)\cdot S_{m,e}(\eta),$$

together with the factorization

$$R^*(f_{m,e}(p)\cdot S_{m,e}(\eta))(r,\theta) = T_m(f_{m,e})(r)\cdot S_{m,e}(\eta),$$

due to LUDWIG [1] (where T_m denotes the m-th Gegenbauer polynomial), yields a decomposition of V, by which the proof is reduced to a suitable characterization of the class of functions f for which

$$T_mg(r) = 0 \quad \forall\ |r|\leq R \quad (m\in\mathbb{Z}^+),$$

which also was provided by LUDWIG [1]. - For a similar treatment of Lévy's S^n- or ℓ^2-parameter Brownian motion see NODA [2] resp. [3].

6. The Radon transform on a separable Hilbert space

The following results are due to HERTLE [1] and [4]. Let H denote a real separable Hilbert space, S the unit sphere on H, and μ a fixed Gaussian measure on H which is non degenerate, i.e., with a strictly positive covariance operator. Analogously to (9), the Radon transform on H is defined by

$$(10)\quad Rf(x,p) = \frac{\partial}{\partial p}\int_{x\cdot y<p} f(y)d\mu(y) \quad \forall\ (x,p)\in S\times\mathbb{R} \quad \forall\ f\in C_{b,\omega}(H).$$

Next, the surface measure μ_S, induced by μ, is defined by

$$(11)\quad \mu_S(f) = \frac{d}{dr}\int_{\|x\|\leq r} f\,d\mu\Big|_{r=1} \quad \forall\ f\in C_\omega(\mathbb{R})$$

By this, the adjoint Radon transform R* is defined, analogously to (5), as a map on $E(S\times\mathbb{R})$ to $E(H)$.

The adjoint Radon transform is used to define the Radon transform on the space $M(H)$ of finite signed measures to the space $M(S\times\mathbb{R})$,

analogously to (7). HERTLE [1] and [3] proved the existence of the derivatives, appearing in the definitions (10) and (11), by applying the inversion theorems for the Fourier transform and for the Laplace transform, respectively. The Radon transform on $C_{b,\omega}(H)$ resp. on $M(H)$ turns out to have similar properties as the Radon transform on $\mathbb{R}^n$ with which it is consistent. But, analogs to the inversion formulas (6) and (8) do not exist for infinite dimensional Hilbert spaces. To get inversion formulas, also in this case, the only way known seems to be via Fourier transform.

Finally, it has to be mentioned that the Radon transform has not been established on Banach spaces, in spite of the announcement in HERTLE [4]. The crucial question, not settled, is whether, on a (separable) Banach space, a non degenerate Gaussian measure induces a surface measure on hyperplanes and on hyperspheres.

7. Funk's grand circle transform, revisited

Concerning the unit sphere, even in the finite-dim. case, new discoveries are still possible: For the grand circle transform on S^n

$$Tf(x) = \int_{x\cdot y=0} f(y)\,d\sigma^{n-1}(y),$$

introduced by FUNK [1], the L^2-version was established by FLEISCHER [1], who proved:

> The grand circle transform on $L^2(S^n)$ is absolutely p-summing iff $4 < p < \infty$.

8. Concluding remarks

It was the ref.'s intention to report on some recent results probably less known than those in the traditional directions of research on the Radon transform. For recent results of the latter kind as well as for other new ideas, not mentioned in this survey, the reader is referred to the monographs DEANS [1], HELGASON [2] and to the expositions SHEPP [1], STRICHARTZ [1], ZALCMAN [1].

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RECENT DEVELOPMENTS OF GENERAL TOPOLOGY IN JAPAN

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I feel highly honored by Circolo Matematico di Palermo's invitation to give the lecture on general topology for the first centenary of the Circolo, that made an important contribution to the foundation of general topology by publishing in its 'Rendiconti' M.Fréchet's early paper on the concept of metric space.

The objective of this lecture is to survey recent developments of general topology while putting emphasis on Japanese School's activities for the last few years.

What we investigate in general topology are various properties of topological spaces and especially

a) three basic properties of topological spaces, separation axioms, compactness and metrizability as well as their modifications,

b) special properties like dimension, connectedness and cardinality invariants of topological spaces,

c) continuous mappings between topological spaces.

Some one may prefer to count cardinality invariants (like weight $w(X)$=the minimum cardinality of base for a topological space X, and character $\chi(x,X)$=the minimum cardinality of nbd base of $x \in X$) as a basic property. Continuous maps are usually studied in conjunction with properties a) and/or b) of topological spaces, and thus one may not prefer to count it as an independent theme.

In the following text the reader is refered to J.Nagata's book [39] for standard terminologies in general topology and also for classical theorems that are quoted in this lecture without revealing the sources. But many concepts, which are popular among general topologists, will be defined in the text for the convenience of general readers.

Note that all maps in the text are continuous, and 'space' means a Hausdorff space (= T_2-space) while 'topological space' means

a topological space with no separation axiom, and a T_1-space a topological space with T_1-axiom. ('Nbd' means neighborhood.)

The text is divided into eight small sections, 1.dimension theory, 2. metrization, 3.special metrics, 4.generalized metric spaces, 5. universal spaces, 6. normality of product spaces, 7. extension of topological structures, 8. other aspects.

1. Dimension theory

(Refer to R.Engelking [8], J.Nagata [40] for standard terminologies in dimension theory and J.Nagata [41] for a more extensive survey.)

General situation.

A satisfactory dimension theory was established for general metric spaces by M.Katětov and K.Morita in 1952-54. Recently more efforts have been concentrated to the theory of non-metrizable spaces and infinite-dimensional spaces, where many important questions have been settled one by one. A good example is M.Wage-T.Przymusiński's work (1978-79) which showed:

The product theorem, $\dim X\times Y \leq \dim X + \dim Y$ does not necessarily hold even if $X\times Y$ is normal,

where $\dim X$ denotes the covering dimension (= Lebesgue dimension) of X. (As proved by Katětov-Morita, the product theorem holds if X and Y are metric, and either X or Y is non-empty.)

Studies in Japan.

Studies of dimension of metric spaces were pioneered by K.Morita and kept up by J.Nagata, K.Nagami and others in Japan. Y.Kodama also did good works on homological dimension. Dimension theory is still one of the main themes for Japanese School. Let us review some of their recent results putting emphasis on young generation's activity though Morita, Nagami and Nagata are still working in this field. (See K.Morita [35], T.Hoshina-K.Morita [16], K.Nagami [36], [37], [38] and J.Bruijning-J.Nagata [2].)

While Wage-Przymusiński showed the product theorem may fail for non-metrizable normal spaces, Japanese topologists were interested in finding out conditions for the theorem to hold, and some good results were obtained. For example, K.Morita proved in 1977:

The product theorem, $\dim X\times Y \leq \dim X + \dim Y$ holds if X is paracompact T_2 and a countable sum of locally compact closed sets and if $X\times Y$ is normal ($X\neq\emptyset$ or $Y\neq\emptyset$.)

Recently Y.Yajima [63] has proved

Theorem. The product theorem holds if X is collectionwise normal and has a σ-closure-preserving cover by compact sets, Y is subparacompact, and $X\times Y$ is normal. ($X\neq\emptyset$ or $Y\neq\emptyset$.)

Definition. A T_1-space X is called collectionwise normal if every discrete closed collection (= collection of closed sets) $\{F_\alpha \mid \alpha\in A\}$ of X can be expanded to a discrete open collection $\{U_\alpha \mid \alpha\in A\}$ satisfying $U_\alpha \supset F_\alpha$ for all $\alpha\in A$. A topological space X is called subparacompact if every open cover can be refined by a σ-discrete (= a countable sum of discrete collections) closed cover.

A collection $\mathcal{U}$ of subsets of a topological space is called closure-preserving if $\cup\{\bar{U}\mid U\in\mathcal{U}'\} = \overline{\cup\{U\mid U\in\mathcal{U}'\}}$ for every $\mathcal{U}'\subset\mathcal{U}$. '$\sigma$-closure-preserving' means that the collection is a sum of countably many closure-preserving collections.

Yajima's condition may seem somewhat similar to Morita's. But his method of proof is interesting. To prove his theorem, he used the concept of topological game that was extensively studied by R.Telgársky [58] as a convenient tool for general topology.

Definition. Let $\mathcal{C}$ be a class of topological spaces and X a topological space. In the topological game $G(\mathcal{C},X)$ there are two players, player I and player II. Player I chooses a closed set F_1 of X such that $F_1\in\mathcal{C}$. Then player II chooses an open set U_1 such that $F_1\subset U_1$. Then player I chooses a closed set F_2 such that $F_2\in\mathcal{C}$. Then player II chooses an open set U_2 such that $F_2\subset U_2$. Continue the same process; then the sequence $\langle F_1,U_1,F_2,U_2,\ldots\rangle$ is a play of $G(\mathcal{C},X)$. Player I wins the play if $\cup\{U_i\mid i=1,2,\ldots\} = X$, and player II wins otherwise. A strategy s for player I in $G(\mathcal{C},X)$ is a map from the collection $\{\emptyset, \langle U_1,\ldots,U_n\rangle \mid U_i$ is an open set of X, $i=1,\ldots,n$; $n=1,2,\ldots\}$ into the collection $\{F\mid F$ is a closed subset of X with $F\in\mathcal{C}\}$. A strategy s for player I is winning if he wins each play $\langle F_1,U_1,F_2,U_2,\ldots\rangle$ whenever $F_1 = s(\emptyset)$, $F_{n+1} = s(U_1,\ldots,U_n)$ for $n=1,2,\ldots$.

For example, if $\mathcal{D}_n$ is the class of all normal spaces with $\dim \leq n$ and X a normal space, then player I has a winning strategy in $G(\mathcal{D}_n,X)$ iff $\dim X\leq n$. To prove his product theorem, Yajima succeeded in replacing dimensional arguments with arguments of winning strategies in several topological games.

Y.Hattori [11] also characterized some infinite-dimensional property in terms of topological game.

'Refinable map' is another new tool that may be useful in dimension theory. It is a modification of the concept of 'ε-map'

which has already proved its usefullness in dimension theory.

Definition. A map r from a metric space X onto a metric space Y is called refinable if for each $\varepsilon > 0$ there is an ε-map f from X onto Y such that

$$\sup \{ \rho(r(x), f(x)) \mid x \in X \} < \varepsilon ,$$

where f is called an ε-map if the diameter of $f^{-1}(y) < \varepsilon$ for each $y \in Y$.

According to P.S.Alexandroff's Theorem, a compact metric space X is of $\dim \leq n$ iff for every $\varepsilon > 0$ it can be mapped onto a polyhedron of $\dim \leq n$ by an ε-map. Thus it is natural to suppose that refinable maps may be also useful in dimension theory. H.Kato [24], [25], [26] has done an extensive study of refinable maps in shape theory, connectivity theory anddimension theory. Especially H.Kato has proved

Theorem. Suppose that there is a refinable map from a compact metric space X onto a compact metric space Y. If $\dim X < \infty$, then $\dim X = \dim Y$. If $\dim X = \infty$, then $\dim Y = \infty$. If X is weakly infinite-dimensional, then so is Y,

where X is called weakly infinite-dimensional in the sense of Alexandroff (Smirnov) if for any sequence $\{(F_i, G_i) \mid i = 1,2,\ldots\}$ of pairs of disjoint closed sets there is a sequence $\{B_i \mid i = 1,2,\ldots\}$ of closed sets such that B_i separates F_i and G_i, and $\bigcap_{i=1}^{\infty} B_i = \emptyset$ ($\bigcap_{i=1}^{k} B_i = \emptyset$ for some natural number k).

(This theorem is partially due to P.Patten [50].)

A.Koyama [29] has generalized the concept of refinable map to the one of 'c-refinable map' between non-metrizable spaces and proved that a c-refinable map between normal spaces preserves dim and weak infinite-dimensionality in the sense of Smirnov.

Generally, even a perfect map (= a closed continuous map such that the inverse image of each point of the range is compact) may not preserve dim of a normal space while the closed interval I can be mapped onto I^n and vice versa. In view of this fact the above results of H.Kato and A.Koyama are quite interesting.

In another aspect of study S.Oka [49] has studied dimension of generalized metric spaces. In yet another aspect Y.Hattori has proved that for every transfinite cardinal α there is a metric space whose small transfinite inductive dimension is α. This is interesting in comparison with the fact (proved by Yu.Smirnov and E.Sklyarenko) that if a metric space X has a large transfinite inductive dimension Ind, then $\text{Ind } X < \omega_1$. (B.Pasynkov proved the same independently and alittle earlier than Hattori.)

(Definition. Small inductive dimension ind X of a topological space X is inductively defined as $\text{ind } \emptyset = -1$, and $\text{ind } X \leq n$ $(n=0,1,\ldots)$ if for any point x of X and any closed set F with $x \notin F$, there is an open set U with $x \in U$, $U \cap F = \emptyset$ and $\text{ind Br } U \leq n-1$.

Large inductive dimension Ind is defined similarly but replacing x and F with disjoint closed sets G and F. By using a transfinite induction in place of the ordinary induction, we obtain the definitions of small and large transfinite inductive dimensions.)

2. Metrization.

(Refer to J.Nagata [42] for a more extensive survey of recent developments.)

General situation.

Since it turned out that the famous normal Moore space question (Is every normal Moore space metrizable ?) was deeply related with axioms of set theory, many results of general topology have been obtained in relation with set theory. Some results of this type concerning normal Moore space question are as follows.

T.Przymusiński-F.D.Tall [51] proved:

The statement that a normal Moore space satisfying the countable chain condition is metrizable is independent of the ordinary axioms of set theory.

(Definition. A topological space X is called developable if it has a sequence $\{\mathcal{U}_n \mid n = 1,2,\ldots\}$ of open covers such that $\{St(x, \mathcal{U}_n) \mid n = 1,2,\ldots\}$ is a nbd base of each point x of X. A regular developable space is called a Moore space.)

W.G.Fleissner [9] gave an example of a normal non-metrizable Moore space under the assumption of CH (=continum hypothesis). He also proved in another paper that the consistency of the statement 'All normal Moore spaces are metrizable' implies the consistency of the statement 'There is a measurable cardinal'.

Studies in Japan.

Since some basic results of metrization theory were obtained in the initial stage of its developments by S.Kakutani, J.Nagata, K.Morita and S.Hanai, this aspect of topology was also a main theme of Japanese School for some time. But now, not so many people in Japan are occupied with this specific area though sporadic efforts are being made now and then. Among them Y.Tanaka's results concerning metrizability of spaces covered by metric sets are interesting. Y.Tanaka and Zhou Hao-xuan [57] has proved

Theorem. If a space X is dominated by metrizable closed sets, then X is metrizable iff it has no closed copy of S_2 or S_ω, where S_ω is the quotient space obtained from the topological sum of countably many converging sequences by identifying all the limit points and S_2 the space consisting of the points $(N\times N)\cup N\cup\{0\}$ with the strongest topology such that $\{(m,n) \mid m = 1,2,\ldots\}$ converges to n for each $n\in N$, and $\{n \mid n = 1,2,\ldots\}$ converges to 0.

(Definition. A topological space X is dominated by a closed cover $\mathcal{F}$ if the following holds: $F\subset X$ is a closed set iff $F\subset\cup\mathcal{F}'$ for some $\mathcal{F}'\subset\mathcal{F}$, and $F\cap F'$ is closed for each $F'\in\mathcal{F}'$. E.g. every CW-complex is dominated by metrizable sets.)

This theorem generalizes S.Franklin-B.V.Smith Thomas' [10] theorem as well as J.Nagata's classical theorem: A topological space covered by a locally finite cover consisting of closed metrizable sets is metrizable, because in this case the space obviously contains no copy of S_2 or of S_ω. It is interesting that the metrizability of abstract spaces is characterized in terms of such concrete spaces like S_2 and S_ω. (See also Y.Tanaka [55].)

Y.Tanaka also obtained in [56] the following theorems, which are interesting in comparison with A.H.Stone-K.Morita-S.Hanai's classical theorem on metrizability of the image of a metric space by a closed map.

Theorem. Assume MA (= Martin's axiom). If f is a closed map from a metric space X onto Y and if $\mathrm{Br}\, f^{-1}(y)$ is 2^ω-compact (every subset with cardinality 2^ω has a cluster point), then $\chi(y,Y)\le\omega$ or $\chi(y,Y) = 2^\omega$.

Theorem. Assume MA. Let f be a quotient map from a metric space X onto a space Y. If $\mathrm{Br}\, f^{-1}(y)$ is separable, then $\chi(y,Y)\le\omega$ iff $\chi(y,Y) < 2^\omega$. If Y is Frechet-Urysohn, then $\chi(y,Y)\le\omega$ or $\chi(y,Y) = 2^\omega$, where Y is called Frechet-Urysohn if $y\in\bar{A}$ for $A\subset Y$ implies $y_n\to y$ for some point sequence $\{y_n \mid n = 1,2,\ldots\}\subset A$.

K.Tsuda [61] generalized E.V.Ščepin's important theorem (Every finite-dimensional ANR(compact) is metrizable) to locally compact paracompact spaces, where ANR(compact) means absolute nbd retract for the compact spaces.

In the following two sections we are going to discuss two topics that arose from the study of metrization theory.

3. Special metrics.

(Refer to J.Nagata [43] for a survey of this aspect.)

The main objective of the study in this aspect is to characterize topological properties of a metrizable space by special properties of metrics that are admitted by the space and vice versa. Perhaps the first important result of this aspect is the classical theorem of E.Čech that says:

A metrizable space X admits a complete metric iff X is Čech complete, i.e. it is G_δ in its Čech-Stone compactification βX.

Another example is the special metric that characterizes $\dim X$ of a metric space X. Namely, as proved by J.Nagata and P.A.Ostrand in 1963-65,

a metrizable space X has $\dim\le n$ iff X admits a metric ρ such that for any $x,y_1,\ldots,y_{n+1}\in X$, there are distinct i,j satisfying

$$\rho(y_i,y_j) \le \rho(x,y_j).$$

(Note that this condition is not satisfied by the Euclidean metric of R^n if $n\ge 2$, so this is a rather peculiar metric.)

Studies in Japan.

Y.Hattori [12] has furthered de Groot, Nagata, Ostrand's studies on special metrics in dimension theory, especially for infinite-dimensional spaces. He characterized in [13] strongly metrizable spaces also in terms of a special metric.

Definition. A metrizable space is called strongly metrizable if it has a base that is the sum of countably many star-finite open covers. (The strongly metrizable spaces form an important class between the separable metric spaces and the general metric spaces.)

Theorem. A metrizable space X is strongly metrizable iff X admits a metric ρ such that for every $\varepsilon>0$, every $x\in X$ and every sequence $y_1,y_2,\ldots\in X$ with

$$\rho(S_\varepsilon(x),y_i) < \varepsilon \quad , \quad i = 1,2,\ldots,$$

there are distinct i,j for which

$$\rho(y_i,y_j) < \varepsilon \quad ,$$

where $S_\varepsilon(x) = \{y\in X \mid \rho(x,y) < \varepsilon\}$.

(This condition is satisfied by any Euclidean metric and totally bounded metric. Thus this metric is special but not peculiar.)

H.Kato [27] has initiated and studied another interesting special metric.

Definition. A metric ρ of X is called a scalene metric if for any $x_1,x_2\in X$ $(x_1\neq x_2)$ there is $x_0\in X$ such that for every $x\in X$ either $\rho(x,x_1) > \rho(x,x_0)$ or $\rho(x,x_2) > \rho(x,x_0)$ holds.

Theorem. If a compact metric space X has a scalene metric, then it is AR (= absolute retract) for the separable metric spaces. The converse is not true.

He also showed that scalene metrics are useful for the construction of continuous selections.
(Definition. Generally, let φ be a map from a topological space X into the collection 2^Y of non-empty subsets of a topological space Y. If f is a continuous map from X into Y satisfying $f(x) \in \varphi(x)$ for all $x \in X$, then f is called a continuous selection for φ.)
Moreover he showed that the metric of Hilbert space is a scalene metric, but he did not specify exactly what are the spaces admitting a scalene metric. Perhaps it is a problem for the future.

4. Generalized metric spaces.

(Refer to J.Nagata [44] and D.Burke-D.Lutzer [3] for surveys.

General situation.

Various factors have activated the study of this aspect.

a) The class of all metrizable spaces does not behave so well despite that metrizable spaces have very nice properties. E.g. either a space dominated by closed metrizable sets or the closed continuous image of a metrizable space need not be metrizable, As a consequence there are many nice spaces that are 'near' but not metrizable. Simple examples are S_2, S_ω and a hedgehog, the quotient space obtained from a topological sum of closed segments $[0,1]$ by identifying all zeros.

b) Such 'near' metrizable spaces inherit good properties from the metric spaces. In fact some deep results in dimension theory and in extension theory of maps can be extended from metric spaces to some generalized metric spaces. (An example is C.J.R.Borges' generalization of Dugundji's extension theorem to M_3-spaces.) Some classes of generalized metric spaces behave better than the class of metrizable spaces does.

c) Recently metrizability conditions are often studied while being decomposed into several subconditions, that lead us to the study of generalized metric spaces.

d) P.S.Alexandroff's problem in 1961 (Which spaces can be expressed as the images or inverse images of nice spaces by nice maps ?) have been vigorously studied by many topologists. If we understand 'nice spaces' in the problem as metric spaces , then we reach again the concept of generalized metric spaces.

e) The product of two paracompact spaces need not be paracompact. But if the paracompact spaces are endowed with some 'near' metric condition, then the product (even the countable product) is paracompact, which has also stimulated the study of this aspect.

In the following are the definitions of some popular generalizations of metric spaces.

Definition. A regular space X is called an M_1-space if it has a σ-closure-preserving base. A space X is called an M_3-space (or a stratifiable space) if there is a map $g: N \times \tau \to \tau$ (τ denotes the topology of X) such that

i) $U = \bigcup_{n=1}^{\infty} g(n,U) = \bigcup_{n=1}^{\infty} \overline{g(n,U)}$ for each $U \in \tau$,

ii) $g(n,U) \subset g(n,V)$ whenever $U,V \in \tau$ and $U \subset V$.

A topological space X is called a σ-space if there is a σ-locally finite network $\mathcal{F}$,i.e. each open set of X is a sum of members of $\mathcal{F}$.

A topological space X is called an M-space if it has a sequence $\{\mathcal{U}_i \mid i = 1,2,\ldots\}$ of open covers such that

i) $\mathcal{U}_{i+1}^* < \mathcal{U}_i$, $i = 1,2,\ldots$,

ii) if $x_i \in St(x, \mathcal{U}_i)$, $i = 1,2,\ldots$, then the point sequence $\{x_i \mid i = 1,2,\ldots\}$ has a cluster point (which is not necessarily x).

One of the most famous open questions in this aspect is

$$M_1 = M_3 \ ?$$

posed by J.Ceder in 1961. One can define M_3-space in terms of a base satisfying a condition somewhat similar to the one for M_1-space, and it is easy to see that every M_1-space is M_3. We know much about M_3-spaces. In fact the M_3-spaces are hereditary and countably productive. The closed continuous image of an M_3-space is M_3, and a space dominated by closed M_3-subsets is M_3. Thus it is curious that little is known about basic properties of M_1-spaces For examples, we have no answer to any of the following seemingly simple questions.

i) Is every closed set (every subset) of an M_1-space M_1 ?

ii) Is the perfect image (closed continuous image) of an M_1-space M_1 ?

R.Heath-H.Junnila [14] proved:
Every M_3-space is the image of an M_1-space by a perfect retraction. Thus i) and ii) are both equivalent to J.Ceder's question. In other words, if the answer to any one of those questions is yes, then so is the answer to all other questions as well.

Studies in Japan.

M-space was originated by K.Morita, and σ-space by A.Okuyama

that generalized M_3-space, and Σ-space by K.Nagami that generaized bothM- and σ-spaces. Since then this aspect has attracted many Japanese topologists, who obtained good results on various generalized metric spaces. As for J.Ceder's question, a very important contribution was made by M.Itō [21] who proved

Theorem. Every M_3-space whose every point has a closure-preserving open nbd base is M_1. Thus every first countable M_3-space is M_1.

However, it is not known if every M_3-space satisfies the said condition. T.Mizokami [32] studied M_3-spaces satisfying the condition.

Definition. (C.J.R.Borges) A regular space X is called WCP-expandable (CP-expandable) if for each collection $\{U_\alpha \mid \alpha \in A\}$ of open sets of X and closure-preserving collection $\{F_\alpha \mid \alpha \in A\}$ with $\bar{F}_\alpha \subset U_\alpha$ there is a closure preserving collection (open collection) $\{G_\alpha \mid \alpha \in A\}$ such that $\bar{F}_\alpha \subset \mathrm{Int}\, G_\alpha$, $\bar{G}_\alpha \subset U_\alpha$.

In reply to Borges' question, K.Yamada [64] has proved

Theorem. A space X is an M_3-space whose every point has a closure-preserving open nbd base iff X is a σ-space and CP-expandable.

X is an M_3-space iff it is a σ-space and WCP-expandable.

(G.Gruenhage proved independently that M_3-spaces are WCP-expandable. Note that this theorem implies that $M_1 = M_3$ is true iff CP-expandability = WCP-expandability for the σ-spaces.)

M.Itō [22] also proved

Theorem. The closed continuous image of a hereditary M_1-space is M_1.

This theorem partially answers the above question ii).

There are various other conditions which combined together with M_3 implies M_1. Extending G.Gruenhage's theorem, T.Mizokami [33] proved

Theorem. Every M_3- $\mathcal{M}$-space is M_1, where a space is called a $\mathcal{M}$-space if it can be topologically embedded in a countable product of paracompact F_σ-metrizable (= a countable sum of closed metrizable sets) spaces.

K.Tamano [52] also introduced two conditions that turned out to be equivalent with Mizokzmi's. Mizokami and Tamano also studied dimension theoretical properties of M_3-spaces satisfying the condition. Y.Kotake and T.Miwa also study generalized metric spaces.

5. Universal spaces.

General situation.

Let $\mathcal{C}$ be a class of topological spaces. Then a fixed member X_0 of $\mathcal{C}$ is called a universal space for $\mathcal{C}$ if the following is satisfied: $X \in \mathcal{C}$ iff X can be topologically embedded in X_0. For a non-hereditary class $\mathcal{C}$ we may modify the definition as follws. A fixed member X_0 of $\mathcal{C}$ is called a c-universal space for $\mathcal{C}$ if the following is satisfied: $X \in \mathcal{C}$ iff X is homeomorphic to a closed subset of X_0. We may generalize the definition further by replacing the closed sets with any family of subsets of X_0. Well-known classical examples are the product I^m of closed segments which is a universal space for the Tychonoff spaces with weight $\leq m$ and also a c-universal space for the compact T_2-spaces with weight $\leq m$, and Hilbert cube which is a universal space for the separable metric spaces. We also know universal spaces for the classes of the metrizable spaces, the strongly metrizable spaces, (at most) n-dimensional metric spaces, all with a restricted weight, and a c-universal space for the class of complete metric spaces with a restricted weight. Universal spaces help us to visualize abstract spaces and are actually useful in proving some theoremson the concerned spaces. But we know few universal spaces for generalized metric spaces.

H.Brandenburg [1] gave a universal space for the second countable developable T_1-spaces. J.Chaber [4] constructed a universal space for the class of metacompact developable T_1-spaces of weight $\leq m$, where a topological space is called metacompact if every open cover has a point-finite open refinement. On the other hand F.v.Douwen [7] showed:

There is no universal space for the separable Moore spaces.

Studies in Japan.

K.Tsuda [62] proved

Theorem. There is no universal space for the class of M_3-spaces (of $\dim \leq 0$) with a network weight (the minimum cardinality of network) $\leq m$, where m is a given cardinal,

which implies that

for each cardinal m there is $m' > m$ such that there is no universal space for the class of M_3-spaces with weight $\leq m'$. He also proved

Theorem. There is no universal space for the class of first countable, separable, M_3-spaces with $\dim \leq 0$.

On the other hand A.Kato [23] has proved, in reply to Nagata's question,

Theorem. There is no c-universal space for the class of paracompact M-spaces with weight $\leq m$, where m is any cardinal satisfying $\omega \leq m \leq 2^{\omega}$.

He has proved it by reducing the problem to the one on c-universal spaces for metric spaces. Thus we are inclined to suppose that most classes of generalized metric spaces, that satisfy T_2 or a stronger separation axiom, may have no universal space. So far proofs for the non-existence of universal spaces rely on cardinality arguments. Namely one can prove that $\mathcal{C}$ has no universal space (or no c-universal space) by showing that for each $X_0 \in \mathcal{C}$ $|\mathcal{C}| > |2^{X_0}|$ (or $|\mathcal{C}| > |$the family of all closed subsets of $X_0|$). This is a rather rough argument in some sense, and I hope more delicate method will be found in future to push the study of this aspect further.

6. Normality of product spaces.

General situation.

Generally speaking, 'separation axioms' are no more subjects of serious investigations, perhaps with the only exception of normality of product spaces. Since the product of two normal spaces need not be normal, it is an interesting problem to find conditions for the product to be normal. This problem can be formulated as follows. Let $\mathcal{C}$ be a class of normal spaces. Then find the class $\mathcal{C}'$ satisfying $X \in \mathcal{C}'$ iff $X \times Y$ is normal for every $Y \in \mathcal{C}$. C.H.Dowker was a pioneer of this aspect, who proved in 1951 that

$\mathcal{C}'$ is the class of all countably paracompact normal spaces if $\mathcal{C}$ is the class of all compact metric spaces.

Studies in Japan.

After C.H.Dowker's initial work, many remarkable results were obtained by Japanese School. H.Tamano proved

$\mathcal{C}'$ = all paracompact T_2-spaces if $\mathcal{C}$ = all compact T_2-spaces.

K.Morita furthered Tamano's research and also proved

$\mathcal{C}'$ = all normal P-spaces if $\mathcal{C}$ = all metric spaces.

Definition. A topological space X is called a P-space if for any open collection $\{U(\alpha_1,\ldots,\alpha_i) \mid \alpha_k \in A,\ k = 1,\ldots,i;\ i = 1,2,\ldots\}$ satisfying $U(\alpha_1,\ldots,\alpha_i) \subset U(\alpha_1,\ldots,\alpha_{i+1})$, there is a closed collection $\{F(\alpha_1,\ldots,\alpha_i) \mid \alpha_k \in A,\ k = 1,2,\ldots i;\ i = 1,2,\cdots\}$ such that

i) $F(\alpha_1,\ldots,\alpha_i) \subset U(\alpha_1,\ldots,\alpha_i)$,

ii) for any $(\alpha_1,\alpha_2,\ldots) \in A^{\omega}$ satisfying $\bigcup_{i=1}^{\infty} U(\alpha_1,\ldots,\alpha_i) = X$,

$\bigcup_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i) = X$ holds.

Y.Katuta determined $\mathcal{C}'$ for $\mathcal{C}$ = all paracompact T_2-spaces. M.Atsuji and M.E.Rudin's works led to the conclusion that

$\mathcal{C}'$ = all discrete spaces if $\mathcal{C}$ = all normal spaces.

This tradition is kept up by younger generation in Japan. T.Hoshina [15] has proved, extending K.Morita, M.Rudin and M.Starbird's results,

Theorem. If X is normal and countably paracompact and Y is a Lašnev space (= a closed continuous image of a metric space), then $X \times Y$ is normal iff it is countably paracompact.

K.Chiba [5] studied a condition under which $X \times Y$ is collectionwise normal and also some conditions under which $X \times Y$ is collectionwise normal iff it is normal. K.Chiba-T.Przymusiński-M.E.Rudin's recent paper [6] contains some extremely interesting results of this aspect. They studied $\mathcal{C}'$ for $\mathcal{C}$ = all perfectly normal spaces and showed

Theorem. All members of $\mathcal{C}'$ are M_3, and the weight, cellularity and Lindelöf number (See [39] for the definitions) coincide for every member of $\mathcal{C}'$. $\mathcal{C}'$ is hereditary and countably productive and contains non-metrizable members. All first countable members of $\mathcal{C}'$ are metrizable,

They also gave a consistency proof of Morita's conjectures:

$\mathcal{C}'$ = the class of all metrizable spaces if $\mathcal{C}$ = the class of all normal P-spaces.

$\mathcal{C}'$ = all metrizable σ-locally compact spaces if $\mathcal{C}$ = all countably paracompact normal spaces.

7. Extension of topological structures.

General situation.

The most significant recent development in this aspect is V.M. Ul'yanov's solution of O.Frink's problem concerning Wallman type compactification. Ul'yanov showed in 1977:

There is a T_2-compactification which is not of Wallman type.

I am not going to give here the precise definition of Wallman type compactification, but H.Wallman originally constructed a T_1-compactification of a T_1-space X by introducing a special topology into the set of all maximal closed filters of X (1938). Then in 1943 N.A,Shanin furthered Wallman's work by showing that if the method of Wallman is applied on the set of maximal filters in $\mathcal{F}$,

a closed base of X satisfying certain condition, then it produces various compactifications including Čech-Stone compactification. Later in 1964 O.Frink asked if every T_2-compactification of a Tychonoff space can be obtained in that manner, which had been unsettled until 1977. On the other hand various conditions for a compactification to be of Wallman type are being obtained. Extensions other than compactification are also being studied. An interesting result in this respect is J.v.Mill's [30] that the superextension of the closed interval is the Hilbert cube.

Studies in Japan.

T.Kimura [27] has obtained some interesting conditions in order that T_2-compactifications be regular Wallman compactifications (= Wallman type compactifications satisfying certain special condition). T.Isiwata, who did good works on realcompact spaces, generalized metric spaces, etc., studies mappings, realcompact spaces and realcompactifications in [19],[20]. Y.Ikeda also works in a nearby field. T.Nogura[45] has solved A.V.Arhangelskii's problems 'Can each regular space X be embedded in a regular space $\tilde{X}$ with $t(\tilde{X}) = t(X)$?' and 'Can each Tychonoff first countable space be embedded in a Tychonoff first countable countably compact space ?', where t(X) denotes the tightness of $X = \min\{m \mid$ for each $x \in X$ and $A \subset X$ with $x \in \bar{A}$, there is $B \subset A$ with $x \in \bar{B}$ and $|B| \leq m\}$. Namely he has proved

Theorem. For a Lašnev space X the following are equivalent: i) X is metrizable, ii) X can be embedded in a normal countably compact first countable space, iii) X can be embedded in a regular countably compact space with a countable tightness, (which implies a negative answer to the first question of Arhangelskii) and pointed out that S.Mrówka's space $N \cup \mathcal{R}$ gives a counter example for the second question. (See [60] for the space $N \cup \mathcal{R}$.)

T.Terada-J.Terasawa [59] has studied first countable maximal extensions assuming Tychonoff separation axiom for the concerned spaces. E.g. they have proved

Theorem. Every locally compact first countable space as well as every metrizable space has a first countable maximal extension. (Every first countable T_2-compactifiation is a first countable maximal extension, but the converse is not true. A first countable space is maximal iff it is pseudocompact.)

8.Other aspects.

There are many other aspects where Japanese School is showing its strength. K.Morita in cooperation with T.Hoshina studies extension theory of maps by use of the concept of P^{κ}-embedding.

Definition. Let κ be a cardinal number. A subset A of a topological space X is called P^{κ}-embedded if for every normal open cover $\mathcal{U}$ of A with $|\mathcal{U}| \leq \kappa$, there is a normal open cover $\mathcal{V}$ of X such that $\{V \cap A \mid V \in \mathcal{V}\} < \mathcal{U}$, where a cover $\mathcal{U}$ is called normal if there is a sequence $\{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ of open covers such that $\mathcal{U} > \mathcal{U}_1^* > \mathcal{U}_1 > \mathcal{U}_2^* > \ldots$

Previously they furthered Tietze's extension theorem and Dowker's extension theorem by proving

Theorem. Let $A \subset X$; then the following are equivalent: i) A is P^{κ}-embedded in X, ii) for any Čech complete AR(metric) space Y with $w(Y) \leq \kappa$ and for any map $f: A \to Y$, there is an extension of f over X, iii) for any Čech complete ANR(metric) space Y with w(Y) and any continuous map $f: X \times \{0\} \cup A \times I \ (\subset X \times I) \to Y$, there is an extension of f over $X \times I$.

In [34] Morita pushed their work further by generalizing the concepts of AR, ANR and considering dimensional conditions. T.Ishii, who did important works on realcompact spaces, generalized metric spaces, etc., studies in [17],[18] w-compactness and w-paracompactness, both concepts originated by himself, and he shows that these concepts are useful in the study of Tychonoff functor and rectangular product, which are important e.g. in the study of product theorem in dimension theory. N.Kimura, who worked previously in dimension theory, also studies some modification of compactness. T.Tanaka is successively working on connectivity of topological spaces. In [54] he has got an interesting result to answer J.Ceder's question concerning connected selections. Y.Yasui has studied in [65] some modifications of paracompactness and solved E.Michael's problem in [66] by proving

Theorem. Every open cover $\mathcal{U}$ of a regular space can be shrunk to a closed cover provided each member of $\mathcal{U}$ has a perfectly normal closure.

H.Ohta [48] has settled a number of questions concerning realcompact spaces and extensions of topological structure under the set theoretical assumption, 'the cofinality of $c > \aleph_1$' (which is implied by the negation of CH plus MA); e.g.

Theorem. The clopen image of a realcompact space need not be realcompact (answering J.Ginsburg's question). The hyper space of the closed sets of a realcompact space need not be realcompact (answering Ginsburg's another question). For a clopen map $f: X \to Y$ the extension $\mu f: \mu X \to \mu Y$ need not be clopen, where μX is the completion

of X with respect to the strongest uniformity (answering K.Morita's question). (All spaces are at least Tychonoff.)

A recent trend of general topology is that many new theorems and counter examples are being obtained under certain set theoretical assumption. This trend can be seen in the previously mentioned results of Y.Tanaka, H.Ohta, etc. Besides, M.Hanazawa, N.Kemoto and others are working in general topology with special interests in set theory. Another trend, perhaps related with the above, is remarkable interests in cardinality invariants. In fact, since 1969 when A.V.Arhangelskii solved P.S.Alexandroff's famous question of 1923 by proving

$|X| \leq c$ for every first countable compact T_2-space X,

the study of relations between various cardinality invariants has flourished as one of the most active fields of general topology. Although relatively few Japanese topologists are specialized in this field, the influence of that trend is eminent in many papers produced by Japanese School. T.Nogura's result and Y.Tanaka's result are some examples to show the influence.

'Categorical topology' and 'Infinite-dimensional topology' are relatively new fields that are deeply related with general topology. Among the few Japanese topologists working in those aspects, R.Nakagawa in categorical topology and K.Sakai in infinite-dimensional topology are achieving notable success.

As for other aspects, H.Tanaka is working on Pixley-Roy space. (See [53].) T.Tani and T.Yagi are working on function spaces from various points of view. T.Noiri is doing extensive research on modifications of basic conceps like closedness and continuity. Namely S-closedness, super continuous maps, etc. are his subjects of study. (See e.g. [46],[47].) Ranked spaces, which were originated by K.Kunugi, are so far not so popular in the international level, but they are eagerly studied by several people in Japan including F.Ishikawa and S.Nakanishi. The latter's works indicate usefulness of the concept in analysis.

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LES DÉBUTS DE LA TOPOLOGIE ALGÉBRIQUE

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L'histoire de la Topologie algébrique à ses débuts est très particulière; on n'y trouve pas d'évolution lente, par accumulation de nouvelles acquisitions, soumises à une exigence logique de plus en plus serrée, telle qu'on peut la suivre, par exemple, en Théorie des nombres, en Algèbre, en Géométrie (algébrique ou différentielle), en Analyse, et même en Analyse fonctionnelle et en Topologie " générale ". Au contraire, la Topologie algébrique, en tant que discipline mathématique, a une date de naissance précise: 1900, et est sortie du cerveau d'un seul homme, Henri Poincaré, dans un mémoire publié aux Rendiconti del Circolo Matematico di Palermo .

Les historiens des sciences ont montré de façon convaincante qu'avant la Géométrie " démonstrative " des Grecs, basée sur des déductions logiques à partir d'un petit nombre de propositions explicitées, il y a eu une " prégéométrie ", où les figures géométriques sont des objets physiques, sur lesquels on fait des constatations expérimentales faisant appel à nôtre sense de la vue. De même, de 1850 à 1900, il y a une " prétopologie algébrique " non mathématique, c'est-à-dire sans démonstration . Les notions qui y entrent portent sur des objets non dégagés d'un support matériel, et il n'y a aucun effort pour les définir mathématiquement, en les rattachant, avec plus ou moins de précision, à des notions mathématiques antérieures, telles que les nombres ou les objets de la Géométrie classique. Ce n'est donc pas du tout comparable aux débuts du Calcul infinitésimal ou de la Géométrie algébrique. Par exemple, les mathématiciens de la seconde moitié du XIX siècle qui s'occupent de ces questions parlent libre

ment de courbes, de surfaces, de déformations (voire de " déformations infiniment petites "!) , sans jamais dire ce qu'ils entendent par ces mots. Lorsque les successeurs de Riemann, appliquant ses " définitions ", affirment que le " bretzel " à 2 trous est une surface de genre 2, tout ce qu'ils font pour appuyer leur assertion est de montrer au lecteur une figure où l'on " voit " 4 courbes fermées sur la surface, ayant un point commun, et dont il doit s'assurer par une " expérience mentale ", d'une part que le complémentaire est connexe, et d'autre part que toute autre courbe passant par le point commun morcellerait ce complémentaire!!

Encore les successeurs de Riemann, après maintes réflexions sur les surfaces et leur réduction à une forme " normale ", eussent-ils pu donner des raisons plausibles appuyant de tels " théorèmes ". Mais Betti, qui élabore des notes laissées par Riemann, pour une " variété " V à un nombre quelconque n de dimensions, introduit, pour chaque entier p tel que $1 \leq p \leq n-1$, le nombre maximum b_p de sous-variété de V compactes, connexes, sans bord, de dimension p, dont la réunion ne peut " former frontière " d'une sous-variété connexe à bord, contenue dans V et de dimension p+1 (*) . On se demande avec effarement ce qu'une telle " définition " pouvait évoquer dans l'esprit d'un mathématicien de cette époque, et surtout comment il aurait pu s'en servir; comment, par exemple, montrer que pour un tore plein dans l'espace ordinaire, on a $b_1=1$ comme l'affirme Betti?

Nous allons décrire sommairement, dans ce qui suit, les notions " combinatoires " imaginées par Poincaré,et comment il concevait que ces notions permettaient d'attacher à certains types d'espaces des nombres invariants par homéomorphie. Mais il ne possédait pas de technique qui lui eût permis de démontrer cette invariance; elle fut créée et appliquée par deux de ses successeurs immédiats , Brouwer et Alexander, dont nous résumerons aussi les contributions essentielles.

(*) - Cette définition est formulée de façon beaucoup moins précise chez Betti, qui ne dispose pas des notions topologiques actuelles .

I. H.Poincaré (1854-1912)

Dans la première tentative de Poincaré (1895) pour donner des fondements mathématiques à ce qu'il appelait l'Analysis Situs, on est encore dans la "préhistoire " de la Topologie algébrique, car le point de vue est resté fondamentalement le même que chez Riemann et Betti. Il s'agit toujours de " variétés ", en fait ce que nous appelons des variétés de classe C^1 ou analytiques, et Poincaré en donne une définition équivalente à la nôtre par des " cartes "; il essaie aussi de préciser la notion d'orientation, et de ce qu'il faut entendre par " former frontière " d'un ouvert, de façon fort peu claire à la vérité. Mais il y a dans son mémoire plusieurs idées nouvelles, dont une tout à fait fondamentale: au lieu de se borner, comme ses predécésseurs, à attacher des nombres aux variétés, il calcule sur ces variétés elles-mêmes, d'une manière peut-être inspirée (bien qu'il n'en dise rien) par le calcul des " diviseurs " de Dedekind-Weber sur les courbes algébriques, ou des "systèmes linéaires " de la Géométrie algébrique italienne sur les surfaces algébriques. Au lieu de dire, comme Betti, que la réunion de sous-variétés distinctes V_1, V_2, ..., V_r , orientées, compactes, connexes, sans bord et de dimension p " forme frontière " d'un ouvert orienté connexe U dans une sous-variété de V de dimension p+1, il écrit une relation algébrique

$$(1) \qquad V_1 + V_2 + \ldots + V_r \sim 0$$

qu'il appelle une " homologie " . Mais ce qui est beaucoup plus remarquable, c'est qu'il ajoute aussitôt que les homologies " se combinent comme des équations " ; autrement dit, il considère qu'une relation

$$(2) \qquad k_1V_1 + k_2V_2 + \ldots + k_rV_r \sim 0$$

où les k_j sont des entiers quelconques (positifs ou negatifs) a un sens. A vrai dire, ce sens ne ressort pas très clairement des explications qu'il donne. Si une relation telle que (1) a lieu, l'orientation de chaque V_j est " en relation directe " avec celle de U, notion classique dans l'espace ordinaire, et qui avait été étendue par

Klein et ses élèves aux dimensions quelconques; une relation telle que (1) où certains termes $-v_j$ sont affectés du signe " moins " signifie donc que c'est l'orientation opposée à celle de v_j qui est " en relation directe " avec celle de U. Mais le sense à donner à (2) lorsque les k_j sont des entiers ≥ 2 est beaucoup moins évident; Poincaré se borne à dire qu'il faut, dans (2), remplacer $k_j v_j$ par une somme

$$v_{j1} + v_{j2} + \dots + v_{j,k_j}$$

de k_j sous-variétés distinctes " peu différentes " de v_j; un commentaire de Picard explique que ces k_j sous-variétés s'obtiennent (sans doute avec leur orientation) par de " petites déformations " de v_j dans V (Picard dit " infiniment petites " !) .

Toutefois, dès qu'on y réfléchit un peu, ces " définitions " révèlent des difficultés qui paraissent insurmontables. Comment prouver une relation (1) entre des sous-variétés distinctes données, c'est-à-dire construire l'ouvert U ? Si, dans (2), avec les $k_j > 0$, on remplace chaque v_j par k_j sous-variétés " peu différentes " de v_j, la relation d'homologie entre les $k_1 + k_2 + \dots + k_r$ sous-variétés distinctes obtenues est-elle indépendante de la façon dont on a fait le " remplacement " ? Comment même justifier l'idée de " combiner " des homologies? Si l'on a trouvé des ouverts U_1, U_2 de sous-variétés de dimension p+1 tels que $v_1, v_2, \dots, v_r$ " forment frontière " de U_1, et $v_{r+1}, \dots, v_{r+s}$ " forment frontière " de U_2 , comment prouve-t-on qu'il y a un ouvert U d'une sous-variété de dimension p+1 tel que la réunion des sous-variétés $v_1, \dots, v_r, v_{r+1}, \dots, v_{r+s}$ " forme frontière " de U? On ne peut certainement pas en général prendre $U = U_1 \cup U_2$, car $U_1 \cap U_2$ ne sera pas nécessairement vide, et les points de cet ensemble n'ont pas en général un voisinage dans $U_1 \cup U_2$ homéomorphe à R^{p+1}. Même avec toutes les ressources de la Topologie différentielle actuelle, il n'est pas évident que l'on pourrait mettre sur pied une théorie qui justifierait l'intuition de Poincaré.

Je pense que Poincaré avait peut-être dans l'esprit comme

une confuse prémonition de quelque chose qui serait réalisé 60 ans plus tard par R.Thom dans la théorié du cobordisme des variétés; il n'y avait évidemment pas la moindre possibilité de trouver dans les mathématiques de 1900 le vocabulaire qui eût permis de formuler une telle théorie, ni la technique qui eût pu la concrétiser.

On peut donc dire en résumé que chez Riemann, chez Betti et dans le premier mémoire de Poincaré, non seulement il n'y a aucune démonstration mathématique au sens où l'entendaient à cette époque Weierstrass et ces auteurs eux-mêmes quand ils faisaient de l'Analyse, mais leur définitions ne sont pas opérationnelles : il n'y a aucun algorithme qui, au moins théoriquement, pourrait conduire à calculer les nombres qu'ils définissent. Comme pour Betti, il faut croire Poincaré sur parole quand, sur un exemple explicité, il prétend calculer des " nombres de Betti ".

En tout cas, s'il a songé aux difficultés mentionnées ci-dessus, il a dû penser qu'on pouvait les surmonter (peut-être par des raisonnements de " position générale ", comme on faisait à cette époque en Géométrie algébrique), car il n'hésite pas à aller de l'avant. Ayant défini le nombre b_p comme le plus grand nombre tel qu'il existe toujours une homologie (2) à coefficients non tous nuls entre b_p sous-variétés connexes, compactes, sans bord, orientées et de dimension p, il se propose de démontrer son célèbre théorème de dualité affirmant que pour une variété compacte, connexe, orientable et de dimension n on a $b_p = b_{n-p}$.

Il introduit à cet effet une nouvelle idée, celle d'utiliser le nombre algébrique d'intersection $N(V,W)$ de deux sous-variétés orientées V, W de M , de dimensions complémentaires p, n-p , qui généralise un concept de Kronecker. Il tente alors de démontrer que une " homologie " (2) équivaut à la propriété suivante (d'où il tirerait le théorème de dualité); pour toute sous-variétés W de dimension n-p dans M, on a

$$(3) \qquad \sum_j k_j N(W, v_j) = 0 \quad .$$

Mais dès que p ou n-p n'est pas égal à 1, il est hors d'état de mener

à bien une démonstration convaincante (elle ne sera obtenue que beaucoup plus tard par Lefschetz) .

Poincaré en serait peu-être resté là si, en 1898, le jeune mathématicien danois P.Heegaard n'avait exhibé un exemple de variété M à 3 dimensions, vérifiant les hypothèses de Poincaré, et dont les nombres de Betti, pour la définition de Betti, ne vérifiaient pas la relation $b_1 = b_2$. Cela amena Poincaré à se rendre compte de deux choses:

1) Sa définition de b_p n'était pas équivalente à celle de Betti. Il en vit aussitôt la raison, c'est qu'on peut avoir une relation d'homologie $2v \sim 0$ alors que $v \not\sim 0$ (l'exemple plus simple est la droite projective considérée comme sous-variété du plan projectif).

2) Cependant, sa prétendue démonstration du théorème de dualité se serait appliquée sans changement à la définition de Betti!

A ce moment-là, il a dû comprendre qu'il n'arriverait jamais à des démonstrations satisfaisantes, en suivant Riemann et Betti. En 1900, dans deux mémoires intitulés " Compléments à l'Analysis Situs " , il changea entièrement de point de vue, en créant de toutes pièces la Topologie combinatoire .

Il est probable qu'il inventa cette nouvelle approche en réfléchissant à une généralisation de la classique formule d'Euler pour les polyèdres, qu'il avait énoncée et tenté de prouver dans son mémoire de 1895. En 1855, Schläfli avait prouvé que pour un polyèdre borné convexe dans R^n , on a la formule générale

$$(4) \qquad \alpha_0 - \alpha_1 + \alpha_2 - \ldots + (-1)^n \alpha_n = 1$$

où α_j est le nombre des " facettes " de dimension j dans le polyèdre (l'intérieur du polyèdre est considéré comme l'unique " facette " de dimension n). D'autre part, Jordan avait énoncé que, dans R^3, si un polyèdre considéré comme surface compacte connexe , est orientable et de genre g, on a la relation

$$(5) \qquad \alpha_0 - \alpha_1 + \alpha_2 = 2-2g$$

qui se réduit à la formule d'Euler pour g = 0. Poincaré se proposa de montrer que pour un polyèdre X quelconque de dimension p, on a

$$(6) \qquad \alpha_0 - \alpha_1 + \ldots + (-1)^p \alpha_p = b_0 - b_1 + b_2 - \ldots + (-1)^p b_p$$

où les b_j sont les " nombres de Betti " de X tels qu'il les avait définis. Il avait dû pour cela commencer par définir (apparamment pour la première fois!) la notion générale de polyèdre (curviligne) de dimension p: c'est, dans un R^N, une réunion disjointe X d'un nombre fini de " cellules " a_i^q $(0 \leq q \leq p,\ 1 \leq i \leq \alpha_q)$, telles que chaque adhérence $\overline{a_i^q}$ dans R^N soit homéomorphe à une boule fermée dans R^q , a_i^q étant l'image par cet homéomorphisme de l'intérieur de la boule, et la frontière $\overline{a_i^q} - a_i^q$ réunion (disjointe) de cellules a_j^r de dimension $r \leq q-1$. Poincaré impose en outre la condition supplémentaire que toute cellule de dimension p-1 est sur la frontière de deux cellules de dimension maxima p. Les cellules a_i^q des diverses dimensions $\leq p$ constituent ce qu'on a appelé après Poincaré une triangulation T de l'espace X; naturellement, un même espace X peut avoir plusieurs triangulations distinctes, et il est très facile de construire des triangulations d'espaces usuels tels que S_n, T^n ou $P_n(R)$.

L'idée qui a dû guider Poincaré est que, dans sa conception de 1895, pour évaluer les nombres de Betti, il fallait considérer des expressions (2) , où les v_j peuvent être des sous-variétés compactes, connexes et orientées quelconques; dans une variété X il y a " beaucoup trop " de ces sous-variétés pour qu'un calcul sur elles soit praticable. La grande découverte qui a permis la naissance de la Topologie algébrique est qu'on peut remplacer cet immense ensemble par les seules cellules d'une triangulation fixe T , et obtenir ainsi, non seulement le seconde membre de (6) (la " caractéristique d'Euler-Poincaré "), mais aussi les nombres de Betti b_j eux-mêmes.

L'idée de base (qui est déjà en filigrane dans la tentative de 1895 pour prouver la formule (6)) est certainement qu'avec des combinaisons de " cellules " de dimension q, on doit pouvoir " approcher " des variétés différentielles de dimension q, de même qu'on

" approche " une courbe par un polygone rectiligne dans le plan; c'est seulement Brouwer que la rendra opérationnelle (voir ci-dessus). Mais pour créer son algorithme, Poincaré écarte délibérément toute notion de variation continue; les seules données géométriques qu'il utilise sont les positions respectives des cellules et des orientations qu'on leur attribue (arbitrairement). Tout le reste est purement algébrique.

Les objets que l'on manipule sont en effet les combinaisons linéaires formelles (pour chaque dimension p) $z = \sum_i \lambda_i a_i^p$ des cellules de dimension p, à coefficients entiers, que l'on appellera plus tard les p-chaînes . Elles sont destinées à se substituer aux sous-variétés de dimension p, mais on ne cherche pas à leur donner une interprétation géométrique; toutefois, il faut introduire la notion algébrique qui se substitue à la propriété de " former frontière ". Pour cela, Poincaré attache à toute couple de cellules (a_i^p, a_j^{p-1}) un entier ε_{ij}^p , egal à 0 si a_j^{p-1} n'est pas contenu dans la frontière de a_i^p, à +1 (resp. -1) s'il est contenu dans cette frontière et si son orientation est en relation directe (resp. inverse) avec celle de a_i^p. Il forme alors la (p-1)-chaîne

$$(7) \qquad bz = \sum_{i,j} \lambda_i \varepsilon_{ij}^p a_j^{p-1} \quad .$$

A partir de là, les p-cycles et p-bords sont définis comme nous faisons, et aussi en substance les groupes d'homologie $H_p(T;Z)$ de la triangulation T, à coefficients entiers, bien que Poincaré n'utilise que le langage des " matrices d'incidence " (ce sont les matrices des opérateurs " bord " $b: C_p \to C_{p-1}$ entre les Z-modules de chaînes, relativement aux bases canoniques formées par les cellules de T).

Poincaré a ainsi forgé un algorithme régulier pour le calcul des invariants du Z-module $H_p(T;Z)$ (le " nombre de Betti " b_p est son rang , et les " coefficients de torsion " découverts par Poincaré, les facteurs invariants $\neq 1$); c'est un progrés considérable, dont on peut se convaincre en calculant l'homologie des exemples données par Betti à l'aide de triangulations simples. Poincaré en déduit aussi la formule (6), cette fois de façon entiérement correcte. Il

convient en outre de mentionner deux nouvelles techniques qu'il invente dans les " Complements ", et qui resteront fondamentales en Topologie combinatoire: la subdivision barycentrique et la triangulation duale d'une triangulation; elles lui permettent de donner une démonstration rigoureuse de son théorème de dualité (pour deux triangulation " duales ").

Mais naturellement cette nouvelle méthode soulevait aussitôt deux problèmes:

1) Quels sont les espaces susceptibles d'une triangulation?

2) Deux triangulation T, T' d'un même espace X donnent-elles la même homologie (c'est seulement à cette condition qu'on peut prouver que les modules $H_p(T;Z)$ sont invariants par homéomorphisme)?

Poincaré fit quelques tentatives pour donner des résponses à ces questions, pour les variétés différentielles, mais sans succès, en raison des " pathologies " liées aux questions des intersection de sous-variétés (même pour des variétés analytiques, une telle intersection admet en général des " singularités ").

On peut donc dire qu'il a défini toutes les notions de base concernant l'homologie et le groupe fondamental (aucune autre ne sera introduite en Topologie algébrique avant 1930), mais qu'il n'a pu trouver aucun moyen de démontrer des théorèmes de Topologie où entrent ces notions.

II. L.E.J.Brouwer (1881-1966)

L'outil technique fondamental qui permit ces démonstrations, l'approximation simpliciale, fut imaginé par Brouwer en 1911, mais c'est seulement en 1915 qu'Alexander eut l'idée de l'appliquer à la preuve du théorème d'invariance.

La notion d'approximation simpliciale est très simple: elle généralise l'idée d'approcher une fonction continue réelle d'une variable réelle (qui, comme on sait, peut présenter des aspects très " pathologiques ") par des fonctions linéaires par morceaux. On travaille sur un couple (X,T) formé d'un espace topologique X muni d'une triangulation T, qui sont plus généraux que les " polyèdres " de Poincaré en ce qu'on ne suppose plus que si n est la dimension maxima

des cellules de T, toute cellule de dimension n-1 est dans la frontière de deux cellules de dimension n. En outre, on montre qu'on peut se limiter au cas où les cellules sont des simplexes rectilignes (chacun étant ouvert dans la variété linéaire affine qu'il engendre), et où toute facette d'un simplexe de T est elle-même un simplexe de T; de tels couples (X,T) sont appelés complexes simpliciaux euclidiens.

Soient (X,T), (X',T') deux complexes simpliciaux euclidiens. Une application simpliciale $f: (X,T) \to (X',T')$ est une application de X dans X' telle que, pour tout p et tout p-simplexe σ de T, $f(\sigma)$ est un q-simplexe de T' avec $q \leq p$, et que la restriction f/σ est une application affine de σ sur $f(\sigma)$. L'application f est donc entièrement déterminée par les images f(v) des sommets de T, qui doivent être des sommets de T', tel que si $v_0, v_1, \ldots, v_p$ sont les sommets d'un p-simplexe σ de T, $f(v_0), f(v_1), \ldots, f(v_p)$ sont les sommets (non tous distincts si $q < p$) d'un q-simplexe de T' avec $q \leq p$; f/σ n'est injective que si les $f(v_j)$ sont tous distincts, auquel cas f/σ est une bijection sur $f(\sigma)$. Si $q < p$, on dit que $f(\sigma)$ est un p-simplexe dégénéré.

Si maintenant $f: X \to X'$ est une application continue quelconque et ε un nombre > 0, une approximation simpliciale de f à ε près est une application simpliciale $g: (X,T_1) \to (X',T'_1)$, où T_1 (respect. T'_1) est obtenue à partir de T (resp. T') par subdivisions barycentriques successives, et on a

$$|f(x) - g(x)| \leq \varepsilon \quad \text{dans } X. \tag{8}$$

La notion a été utilisée par Brouwer en 1911; elle a été mise sous sa forme définitive par Alexander. On considère une triangulation T'_1 de X' obtenue par subdivisions barycentriques successives de T', répétées suffisamment pour que le diamètre des simplexes de T'_1 soit $\leq \varepsilon/3$. Pour tout sommet v' de T'_1, on considère son étoile St(v') (définie déjà par Poincaré sous le nom d' " aster "): c'est la réunion de $\{v'\}$ et de tous les simplexes de T'_1 qui ont v' pour sommet; c'est un ensemble ouvert dans X' et les St(v') forment un recouvrement ouvert de X'. Par l'uniforme continuité de f, on peut

trouver une subdivision barycentrique répétée T_1 de T telle que pour tout sommet v de T_1, f(St(v)) soit contenu dans au moins une étoile de T'_1. Soient alors a_k ($1 \leq k \leq N$) les sommets distincts de T_1, b_j ($1 \leq j \leq N'$) les sommets distincts de T'_1; pour tout k, soit j(k) un indice tel que $f(St(a_k)) \subset St(b_{j(k)})$. Si p+1 des a_k sont les sommets distincts d'un p-simplexe σ de T'_1, les $b_{j(k)}$ correspondants (qui ne sont pas nécessairement distincts) sont nécessairement les sommets d'un q-simplexe τ de T'_1 avec $q \leq p$, par définition d'une étoile. On définit alors une application simpliciale g de (X,T_1) dans (X',T'_1) en prenant $g(a_k) = b_{j(k)}$; avec les notations précédentes, on a alors $g(\sigma) \subset \tau$, donc $|f(x) - g(x)| \leq \varepsilon$ pour tout $x \in \sigma$. En outre, pour tout $x \in \sigma$, f(x) est dans un simplexe de T'_1 dont $g(\sigma)$ est une facette, donc le segment d'extrémités f(x) et g(x) est contenu dans X', et par suite $(x,t) \mapsto (1-t)f(x) + tg(x)$ est une homotopie entre f et g ($0 \leq t \leq 1$).

Il est très surprenant que Brouwer ne se soit jamais intéressé à la théorie de l'homologie et ne cite aucun des mémoires de Poincaré sur ce sujet. Tous ses travaux sont ramenés, souvent de façon extrêmement détournée, à une seule notion découverte par lui en 1910, celle de degré d'une application continue $f: M \to N$, où M et N sont des " polyèdres " au sens de Poincaré, connexes et orientés, et ont une même dimension n.

L'idée de degré est très simple lorsque M et N sont des variétés de classe C^∞ : on montre qu'il y a des points $y \in N$ tels que $f^{-1}(y)$ soit un ensemble fini $\{x_1, x_2, \ldots, x_r\}$, et qu'en chacun des x_j il y ait un voisinage V_j de x_j dans M tel que f/V_j soit un difféomorphisme de V_j sur un voisinage de y; si alors p (resp. q) est le nombre des x_j tels que f/V_j préserve (resp. renverse) l'orientation, le nombre $c = p-q$ est indépendant du choix de y vérifiant ces conditions; c'est le degré de f. Mais Brouwer veut considérer des polyèdres M, N quelconques, et une application f qu'on suppose seulement continue; or, même si $M = N = S_1$, on peut aisément, à l'aide d'une courbe de Peano, donner des exemples où $f^{-1}(y)$ a la puissance du continu pour tout $y \in S_1$. L'idée de Brouwer est de considérer des approximations simpliciales de f; si l'on peut définir

le degré d'une application simpliciale et montrer que ce degré est le même pour toutes les approximations simpliciales g de f dès que la distance de f à g est assez petite, on prendra ce nombre comme définition du degré de f.

Soient donc T (resp. T') une triangulation de M (resp. N) dont les simplexes sont assez petits, et g une application simpliciale $(M,T) \to (N,T')$. Soient T_{n-1} (resp. T_{n-2}) la réunion des simplexes de T de dimension $\leq n-1$ (resp. $\leq n-2$). L'image $g(T_{n-2})$ est complémentaire d'un ouvert connexe Ω de N, partout dense dans N. Si $y \in \Omega$ n'est pas dans $g(T_{n-1})$, $g^{-1}(y)$ est un ensemble $\{x_1, x_2, \ldots, x_r\}$ d'un nombre fini de points, dont chacun est intérieur à un n-simplexe de T; pour chacun de ces simplexes σ, g/σ est une bijection sur le simplexe $g(\sigma)$ et une application affine. Soit p (resp. q) le nombre de ces simplexes pour lesquels g/σ préserve (resp. renverse) l'orientation. Brouwer montre que p-q est le même pour tous les points de Ω non dans $g(T_{n-1})$; pour cela, il joint deux points y_1, y_2 de Ω non dans $g(T_{n-1})$ par une ligne brisée et prouve que lorsque y se déplace sur cette ligne et franchit $g(T_{n-1})$, le nombre p-q ne change pas; il utilise pour cela le fait que tout (n-1)-simplexe de T est sur la frontière de deux n-simplexes, ce qui entraîne que p et q augment ou diminuent tous deux du même nombre. Il faut ensuite prouver que p-q est le même pour deux applications simpliciales assez voisines.

La théorie de l'homologie permet de donner une définition beaucoup plus simple et plus générale du degré. D'une façon générale, pour un complexe simplicial euclidien (X,T), notons $C_p(T)$ la somme directe des Z-modules $C_p(T)$ des p-chaînes de T, par $H_p(T)$ la somme directe des Z-modules d'homologie $H_p(T)$. Si (X,T) et (X',T') sont deux complexes simpliciaux euclidiens, et $f: (X,T) \to (X',T')$ une application simpliciale, on définit une application Z-linéaire

$$f : C_p(T) \longrightarrow C_p(T') \tag{9}$$

par la formule

$$f\Big(\sum_j \lambda_j \sigma_j\Big) = \sum_j \pm \lambda_j f(\sigma_j) \tag{10}$$

où au second membre, il faut remplacer par 0 les $f(\sigma_j)$ qui sont dégénérés, et pour les autres, le signe est + si f/σ_j préserve l'orientation et - dans le cas contraire. On vérifie aisément que cette application f permute avec l'opérateur bord; il en résulte par passage aux quotients un homomorphisme de Z-modules gradués

$$f_* : H_p(T) \longrightarrow H_p(T') \quad . \tag{11}$$

En particulier, si pour un entier n, $H_n(T)$ et $H_n(T')$ sont tous deux libres et monogènes et si z, z' sont des générateurs de ces modules, on a

$$f_*(z) = c \cdot z' \tag{12}$$

où l'entier c reste le même quand on remplace z par -z et z' par -z'. Cela s'applique aux polyèdres orientés considérés par Brouwer lorsque n est leur dimension, et c est alors le degré de f.

III. J.W.Alexander (1888-1971)

Le mathématicien américain J.W.Alexander a donné 3 preuves du théorème d'invariance, la troisième étant un corollaire de son fameux théorème de dualité. La plus simple est la seconde, dont on peut résumer les grandes lignes comme suit. On a deux complexes simpliciaux euclidiens (X,T), (X',T'), et un homéomorphisme $f: X \xrightarrow{\sim} X'$, dont $g: X' \xrightarrow{\sim} X$ est l'inverse. Il s'agit de prouver que les Z-modules $H_p(T)$ et $H_p(T')$ sont isomorphes.

L'idée est de considérer la suite (T_i) des subdivisions barycentriques successives de T. Il y a un homomorphisme canonique

$$sd_i : C_p(T) \longrightarrow C_p(T_i) \tag{13}$$

qui, à chaque p-simplexe σ de T, fait correspondre la somme $\Sigma\tau$ des p-simplexes de T_i contenus dans σ, et munis de l'orientation induite par celle de σ. Un résultat simple qui remonte à Poincaré est que

cet homomorphisme de chaînes commute avec l'opérateur bord, et que l'homomorphisme

$$(14) \qquad (sd_i)_* : H_\bullet(T) \longrightarrow H_\bullet(T_i)$$

qu'on en déduit est bijectif.

Soit de même (T'_j) la suite des subdivisions barycentriques successives de T'. On peut trouver j assez grand et une approximation simpliciale $g_j : T'_j \to T$ de g par le procédé d'Alexander rappelé plus haut. Puis on peut trouver une approximation simpliciale $f_{ji} : T_i \to T'_j$ de f par le même procédé; on en deduit une approximation simpliciale

$$h_i = g_j \circ f_{ji} : T_i \longrightarrow T$$

de l'identité $g \circ f = 1_X$, et par suite un homomorphisme composé

$$(h_i)_* : H_\bullet(T_i) \xrightarrow{(f_{ji})_*} H_\bullet(T'_j) \xrightarrow{(g_j)_*} H_\bullet(T) \quad .$$

Il suffit de prouver que $(h_i)_*$ est bijective, car on en déduira que $(g_j)_*$ est surjective et $(f_{ji})_*$ injective; en échangeant les rôles de g et f, et de T et T'_j, on voit de même que $(f_{ji})_*$ est surjective, ce qui prouvera le théorème d'invariance.

La démonstration se fait en montrant que l'application composée

$$(15) \qquad C_\bullet(T) \xrightarrow{sd_i} C_\bullet(T_i) \xrightarrow{\tilde{h}_i} C_\bullet(T)$$

donne en homologie l'application identique

$$(16) \qquad H_\bullet(T) \xrightarrow{(sd_i)_*} H_\bullet(T_i) \xrightarrow{(h_i)_*} H_\bullet(T) \quad .$$

Pour cela, il suffira prouver le lemme suivant : pour tout p-simplexe σ_p de T, on a, compte tenu de la définition de $\tilde{h}_i$,

$$(17) \qquad \tilde{h}_i(sd_i(\sigma_p)) = \sigma_p \quad .$$

On raisonne par récurrence sur p, le lemme étant évident pour p=0. Pour tout sommet v d'un p-simplexe τ de T_i contenu dans σ_p, il résulte de la construction d'Alexander et du fait que $g \circ f$ est l'identité, que $h_i(v)$ ne peut être qu'un des sommets de σ_p ; donc $h_i(\tau)$ ne peut être que σ_p ou un p-simplexe dégénéré; on a par suite

$$(18) \qquad \tilde{h}_i(sd_i(\sigma_p)) = c \cdot \sigma_p$$

pour un entier $c \in \mathbb{Z}$. Mais comme $\tilde{h}_i$ et sd_i permutent aux opérateurs bord, on a aussi

$$(19) \qquad \tilde{h}_i(sd_i(b\sigma_p)) = c \cdot b\,\sigma_p \quad .$$

Toutefois, l'hypothèse de récurrence entraîne $\tilde{h}_i(sd_i(b\,\sigma_p)) = b\,\sigma_p$, donc $c = 1$, ce qui achève la démonstration.

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KALUZA - KLEIN THEORIES

Yvonne Choquet-Bruhat

Introduction.

The last decade has seen impressive progress in the unification of the three fundamental interactions of elementary particles, electromagnetic, weak and strong into one geometric structure, a Yang-Mills field. Such a field is indeed the curvature of a connection on a principal fiber bundle with base the space time. The structure group is some Lie group whose choice is still debated for the grand unification. It has been known since H. Weyl that the electromagnetic field, on a given space time manifold, can be considered as such a curvature with the structure group U(1) and it has been suggested by Weinberg and Salam that the electromagnetic and weak interactions are two aspects of the curvature of a principal bundle over space time with structure group the semi direct product SU(2) X U(1). This conjecture has received recently an experimental confirmation by the observation of the W and the Z bosons whose existence was predicted by the theory. The appropriate structure group for the so called G.U.T , unification of the electromagnetic, weak and strong interaction is still under discussion ([1]) : if the quark's model ([2]) for the building up of hadrons is accepted the group G must contain SU(3) X SU(2) X U(1) - other groups can be chosen to accomodate other proposals suggested by experimental results.

The gauge theories - other name for the Yang-Mills formulation - are a beautiful geometrization of these three fundamental interactions, but they do not include the fourth, most anciently known and geometrized by Einstein, gravitation. In the theory of General Relativity the gravitational field is the connection of a hyperbolic metric on the space time manifold. Einstein's equations in vacuum express that the Ricci tensor of thi s connection vanishes. General Relativity is said sometimes to be a gauge theory of the Lorentz group - but it is not in the same sense ([3]) as the Yang-Mills theories we spoke of before.

In fact there is an opposite way stemming from General Relativity for the Unification in one Geometric structure of the four fundamental interactions - that is to increase the number of dimensions of the space-time which bears the hyperbolic metric. We give in this paper a self contained exposition of the mathematics of the Kaluza-Klein theory, and of the Lichnerowicz proof of the non existence of

solitons in the five dimensional case, for trivial U(1) bundles. We indicate some results for the higher dimensional case, which remains however essentially unsolved for a general non abelian Lie group.

We do not in this paper consider the more general case of a non principal bundle, with fiber a homogeneous space G/H : it has been demonstrated by Coquereaux and Jadczyk [5] that the elements of a Kaluza-Klein theory are then a metric on the basis V_4, a connection on a principal bundle with basis V_4 and group $N(H)/H$ (with $N(H)$ the normalizer of H in G) and a G-invariant metric on the fibers. With appropriate conventions the equations obtained look like equations of §2.

1 - Five dimensional Kaluza-Klein theory and its Jordan-Thiry extension.

It was already discovered by Kaluza and Klein in the twenties that the coupled system of Einstein and Maxwell equations can be obtained by considering the Ricci tensor of a 5-dimensional manifold V_5, endowed with a metric $\hat{g}$ of hyperbolic signature. They supposed that V_5 was diffeomorphic to a product $V_4 \times T^1$, and that the metric $\hat{g}$ was invariant under the 1-parameter group of translations of T^1, isomorphic to T^1 itself (equivalently $U(1)$). They supposed moreover that the submanifolds $\{x\} \times T^1$, $x \in V_4$ were all isometric. This last restriction has been removed by Jordan and, independantly Thiry. It is also clear from Lichnerowicz [13] account of the theories that V_5 does not need to be a product $V_4 \times T^1$ but only to be endowed by its isometry group with a principal fiber bundle structure, with group and typical fiber $U(1)$. The base V_4 of this bundle, quotient of V_5 by the isometry group, is endowed by this quotient of a metric g, which is interpreted as the space-time metric (^{4}). The hyperplanes (subspaces of the tangent spaces to V_5) orthogonal to the orbits of the isometry group are obviously invariant by this group : they can be interpreted as the horizontal spaces of a $U(1)$ connexion on V_5.

The metric induced on an orbit (1-dimensional) is determined by a scalar function $x \mapsto \xi(x)$ on the base manifold. We thus see appear the geometrical elements of an Einstein-Maxwell field on V_4 (space-time metric and $U(1)$ connexion), and as a supplementary unknown, a scalar field. The physical interpretation of this field by Jordan was a varying gravitational "constant", idea introduced in another context by Dirac : this interpretation seems in contradiction with very recent experimental results (Canuto and Al., Nasa reports). Thiry and Lichnerowicz, in strict accord with its appearance in the equations, interpret it as a "vacuum permeability" - with the emphasis now on elementary particles this field is considered as associated to a neutral, scalar boson (cf Pigeaud [14]).

Equations.

We take local coordinates adapted to the fibered structure of V_5, that is we consider a local trivialisation of V_5 :

$$\phi(\pi^{-1}(U)) \to U \times T^1$$

with U an open set of V_4, domain of a coordinate chart, and π the projection $V_5 \to V_4$, and we take as coordinates in $\pi^{-1}(U)$ the coordinates x^α, $\alpha = 0,1,2,3$ in U and $x^5 \in T^1$; a generic metric on V_5 reads

$$d\hat{s}^2 = \hat{g}_{MN}\, dx^M\, dx^N = \hat{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2\, \hat{g}_{\alpha 5}\, dx^\alpha\, dx^5 + \hat{g}_{55}(dx^5)^2 \quad .$$

The isometry hypothesis is equivalent to the fact that all coefficients $\hat{g}_{MN}$ depend only on x^α variables.

The space time metric on the quotient V_4 (distance between orbits) is

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \quad \text{with} \quad g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \hat{g}_{\alpha 5}\, \hat{g}_{\beta 5}\, (g_{55})^{-1}$$

By choice of a moving frame with e_5 axis tangent to the orbit the metric $d\hat{s}^2$ can be written

$$d\hat{s}^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + \xi^2(\hat{\omega}^5)^2$$

with

$$\hat{\omega}^5 = dx^5 + A_\alpha\, dx^\alpha$$

$$\xi^2 = \hat{g}_{55} \quad , \quad A_\alpha = \xi^{-2}\, \hat{g}_{\alpha 5}$$

In the local coordinates $A = A_\alpha\, dx^\alpha$ can be considered as a 1-form on V_4, but by a change of adapted coordinates on the fiber bundle V_5

$$x'^5 = x^5 + f(x^\alpha) \quad , \quad x'^\alpha = x^\alpha$$

it obeys the transformation law of a $U(1)$ connection (equivalently an electromagnetic potential) since then

$$\hat{g}'_{\alpha 5} = \hat{g}_{\alpha 5} + \hat{g}_{55}\, \frac{\partial f}{\partial x^\alpha} \quad , \quad \hat{g}'_{55} = \hat{g}_{55}$$

A straightforward computation gives that in the moving frame

$$\hat{\omega}^\alpha = dx^\alpha \quad , \quad \hat{\omega}^5$$

the components with space time ($\alpha = 0,1,2,3$) indices of the riemannian connection of the metric $\hat{g}$ coincide with the riemannian connection of the space-time metric g

$$\hat{\omega}^\beta{}_{\alpha\gamma} = \omega^\beta{}_{\alpha\gamma}$$

while the components

$$\hat{\omega}^5{}_{\alpha\beta} = F_{\alpha\beta} \equiv \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$$

coincide with the pull back on V_4 of the $U(1)$ curvature (electromagnetic field) of the connection with pull back the 1-form A.

The $(V_5, \hat{g})$ Ricci tensor r can be decomposed as follows

1-1 $$\hat{R}_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}\, \xi^2\, F_\alpha{}^\lambda\, F_{\beta\lambda} - \xi^{-1}\, \nabla_\alpha\, \partial_\beta\, \xi$$

1-2 $$\hat{R}_{\alpha 5} \equiv \frac{1}{2}\, \xi^{-1}\, \nabla_\lambda\, (F_\alpha{}^\lambda\, \xi^3)$$

1-3 $$\hat{R}_{55} \equiv \frac{\xi^4}{4}\, F_\alpha{}^\beta\, F^\alpha{}_\beta - \xi\, \nabla_\mu\, \partial^\mu\, \xi$$

We deduce from these equations :

$$\hat{R} = R - \frac{\xi^{-2}}{4} F_{\alpha\beta} F^{\alpha\beta} - 2 \xi^{-1} \nabla_\alpha \nabla^\alpha \xi$$

In the case ξ = constant (Kaluza-Klein ansatz) we recognize in $\hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R} = 0$ the Einstein equations with sources an electromagnetic field, in $\hat{R}_{\alpha 5} = 0$ the second set of Maxwell equations ; the first set $dF = 0$ is a consequence of the definition (5) $F = dA$.

When ξ is not constant it appears as a source in Einstein and Maxwell equations and satisfies, by $\hat{R}_{55} = 0$, a wave type partial differential equation.

2 - Kaluza-Klein theories on an arbitrary principal bundle.

A generalization of the Kaluza-Klein ansatz to a riemannian fiber bundle of arbitrary dimension d and Lie group G has been proposed already twenty years ago by DeWitt |8| and Kerner |12|. Its Jordan-Thiry type extension has been worked out more recently |1|,|5|. We shall in the following call this extension "Kaluza-Klein", as it has become usual.

In such theories the (pseudo) riemannian manifold $(V_d, \hat{g})$ has a Lie group G of isometries which is supposed (6) to endow it with a principal fiber bundle structure with typical fiber and structure group G : the fiber at $\hat{x} \in F_d$ is a submanifold diffeomorphic to G, it is the orbit of $\hat{x}$ under the action of G which acts transitively in each orbit, by a right action. The base manifold V_4 is the space of orbits. In fibered coordinates, associated to a local trivialisation

$$\phi(\pi^{-1}(U)) \rightarrow U \times G$$

with U domain of a coordinate chart in V_4, π projection $V_d \rightarrow V_4$, we write the metric $\hat{g}$:

$$d\hat{s}^2 = \hat{g}_{MN} dx^M dx^N = \hat{g}_{\alpha\beta} dx^\alpha dx^\beta + 2 \hat{g}_{\alpha m} dx^\alpha \theta^m + \hat{g}_{mn} \theta^m \theta^n$$

where x^α, $\alpha = 0,1,2,3$ are coordinates in U and θ^m, $m = 5,..,d$ are right invariant 1-forms on G, a basis of the dual of the right Lie algebra $\mathcal{G}$ of G.

This metric is invariant under the right action of G if $\hat{g}_{\alpha\beta}$, $\hat{g}_{\alpha m}$, $\hat{g}_{mn}$ depend only on the "space time coordinates" x^α. We can also write, by choice of a moving frame with axes $\hat{e}_m$ tangent to the orbit, $\hat{e}_\alpha$ orthogonal to it (7)

2-1 $$d\hat{s}^2 = g_{\alpha\beta} dx^\alpha dx^\beta + \xi_{mn} \hat{\omega}^m \hat{\omega}^n$$

with, ξ^{mn} denoting the inverse matrix of ξ_{mn},

$$\xi_{mn} = \hat{g}_{mn}$$

2-2 $$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \xi^{mn} \hat{g}_{\alpha m} \hat{g}_{\beta n}$$

2-3 $$\hat{\omega}^m = \theta^m + A^m \quad , \quad A^m = A^m_\alpha dx^\alpha \quad , \quad A^m_\alpha = \xi^{mn} \hat{g}_{\alpha n}$$

The metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

determines the distance between orbits - it is interpreted as the space time metric on V_4.

The 1-form $A = A^m e_m$, e_m basis of $\mathcal{G}$, is interpreted as the representant on U in the chosen coordinates of a connection on the principal fiber bundle V_d, whose horizontal spaces (orthonormal to the orbits) are determined by the Pfaff system of equations

$$\hat{\omega}^m = 0 \quad , \quad m = 5,\ldots, d \ .$$

The coefficients ξ_{mn} constitute a "scalar multiplet", they take their value in the space of non degenerate quadratic forms in $\mathcal{G}$. In realistic models this space is restricted to the cone of positive definite quadratic forms, the metric $\hat{g}$ has signature (-,+..+).

The metric g does not depend on the trivialisation.

The 1-form A transforms upon a change of trivialisation determined by a mapping :

$$u : G \rightarrow G \quad \text{by} \quad g \mapsto h(x)\, g \quad , \quad x \in V^4$$

by the usual law $\mathcal{G} \rightarrow \mathcal{G}$ with :

2-4 $$A \rightsquigarrow Ad(h^{-1})\, A + \theta_{M.C}$$

the scalar multiplet ξ transforms by

2-5 $$\xi \rightsquigarrow \varrho^2 \, Ad\, h \, \xi \ .$$

The form 2-1 of the metric leads to a relatively simple computation of the corresponding riemannian connection and curvature. We use now the fact that, due to their definition :

2-6 $$d\theta^m = -\frac{1}{2} C^m_{pq} \theta^p \wedge \theta^q$$

where $C_p{}^m{}_q$ are the structure constants of $\mathcal{G}$.

The results are as follows :

The Ricci coefficients $\hat{C}^M{}_{PQ}$ of the moving frame

$$\hat{\omega}^M = (\hat{\omega}^\alpha = dx^\alpha \ , \ \hat{\omega}^m = \theta^m + A^m_\alpha dx^\alpha) \quad , \quad d\hat{\omega}^M = -\frac{1}{2} C^M{}_{PQ} \hat{\omega}^P \wedge \hat{\omega}^Q$$

are :

$$\hat{C}_p{}^m{}_q = C_p{}^m{}_q \quad , \quad \hat{C}_M{}^\alpha{}_N = 0 \ ,$$

$$\hat{C}_\alpha{}^m{}_n = -C_p{}^m{}_n A^p_\alpha = -\hat{C}_n{}^m{}_\alpha \quad , \quad \hat{C}_\alpha{}^m{}_\beta = -F_\alpha{}^m{}_\beta$$

Where we have set :

2-7 $$F_\alpha{}^m{}_\beta = \partial_\alpha A^m_\beta - \partial_\beta A^m_\alpha + C_p{}^m{}_q A^p_\alpha A^q_\beta$$

thus $F = (F_\alpha{}^m{}_\beta)$ is the $\mathcal{G}$-valued 2-form representant in the chosen coordinates of the curvature of the connection with representant A.

The coefficients of the riemannian connection of $\hat{g}$ are given by the formula (cf |2|) :

$$\omega^M{}_{Pq} = \frac{1}{2} g^{MN}(\partial_P g_{NQ} + \partial_Q g_{NP} - \partial_N g_{PQ}) - \frac{1}{2}(C^M{}_{QP} + g^{MN} g_{LP} C^L{}_{QN} + g^{MN} g_{LQ} C^L{}_{PN}) .$$

They are for the indices 0,1,2,3 identical with those of the connection of the space time metric :

$$\hat{\omega}^\alpha{}_{\beta\gamma} = \omega^\alpha{}_{\beta\gamma}$$

while the others are :

$$\hat{\omega}^\alpha{}_{mn} = -\frac{1}{2} g^{\alpha\beta} D_\beta \xi_{mn}$$

with D_β the covariant derivative of ξ_{mn} with respect to the connection A, given by, in agreement with 2-5 :

$$D_\beta \xi_{mn} = \partial_\beta \xi_{mn} + \xi_{\ell m} C^\ell{}_{pn} A^p_\beta + \xi_{\ell n} C^\ell{}_{pm} A^p_\beta$$

$$\hat{\omega}^\alpha{}_{\beta m} = \frac{1}{2} F_{m,\beta}{}^\alpha = \hat{\omega}^\alpha{}_{m\beta} \; ; \; \hat{\omega}^m{}_{\alpha\beta} = \frac{1}{2} F_\alpha{}^m{}_\beta$$

$$\hat{\omega}^m{}_{\beta n} = \frac{1}{2} \xi^{mp} D_\beta \xi_{np} - C_r{}^m{}_n A^r_\beta \; ; \; \hat{\omega}^m{}_{n\beta} = \frac{1}{2} \xi^{mp} D_\beta \xi_{np}$$

and $\hat{\omega}^m{}_{np}$ is equal to the riemannian connection of the metric induced by $\hat{g}$ on the orbit G_x , that is :

$$\hat{\omega}^m{}_{np} = \omega^m{}_{pn}$$

The components of the curvature tensor are given by the general formula :

$$\hat{R}_M{}^N{}_{PQ} = \hat{\omega}^N{}_{PR} \hat{\omega}^R{}_{QM} - \hat{\omega}^N{}_{QR} \hat{\omega}^R{}_{PM} - \hat{C}^R{}_{PQ} \hat{\omega}^N{}_{RM} + \partial_\ell \hat{\omega}^N{}_{QM} - \partial_Q \hat{\omega}^N{}_{PM}$$

We are concerned with the Ricci tensor $R_{MN} = R_M{}^P{}_{NP}$ whose components are found to be :

2-8-a $$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} F_\alpha{}^\lambda{}_{,m} F_\beta{}^m{}_\lambda - \frac{1}{4} \xi^{mp} \xi^{nq} D_\alpha \xi_{mn} D_\beta \xi_{pq} - \frac{1}{2} D_\alpha(\xi^{mn} D_\beta \xi_{mn})$$

2-8-b $$\hat{R}_{\alpha m} = \frac{1}{2} D^\lambda F_{\alpha\lambda,m} + \frac{1}{4} F_\alpha{}^\lambda{}_{,m} \xi^{np} D_\lambda \xi_{np} - \frac{1}{2} C_m{}^p{}_n \xi^{nq} D_\alpha \xi_{pq}$$

2-8-c $$\hat{R}_{mn} = R_{mn} + \frac{1}{4} F_{\alpha\beta;m} F^\alpha{}_n{}^\beta + \frac{1}{2} \xi^{pq} D_\alpha \xi_{mp} D^\alpha \xi_{nq} - \frac{1}{4} \xi^{pq} D^\alpha \xi_{mn} D_\alpha \xi_{pq}) - \frac{1}{2} D_\alpha D^\alpha \xi_{mn} .$$

In these formulas all fields are defined on the space time V_4 . The Ricci curvature (R_{mn}) of G_x, endowed with the induced metric $\xi_{mn} \theta^m \theta^n$ depends on x through ξ^{pq}, but not through their derivatives.

The scalar curvature is :

2-9 $$\hat{R} = R - \frac{1}{4} F_\alpha{}^m{}_\beta F^\alpha{}_m{}^\beta + \frac{1}{4} \xi^{mn} \xi^{pq} D_\alpha \xi_{mp} D^\alpha \xi^{nq} + \frac{1}{4}(\xi^{mn} D^\alpha \xi_{mn})(\xi_{pq} D_\alpha \xi_{pq}) - g^{\alpha\beta} D_\alpha (\xi^{mn} D_\beta \xi_{mn}) + \Xi$$

where $\Xi \equiv R(G_x) \equiv \xi^{mn} R_{mn}$ depends on x through the ξ_{pq} , but not through their derivatives.

The equation ($\hat{R}_{MN} = 0$) reduce to the ones 1-1,1-2,1-3 of the Jordan Thiry theory in the case where G is U(1) (note that in the notations of §2, $\xi^2 = \xi_{55}$).

The pull back on V_4 of the corresponding Einstein tensor is given by

$$\hat{S}_{\alpha\beta} \equiv \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R} \equiv S_{\alpha\beta} - \tau_{\alpha\beta} - s_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \Xi$$

where

$$\tau_{\alpha\beta} = \frac{1}{2}(F_\alpha{}^\lambda{}_m F_\beta{}^m{}_\lambda - \frac{1}{4} g_{\alpha\beta} F_\lambda{}^m{}_\mu F^\lambda{}_m{}^\mu)$$

is the stress energy tensor of the Yang-Mills field F and

$$s_{\alpha\beta} = \frac{1}{4} \xi^{mn} \xi^{pq}(D_\alpha \xi_{mn} D_\beta \xi_{pq} + \frac{1}{2} g_{\alpha\beta} D_\lambda \xi_{mp} D^\lambda \xi_{nq}) + \frac{1}{2} g_{\alpha\beta} D^\lambda(\xi^{mn} D_\lambda \xi_{mn}) + \frac{1}{8} g_{\alpha\beta} (\xi^{mn} D_\lambda \xi_{mn})(\xi^{pq} D^\lambda \xi_{pq}) - \frac{1}{2} D_\alpha(\xi^{mn} D_\beta \xi_{mn})$$

is the stress-energy tensor of the scalar-multiplet.

We also have

$$\hat{S}_{m\alpha} \equiv \hat{R}_{m\alpha}$$

If we take for ξ_{mn} the coefficients of a fixed biinvariant metric (8) on G we have $D_\alpha \xi_{mn} = 0$. The equation $\hat{S}_{\alpha\beta} = 0$ and $\hat{R}_{\alpha m} = 0$ reduce then to the coupled Einstein-Yang-Mills system with a cosmological constant Ξ . This system is a well posed causal system with constraints (cf |4|).

3 - Cauchy problem.

We shall now study the general case of the system

$$\hat{R}_{\alpha\beta} = 0 \quad , \quad \hat{R}_{\alpha m} = 0 \quad , \quad \hat{R}_{mn} = 0 \quad .$$

The Cauchy problem is that of determination of the fields, g, A, ξ on V_4 when they are given on a three dimensional space like submanifold S, together with their derivatives transversal to S. The problem can be formulated in an intrinsic way (see for instance for Einstein-Maxwell equations the review article |4|), and the solution can atbest be geometrically unique. The same general methods apply here, and the system can be proved to be a well posed causal system for the unknown g, A, ξ with constraints on the Cauchy data given by the equations (they contain no second derivatives ∂^2_{oo} of the unknown, $x^o = 0$ is the local equation of S)

$$\hat{S}^{o\alpha} \equiv \hat{R}^{o\alpha} - \frac{1}{2} \hat{g}^{o\alpha} \hat{R} = 0 \quad , \quad \hat{S}^{om} = 0 \quad .$$

The constraints are conserved by time evolution due to identities which are consequences of the d-dimensional Bianchi identities.

The choice of an appropriate gauge turns, on the other hand, the equations into a strictly hyperbolic system in the sense of Leray. We can take for instance the harmonic gauge for the space time metric

$$g^{\alpha\beta} \omega^\lambda{}_{\alpha\beta} = 0$$

and the generalized Lorentz gauge for the Yang-Mills potential

$$\nabla_\lambda A^{\lambda,m} = 0$$

(the covariant derivative is taken only for the index λ - it is not a gauge covariant condition). The higher order (that is second) terms of $\hat{R}_{\alpha\beta}$, $\hat{R}_{\alpha m}$ and $\hat{R}_{mn}$ reduce then respectively, up to multiplicative constants, to $g^{\lambda\mu}\,\partial^2_{\lambda\mu} g_{\alpha\beta}$, $g^{\lambda\mu}\,\partial^2_{\lambda\mu} A_{\alpha m}$, and $g^{\lambda\mu}\,\partial^2_{\lambda\mu}\,\xi_{mn}$: the characteristic matrix is diagonal, with diagonal element $g^{\lambda\mu}\,p_\lambda\,p_\mu$: the system is therefore strictly hyperbolic in the sense of Leray if $g^{\lambda\mu}$ is a hyperbolic metric.

This result can also be deduced from the general properties of an Einstein system $\hat{R}_{MN} = 0$ in arbitrary dimensions, and the propagation of isometry properties (cf |6|).

4 - Solitons.

In the recent physics literature a soliton is a global solution of the classical equations which is stationnary, i.e. independant of time and has finite energy. This last requirement is equivalent in the case of non compact space time and smooth solutions to some properties of "fall off at infinity".

With this point of view a Kaluza-Klein soliton is a global stationnary solution of the equations 2-8 on a space-time manifold V_4. A stationnary solution consists of a metric g on V_4 which admits a 1-parameter, time oriented, isometry group T, a $\mathcal{G}$-valued connection A on V_4 and a $\mathcal{G}^* \otimes \mathcal{G}^*$ valued scalar field ξ, both invariant by the 1-parameter group T. Moreover we suppose that the isometry group T endows V_4 with a fiber bundle structure with $\mathbb{R}$ as a typical fiber and structure group. One usually calls solitons only such solutions where the base $(V_3,\bar{g})$, quotient of (V_4,g) by the action of T is a complete riemannian manifold and the fields have some finite energy property. This last concept is well understood when V_3 is compact or when $(V_3,\bar{g})$ is asymptotically euclidean (9). It raises some difficulties in the Kaluza-Klein theory with non abelian isometry group whose ground state is not the Minkowski metric, but a solution of Einstein equations with a cosmological constant.

The existence of non trivial solitons is a much debated question of great physical interest, in classical and quantum developments. It was conjectured by Einstein that in pure, 4-dimensional relativity there should exist no solitons (10).This conjecture has been proved to be true by Lichnerowicz (1939). He has also proved (cf |13|), that there are no solitons in 5-dimensional Kaluza-Klein theory (equivalent to the Einstein-Maxwell system) nor in its Jordan-Thiry extension when V_3 is compact and also when V_3 is complete, asymptotically euclidean and moreover the U(1) bundle is a trivial bundle.

The Sorkin-Gross-Perry solutions.

Recently, and independantly, Sorkin and Gross-Perry have found stationnary, and even static, solutions of the (Jordan-Thiry extension of) Kaluza-Klein theory in 5-dimensions, Gross and Perry call these solutions solitons but Sorkin remarks accurately that though his solution isdefined by a smooth five dimensional (pseudo) riemannian manifold "it appears singular from a four dimensional perspective". In fact it does not define an S^1 bundle - even a non trivial one. The conclusion of Sorkin is not that its solution is to be rejected, but that it provides an example where the fifth dimension could have physically observable consequences, and so is more than a mathematical ingenious device.

The Sorkin solution is of the type :

$$d\hat{s}^2 = -(dx^o)^2 + d\tilde{s}^2$$

where $d\tilde{s}^2$ is the four dimensional, properly riemannian metric corresponding to the Taub-Nut solution of ordinary, vacuum, Einstein equations. He thus obtains a smooth static solution (V_5,g), with a space-like 1-parameter, S^1 isometry group. But V_5 is not a globally S^1 fiber bundle over the quotient V_4 of V_5 under these isometries, since - as Sorkin himself points out - the orbit over one point (the origin in the chosen coordinates) shrinks to one point. We remark also that the space V_3 of Sorkin's solution (the quotient of Taub-Nut instanton by the U(1) isometry group) is not asymptotically euclidean.

Sorkin interprets his solution as a magnetic monopole (11), because it has a non vanishing total magnetic charge, and a zero total electric charge. It has also a scalar charge, and a non vanishing gravitational "mass".

In fact Sorkin's procedure for construction of 5-dimensional solutions under the form

$$-(dx^o)^2 + d\tilde{s}^2$$

can be applied with $d\tilde{s}^2$ any "gravitational instanton" that is any properly riemannian metric g on a 4-dimensional manifold W_4 with vanishing Ricci tensor. By the theorem §5 such a riemannian manifold is flat if it is a trivial bundle $W_4 \to W_3$ under a 1-parameter S^1 isometry group and W_3 is asymptotically euclidean.

Gross and Perry also look for solutions of the type :

$$d\hat{s}^2 = -(dx^o)^2 + d\tilde{s}^2 \quad , \quad \text{with} \quad \text{Ricc}(\tilde{g}) = 0 \quad .$$

They also consider the Taub-Nut metric, but they now construct an S^1 fiber bundle $V_5 \to W_4$, endowed with this local metric in two local trivialisations : this bundle is not trivial, the potential has two local-representants linked by (all other components unchanged)

$$A'_\varphi = A_\varphi + 8\,m \quad .$$

It is another way of constructing a magnetic monopole, more in agreement with the spirit of the Kaluza-Klein theory.

5 - Solitons for a general compact Lie group.

In the case where G is compact we can always choose the basis of $\mathcal{G}$ such that $C_p{}^q{}_q = 0$. We remark then that we have

$$\xi^{mn}\, D_\mu\, \xi_{mn} = \xi^{mn}\, \partial_\mu\, \xi_{mn} = 2\, a^{-1}\, \partial_\mu\, a$$

with

$$a^2 = \det(\xi_{mn})$$

a scalar [12] on V_4.

When V_d possesses, in addition to the compact Lie group G a 1-parameter group of time like isometries which commutes with G and endows V_4 with a fiber bundle structure, the equation $\hat{R}_{oo} = 0$ reads when the space time metric is static, that is

$$g_{\alpha\beta}\, dx^\alpha\, dx^\beta = - V^2 (dx^o)^2 + \overline{g}_{ij}\, dx^i\, dx^j$$

5-1
$$\hat{R}_{oo} \equiv V\, \overline{\Delta}\, V - \frac{1}{2}\, a^{-1}\, \partial_i\, a\, \partial^i\, V - \frac{1}{2}\, \xi_{mn}\, F^m{}_o{}^i\, F_o{}^n{}_i - \frac{1}{4}\, \xi^{mn}\, \xi^{pq}\, D_o\, \xi_{mp}\, D_o\, \xi_{nq} = 0$$

(the term $D_o(\xi^{mn}\, D_o\, \xi_{mn})$ reduces to $- \Gamma_o{}^i{}_o\, \xi^{mn}\, D_i\, \xi_{mn}$).

We deduce from 5-1 through the maximum principle that V = constant if the space manifold V_3 is compact, thus also

5-2
$$F_o{}^m{}_i = 0 \quad , \quad D_o\, \xi_{mp} = 0$$

(we have supposed ξ_{mn} a positive definite metric).

The same results hold for V_3 asymptotically euclidean if we suppose moreover that V attains its maximum on V_3, which will be the case if V tends uniformly at infinity to a constant by larger or equal values.

We now consider $\xi^{mn}\, \hat{R}_{mn}$. We have

$$\xi^{mn}\, D_\alpha\, D^\alpha\, \xi_{mn} = 2\, \nabla^\alpha(\partial_\alpha \log a) + \xi^{mn}\, \xi^{nq}\, D^\alpha\, \xi_{pq}\, D_\alpha\, \xi_{mn}$$

thus

5-3
$$\xi^{mn}\, \hat{R}_{mn} = \xi^{mn}\, R_{mn} + \frac{1}{4}\, F_\alpha{}^m{}_\beta\, F^\alpha{}_m{}^\beta + \frac{1}{2}\, \xi^{pq}\, \xi^{mn}\, D_\alpha\, \xi_{mp}\, D^\alpha\, \xi_{nq} - a^{-1}\, \nabla^\alpha\, \partial_\alpha\, a$$

which reads, in our stationary case, and under the previous conclusions :

5-4
$$\xi^{mn}\, \hat{R}_{mn} \equiv \Xi + \frac{1}{4}\, F_i{}^m{}_j\, F^i{}_m{}^j + \frac{1}{2}\, \xi^{pq}\, \xi^{mn}\, D_i\, \xi_{mp}\, D^i\, \xi_{nq} - a^{-1}\, \Delta\, a = 0$$

This equation is independant of the local trivialisation of V_d, and therefore is globally defined on V_4, even when V_d is not a trivial bundle.

If V_3 is a compact manifold and if we choose G and the metric ξ such that $\Xi \geq 0$, the equation 5-4 implies :

5-5
$$\Xi = 0 \quad , \quad F_i{}^m{}_j = 0 \quad , \quad D_i\, \xi_{mp} = 0 \quad , \quad a = \text{constant}$$

the equations $\hat{R}_{ij} = 0$ imply then, as in the pure gravitational stationary case (cf §1) that V_3 is flat, as well as the whole V_d. We have proved :

Theorem - The Kaluza-Klein theory admits no static soliton with V_3 compact and $\Xi \geq 0$. For asymptotically euclidean $(V_3, \overline{g})$ the theorem still holds if we suppose that at infinity V tends to a constant by larger or equal values (then 5-2 holds) and a tends to a constant by smaller or equal values (then 5-5 holds). These conditions however have no physical reason to be satisfied.

As a step towards a solution of the general (non necessarily static) case we have the following :

Theorem - Let $V_d \to W_{d-1}$ be a principal fiber bundle, with typical fiber and structure group a 1-dimensional Lie group ($\mathbb{R}$ or S^1), and let $\hat{g}$ be a metric on V_d invariant under the right action of G, such that the quotient metric $\tilde{g}$ on W_{d-1} is elliptic. Then if $\mathrm{Ricc}(\hat{g}) = 0$ the metric $\tilde{g}$ is such that $\mathrm{Ricc}(\tilde{g}) = 0$, the $\mathbb{R}$-curvature H is zero and the $\mathbb{R}$-scalar ζ constant in either of the two cases :

1) W_{d-1} is compact

2) The bundle $V_d \to W_{d-1}$ is trivial, $(W_{d-1}, \tilde{g})$ is asymptotically flat, the $\mathbb{R}$-connection asymptotically zero and the $\mathbb{R}$-scalar ζ bounded.

Proof : We write the metric $\hat{g}$ in coordinates adapted to a trivialisation :

$$d\hat{s}^2 = \tilde{g}_{AB}\, dx^A\, dx^B - \zeta^2 (dx^o + \mu_A\, dx^A)^2 \quad .$$

the scalar ζ is real if $(V_d, \hat{g})$ is a hyperbolic manifold (the fibers are then time-like) and pure imaginary if $(V_d, \hat{g})$ is properly riemannian. We suppose $\zeta \neq 0$, for instance $\zeta > 0$ if real [resp. $i\zeta > 0$ if ζ imaginary].

We denote by $\tilde{H}$ the $\mathbb{R}$-valued curvature

$$\tilde{H}_{AB} = \tilde{\nabla}_A\, \mu_B - \tilde{\nabla}_B\, \mu_A \quad .$$

The scalar multiplet, denoted ξ in the general case is also $\mathbb{R}$-valued and equal to $-\zeta^2$, its inverse is $-\zeta^{-2}$: we have $\tilde{H}_{AB}\, \tilde{H}^{AB} = -\zeta^2\, \tilde{H}_{AB}\, \tilde{H}^{AB}$.

Since the group G is abelian the structure constants vanish and $D_A = \tilde{\nabla}_A$. The application of formulas §4 give

$$\hat{R}_{AB} = \tilde{R}_{AB} + \frac{1}{2}\, \zeta^2\, H_{AC} - \zeta^{-1}\, \tilde{\nabla}_A\, \tilde{\nabla}_\beta\, \zeta$$

$$\hat{R}_{AO} = -\frac{1}{2}\, \tilde{\nabla}^B(\zeta^2\, H_{AB}) - \frac{1}{4}\, H_A{}^B \qquad \tilde{\nabla}_B\, \zeta^2 = -\frac{1}{2}\, \zeta^{-1}\, \tilde{\nabla}^B(\zeta^3\, H_{AB})$$

$$\hat{R}_{OO} = \frac{\zeta^4}{4}\, H_{AB}\, H^{AB} + \zeta\, \tilde{\nabla}^A\, \tilde{\nabla}_A\, \zeta$$

1) We suppose W_{d-1} compact.

The equation $\hat{R}_{OO} = 0$ implies :

$$H_{AB} = 0 \quad , \quad \zeta = \text{constant}$$

and then $\hat{R}_{AB} = 0$ implies :

$$\tilde{R}_{AB} = 0$$

2) If W_{d-1} is non compact the equation $\hat{R}_{OO} = 0$ will imply ζ = constant, $H_{AB} = 0$ only if we supposed that ζ attains its minimum [resp. its maximum] on W_{d-1},

if real [resp. imaginary]. Instead of using this equation we shall follow the idea used by Lichnerowicz in the five dimensional case, namely the combination, defined globally if V_d is a trivial bundle (which is always the case when the fiber is $\mathbb{R}$)

$$\mu^A \hat{R}_{AO} = -\frac{\zeta^{-1}}{2} \overset{\sim}{\nabla}_B(\mu^A \zeta^3 H_{AB}) - \frac{\zeta^2}{4} H_{AB} H^{AB}$$

This equation will imply $H_{AB} = 0$ on W_{d-1} if $H_{AB} H^{AB}$ and $\overset{\sim}{\nabla}_B(\mu^A \zeta^3 H_{AB})$ are integrable on W_{d-1}, with μ_A and H_{AB} such that, by application of the Stokes formula the second integral is found to be zero.

If, for instance, W_{d-1} is outside of a compact set diffeomorphic to the exterior of a ball of $\mathbb{R}^{d-1}$ a sufficient condition for this annulation is

$$\mu^A = 0\left(\frac{1}{r^{(d-1)/2}}\right) \quad , \quad H_{AB} = 0\left(\frac{1}{r^{(d+1)/2}}\right) \quad , \quad \zeta \text{ bounded}$$

If $W_{d-1} = V_3 \times K$, with K some compact manifold and V_3 asymptotically euclidean, the sufficient condition is the usual one, $\mu_A = 0(\frac{1}{r})$, $H_{AB} = 0(\frac{1}{r^2})$, uniformly on K.

Remark : Ricci$(\tilde{g}) = 0$ with W_{d-1} strongly asymptotically euclidean, that is diffeomorphic, outside of a compact set to the exterior of a ball of $\mathbb{R}^{d-1}$, with a strongly asymptotically euclidean metric $\tilde{g}$, (that is of zero action) implies, at least when d-1 = 4 that $\tilde{g}$ is flat (it is part of the "positive action conjecture" proven by Schoen and Yau).

The existence of $(W_{d-1},\tilde{g})$ of a Lie group G of isometries of dimension d-4 does not seem to imply necessarily that $(W_{d-1},\tilde{g})$ is flat if G induces on $(W_{d-1},\tilde{g})$ a trivial principal bundle structure $(W_{d-1},\tilde{g}) \to (V_3,\bar{g})$. It is clear that it does if G is abelian : G being a direct product of tori an induction proof can be used.

Theorem - There exists no solutions of the extended Kaluza-Klein theory with $(V_d,\hat{g})$ a riemannian manifold of signature (- +..+) endowed with a trivial principal bundle structure $(V_d,\hat{g}) \to (V_3,\bar{g})$ by a d-3 dimensional, abelian, isometry group $G = G_o \times G_1 \times .. G_{d-4}$, with the 1-dimensional orbits of G_o, [resp. $G_1,\ldots,G_{d-4}$] time like [resp. space like], $(V_3,\bar{g})$ an asymptotically zero on V_3 (in $0(1/r)$ and $0(1/r^2)$ respectively), together with bounded scalar fields.

When G is non abelian, and V_3 non compact, the problem is still essentially open.

It has been demonstrated by Deser that the Yang-Mills equations on Minkowski space time admit no soliton. It can be proved, by an extension of his method that the coupled Eisntein-Yang-Mills equations admit no static soliton where the space $(V_3,\bar{g})$ has an expanding conformal Killing field [14] ($L_X\bar{g} = k\bar{g}$, $k>0$), and V =const.

The proof is as follows : the Einstein-Yang-Mills equations are :

$$S_{\alpha\beta} - T_{\alpha\beta} = 0$$

with

$$T_{\alpha\beta} = F_\alpha{}^\lambda{}_m F_{\beta\lambda}{}^m - \frac{1}{4} g_{\alpha\beta} F_\lambda{}^m{}_\mu F^\lambda{}_m{}^\mu$$

and

$$\hat{S}_{\alpha m} \equiv D_\lambda F_\alpha{}^\lambda{}_m = 0$$

the stationary metric of space time is written, using its bundle structure

$$ds^2 = \bar{g}_{ij} dx^i dx^j - V^2(dx^o)^2$$

If V = constant we have the identities

$$S_{ij} \equiv \bar{S}_{ij}$$

the equation $\hat{S}_{ij} = 0$ read therefore

$$\bar{S}_{ij} = T_{ij}$$

and we remark that :

$$T \equiv \bar{g}^{ij} T_{ij} = \frac{1}{4} F_i{}^h F^i{}_h - \frac{1}{2} F_i{}^o F^i{}_o$$

is positive [15] or zero, and zero only if $F_{\alpha\beta} = 0$.

We use the Bianchi identities, together with the conformal Killing vector X, to write :

$$\bar{\nabla}_i (\bar{S}^{ij} X_j) \equiv \frac{1}{2} k \bar{S}$$

thus, by integration [16] on V_3, and if the Einstein Yang-Mills equations are satisfied :

$$\int_{V_3} k T \, d\mu = 0$$

This equality implies if $k > 0$ on V_3, that $F_{\alpha\beta} = 0$, and therefore $\bar{S}_{ij} = 0$.

A proof had been given previously by Hu and Pau (cf [11], cf also Sealey [15]) in the case where $(V_3,\bar{g})$ is conformally flat, with less restrictive conditions at infinity which could also probably be extended to the more general case treated here.

The general problem for the Einstein-Yang-Mills system is still unsolved.

For the Kaluza-Klein equations, equivalent to an Einstein-Yang-Mills system with a cosmological constant, non zero for a non abelian group, the formulation of the problem itself has to be changed since much a system does not admit asymptotically euclidean solutions in the sense previously given.

Acknowledgement - I thank A. Ashtekar and A. Taub for enlightening conversations on the subject.

Footnotes.

(1) An experiment in favour of the Grand Unification would be an observation of proton decay : active experimental research is on the way, but inconclusive as yet.

(2) Other models - with a new generation of "elementary" particles are now proposed.

(3) True gauge theories of gravity - different from General Relativity - have been constructed, but there is no experimental evidences to favor them, while there are for General Relativity.
(4) In physical interpretations the physical metric is often given by a conformal metric ξ g, with ξ the fifteenth (scalar) potential.
(5) Note that A does not need to be a globally defined 1-form on V_4 though F is a globally defined 2-form.
(6) This property is a consequence of the existence of the group of isometries, under some further hypothesis (G compact, all orbits of the same type, etc..).
(7) That is $\hat{e}_m = e_m$ (vector dual to θ^m on the orbit, right invariant vector field) and $\hat{e}_\alpha = \partial_\alpha - A^m_\alpha \hat{e}_m$.
(8) Such a properly riemannian metric exists if G is compact. It is unique up to a numerical factor if G is semi-simple (cf Helgason |10|).
(9) Manifold diffeomorphic outside of a compact set to the exterior of a ball of $\mathbb{R}^3$ (or a finite disjoint union of such balls), with a metric which tends to the euclidean metric at infinity of $\mathbb{R}^3$ (in some relevant functional sense).
(10) Remark the Kruskal-Schwarzschild metric is not a counter example to the theorem because the isometry group is not time like at r = 2m.
(11) The Dirac magnetic monopole, whose physical existence has been conjectured, has not been observed yet, with non vanishing total magnetic charge (and zero total electric charge).
(12) Invariant by change of trivialisation of V_d (consequence of the compactness of G and its representation as a subgroup of the orthonormal group).
(13) We mean by his hypothesis that W_{d-1} is outside of a compact set diffeomorphic to the product of a compact manifold by the exterior of a ball in a space $\mathbb{R}^n$ and that the asymptotic behaviour of the various fields insure the vanishing of the integrals at infinity which one considers.
(14) $(\mathbb{R}^3,e)$ with e the euclidean metric admits such a Killing field, $X^i = x^i$, generating scale transformations.
(15) Recall that our signature is (-,+,+,+).
(16) We suppose $S^{ij} X_j$ to be $O(r^{-2-\varepsilon})$, $\varepsilon > 0$ on V_3, which will be satisfied under usual physical assumptions on the fields, F, H = $O(r^{-2})$ (cf Deser |7|).

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NON-EXISTENCE THEOREMS FOR QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

Wei-Ming Ni and James Serrin

1. Introduction

The semilinear elliptic equation

$$\Delta u + f(u) = 0 \tag{1.1}$$

in $\mathbb{R}^n$, where Δ is the n-dimensional Laplace operator, f is continuous, and $f(0) = 0$, arises in various brances of applied mathematics. Positive solutions of (1.1) which tend to zero at ∞, known as ground states, have received special attention and have been studied by many authors (see e.g. by Berestycki & Lions [BL], Strauss [S] and the references therein).

The case $f(u) = -u + \beta u^s + \gamma u^q$, where $\beta, \gamma \in \mathbb{R}$ and $s, q > 1$ was studied in [S]. The Sobolev critical exponent $(n+2)/(n-2)$ plays an important role as far as existence and non-existence is concerned, and Strauss obtained results in this direction under various circumstances. The case where $\beta < 0 < \gamma$ and $q > s > (n+2)/(n-2)$ was however left open in [S], and even though Strauss' results were later generalized by Berestycki & Lions [BL] this case still remains untouched. This problem serves as the motivation for our present study. In particular, we establish for this case the non-existence of both positive and radial solutions of (1.1) which are small at ∞ (see §§ 3 and 4).

It is well-known that any positive solution of (1.1) which tends to zero at infinity must be radial if $f \in C^{1+\epsilon}$ and $f'(0) < 0$. We shall therefore particularly consider the case of radial solutions of (1.1). In fact, the methods we use allow us more generally to study radial solutions of the general equation

$$\operatorname{div}[A(|Du|)Du] + f(u) = 0 \tag{1.2}$$

in $\mathbb{R}^n$, where Du is the gradient of u and the function A need satisfy only very mild conditions. A natural and simple example of (1.2), besides the case of the Laplacian where $A \equiv 1$, is given by

$$A = (1+|Du|^2)^{-1/2} ,$$

a case which arises for the mean curvature equation. A further unusual feature of our method is that it applies even in situations where A need not be elliptic (see §3 below).

We first obtain a generalized version of the well-known Pohozaev identity for radial solutions of (1.2), this identity in turn yielding non-existence of radial solutions of (1.2) in $\mathbb{R}^n$ which are small at infinity (as well as in finite balls with zero Dirichlet data). In §2 we establish our main inequality and identity, and in §3 we apply the results of §2 to equations (1.1) and (1.2), including several examples of (1.2) of special and independent interest. In §4, using techniques developed by Ni [N], we are able to determine all positive radial solutions of (1.1) in $\mathbb{R}^n$ when $f < 0$ for u small and f grows sufficiently rapidly for u large.

In a forthcoming paper [NS], we extend the results in this paper to the case where $f > 0$ for u near 0.

We should like to thank W.A. Strauss for bringing the non-existence question concerning (1.1) to our attention during a visit to Minnesota.

2. A Generalized Pohozaev Inequality

In this section we consider the equation

$$\text{(2.1)} \qquad \operatorname{div}[A(|Du|)Du] + f(u) = 0 .$$

Let $u = u(r)$ be a radial solution of (2.1); then u satisfies the equation

$$\text{(2.2)} \qquad Eu'' + \frac{n-1}{r} Au' + f(u) = 0 ,$$

where

$$\text{(2.3)} \qquad E = A + pA_p = (pA)_p , \quad p = |Du| = \pm u' .$$

Our main result here is the following

Theorem 2.1. Let u be a solution of (2.2) in $[0,R)$, $R \le \infty$. If there exists a constant c such that

$$\text{(2.4)} \qquad \int_0^p \rho A \, d\rho \ge c p^2 A$$

holds for all $p \ge 0$, then for any $k \ge 0$ and any constant a we have

$$\text{(2.5)} \qquad \frac{d}{dr}\Big[r^k \Big\{ \int_0^p \rho E \, d\rho + F(u) + a \frac{A(p)uu'}{r} \Big\} \Big]$$

$$\le r^{k-1}\Big[kF(u) - auf(u) + bA(p)p^2 + a(k-n)\frac{A(p)uu'}{r} \Big]$$

for $r \in [0,R)$, where

$$b = a + 1 - n + (1-c)k$$

and

$$F(u) = \int_0^u f(t)\,dt .$$

Equality in (2.5) holds if and only if equality holds in (2.4).

Proof. First observe by an integration by parts that (2.4) is equivalent to

$$\text{(2.6)} \qquad \int_0^p \rho E \, d\rho \le (1-c)Ap^2$$

for all $p \ge 0$. Since $\frac{dp}{dr} = \pm u''$, we have also

$$\frac{d}{dr}\Big[r^k \Big\{ \int_0^p \rho E \, d\rho + F + a\frac{Auu'}{r} \Big\} \Big]$$

$$= kr^{k-1}\Big[\int_0^p \rho E \, d\rho + F + a\frac{Auu'}{r} \Big] + r^k\Big[Eu'u'' + u'f + a\Big(\frac{Euu''}{r} + \frac{Ap^2}{r} - \frac{Auu'}{r^2} \Big) \Big]$$

$$= r^{k-1}\Big[k\Big(\int_0^p \rho E \, d\rho + F \Big) + a(k-1)\frac{Auu'}{r} + (1-n+a)Ap^2 - au\Big(\frac{n-1}{r}u' + f \Big) \Big]$$

$$= r^{k-1}\Big[k\Big(\int_0^p \rho E \, d\rho + F \Big) + a(k-n)\frac{Auu'}{r} + (a+1-n)Ap^2 - auf \Big]$$

and the result now follows by application of (2.6).

A sufficient condition for (2.4) to hold is that there exists a constant c such that the inequality

$$(2.7) \qquad (1-c)A \geq cE$$

holds for all $p \geq 0$. In particular, if $A_p \leq 0$ for all p then we can always choose $c = 1/2$.

Similarly, if A is bounded both from 0 and ∞, say

$$0 < c_1 \leq A \leq c_2 \quad \text{for all } p \geq 0,$$

then we can take $c = c_1/2c_2$. A related case occurs when equation (2.1) is uniformly elliptic; since the eigenvalues of the quadratic form associated with (2.1) are A and E (A repeated n-1 times), the condition of uniform ellipticity is just

$$\frac{A}{E} \leq \Lambda, \quad \frac{E}{A} \leq \Lambda.$$

The condition $E/A \leq \Lambda$ (alone) yields the value $c = 1/(1+\Lambda)$ for (2.7). If A is the larger eigenvalue throughout, then of course $A_p \leq 0$ and we can use the value $c = 1/2$.

We note finally that if $A > 0$ and $E > 0$, conditions which are characteristic of elliptic equations, then necessarily $c \leq 1/2$, while of course one may assume also that $c \geq 0$.

3. Applications

For simplicity we assume that f is of class C^1 in $\mathbb{R}$. We recall also that $f(0) = 0$.

(I) In the special case $A \equiv 1$, we have $E \equiv 1$ and $\operatorname{div}[A(|Du|)Du] \equiv \Delta u$, leading to equation (1.1). Choosing $c = 1/2$, we have (2.4) since

$$\int_0^p \rho A \, d\rho = (1/2)p^2 A \quad \text{when} \quad A \equiv 1.$$

It follows that equality in (2.5) holds, and in particular for $k = n$ we get

$$(3.1) \qquad \frac{d}{dr}\left[r^n\left(\frac{1}{2}u'^2 + F(u)\right) + auu'r^{n-1}\right] = r^{n-1}[nF(u) - auf(u) + (a+1-n/2)p^2].$$

Our first application is the following

Theorem 3.1. Suppose $n > 2$. If there exists $\alpha \geq (n+2)/(n-2)$ such that the condition

$$(3.2) \qquad (\alpha+1)F(u) \leq uf(u)$$

holds for all $u > 0$ ($u \in \mathbb{R}$), then the Dirichlet problem

$$(3.3) \qquad \begin{aligned} &\Delta u + f(u) = 0 \ \text{in } \Omega \\ &u = 0 \ \text{on } \partial\Omega, \quad u \not\equiv 0 \ \text{in } \Omega \end{aligned}$$

where Ω is a ball, does not possess any positive (radial) solutions.

Proof. Suppose (3.3) has a positive solution u for some ball Ω of radius R. Then u must be radial ([GNN1]). Integrating (3.1) from 0 to R we obtain

$$(3.4) \qquad \int_0^R [nF(u) - auf(u) + (a+1-n/2)p^2] r^{n-1} dr = \frac{1}{2}R^n u'^2(R).$$

(Note that the integrated term vanishes at $r = 0$ because of the factor r^n; at $r = R$ we have $u = 0$, so that $F = 0$ and the integrated term reduces simply to $\frac{1}{2}R^n u'(R)^2$.) The right hand side of the above relation is positive in view of Hopf's boundary point lemma. Now choose $a = n/(\alpha+1)$. Then $a+1-\frac{n}{2} \leq 0$ by assumption on α, and by (3.2) the left-hand side is non-positive, which is a contradiction.

The coefficient condition $\alpha \geq (n+2)/(n-2)$ is best possible, as is well known; the function

$$f(u) = -u + u^{(n+2)/(n-2)-\varepsilon}, \quad \varepsilon > 0$$

for example, allows positive solutions of (3.3) in an arbitrary ball, while at the same time satisfying (3.2) with $\alpha = (n+2)/(n-2) - \varepsilon$.

Our second application concerns non-existence in $\mathbb{R}^n$. We first derive

Proposition 3.1. Suppose $n \geq 2$. Assume that $f \leq 0$ in $(0,\varepsilon)$ for some $\varepsilon > 0$ and that u is a positive radial solution of (1.1) which tends to zero at ∞. Then

$$(3.5) \qquad \int_{\mathbb{R}^n} [nF(u) - auf(u)]\,dx = \left(\frac{n}{2} - 1 - a\right) \int_{\mathbb{R}^n} \frac{p^2}{r}\,dx \ .$$

Proof. Integrating (3.1) from 0 to R, we obtain

$$(3.6) \qquad R^n\left(\frac{u'^2(R)}{2} + F(u(R))\right) + au(R)u'(R)R^{n-1}$$

$$= \int_0^R [nF(u) - auf(u) + (a+1-\frac{n}{2})p^2]\, r^{n-1}dr \ .$$

The left-hand side tends to zero as R tends to ∞ by Lemma 5 in [PS], since $n \leq 2(n-1)$. The required result now follows by letting R tend to ∞ in (3.6).

Choosing a so that $n/a = \alpha + 1$ in (3.5), we easily deduce the following results.

Theorem 3.2. Suppose $n > 2$ and that there exists $\alpha \geq (n+2)/(n-2)$ such that (3.2) holds for all $u \in \mathbb{R}$. If $f \leq 0$ in $(0,\varepsilon)$ and $f \geq 0$ in $(-\varepsilon,0)$, for some $\varepsilon > 0$, then equation (1.1) does not possess any radial solution in $\mathbb{R}^n$ which tends to 0 at ∞.

Proof. There are three cases to consider, solutions which are everywhere positive, solutions which are everywhere negative, and solutions which have a zero at some positive value of r. In the first case non-existence follows directly from Proposition 3.1. [The left hand side of (3.5) when $a = n/(\alpha+1)$ is non-positive in view of (3.2); moreover with the help of the conditions $f \leq 0$ for $u \in (0,\varepsilon)$ and $f \geq 0$ for $u \in (-\varepsilon,0)$ one sees that the left hand side in fact is strictly negative, giving a contradiction.] In the second case, replacing u by $-u$ gives the first situation and hence non-existence again applies. Finally, the third case leads to a radial solution of the Dirichlet problem (3.3) in some ball, and hence again there can be no solution.

Theorem 3.3. Suppose $n > 2$ and that there exists $\alpha \geq (n+2)/(n-2)$ such that (3.2) holds for all $u > 0$ $(u \in \mathbb{R})$. If $f'(0) < 0$, then equation (1.1) does not possess any positive (radial) solutions in $\mathbb{R}^n$ which tends to zero at ∞.

The first part of Theorem 3.3 is a consequence of Theorem 3.2 and the well-known fact that any positive solution of (1.1) in $\mathbb{R}^n$ which tends to 0 at ∞ must be radial if $f'(0) < 0$ ([GNN2]). The second half follows directly from Theorem 3.2.

Remark. Note that if $f(u) = u^\alpha g(u)$ for some α, then

$$(\alpha+1)F(u) - uf(u) = -\int_0^u t^{\alpha+1} g'(t)\,dt \ .$$

Consequently Theorems 3.2 and 3.3 apply to nonlinearities of the form $f(u) = |u|^{\alpha-1}ug(u)$ where $\alpha \geq (n+2)/(n-2)$, $g' \leq 0$ in $(-\infty,0)$, $g' \geq 0$ in $(0,\infty)$ and $g \leq 0$ in $(-\varepsilon,\varepsilon)\setminus\{0\}$ for some $\varepsilon > 0$. (The additional condition $f'(0) < 0$ naturally being required in the case of Theorem 3.3.)

Examples. We consider particularly the cases

(i) $f(u) = -u + |u|^{q-1}u$, $\quad q \geq (n+2)/(n-2)$.

(ii) $f(u) = -u - \beta|u|^{s-1}u + \gamma|u|^{q-1}u$, $\beta,\gamma > 0$ and $q > s \geq (n+2)/(n-2)$.

(iii) $f(u) = -\beta|u|^{s-1}u + \gamma|u|^{q-1}u$, β,γ,s,q as in (ii) above.

(iv) $f(u) = -u + |u|^{t-1}u - \beta|u|^{s-1}u + \gamma|u|^{q-1}u$, where β,γ,q,s are as in (ii), while $1 < t < (n+2)/(n-2)$.

The fact that equation (1.1) with the nonlinearity (i) does not have any positive solution (or any radial solution in $\mathbb{R}^n$ which tends to zero at ∞ follows from the Pohozaev identity (see e.g. Strauss [S]), this having been known for some time. The assertion that equation (1.1) with the nonlinearity (ii) possesses neither positive nor radial solutions in $\mathbb{R}^n$ which tend to 0 at ∞ follows from Theorem 3.3 (choose $\alpha = q$) and is new. In fact this particular question was stated explicityly by Strauss [S] and by Berestycki and Lions [BL], and was

brought to our attention by W.A. Strauss. For equation (1.1) with the nonlinearity (iii) our conclusion is weaker; namely, we can only conclude that there are no radial solutions which tend to 0 as r approaches ∞ (the proof is immediate from Theorem 3.2 with $\alpha = q$ - indeed α can be chosen as any number between s and q, endpoints included).

In §4 we shall classify all positive radial solutions of (i), (ii) and (iii).

Example (iv) falls neither in the standard category for existence of ground states found by Berestycki & Lions, (that is, the highest exponent being less than $(n+2)/(n-2)$) nor in the non-existence regime of Theorem 3.3. The situation in fact depends on the values of β and γ. If these are small then f is simply a perturbation from the case $-u + u^t$, for which existence is known; consequently existence continues to hold by virtue of continuity arguments. On the other hand, if either β or γ is suitably large then we have nonexistence. To see this, choose $\alpha = s$ and observe that then

$$(\alpha+1)F - uf(u) = -\{\frac{s-1}{2} - \frac{s-t}{t+1}|u|^{t-1} - \gamma\frac{s-q}{q+1}|u|^{q-1}\}u^2 ,$$

which can be made negative by taking γ suitably large (depending only on the values of q and s). Alternately, choose $\alpha = q$ so that

$$(\alpha+1)F - uf = -\{\frac{q-1}{2} - \frac{q-t}{t+1}|u|^{t-1} - \beta\frac{s-q}{s+1}|u|^{s-1}\}u^2$$

which again can be made negative by taking β suitably large (depending only on q and s).

(II) We next consider the case of the mean curvature operator, to illustrate the wider applicability of the main inequality in §2. Much of the discussion is the same as that in (I) for the Laplace operator, so we shall be brief.

Set

$$A(\rho) = (1+\rho^2)^{-1/2}$$

so that $E(\rho) = (1+\rho^2)^{-3/2}$. We may choose $c = 1/2$ in (2.4) since

$$\int_0^\rho \rho A\,d\rho = \sqrt{1+\rho^2} - 1 \geq \frac{1}{2}\rho^2/\sqrt{1+\rho^2}$$

(alternately this choice of c holds since $A_\rho \leq 0$). The inequality (2.5) now takes the form, with $k = n$,

$$\frac{d}{dr}\left[r^r\left(1 - \frac{1}{\sqrt{1+u^2}} + F(u) + \frac{auu'}{r\sqrt{1+u'^2}}\right)\right] \leq r^{n-1}\left[nF(u) - auf(u) + (a+1-\frac{n}{2})\frac{u'^2}{r\sqrt{1+u'^2}}\right] . \tag{3.7}$$

The following result is the analogue of Theorem 3.1.

Theorem 3.4. Suppose $n > 2$. If there exists $\alpha \geq (n+2)/(n-2)$ such that (3.2) holds for all $u > 0$ ($u \in \mathbb{R}$) then the Dirichlet problem

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + f(u) = 0 \text{ in } \Omega ,$$
$$u = 0 \text{ on } \partial\Omega , \quad u \neq 0 \text{ in } \Omega \tag{3.8}$$

where Ω is a ball, does not possess any positive (radial) solutions.

The proof is the same as that of Theorem 3.1, once we observe that the function

$$\sum_{i=1}^n p_{ii} - \sum_{i,j=1}^n \frac{p_i p_j p_{ij}}{1+p^2} + f(u)\sqrt{1+p^2} ,$$

where $p = |(p_1,\ldots,p_n)|$, satisfies the hypotheses of Corollary 1, p. 221 in [GNN1].

Similarly, Proposition 3.1 and Theorem 3.2 hold for the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + f(u) = 0 \tag{3.9}$$

in $\mathbb{R}^n$. The proofs are the same as before except that we now must use the asymptotic behavior of radial solutions of (3.9). This behavior is somewhat similar to that of the Laplace operator case and is derived in Proposition 5.1. Using that we see that again no change in the proof is necessary.

(III) The case of the generalized mean curvature operator

$$\operatorname{div}\{(1+|Du|^2)^{\frac{m}{2}-1}Du\} + f(u) = 0 \quad , \quad (m \geq 1)$$

follows the same pattern (note that $m = 2$ gives (I) and $m = 1$ gives (II)). We find after an easy computation that $c = 1/2$ if $1 \leq m \leq 2$ and $c = 1/m$ if $m > 2$. Since $c \neq 1/2$ when $m > 2$ the criterion (3.2) for this case now applies when α satisfies

$$a = n/(\alpha+1) \quad , \quad a+1-cn \leq 0 \quad ,$$

that is, when

$$\alpha \geq \frac{(1-c)n+1}{cn-1} = \frac{(m-1)n+m}{n-m} \quad .$$

This in turn shows that our conclusions in the case $m > 2$ hold only when $n > m$.

(IV) From the two preceding examples, it is clear that the results of case (I) carry over in general to the equation

$$\operatorname{div}(A(|Du|)Du) + f(u) = 0 \tag{1.2}$$

whenever A is of class C^1 for $p > 0$. In particular, Theorem 3.1 remains valid for radial solutions provided that

(i) $\quad A(p) > 0 \ , \ \int_0^p \rho E \, d\rho > 0 \qquad (p > 0)$

and

(ii) the α inequality (3.2) is replaced by

$$\alpha \geq \frac{(1-c)n+1}{cn-1} \qquad (n > 1/c) \quad .$$

The only difference to be noted is in the application of the boundary point lemma, which, because of the generality of the assumptions, may no longer apply. On the other hand, if equality does hold in the present analogue of (3.4) it is easy to see that necessarily $f(u) \equiv \text{const.}\, u^{\alpha}$ and $A(p) \equiv p^{-2+1/c}$. In this case one easily finds by direct consideration of (1.2) that $u'(R) < 0$, a contradiction.

Similarly, Theorem 3.2 continues to hold for radial solutions of (1.2) in

$\mathbb{R}^n$. The proofs are essentially the same as before except that now we must apply the asymptotic behavior given in Section 5, Lemmas 5.1 and 5.2. Using these results, it is evident that the boundary term leading to Proposition 3.1 can now only be asserted to have a non-negative limit, rather than being zero. The conclusion of Proposition 3.1 then must be modified be replacing the equality sign with inequality ($\geq$), and by changing the integrand on the right side to the new form $p^2 A(p)/r$. The rest of the proof then proceeds exactly as before.

(V) The generalized Laplace equation

$$\operatorname{div}\{|Du|^{m-2}Du\} + f(u) = 0 \ , \qquad 1 < m < n \ , \tag{3.10}$$

fails to be elliptic at $p = 0$ since $A(0) = 0$ when $m > 2$ and $A(0) = \infty$ when $1 < m < 2$. Nevertheless condition (i) of Part (IV) holds, and accordingly Theorems 3.1 and 3.2 are valid with

$$\alpha \geq \frac{(m-1)n+m}{n-m} \qquad (n > m)$$

(note that $c = 1/m$). In fact, Proposition 3.1 also holds, if we take into account the final remarks in Section 5.

4. More about Equation (1.1)

Consider the initial value problem

$$\begin{aligned} &u'' + \frac{n-1}{r}u' + f(u) = 0 \ , \\ &u'(0) = 0 \ , \quad u(0) = u_0 > 0 \ , \end{aligned} \tag{4.1}$$

where f is a C^1 function defined on $[0,\infty)$ which satisfies the following hypotheses: there exists $\xi > 0$ such that

$$f < 0 \text{ in } (0,\xi) \text{ with } f(0) = f(\xi) = 0 \text{ and } f'(\xi) > 0 \ , \tag{4.2}$$

$$\text{there exists } \eta > \xi \text{ such that } \int_0^{\eta} f(t)\,dt = 0 \text{ and } f > 0 \text{ in } (\xi,\infty) \ . \tag{4.3}$$

The following result completely determines the behavior of solutions of (4.1) on $[0,\infty)$ when we have $\alpha \geq (n+2)/(n-2)$.

Theorem 4.1. Suppose $n > 2$. If f satisfies (4.2), (4.3) and there exists $\alpha \geq (n+2)/(n-2)$ such that (3.2) holds for all $u > 0$, then for every $u_0 > 0$ ($u_0 \neq \xi$) the solution $u(r;u_0)$ of (4.1) exists and stays positive for all $r > 0$. Moreover, u oscillates (i.e. it has infinitely many local maxima and local minima) and $\lim_{r\to\infty} u(r;u_0) = \xi$ in such a way that the local maxima are decreasing (strictly) to ξ at ∞, the local minima are increasing (strictly) to ξ at ∞, and the distance between two consecutive zeros of $u - \xi$ tends to the constant $\pi/\sqrt{f'(\xi)}$.

Observe that Theorem 4.1 applies to examples (i), (ii) and (iii) mentioned in §3 and therefore contains Theorem 1.6 in [N]. The proof of Theorem 4.1 moreover is almost identical with that of Theorem 1.6 in [N], except we now use Theorems 3.1 and 3.2 at the appropriate places. The case $u_0 < 0$ in (4.1) may be treated in a similar fashion.

Using results of [FLS] one can extend the conclusion of Theorem 4.1 to the initial value problem for equation (1.2). For brevity we omit the detailed statement of these results.

5. Asymptotic Behavior

The asymptotic estimates used in Parts II - V of Section 3 will be proved in this section. We suppose $f \leq 0$ for $u \in [0,\varepsilon]$.

Lemma 5.1. Let $u = u(r)$ be a positive radial solution of (1.2) which tends to zero as $r \to \infty$. If $A(p) > 0$ for $p > 0$ then

$$\lim_{r\to\infty} r^{n-1} u'A(|u'|) = -L$$

where L is a non-negative constant.*

Moreover, $u' < 0$ for all sufficiently large r, $u' \to 0$ as $r \to \infty$, and necessarily the function $A(p)$ satisfies the condition $pA(p) \to 0$.

Proof. We have

(*) This result was obtained earlier by Franchi, Lanconelli and Serrin. We include the proof here for completeness.

$$\frac{d}{dr}\left(r^{n-1}u'A(|u'|)\right) = -r^{n-1}f(u)$$

so that $r^{n-1}u'A(|u'|)$ is ultimately constant or increasing. On the other hand, since $u \to 0$ as $r \to \infty$ there is certainly a sequence of values $r \to \infty$ for which $u' < 0$. It follows that $r^{n-1}u'A(|u'|)$ is everywhere negative and increases to a non-positive limit.

In turn, it is now evident that $u'A(|u'|) \to 0$ as $r \to \infty$. On the other hand $pA(p)$ is a continuous function, positive for all $p > 0$. Consequently either $u' \to 0$ as $r \to \infty$ or $u' \to \infty$ as $r \to 0$. The second possibility is, however, absurd. The necessity of the condition $pA(p) \to 0$ as $p \to 0$ is a further immediate consequence of this argument.

Finally, since $u'A(|u'|) < 0$ for all sufficiently large r we see that $u' < 0$ for all large r.

Lemma 5.2. Under the hypotheses of Lemma 5.1 we have

$$\int_0^{|u'(r)|} \rho E(\rho)\, d\rho + F(u) > 0 \qquad (r \geq 0)$$

and in particular $F(u(0)) > 0$.

Proof. By differentiation

$$\frac{d}{dr}\left\{\int_0^p \rho E(\rho)\, d\rho + F(u)\right\} = -(n-1)\frac{p^2 A(p)}{r} \qquad (5.1)$$

where $p = |u'(r)|$. It follows that the quantity in braces is a strictly decreasing function of r. Since its limit value as $r \to \infty$ is zero (see Lemma 5.1) the result follows at once.

Note that (5.1) is the special case $k = a = 0$ of the proof of Theorem 2.1.

In the important case where $A(p) \to$ Pos. Const. as $p \to 0$ the previous results can be made considerably more precise. In fact in this case we have the following.

Proposition 5.1. Suppose that $u = u(r)$ is a positive radial solution of (1.2) which tends to zero as $r \to \infty$. Suppose also that $A(p) \to C > 0$ as

$p \to 0$. Then

$$r^{n-1} p \to L/C, \quad r^{2(n-1)} \int_0^p \rho E(\rho)\, d\rho \to \frac{1}{2} L^2/C$$

and

$$r^{2(n-1)} F(u) \to 0$$

as $r \to \infty$.

Proof. The first conclusion is immediate from Lemma 5.1, and the second follows from the relation

$$\int_0^p \rho E(\rho)\, d\rho = p^2 A(p) - \int_0^p \rho A(\rho)\, d\rho .$$

To obtain the final part we use L'Hopital's rule; thus

$$\lim r^{2(n-1)} \left[\int_0^p \rho E(\rho)\, d\rho + F(u) \right]$$

$$= \lim \frac{-p E(p) u'' + f(u) u'}{-2(n-1) r^{-2n+1}}$$

$$= \frac{1}{2} \lim r^{2(n-1)} p^2 A(p) = \frac{1}{2} L^2/C$$

as required.

For the case of the degenerate Laplacian, Section 3, Part (V), the results are similar, namely

$$r^{n-1} p^{m-1} \to L, \quad r^{(n-1)m/(m-1)} \int_0^p \rho E(\rho)\, d\rho \to \frac{m-1}{m} L^{m/(m-1)}$$

and

$$r^{(n-1)m/(m-1)} F(u) \to 0$$

as $r \to \infty$. We leave the proof to the reader.

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ON FUNCTIONS, WHOSE LEVEL LINES BEND PROPORTIONALLY TO FALL LINES

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In this paper we consider

$$(1)\qquad u_y(u_x + \lambda u_y)u_{xx} - (2\lambda u_x u_y + u_x^2 - u_y^2)u_{xy} + u_x(\lambda u_x - u_y)u_{yy} = 0\ ,$$

a quasi-linear partial differential equation of the second order with quadratic nonlinearities. Here λ is a constant, x and y are the independent variables. Equation (1) is hyperbolic, since

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = -\frac{1}{4}\,|Du|^4$$

if $a = u_y(u_x + \lambda u_y)$, $2b = \dots$, $c = u_x(\lambda u_x - u_y)$ denote the coefficients of u_{xx}, u_{xy}, u_{yy}.

Equation (1) reads

$$(2)\qquad h = \lambda k\ .$$

Throughout this paper,

$$(3a)\qquad k = -\operatorname{div}\frac{Du}{|Du|}$$

= curvature of the level lines of u,

$$(3b)\qquad h = -\operatorname{div}\begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix}\frac{Du}{|Du|}$$

= curvature of the fall lines of u.

More precisely, the value of $|k|$ at any point (x,y), where the gradient of u does not vanish, is the curvature at (x,y) of $u^{-1}(u(x,y))$, the level line of u passing through (x,y); the sign of k is that which makes $k\,Du$, a normal vector field to the level lines of u, orientate towards the center of curvature. Analogously, $|h(x,y)|$ is the curvature at (x,y) of the fall line passing through (x,y). For us, the fall lines of u (also called lines of steepest descent) are the trajectories of Du, i.e. the orbits of the dynamical system

$$dx : u_x(x,y) = dy : u_y(x,y) .$$

Our main results are theorems 1 and 2 below. Theorem 1 tells us that solutions to equation (1), which are not plane waves, either have critical points or develop singularities. Theorem 1 is a corollary of theorem 2. The latter asserts that the curvature of the level lines and the curvature of the fall lines of a function u are automatically constrained by a system of two partial differential equations of the second order, whatever u is (provided u is smooth enough and has no critical points in the region of interest).

Theorem 1. Let u be a solution to equation (1). Suppose u has not critical points; suppose further u is smooth near a bounded closed set E. Then the largest r such that u is twice continuously differentiable in

$$E + \left\{ (x,y) : x^2 + y^2 \leq r^2 \right\}$$

cannot exceed

$$\left[\sqrt{1+\lambda^2} \max \{ |k(x,y)| : (x,y) \in E \} \right]^{-1} .$$

Theorem 2. Let u be any smooth real-valued function of two real

variables x and y. Let $\phi = k+ih$, where $i = \sqrt{-1}$ and k and h are defined by (3). Then

$$2 \left| \frac{\partial \phi}{\partial \bar{z}} \right| \geq |\phi|^2$$

at any point where Du is not zero,

$$\frac{\partial}{\partial z} \frac{|\phi|^2 \pm i \sqrt{4 \left| \frac{\partial \phi}{\partial \bar{z}} \right|^2 - |\phi|^2}}{\frac{\partial \bar{\phi}}{\partial z}} + \phi = 0$$

at any point where Du and $\partial\bar{\phi}/\partial z$, are not zero.

As usual, we have set

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) , \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) .$$

Theorem 2 implies easily lemma 1 below. Lemma 1, and a number of ingredients from geometric measure theory, yields lemma 2. Lemma 2 implies theorem 1.

Lemma 1. Suppose u is a smooth solution to equation (1) and u has no critical point. Then

$$|Dk| \geq \sqrt{1+\lambda^2}\, k^2 ,$$

and

$$\frac{\partial}{\partial x} \cdot \frac{\sqrt{1+\lambda^2}\, k^2 k_x \mp k_y \sqrt{|Dk|^2 - (1+\lambda^2) k^4}}{|Dk|^2} +$$

$$\frac{\partial}{\partial y} \frac{\sqrt{1+\lambda^2}\, k^2 k_y \pm k_x \sqrt{|Dk|^2 - (1+\lambda^2) k^4}}{|Dk|^2} + \sqrt{1+\lambda^2}\, k = 0$$

at any point where $k \neq 0$.

Lemma 2. Assume u is a smooth solution to equation (1) and u has no critical point. Let p be any exponent > 2 and let G be any bounded open set with a rectifiable boundary. Then

$$\int_G |k|^p \, dx\,dy \leq \frac{(1+\lambda^2)^{-\frac{1}{2}}}{p-2} \int_{\partial G} |k|^{p-1} \sqrt{dx^2 + dy^2} \quad .$$

A straightforward variant of theorem 1 has the following corollary.

Corollary. Let u be a solution to the equation of geometrical optics

$$u_x^2 + u_y^2 = 1$$

and suppose that the restriction of u to a disk $x^2 + y^2 < \varepsilon^2$ belongs to Sobolev space $W^{2,p}$ for some $p > 2$. Then u cannot belong to $W^{2,p}$ in a larger disk $x^2 + y^2 < r^2$ unless r is sufficiently small, namely

$$\int_{\{(x,y):x^2+y^2<\varepsilon^2\}} \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right)^{p/2} dx\,dy \leq$$

$$\frac{2\pi}{p-2} \left[r^{\frac{p-2}{p-1}} - \varepsilon^{\frac{p-2}{p-1}} \right]^{1-p}$$

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LOCAL LIE ALGEBRAS OF KIRILLOV AND GEOMETRY OF THE JACOBI MANIFOLDS.

André Lichnerowicz (Oct 84)

Consider a real line bundle and the space of its differentiable sections. Introduce on this space a local Lie algebra, the bracket of two sections having a local character. Kirillov has given in 1976 an interesting study of such local Lie algebras. If the bundle is trivial, it is equivalent to study the local Lie algebras on the space of the real-valued functions defined on the base. This "scalar case" is in fact the more interesting case. Methods and results can be extended in a natural manner to the Kirillov case [2].

Independently from Kirillov, I have introduced in 1975 the notion of general Poisson structure and in 1976-1977 the notion of general Jacobi structure [3]. Such a structure is a contravariant generalization together of the symplectic or Poisson structures and of the contact structures. To each local Lie algebra is canonically associated such a geometrical structure and conversely. In this talk, I will analyze the geometry of these Jacobi structures and the properties of the corresponding Lie algebras of vector fields ([3] and [4]. The corresponding work has been partially in collaboration with F. Guedira.

1 - The Schouten bracket [7].

I will use the following technical instrument which is called the Schouten or Schouten-Nijenhuis bracket.

Let W be a connected, paracompact, differentiable manifold of dimension m and class C^∞; all the considered elements are C^∞. We set $N = N(W) = C^\infty(W; \mathbb{R})$. For brevity we call p-tensor an antisymmetrical contravariant tensor of order p. Let T be the space of all these tensors; Schouten and Nijenhuis have introduced on T a bracket that can be defined in the following way : if A is a p-tensor, B a q-tensor, $[A,B]$ is a (p+q-1)-tensor such that, for any closed (p+q-1)-form γ, we have :

$$(1\text{-}1) \qquad i([A,B])\gamma = (-1)^{pq+q}\, i(A)\, d\, i(B)\gamma + (-1)^{p}\, i(B)\, d\, i(A)\gamma$$

where i(.) is the inner product. If p = 1, $[A,B] = \mathcal{L}(A)B$, where $\mathcal{L}(.)$ is the Lie derivative. If C is an r-tensor, we have :

(1-2) $\quad [A,B] = (-1)^{pq}[B,A] \qquad S(-1)^{pq}\,[[B,C],A] = 0$

where S is the summation after cyclic permutation; (1-2) expresses that the Schouten bracket defines on T a natural structure of Z_2-graded Lie algebra. Moreover :

(1-3) $\quad [A, B \wedge C] = [A,B] \wedge C + (-1)^{pq+q}\, B \wedge [A,C]$

2 - Jacobi structure and conformal Jacobi structure associated with a local Lie algebra.

a) A Lie algebra $(N,[\ ,\ ])$ defined on N is said to be local if for any $u,v \in N$

$$\mathrm{Sup}\,[u,v] \subset \mathrm{Sup}\,u \cap \mathrm{Sup}\,v$$

Let U be the domain of a chart $\{x^i\}$ of W. It follows from the Peetre theorem that $[u,v]$ is given on U by a bidifferential, bilinear operator. If this operator is of bidifferential type (r,s) (with $r \geq s$), its principal symbol in $x \in U$ can be written in the chart $\{x^i\}$:

$$A^{i_1..i_r,j_1..j_s}(\xi_{i_1}..\xi_{i_r}\,\eta_{j_1}\,\eta_{j_s} - \eta_{i_1}..\eta_{i_r}\,\xi_{j_1}..\,\xi_{j_s})$$

Suppose $r \geq 2$. The principal symbol of the left member of the Jacobi identity is of type (2r-1,s+1,s) and the vanishing of this symbol implies A = 0. It follows :

Proposition - Each local Lie algebra $(N,[\ ,\ ])$ is given by an alternate bidifferential operator of maximum order 1 in each argument.

b) There exist thus a 2-tensor Λ and a vector E such that :

(2-1) $\quad [u,v] = i(\Lambda)(du \wedge dv) + u\, i(E)\, dv - v\, i(E)\, du$

The Jacobi identity can be translated in terms of Schouten brackets by :

(2-2) $\quad [\Lambda,\Lambda] = 2\,E \wedge \Lambda \qquad [E,\Lambda] = \mathcal{L}(E)\,\Lambda = 0$

A pair (Λ,E) satisfying (2-2) defines on W a Jacobi structure. If E = 0, that is if Λ satisfies $[\Lambda,\Lambda] = 0$, Λ defines a Poisson structure.

Let h be an element of N, $\neq 0$ everywhere; we set $\rho_h : u \in N \to hu \in N$. Two local Lie algebras $(N,[\ ,\ ])$ and $(N,[\ ,\ ]')$ are said to be equivalent if we have :

(2-3) $\quad \rho_h\,[u,v]' = [\rho_h u, \rho_h v]$

The corresponding Jacobi structures are connected by :

(2-4) $\quad \Lambda' = h\,\Lambda \qquad E' = h\cdot E + [\Lambda,h]$

The equivalence class of structures defined by (2-4) is a conformal Jacobi structure. We denote by L the Lie algebra of the infinitesimal automorphisms of the conformal Jacobi structure defined by (Λ,E). A vector field Z of W belongs to L iff there is $a_Z = a \in N$ such that :

(2-5) $\quad \mathcal{L}(Z)\,\Lambda = a\,\Lambda \qquad \mathcal{L}(Z)\,E = a\,E + [\Lambda,a]$

c) If $u \in N$ consider the vector field (called generalized hamiltonian vector field, or briefly hamiltonian vector field) given by $X_u = u\,E + [\Lambda,u]$. It is easy to see that if $u,v \in N$

$$[X_u, X_v] = X_{[u,v]}$$

and that X_u satisfies (2-5) with $a = -\,i(E)\,du$. The X_u are thus the elements of a Lie subalgebra L^* of L. Moreover, if $Z \in L$ and $X_u \in L^*$, we have :

(2-6) $\quad [Z, X_u] = (\mathcal{L}(Z)u + a_Z u)\,E + [\Lambda, \mathcal{L}(Z)u + a_Z u]$

and L^* is an ideal of L.

The mapping $\sigma : u \in N \to X_u \in L^*$ is a homomorphism of $(N,[\ ,\ ])$ onto L^*. Study the kernel C of σ. The bracket (2-1) writes :

(2-7) $\quad [u,v] = \mathcal{L}(X_u)v - v\,i(E)\,du$

If $u \in C$, we have $X_u = 0$ and $u,v = -\,v\,i(E)\,du$. In particular $u\,i(E)\,du = 0$. If $x \in W$ such that $u(x) \neq 0$, we have $(i(E)du)(x) = 0$ and thus $[u,v](x) = 0$ for each $x \in \mathrm{Sup}\,u$ and for any v. It follows that if $u \in C$, u belongs to the center of $(N,[\ ,\ ])$. The converse is evident. We have :

Proposition - The Lie algebra L^* of the hamiltonian vector fields is an ideal of L. The mapping $\sigma : u \subset N \to X_u \subset L^*$ is a homomorphism of $(N,[\ ,\ ])$ onto L^* and the kernel of σ is the center C of $(N,[\ ,\ ])$.

3 - Generalized foliations.

a) Let M be a smooth manifold of dimension m. A field P of planes is a map $x \in M \to P_x$, subspace of T_xM; we say that P is regular if $\dim P_x = \mathrm{const}$.

A C^∞-vector X of M belongs to P if, for any $x \in M$, the value X(x) of X belongs to P_x. A field P is said to be of class C^∞ if there is a set of C^∞-vectors belonging to P and generating P_x for all x of M. We say that this set generates P.

An integral manifold of P is an immersed connected submanifold S of M such that for all $x \in S$, we have $T_xS \subset P_x$. An integral manifold S is maximal if each integral manifold that includes S coincides with S.

We say that P is invariant under a vector X if it is invariant under the flow of X. Sussmann has proved [8].

Theorem - Let P be a C^∞-field of planes generated by a set E of C^∞-vectors. Suppose that P is invariant under each vector of E. Then, through every point $x \subset M$, there passes a unique maximal integral manifold S(x) of P. For all $y \in S(x)$, we have :

$$T_y\,S(x) = P_y$$

and the dimension of the elements of P is constant along S(x). Conversely if such is the situation, P is invariant under each C^∞-vector belonging to P.

b) In this case, P. Stefan and P. Dazord have given more precise results. We are led to the following definition [8] .

Definition - A generalized foliation (or briefly a foliation) on M is defined by a field P of planes invariant under each vector field belonging to P. If $x \in M$, the leaf S(x) is the unique maximal integral manifold of P passing through x; S(x) is a weakly imbedded submanifold of M.

A usual Reeb foliation is called here a regular foliation. We have :

Theorem - Let P be a (generalized) foliation on M. Each point x_o of M, with dim $S(x_o) = p$, admits a chart f of domain U with values in $V_1 \times V_2$, where V_1 is a neighborhood of 0 in R^p, V_2 a neighborhood of 0 in $R^q (p+q = m)$, with the following properties : $f(x_o) = 0$ and the image by f of the foliation P_U induced on U is the foliation $V_1 \times Q$, where Q is a foliation of V_2 null at the origin.

We say that f is an adapted chart; for $x \in U$, $f(x) = \{x^\alpha ; x^a\}$ where $\alpha, \beta, \ldots = 1, \ldots, p$ and $a, b, \ldots = p+1, \ldots, m$. The connected component of x_o in $S(x_o) \cap U$ is isomorphic with V_1 and satisfies $\{x^a = 0\}$.

We see that the leaves S(x) corresponding to the points x in a suitable small neighborhood of x_o have a dimension superior or equal to dim.$S(x_o)$. A point x is called regular if there is a neighborhood U of x for which P_U is regular; all the points of S(x) are regular. We have thus notions of regular leaf and singular leaf. The set of the regular points of M is open and dense on M; on each connected component of this open set, P defines a regular foliation.

4 - The Kirillov theorem.

Let $(N, [\ ,\ ])$ be a local Lie algebra on $N = N(W)$, (Λ, E) the corresponding Jacobi structure. If $x \in W$, let P^* be the plane generated by the values at x of the hamiltonian vector fields, elements of L^*; P_x is the plane determined by E(x) and $\Lambda(x)$. We call characteristic field of $(N, [\ ,\ ])$ or of (Λ, E) the field P of the planes P_x. It is clear that P depends only upon the conformal Jacobi structure determined by (Λ, E); the algebra $(N, [\ ,\ ])$ is said to be regular if dim P_x = const, transitive if $P_x = T_x W$; it is the same for the corresponding Jacobi structure.

a) We have the following theorem which is substantially equivalent to the main statement of Kirillov [2] :

Theorem - Let $(N, [\ ,\ ])$ be a local Lie algebra of characterized field P.

1° The field P defines on W a (generalized) foliation.

2° For each leaf S, the restrictions of Λ and E to S determine on S a transitive local Lie algebra which is said to be induced by $(N, [\ ,\ ])$.

According to the Sussmann condition, it is sufficient to prove that P is invariant under each hamiltonian vector field. The 2 is an immediate consequence of the existence of adapted charts.

b) The geometrical structure of the leaves of P is simple. A leaf S of odd dimension satisfies my contravariant definition of a pfaffian manifold (or manifold with a global contact 1-form). A leaf S of even dimension for which $E = 0$ is symplectic ($[\Lambda, \Lambda] = 0$). It is the case for all the leaves of a Poisson manifold. A leaf S of a Jacobi manifold is globally conformally symplectic if we can come down the situation to $E = 0$ by equivalence. If such is not the case, the leaf is a true locally conformally symplectic manifold (t.l.c.s) |4|.

The origin of this denomination is the following : consider a manifold W of dimension 2p with a 2-tensor Λ of rank 2p and a vector E satisfying (2-2). Let F be the 2-form obtained by inversion of Λ and set $\omega = -i(E)F$. For (W, Λ, E), the relations (2-2) are equivalent to the following relations :

$$(4\text{-}1) \qquad dF + \omega \wedge F = 0 \qquad\qquad d\omega = 0$$

Let U be a contractile domain of W. There is on U a function $\varphi_U > 0$ such that $\omega|_U = \varphi_U^{-1} d\varphi_U$; (4-1) shows that 2-form $F'_U = \varphi_U F|_U$ is symplectic on U. We see why (W, Λ, E) is l.c.s.

If ω is exact on W, we have $\omega = \varphi^{-1} d\varphi$, where φ is > 0; $F' = \varphi F$ or $\Lambda' = \varphi^{-1} \Lambda$ defines on W a symplectic structure and $(W, \ , E)$ is globally conformal symplectic.

An example. The most natural example of the situation is the following. Let be a Lie algebra on the real field, $\mathfrak{g}^*$ the dual space, $\langle\ ,\ \rangle$ the corresponding canonical bilinear form. The homothetics vector field Ξ of $\mathfrak{g}^*$ is given by Ξ : $\xi \in \mathfrak{g}^* \to \xi \in \mathfrak{g}^*$. The space $\mathfrak{g}^*$ admits a natural Poisson structure given by the 2-tensor $\hat{\Lambda}$ of $\mathfrak{g}^*$ defined by :

$$(4\text{-}2) \qquad i(\hat{\Lambda})(X \wedge Y) = \langle \Xi, [X, Y] \rangle \qquad \text{for all } X, Y \in \mathfrak{g}$$

It follows from the Jacobi identity that $[\hat{\Lambda}, \hat{\Lambda}] = 0$. Moreover :

$$(4\text{-}3) \qquad \mathcal{L}(\Xi)\, \hat{\Lambda} = -\hat{\Lambda}$$

In the finite-dimensional case, the existence of the symplectic foliation of $(\mathfrak{g}^*, \hat{\Lambda})$ is the essential of the Kirillov-Kostant-Souriau theorem; if G is a connected Lie group admitting $\mathfrak{g}$ as Lie algebra, the orbits of G acting on $\mathfrak{g}^*$ by the coadjoint representation are nothing other but the leaves of $(\mathfrak{g}^*, \hat{\Lambda})$.

5 - Transitive local Lie algebras.

a) It follows from results concerning pfaffian or contact manifolds the following :

Theorem 1 - Let $(N, [\ ,\ ])$ be a transitive local Lie algebra on the manifold W of odd dimension and (Λ, E) the corresponding pfaffian structure. The algebra $(N, [\ ,\ ])$ is the Jacobi Lie algebra of (W, Λ, E) isomorphic by σ with the Lie algebra L^* of the contact infinitesimal transformations and its center is reduced to $\{0\}$. The Lie algebra L^* (resp. N) coincides with its commutator ideal and the derivations

of L* (resp N) are all inner derivations. We have H^2_{dif} (N;N) = {0} and the Lie algebra (N,[,]) is always rigid.

$H^2_{dif}(N;N)$ is here the second differential cohomology space of (N,[,]) for the Chevalley cohomology with values in N, associated with the adjoint representation. Say that (N,[,]) is rigid is say that each formal differential deformation of (N,[,]), in the sense of Gerstenhaber [1] is differential trivial.

b) In a similar way, results concerning the symplectic manifolds give [5].

Theorem - Let (N,[,]) be a transitive local Lie algebra on the manifold W of even dimension such that E = 0 and let Λ be the corresponding symplectic structure tensor. The algebra (N,[,]) is the Poisson Lie algebra of (W,Λ) of center $\mathbb{R}$ and σ is the homomorphism of (N,[,]) on the Lie algebra L* of the hamiltonian vector fields; L* coincides with its commutator ideal. The local derivations of (N,[,]) are given by ($\mathcal{L}(Z) + a_Z$, where Z $\in$ L and a_Z = const. We have H^2_{dif} (N;N) $\neq$ {0} always and (N,[,]) admits always non trivial deformations (given by skewsymmetrization of star-products for example).

Isomorphic results are valid for the Lie algebra (N,[,]) corresponding to a globally conformal symplectic manifold.

c) It is necessary to study the new case where (W,Λ,E) is a true locally conformally symplectic manifold. To obtain results of the type of Theorem 1 or 2, a long autonomous study is necessary where we use cohomology operators on the forms given by $d^\omega = d + e(\omega)$, where $e(\omega)$ is the exterior product by a non-exact closed 1-form ω. It is possible to develop for d^ω a De Rham harmonic theory. We have :

Theorem 3 - Let (N,[,]) be a transitive local Lie algebra on the manifold W of even dimension such that (Λ,E) defines on W a true locally conformally symplectic structure. The algebra (N,[,]) is isomorphic by σ to L* which is an algebra of locally symplectic infinitesimal transformations of (W,Λ,E) and its center is reduced to {0}; L* (resp. N) coincides with its commutator ideal. The derivations of (N,[,]) are given by ($\mathcal{L}(Z) + a_Z$) where Z $\in$ L. The space H^2_{dif}(N;N) is generated by 1-differential 2-cocycles and we can have or not non trivial infinitesimal deformations of (N,[,]), necessarily equivalent to 1-differential deformations.

We see that this situation is intermediary between the situation corresponding to the pfaffian manifolds and the situation corresponding to the symplectic manifolds.

6 - Poisson manifold associated with a Jacobi manifold.

We can associate with an arbitrary Jacobi manifold (W,Λ,E) a Poisson manifold by the same process which leads to the symplectification of a contact manifold.

a) Consider the manifold $\tilde{W}$ = W x $\mathbb{R}$; we denote by z the canonical variable of $\mathbb{R}$, by $\tilde{Z}$ the vector field of $\tilde{W}$ corresponding to $\partial/\partial z$, by π the projection $\pi : \tilde{W} \to W$; Λ and E define on $\tilde{W}$ elements denoted by the same notation such that :

$$\mathcal{L}(\tilde{Z})\,\Lambda = 0 \qquad \mathcal{L}(\tilde{Z})\,E = 0$$

Introduce on $\tilde{W}$ the 2-tensors :

$$(6\text{-}1) \qquad \tilde{\Gamma} = \Lambda + \tilde{Z} \wedge E \qquad \tilde{\Lambda} = e^{-z}\,\Lambda$$

It is easy to see that $\tilde{\Lambda}$ satisfies :

$$(6\text{-}2) \qquad [\tilde{\Lambda},\tilde{\Lambda}] = 0 \qquad \mathcal{L}(\tilde{Z})\,\tilde{\Lambda} = -\tilde{\Lambda}$$

We have :

Proposition - To each Jacobi structure (Λ,E) on W is canonically associated on $\tilde{W}$ = W x $\mathbb{R}$ a Poisson structure $\tilde{\Lambda}$ homogeneous of degree -1 with respect to $\tilde{Z}$, and conversely.

The analysis of the leaves of ($\tilde{W},\tilde{\Lambda}$) gives :

Proposition - Let S(x) be the leaf of (W,Λ,E) passing through x and $\tilde{S}(\tilde{x})$ the leaf of ($\tilde{W},\tilde{\Lambda}$) passing through $\tilde{x}$ such that $\pi\,\tilde{x} = x$; we have $\pi\,\tilde{S}(\tilde{x}) = S(x)$.

1) If dim. S(x) is odd, $\tilde{S}(\tilde{x})$ coincides with S(x) x $\mathbb{R}$.

2) If dim. S(x) is even, π defines $\tilde{S}(\tilde{x})$ as a covering of S(x).

b) An example Come back to a Lie algebra $\mathfrak{g}$ of dimension n and to the standard Poisson manifold ($\mathfrak{g}^*,\tilde{\Lambda}$) where $\mathfrak{g}^*$ is the dual space. Set $\mathfrak{g}^*_0 = \mathfrak{g}^* - \{0\}$. According to (4-3), ($\mathfrak{g}^*_0,\tilde{\Lambda}$) can be considered as the Poisson manifold associated with a Jacobi structure (Λ,E) defined on the sphere S^{n-1} obtained by quotient of $\mathfrak{g}^*_0$ by the one parameter group of homotheties of $\mathfrak{g}^*$. If G is connected Lie group with the Lie algebra $\mathfrak{g}$, acting on $\mathfrak{g}^*_0$ by the coadjoint representation, this action passes to the quotient and G acts on S^{n-1}. The leaves of (S^{n-1},Λ,E) are the orbits of this quotient coadjoint action [6].

The odd-dimensional leaves are universal models, up to a covering, for the proper contact homogeneous spaces of group G; the even-dimensional leaves are similarly universal models for the l.c.s homogeneous spaces.

7 - Distinguished charts.

a) Let (W,Λ) be an arbitrary Poisson manifold. A. Weinstein [9] have proved :

Proposition - Let x be an arbitrary point of W such that dim.S(x) = 2p. There is a chart $\{x^i\} = \{x^\alpha;x^a\} = \{x^\lambda,x^{\bar{\lambda}};x^a\}$ ($\alpha = \lambda,\bar{\lambda}$; $\lambda = 1,\dots,p$; $\bar{\lambda} = \lambda + p$; a,b,c = 2p+1,...,m) of W, of domain U containing x, such that the non-vanishing components of Λ are :

$$\Lambda^{\lambda\bar{\lambda}} = -\Lambda^{\bar{\lambda}\lambda} = 1 \qquad \Lambda^{ab}(x^c)$$

where the Λ^{ab} are null at x. For U, the Poisson manifold is locally isomorphic with the product of a symplectic manifold and of a Poisson manifold of rank 0 at x. The symplectic factor is described by $x^a = 0$ and corresponds to the connected

component of x of $S(x) \cap U$.

Such a chart $\{x^\alpha; x^a\}$ is said to be a distinguished chart in the neighborhood of x. Using this proposition and the propositions of §6, we can obtain a notion of distinguished chart for the neighborhood of a point of an arbitrary Jacobi manifold (W,Λ,E).

b)Let x be a point of W for which the leaf S(x) has an even dimension. It is possible by equivalence to suppose that, for a domain U containing x, the connected component of $S(x) \cap U$ containing x is symplectic. We can deduce from a and §6 :

Proposition - For each point x such that the leaf S(x) has the even dimension 2p and for a structure (Λ,E) such that S(x) is symplectic in the neighborhood of x, there is a chart $\{x^i\} = \{x^\alpha, x^a\} = \{x^\lambda, x^{\bar\lambda}; x^a\}$ ($\alpha = \lambda, \bar\lambda$; $\lambda = 1,\dots,p$; $\bar\lambda = \lambda + p$; a,b,c = 2p+1,...,m) of W, of domain U containing x, such that the non-vanishing components of Λ, and E are respectively :

$$\Lambda^{\lambda\bar\lambda} = -\Lambda^{\bar\lambda\lambda} = 1 \qquad \Lambda^{ab}(x^c) \qquad E^a(x^c$$

where the Λ^{ab} and E^a are null at x. For U, the manifold is locally isomorphic with the product of a symplectic manifold and of a Jacobi manifold of rank 0 at x. The symplectic factor is described by $x^a = 0$ and corresponds to the connected component of $S(x) \cap U$ containing x.

c) Let x be a point of W for which the leaf S(x) has an odd dimension. We have :

Proposition - For each point x such that the leaf S(x) has the odd dimension (2p+1), there is a chart $\{x^i\} = \{x^\alpha; x^a\} = \{x^\lambda, x^{\bar\lambda}, x^{\bar o}; x^a\}$ ($\alpha = \bar 0, \lambda, \bar\lambda; \lambda = 1,..,p$; $\bar\lambda = \lambda + p$; a,b,c = 2p+2,...,m) of W of domain U containing x such that the non-vanishing components of E and Λ are :

$$E^{\bar o} = 1, \quad \Lambda^{\lambda\bar\lambda} = -\Lambda^{\bar\lambda\lambda} = 1 \ , \quad \Lambda^{\bar o\lambda} = x^\lambda \ ; \quad \Lambda^{\bar o a}(x^c) = -\Lambda^{a\bar o}(x^c) \ , \quad \Lambda^{ab}(x^c)$$

where the $\Lambda^{\bar o a}$ and Λ^{ab} are null at x. For U, the manifold is locally isomorphic to the diagonal $x^{\bar o'} = x^{\bar o}$ of the product of the pfaffian leaf described by $x^a = 0$ by the Jacobi manifold $(\Lambda^{ab}, \Lambda^{\bar o' a}, E^{\bar o'} = 1)$ of rank 1 at x described by $x^\lambda = x^{\bar\lambda} = 0$.

We have obtained a notion of distinguished chart in the neighborhood of each point of an arbitrary Jacobi manifold.

We suppose in the following $\Lambda \neq 0$ everywhere.

8 - Derivations of the Lie algebras associated with a local Lie algebra.

a) We denote by Der(L*) (resp Der(L)) the algebra of the derivations of L* (resp. L) for an arbitrary Jacobi manifold (W,Λ,E). It is easy to see that such derivations are always local.

I have proved some years ago that if (W,Λ,E) is a regular Jacobi manifold or Poisson manifold, the elements of Der(L*) or Der(L) are given by the vectors Z of L. We know that if (W,Λ,E) is arbitrary, the set of the regular points is an open

set dense on W. Using on one hand the regular components of W, on the other hand distinguished charts in the neighborhood of limit points of regular components, we can extend the previous results to a general Jacobi manifold.

Theorem - Let $(N,[\ ,\ ])$ be a local Lie algebra. Each derivation D* of L* is given by $D^* : X \in L^* \to [Z,X] \in L^*$ where $Z \in L$. Each derivation D of L is an inner derivation.

b) Let Der(N) be the algebra of all the derivations of $(N,[\ ,\ ])$, $\mathrm{Der}_{loc}(N)$ the algebra of the local derivations. We can prove that if $\mathcal{D}_{loc} \subset \mathrm{Der}_{loc}(N)$; $\mathcal{D}_{loc}$ is given necessarily by a differential operator of order 1 and there is a vector field Z and a scalar a such that :

$$\mathcal{D}_{loc}\, u = \mathcal{L}(Z)\, u + a\, u \tag{8-1}$$

Study the conditions under which (8-1) is a derivation of $(N,[\ ,\ ])$. We obtain :

Proposition - Each local derivation $\mathcal{D}_{loc}$ of $(N,[\ ,\ ])$ is given by (81), where $Z \in L$ and where a is the corresponding scalar. The first space $H^1_{loc}(N;N)$ of the local Chevalley cohomology of $(N,[\ ,\ ])$ is isomorphic to L/L*.

c) Let $\mathcal{D}$ be an element of Der(N); if $u \in N$, associate to the hamiltonian vector $X_u = \sigma(u)$ the hamiltonian vector :

$$D^* X_u = \sigma(\mathcal{D}\, u) = \mathcal{D}_u E + [\Lambda, \mathcal{D}_u] \tag{8-2}$$

(8-2) defines a algebra homomorphism $\rho : \mathcal{D} \in \mathrm{Der}(N) \to D^* \in \mathrm{Der}(L^*)$. If $Z \in L$ defines D*, it defines a local derivation $\mathcal{D}_{loc}$ such that $\rho(\mathcal{D}_{loc}) = D^*$. Therefore we can write in a unique way :

$$\mathcal{D} = \mathcal{D}_{loc} + \mathcal{D}_1 \tag{8-3}$$

where $\mathcal{D}_1$ is a non local derivation such that $\rho(\mathcal{D}_1) = 0$, that is with values in the center C of $(N,[\ ,\ ])$. A map $\mathcal{D}_1 : N \to C$ is a derivation iff $\mathcal{D}_1$ vanishs on $[N,N]$.

Let v be an element of C. We know that if S is a pfaffian leaf or a t.l.c.s leaf we have $v|_S = 0$. Therefore v is $\neq 0$ only on the largest open submanifold W_1 for which all the leaves are globally conformal symplectic. We have :

Theorem - Each derivation $\mathcal{D}$ of $(N,[\ ,\ ])$ is given by :

$$\mathcal{D}_u = \mathcal{L}(Z)u + a_Z u + \mathcal{D}_1 u \qquad (Z \in L)$$

where $\mathcal{D}_1$ is a non local operator with values in C vanishing on $[N,N]$. We have $\mathrm{Sup}(\mathcal{D}_1) \in W_1$.

The typical case for $\mathcal{D}_1 \neq 0$ is the following : suppose that (W,Λ) is a regular Poisson manifold with compact symplectic leaves, such that the foliation is a fibration of W. We have :

$$\mathcal{D}_1 u = A(f_u)$$

where A is an arbitrary linear operator on the center C of $(N,[\ ,\ ])$ and where f_u is given by :

$$f_u = \int_{S(x)} u\,\eta \qquad (\eta \text{ symplectic volume form of the leaves})$$

9 - The case of the line bundles.

a) Let $p : K \to W$ be a real line bundle on W, $(\Gamma(K),[\ ,\ ])$ a local Lie algebra on the space $\Gamma(K)$ of the sections of K. Let σ_U be a section of K without zero over a domain U of W. If $s \in \Gamma(K)$, we have $s_{|U} = u_U \sigma_U$ (where $u_U \in N(U)$); $(\Gamma(K),[\ ,\ ])$ induces on N(U) a local Lie algebra $(N(U),[\ ,\ ])$ for which $[u_U,v_U]$ is given by a bidifferential operator of maximum order 1 in each argument. We have with clear notations :

$$(9\text{-}1) \qquad [u_U,\ v_U] = i(\Lambda_U)(du_U \wedge dv_U) + u_U\, i(E_U)\, dv_U - v_U\, i(E_U)\, du_U$$

Let V be another domain with a section σ_V without zero. For $U \cap V \neq \emptyset$, we have $\sigma_V = h_{UV}\,\sigma_U$ (with $h_{UV} \neq 0$, i.e. transition functions) and it follows :

$$(9\text{-}2) \qquad \Lambda_V = h_{UV}\,\Lambda_U \qquad E_V = h_{UV}\,E_U + [\Lambda_U,\ h_{UV}]$$

$(\Gamma(K),[\ ,\ ])$ defines thus on W a general conformal Jacobi structure, in my sense [3], associated with the bundle K; L is the Lie algebra of the infinitesimal automorphisms of this structure. To each section $s \in \Gamma(K)$ corresponds a hamiltonian vector field X_s, element of an ideal L* of L, such that :

$$(9\text{-}3) \qquad X_{s|U} = u_U\,E_U + [\Lambda_U,\ u_U]$$

We have a homomorphism σ of $(\Gamma(K),[\ ,\ ])$ onto L* ; its kernel is the center C of the considered local Lie algebra.

b) The conformal Jacobi structure defines by means of L* a (generalized) foliation of W. Let S be a leaf of W. If $K|_{p^{-1}(S)}$ is trivial, the geometrical structure of S is the same of course as in the scalar case.

If $K|_{p^{-1}(S)}$ is non trivial, the situation is different. If S is odd-dimensional, S has a non orientable contact structure (that is non generated by a global contact 1-form). If S is even-dimensional, a distinguished covering of S is symplectic; but according to its dimension, S can be or not an orientable manifold. We obtain thus an extension of the notion of l.c.s manifold. In the two cases, the corresponding deformations can be studied. A case appears, for the transitive local Lie algebras, where the algebra admits always non trivial deformations.

c) Let $\overset{\vee}{Z}$ be the main vector field of the vector bundle $K \to W$. We can prove that each local Lie algebra $(\Gamma(K),[\ ,\ ])$ is canonically associated with a Poisson structure $(K,\overset{\vee}{\Lambda})$ on the manifold K, homogeneous of degree -1 with respect to $\overset{\vee}{Z}$, and conversely.

The results concerning the distinguished charts can be extended automatically. It follows that it is the same for the results concerning the derivations of the

Lie algebras L*,L,$(\Gamma(K),[\ ,\ ])$.

In particular a local derivation $\mathcal{D}$ of $(\Gamma(K),[\ ,\ ])$ is given by :

$$\mathcal{D}\,s = \mathcal{L}(X)\,s + a_X\,s$$

where $X \in L$ and a_X a corresponding scalar.

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IL CONTRIBUTO ITALIANO ALLA TEORIA DEI GRUPPI DALLE ORIGINI AL 1939

Guido Zappa

Ho ritenuto opportuno parlare del contributo italiano alla teoria dei gruppi in questo Convegno dedicato alla celebrazione del primo centenario del Circolo Matematico di Palermo perchè, come ricorderemo, diversi eminenti cultori italiani di questa teoria erano siciliani e particolarmente palermitani, e i Rendiconti del Circolo hanno ospitato alcune loro importanti memorie. Mi limiterò alla teoria dei gruppi in senso stretto, lasciando da parte i gruppi continui, i gruppi di omografie, etc. Ho preso come confine superiore del periodo da me esaminato l'anno 1939, in cui morì Gaetano Scorza, uno dei più eminenti algebristi italiani, ed ebbe inizio la seconda guerra mondiale, la quale anche nello sviluppo della matematica e in particolare della teoria dei gruppi separa due epoche contrassegnate da aspetti molto diversi.

L'origine della teoria dei gruppi finiti è legata, come è ben noto tra i matematici, al problema della risoluzione delle equazioni algebriche per radicali. Dopo che i matematici della scuola bolognese avevano fornito, nel cinquecento, le formule risolutive delle equazioni di terzo e quarto grado, l'attenzione degli algebristi si diresse verso la ricerca di formule analoghe per le equazioni generali di grado superiore al quarto. Si mirava a risolvere tali equazioni "per radicali" cioè a determinare per l'equazione generale di grado $n (n>4)$ una formula che permettesse di trovare le radici dell'equazione, a partire dai coefficienti della medesima, mediante operazioni razionali ed estrazioni di radice (ad indice intero

positivo qualunque). Ma, come è ben noto, tali tentativi non ebbero successo, nè potevano averne. Nel settecento Giuseppe Luigi Lagrange, il grande matematico noto a Torino e ivi vissuto nei primi 30 anni di vita, e che quindi può considerasi in parte italiano, sviluppò una teoria generale circa la risoluzione delle equazioni algebriche per radicali. Egli mirava ad arrivare alla determinazione delle radici dell'equazione generale $f(x)=0$ di grado n mediante la costruzione di funzioni delle radici le quali assumessero, per effetto di tutte le possibili permutazioni delle radici stesse, un numero, m, di valori distinti minore di n. I valori assunti da una tale funzione sono allora radici di una equazione $\phi(y)=0$, di grado m, i cui coefficienti sono funzioni razionali simmetriche delle radici della $\phi(y)$ stessa, e quindi anche di quelle della $f(x)$. Supposto di conoscere la formula risolutiva per le equazioni di grado m, sarebbe allora possibile determinare le radici della $\phi(y)$. Mediante una successione di equazioni ausiliarie come la $\phi(y)$ si giungerebbe alla fine alla determinazione delle radici della $f(x)$. Vediamo come si possa ricondurre, per tale via, la risoluzione dell'equazione generale di quarto grado a quella di equazioni di grado 2 e 3. Sia $f(x)=0$ un'equazione generale di quarto grado e si indichino con x_1,x_2,x_3,x_4 le sue radici da determinare. Si consideri l'espressione $x_1x_2+x_3x_4$. Permutando in tutti i 24 modi possibili le radici dell'equazione, tale espressione assume solo 3 valori distinti, e precisamente $y_1=x_1x_2+x_3x_4$, $y_2=x_1x_3+x_2x_4$, $y_3=x_1x_4+x_2x_3$. Si consideri l'equazione $\phi(y)=(y-y_1)(y-y_2)(y-y_3)=0$. I coefficienti di tale equazione sono funzioni simmetriche di y_1,y_2,y_3, e quindi anche di x_1,x_2,x_3,x_4, e pertanto si esprimono razionalmente per mezzo dei coefficienti della $f(x)$. Mediante la formula risolutiva dell'equazione di terzo grado si possono allora determinare y_1,y_2,y_3, cioè $x_1x_2+x_3x_4$, $x_1x_3+x_2x_4$, $x_1x_4+x_2x_3$. Conoscendosi il prodotto $x_1x_2x_3x_4$ (eguale al termine noto della $f(x)$) si possono determinare, risolvendo un'equazione di secondo grado, x_1x_2, x_3x_4 e analogamente x_1x_3, x_2x_4, x_1x_4, x_2x_3. E' poi facile ricavare x_1,x_2,x_3,x_4 risolvendo

altre equazioni di secondo grado.

Un simile procedimento funziona per le equazioni di grado ≤ 4, ma Lagrange non riuscì a risolvere con esso le equazioni generali di grado >4 (nè avrebbe potuto farlo). Infatti il metodo di Lagrange riduce le equazioni di terzo grado ad equazioni di secondo grado, equazioni di quarto grado ad equazioni di terzo grado, ma equazioni di quinto grado ad equazioni di sesto grado.

Va sotto il nome di teorema di Lagrange quello che afferma che, se G è un gruppo d'ordine n, S un sottogruppo d'ordine m e i l'indice di S in G, cioè il numero delle classi laterali di S in G, si ha $n=mi$; in particolare m è un divisore di n. In realtà, Lagrange considerò un caso molto particolare, pur avendo la sua dimostrazione validità anche nel caso generale. Egli era interessato, come si è visto, a determinare il numero dei valori che una data funzione delle radici (ad ex., nel caso delle equazioni di quarto grado, la funzione $x_1x_2+x_3x_4$) può assumere quando si permutino le radici in tutti i modi possibili. Orbene, le permutazioni delle radici dell'equazione di grado n costituiscono un gruppo S_n di ordine $n!$, quelle che mutano in sè una data funzione h costituiscono un sottogruppo H di un certo ordine m. Il numero dei valori distinti che si possono ottenere a partire da h per effetto di S_n eguaglia l'indice i di H in S_n. Lagrange dimostrò, precisamente che $mi=n!$ e che quindi m ed i sono divisori di $n!$

Quindi, benchè in Lagrange non vi sia ancora esplicitamente il concetto di gruppo, a lui si deve un ragionamento che porterà ad un primo importante (benchè semplice) risultato di teoria dei gruppi.

L'importante teorema che afferma l'impossibilità di risolvere per radicali l'equazione generale del quinto grado va comunemente sotto il nome di teorema di Ruffini - Abel. Si deve al Ruffini, matematico e medico modenese, vissuto dal 1765 al 1822, la prima dimostrazione di questo teorema, ritrovato poi indipendentemente da Abel. L'opera di Ruffini è stata valutata in modo molto diverso

dai vari autori che ne hanno parlato; e ciò, soprattutto, perchè si tratta di lavori piuttosto oscuri. Quasi ignorato durante l'ottocento, il Ruffini verso la fine di detto secolo fu fortemente valorizzato dal Burkhardt, e successivamente studiato a fondo da Ettore Bortolotti, che fra l'altro, nel 1915, pubblicò il primo dei tre volumi contenenti i lavori di matematica del nostro autore (gli altri due volumi comparvero nel 1953-54). Ruffini ha dato due diverse dimostrazioni di suddetto teorema, sulla correttezza delle quali sono state fatte diverse riserve; il solo Bortolotti le ritiene ambedue valide. Il matematico australiano Bryce, recentemente, ha ripreso in esame la questione della validità delle dimostrazioni di Ruffini, ed ha affrontato anche il problema di vedere in che misura il Ruffini si possa considerare il fondatore della teoria dei gruppi, come avevano affermato Bortolotti ed altri. Il Bryce giunge a conclusioni che mi sembrano molto equilibrate e che esporrò in breve. Egli ritiene le dimostrazioni di Ruffini inficiate da una lacuna, per colmare la quale sarebbe stata necessaria una nuova idea; anche la dimostrazione di Abel è incompleta, come fu messo in luce successivamente da Hamilton, ma il suo completamento si basa sulle stesse idee elaborate da Abel. Pertanto, egli ritiene sia giusto intitolare al doppio nome Ruffini - Abel detto teorema. Non si può, comunque, a mio parere, non riconoscere il fatto che la prima dimostrazione di Ruffini risale al 1799 e la seconda al 1813, mentre il lavoro di Abel è del 1824, e che il Ruffini per primo intuì l'impossibilità della risoluzione dell'equazione generale del quinto grado per radicali, dandone una dimostrazione, sia pure imperfetta.

Nota ancora il Bryce che il concetto di gruppo di permutazioni, cioè di insieme di permutazioni chiuso rispetto al prodotto, non appare esplicitamente in Ruffini. Egli è interessato all'insieme delle permutazioni operanti sulle radici di un'equazione e fissanti una data funzione razionale delle radici stesse; un tale insieme è effettivamente chiuso rispetto al prodotto, e il Ruffini usa implicitamente questa proprietà, ma non la mette esplicitamente in

evidenza. Comunque, egli ha fatto avanzare considerevolmente la teoria delle permutazioni. In particolare il Bryce mette in luce l'imporanza della scoperta del fatto che non esistono funzioni razionali di cinque variabili capaci di assumere esattamente tre o quattro valori, il che equivale al fatto che nel gruppo S_5 di tutte le permutazioni su cinque oggetti non vi sono sottogruppi di indice 3 o 4. Il Bryce conclude osservando che Ruffini può non avere inventato i gruppi, ma ispirò la loro nascita in misura maggiore di Lagrange.

Il concetto di gruppo compare invece in modo esplicito (anche se con nome diverso) in Cauchy, che del resto conosceva l'opera di Ruffini. E' ben noto come una teoria generale sulla risolubilità delle equazioni algebriche per radicali sia stata data da Evaristo Galois, sulla base della teoria dei gruppi finiti di permutazioni, di cui va considerato il vero fondatore. E' altresì noto come questo geniale matematico sia morto in duello a 21 anni nel 1832, e come egli abbia esposto per sommi capi le linee generali della sua teoria solo in un manoscritto stilato rapidamente nella notte precedente il fatale duello. Esso venne pubblicato da Liouville solo nel 1847, e risultò molto oscuro ai matematici del tempo, perchè molte asserzioni non erano dimostrate e non sembrava affatto facile ricostruirne la dimostrazione. Ci vollero diversi anni per eliminare tutti i punti interrogativi, e a tale scopo si dedicò massimamente il matematico toscano Enrico Betti (1823-1892). Questi, su consiglio di Ottaviano Mossotti, studiò le opere di Abel e di Galois, e, con un lavoro di anni, a partire dal 1850 circa, riuscì a decifrare completamente e a ricostruire in pieno le dimostrazioni di quest'ultimo. Si pose poi alla ricerca di opportune funzioni atte alla risoluzione della equazione generale di quinto grado, e si era portato assai vicino alla soluzione di questo problema, quando fu preceduto dal francese Hermite (nel 1858). Al Betti si deve anche lo studio del cosiddetto gruppo metaciclico, cioè del gruppo delle trasformazioni lineari intere in una variabile in un campo

finito di ordine primo; gruppo legato alle equazioni risolubili per radicali, il cui grado sia un numero primo.

Altro matematico italiano che in quel periodo ottenne risultati molto importanti nella teoria delle equazioni algebriche e dei gruppi di permutazioni ad esse legati fu il lombardo Francesco Brioschi (1824-1897). Anch'egli, quasi contemporaneamente ad Hermite e a Kronecker, giunse alla risoluzione dell'equazione generale di quinto grado: si servì a tale scopo delle funzioni ellittiche. Successivamente egli diede la risoluzione dell'equazione generale di sesto grado servendosi delle funzioni iperellittiche. Oltre al Betti e al Brioschi va ricordato l'algebrista Alfredo Capelli (1855--1910), il quale contribuì anch'egli allo sviluppo della teoria dei gruppi di permutazioni.

Intanto la teoria dei gruppi trovò una prima grandiosa sistemazione nel "Traité des substitutions" del francese Camille Jordan, pubblicato nel 1870; in esso vien fatto il punto di tutti i risultati ottenuti sino a quella data, molti dei quali dovuti allo stesso Jordan. La comparsa di questo libro diede nuovo impulso agli studi sui gruppi. Questi non sono più visti soltanto come gruppi di permutazioni: si comincia a distinguere tra le proprietà legate a tale rappresentazione e quelle che non ne dipendono. Nel 1872 il norvegese Sylow dimostra i noti teoremi che da lui prendono nome. Nel periodo che va fino allo scoppio della prima guerra mondiale, la teoria dei gruppi ha uno sviluppo imponente, specie in Germania e in Inghilterra: basta ricordare i nomi di Frobenius e di Burnside. I contributi forniti dai nostri algebristi alla teoria hanno ora un carattere un po' marginale, pur non mancando alcuni risultati molto significativi. Va anzitutto ricordato il romano Giovanni Frattini (1852-1925) che oggi è di gran lunga il più citato tra tutti gli studiosi italiani di teoria dei gruppi di quel periodo. L'importanza dei suoi risultati non fu subito compresa. I suoi lavori comparvero tutti in un breve intervallo di tempo (1883-85). Si dedicò all'insegnamento in un Istituto Tecnico, da cui non volle mai allontanarsi. Alcune sue ricerche riguardano i gruppi di permutazioni strettamente 1-transitivi; altri il cosiddetto gruppo modulare, cioè il gruppo delle sostituzioni lineari fratte su un campo finito d'ordine primo. Ma i risultati più importanti riguardano il sottogruppo che porta il suo nome. Un elemento x di un gruppo finito G è detto un non-generatore se, comunque si prenda un sottoinsieme H di G che contenga x e sia in grado di generare G, anche $H \setminus \{x\}$ è in grado di generare G. Frattini considerò l'insieme dei non-generatori di un gruppo finito $G \neq 1$ e dimostrò che esso è un sottogruppo di G (evidentemente normale). Esso coincide con l'intersezione dei sottogruppi massimali di G, cioè dei sottogruppi propri non contenuti in sottogruppi propri più ampi: anzi oggi il sottogruppo di Frattini $\Phi(G)$ di un gruppo G è in genere definito come sottogruppo intersezione dei sottogruppi massimali di G. Il Frattini dimostrò che ogni sottogruppo di Sylow di $\Phi(G)$ è normale, ossia che $\Phi(G)$ è prodotto diretto dei suoi sottogruppi di Sylow, vale a dire è nilpotente. Il ragionamento, semplice ma acuto, sviluppato da Frattini per dimostrare ciò si applica a molti altri casi e va sotto il nome di "ragionamento di Frattini" (Frattini's argument). Il sottogruppo di Frattini gioca un ruolo importante in molte questioni sui gruppi, e quindi il risultato del nostro autore è molto citato. Nel 1985 cade il centenario del lavoro di Frattini dedicato a questo sottogruppo, e varrebbe la pena di celebrarlo in qualche modo.

Dopo Frattini va ricordato il siciliano Giuseppe Bagnera (1865--1927), nato a Bagheria, e professore nelle università di Messina, Palermo e Roma. A lui si deve la determinazione di tutti i gruppi d'ordine p^5, con p primo. E' noto come i gruppi il cui ordine sia potenza di un numero primo p (p-gruppi) siano molto numerosi, e quindi come il problema della loro classificazione sia oltremodo difficile, e anche oggi ben lontano dalla soluzione. Altri autori avevano prima di lui determinato i gruppi d'ordine p^3 e p^4 (i casi p e p^2 sono banali), ma il caso p^5 si presentava molto più difficile. Il Bagnera, in un grosso lavoro che gli costò molto

impegno, riuscì a determinare nel 1898 tutti i gruppi d'ordine p^5, benchè gli strumenti posseduti a quell'epoca non fossero troppo raffinati (oggi si sa precedere in modo più rapido).

Va ricordato anche l'insigne matematico Luigi Bianchi (1856-1928) che, dedicatosi principalmente alla Geometria differenziale e alla Teoria dei numeri, si è pure interessato marginalmente di Teoria dei gruppi. Di lui va ricordato un lavoro sul gruppo modulare, e soprattutto il suo volume sui Gruppi finiti di sostituzioni e sulle Equazioni algebriche secondo Galois, scritto con la precisione, chiarezza ed eleganza che gli furono consuete. Tale volume riproduce le lezioni che egli impartiva agli alunni della Scuola normale superiore di Pisa.

Veniamo ora a parlare del palermitano Michele Cipolla (1880-1947) e della sua scuola. Il Cipolla fu professore nelle università di Catania e di Palermo. A partire dal 1909 egli sviluppò la teoria dei cosiddetti sottogruppi fondamentali di un gruppo finito. Se G è un gruppo finito e x un suo elemento, l'insieme degli elementi di G permutabili con x è un sottogruppo, detto normalizzante di x. Cipolla ha chiamato sottogruppi fondamentali quelli che risultano essere normalizzanti di qualche elemento di G. Ad ogni sottogruppo fondamentale S è associato un cosiddetto sistema fondamentale, che consiste nell'insieme degli elementi di G che hanno S come normalizzante. Ogni elemento di G appartiene a uno ed un solo sistema fondamentale. Un gruppo non abeliano ha almeno quattro sottogruppi fondamentali, tra cui è il gruppo stesso. Cipolla chiama tipo di un gruppo finito G non abeliano il numero dei suoi sottogruppi fondamentali, diminuito di 3. Se si mettono nella stessa classe due gruppi G_1, G_2 quando, detti Z_1 e Z_2 i loro centri, $\frac{G}{Z_1}$ risulti isomorfo a $\frac{G}{Z_2}$, i gruppi di dato tipo si distribuiscono, come fu provato da Cipolla, in un numero finito di classi. Il Cipolla mirava ad una classificazione dei gruppi finiti in base al tipo, ma l'opera gli apparve presto oltremodo ardua, poichè, al crescere del tipo, il numero delle classi di gruppi di dato tipo cresce a dismisura. I sottogruppi fondamentali di un gruppo G costituiscono, naturalmente, un insieme (parzialmente) ordinato rispetto all'inclusione, e quindi si può associare a G l'intero che fornisce la massima lunghezza delle catene di tale insieme ordinato. Tale intero è stato chiamato da Cipolla rango del gruppo. Egli ha determinato una disuguaglianza che lega il tipo e il rango del gruppo, la quale è stata successivamente migliorata da lui stesso e da altri autori; e ha trovato tutti i sottogruppi fondamentali dei gruppi di alcune classi notevoli, come quella dei gruppi il cui ordine è libero da quadrati (gruppi di Hölder). La teoria dei sottogruppi fondamentali, pur costituendo un capitolo un po' isolato entro la teoria dei gruppi, si presenta come oltremodo elegante. Attorno ad essa hanno lavorato con successo anche alcuni allievi del Cipolla, tra i quali vanno senz'altro ricordati il palermitano Gaspare Mignosi (1875-1951) e il catanese Vincenzo Amato (1881-1963). In particolare, Amato ha determinato tutti i sottogruppi fondamentali di altri gruppi notevoli.

Dobbiamo da ultimo parlare dei contributi alla teoria dei gruppi apportati da Gaetano Scorza (1876-1939). Questo geniale matematico si era dedicato, negli anni giovanili, a problemi di carattere strettamente geometrico, sulla scia del suo maestro Eugenio Bertini. In seguito, le sue ricerche sulle funzioni abeliane lo hanno condotto ad elaborare, in alcune sue celebri memorie, la teoria generale delle matrici di Riemann, cioè delle matrici che possono pensarsi come tabelle dei periodi primitivi per un corpo di funzioni abeliane. Di qui, la sua attenzione si portò sulla teoria generale delle algebre sopra un campo, in cui egli ravvisò un potente mezzo di ricerca per le proprietà di quelle matrici. Oltre ai suoi numerosi ed importanti risultati originali sulle algebre, è da ricordare a tal proposito il volume: "Corpi numerici ed algebre", pubblicato nel 1921, in cui egli espone, con l'eleganza che contraddistingue sempre il suo stile, i fondamenti della teoria delle algebre. Ai nostri fini, ci interessano alcuni lavori dello

Scorza sopra le algebre legate ai gruppi finiti. Dato un gruppo finito G e un campo K, si dice algebra su K legata a G l'algebra i cui elementi sono le combinazioni lineari degli elementi di G a coefficienti in K, e in cui la moltiplicazione si estende per linearità a partire da quella definita in G. Un allievo di Scorza, G. Fichera, in un lavoro del 1921, si era occupato della rappresentazione di un gruppo come gruppo di sostituzioni lineari. Lo Scorza, dopo aver notato che, ponendo a raffronto le ricerche di Fichera con alcuni classici risultati di Schur e di Speiser, si poteva determinare, entro l'algebra sul campo complesso C legata a un gruppo finito G, una certa sottoalgebra S connessa con una data rappresentazione irriducibile di G su C, dà una costruzione diretta della S, dalla quale si potevano a loro volta ritrovare i suddetti teoremi di Schur e Speiser. Analogo studio egli fa nel caso dell'algebra sul campo reale legata a G.

Ma lo Scorza si dedicò anche ad altre interessanti ricerche sui gruppi. Suo è l'elegante teorema in base al quale un gruppo, finito o infinito, è unione insiemistica di tre suoi sottogruppi propri se e solo se ammette tra le sue immagini omomorfe il gruppo di Klein d'ordine 4. Egli dà anche una bella generalizzazione di un teorema di Chapman, dimostrando che, dati un gruppo finito G e due suoi sottogruppi H, K di eguale indice, è possibile trovare una trasversale sinistra di H che sia anche una trasversale destra di K, e determinando in quanti modi ciò può essere fatto. Inoltre, anche la teoria dei sottogruppi fondamentali di Cipolla ricevette notevoli apporti da parte di Scorza, il quale, fra l'altro, migliorò sensibilmente la diseguaglianza tra il rango e il tipo di un gruppo.

Lo Scorza impartì nelle università in cui si trovò ad insegnare corsi di lezioni di teoria dei gruppi, esposti con la precisione e l'eleganza che caratterizzano tutta la sua opera. Egli era innamorato della bellezza delle teorie matematiche, e vedeva la matematica soprattutto come opera d'arte. Negli ultimi anni attese alla preparazione di un trattato di teoria dei gruppi, e l'aveva quasi

completato quando la morte lo colse prematuramente all'età di 63 anni nel 1939. Assieme al figlio Giuseppe Scorza Dragoni, analista ben noto, io ebbi l'onore di completare detto volume e di curarne l'edizione. La pubblicazione di questo libro, assieme all'azione di maestro svolta dallo Scorza nei suoi ultimi anni di vita a Roma, fu come un seme da lui lasciato. Da tale seme ebbe origine la scuola di teoria dei gruppi che si è sviluppata in Italia dopo la seconda guerra mondiale.

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NEW APPROACHES TO PUZZLING PROBLEMS IN DIFFERENTIATION THEORY

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In his monumental work [33], Zahorski attempted to solve a problem stated by Young [32], namely to obtain necessary and sufficient conditions for a function to be a derivative of some function on an interval. Zahorski was unable to obtain such a characterization, but he did succeed in learning a great deal about the structure of derivatives. In particular, he defined his hierarchy $\{\mathcal{M}_k\}$, $k = 1,2,\ldots,5$ of classes of functions and showed how certain classes of derivatives fit into that hierarchy. This work of Zahorski's has had a major impact on differentiation theory. In particular, it led a number of other authors to study various sorts of generalized derivatives to determine to what extent these generalized derivatives imitated the behavior of the ordinary derivative. These investigations showed that certain generalized derivatives imitate the ordinary derivative very well, and are excellent substitutes for the ordinary derivative when the latter is not known to exist. For example, an approximate derivative has the Darboux property, is in the first class of Baire, possesses the Denjoy property, and is in Zahorski's class $\mathcal{M}_3$ and Weil's class $\mathcal{Z}$. Furthermore, an approximately differentiable function F shares many properties of ordinary primitives: it is of generalized absolute continuity [ACG], is differentiable on a dense-open set, has the Darboux property, assumes local extrema and is determined by its values on any dense set. Finally, any monotonicity theorem that is valid for differentiable functions has an analogue for approximately differentiable functions. Precise formulations and proofs of these statements can be found in [1, 3, 7, 9, 11, 17, 19, 23, 28, 29, 30].

Certain other generalized derivatives possess some of these properties. For example, the Peano derivative has most of the prop-

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This paper is in final form and no version of it will be submitted for publication elsewhere.

erties while the preponderant, selective and sequential derivatives possess only a few and the Dini derivatives may possess none. See [2, 6, 7, 12, 15, 16, 18, 19, 20, 27, 28, 29] for details.

What is it that causes some generalized derivatives to enjoy so many desirable properties, while others possess very few? When a new generalized derivative is invented, can one determine easily which, if any, of the desirable properties it possesses? Recent research has focused on such questions. As one might expect, the most successful approaches involve some abstract view of differentiation which provides conditions under which certain desirable properties are obtained. But other types of studies, accomplished for different purposes, have also shed light on these questions. We shall discuss the abstract approach briefly and then turn to a study of the representation of functions by derivatives, which has corollaries pertaining to our questions.

Abstract Differentiation on the Line

Although various notions of abstract differentiation have been studied for over fifty years, it is only recently that serious attempts have been made to study generalized differentiation of functions of one real variable by abstract means. Prior to 1980 there were a few interesting works that related to this subject, but the main developments have occurred in only the past three or four years. At this point, there appear to be about three levels of generality. The most obvious of these is the notion of Path derivative [4]. This notion contains any theory of differentiation in which the generalized derivative of a function at a point is a derived number. It is suitable for studying the behavior of such generalized derivatives and their primitives and to studying monotonicity conditions, but its applications to other parts of differentiation theory are quite limited. A more general notion involves the concept of systems of sets (called Simple Systems) associated with each point in the domain of the function. This setting provides a useful framework for considering questions involving extreme derivates, for relating differentiation to variation and to measure, but is not convenient for a study of integration. The most general approach involves a study of abstract differentiation bases. This theory allows a complete development of integration theory, from the Riemann viewpoint, and clarifies the relationships that exist among the concepts of differentiation, variation, measure and integration.

Since we are concerned primarily with the behavior of generalized derivatives and their primitives, we will discuss only the most obvious of these approaches, namely the Path derivatives. Interested readers may consult Thomson's recent works [24], [25], [26]. The first of these discusses monotonicity and the last two provide a complete development of abstract differentiation theory.

We begin with a definition [4].

Let $E = \{E_x : x \in R\}$ be a system of sets $E_x \subset R$ such that for each $x \in R$, $x \in E_x$ and x is an accumulation point of E_x.

Such a system is called a system of paths. A function $f : R \to R$ will be called E-differentiable at x if

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{f(y) - f(x)}{y - x} \quad \text{is finite and}$$

we denote this limit by $f'_E(x)$.

We observe first that any notion of generalized pointwise differentiation in which the generalized derivative at a point is a derived number of the function at that point can be described as a path derivative; that is, there is a system of paths E such that the generalized derivative is f'_E. For example, a function f will be differentiable if we can choose $E_x = R$ for every x; it will be approximately differentiable if and only if each path E_x can be chosen to have density one at x.

It turns out that the behavior of a generalized derivative is very closely linked to the geometry of the system E of paths associated with that derivative. If we place no restrictons on E we get no information about the E-derivative f. In fact, given any measurable function f, there exists a continuous function F and a system of paths E such that $F'_E(x) = f(x)$ for every x.

Desirable properties of F'_E derive primarily from two types of conditions on the system E. Intersection conditions provide information related to the "thickness" of the paths, while porosity conditions relate to the size of "gaps" in the paths. When the paths are "thick" and the gaps in the paths are small, one obtains many desirable properties of both F'_E and F.

To give an idea of the kinds of results possible, we give definitions of one type of intersection condition and of the notion of

nonporosity, and we indicate some of the consequences.

Definition: A system of paths is said to satisfy the External Intersection Condition (EIC) if there is a positive function δ defined on R such that if $0 < y - x < \min(\delta(x), \delta(y))$ then

$$E_x \cap E_y \cap (y, 2y-x) \neq \emptyset$$

and

$$E_x \cap E_y \cap (2x-y, x) \neq \emptyset.$$

Thus, a system satisfying EIC has paths sufficiently thick to guarantee that if $x < y$ are sufficiently close (as determined by δ) then E_x and E_y will have points in common near x and near y outside of the interval $[x,y]$. Various generalized derivatives satisfy EIC [4]. This condition on a system E is by itself sufficient to guarantee that every E-differentiable function F is [ACG], is differentiable a.e. on a dense, open set and approximately differentiable a.e., and assumes local extrema. Furthermore, F'_E is in B_1, the first class of Baire. If in addition, for each $x \in R$, the path E_x has x as a bilateral point of accumulation, then F'_E has the Denjoy property and is in Zahorski's class $\mathcal{M}_2$. Thus, a system E satisfying EIC will imply a number of desirable properties for E-differentiable functions and E-derivatives. A comparison with the behavior of approximately differentiable functions and approximate derivatives (see Introduction) reveals that some of the desirable properties are still lacking. This brings us to the notion of a nonporous system of paths.

Definition. A set A is called nonporous at a point x if

$$\lim_{h \to 0} \frac{\ell(x,h,A)}{h} = 0$$ where $\ell(x,h,A)$ denotes the length of the largest open interval contained in the set $(x-h, x+h) \cap (R \setminus E)$.

It is easy to verify, for example, that if x is a point of density of A, then A is nonporous at x.

When each path E_x of a system E is nonporous, we call E a nonporous system of paths. Nonporous systems E have the remarkable property that if an E-differentiable function F is monotonic, it is in fact differentiable in the ordinary sense. It follows readily that if F'_E is bounded above or below (or even dominated by some ordinary derivative), then F is differentiable and $F'_E = F'$. Thus, if in addition, $F'_E \in B_1$, it will follow that F is differentiable on a dense open set.

It is now clear that when a system E satisfies EIC and is nonporous, a great deal of desirable behavior is guaranteed for E-differentiable functions and E-derivatives. The combination of the two conditions gives even more. When both conditons are met, F'_E will be in Zahorski's class $\mathcal{M}_3$. In addition, an analogue of a beautiful theorem of O'Malley and Weil [19] in the setting of approximate differentiation is valid:

Theorem: Let E be nonporous and satisfy EIC. Let F be E-differentiable. If F'_E takes on the values M and $-M$ on an interval I_o, then there is a subinterval of I_o on which F is differentiable and F' assumes both values M and $-M$.

The beauty of this theorem is that it has a number of revealing corollaries [19], [4]. One of these corollaries states, in essence, that any monotonicity theorem valid for differentiable functions has a complete analogue for E-differentiable functions. This corollary may help us understand some of the many monotonicity theorems that have appeared in the literature during the last fifteen or twenty years. To formulate the corollary in precise terms we introduce some notation. Let Δ denote the family of differentiable functions, Δ_E the family of E-differentiable functions, and $\mathcal{N}$ denote the class of nondecreasing functions. For any family $\mathcal{G}$ of functions on R and any interval $I \subset R$, write $\mathcal{G}(I)$ for the family of restrictions of functions in $\mathcal{G}$ to I.

Corollary: Let E be nonporous and satisfy EIC. Let $\mathcal{G}$ be a family of functions on R. If $\Delta(I) \cap \mathcal{G}(I) \subset \mathcal{N}(I)$ for every interval $I \subset R$, then $\Delta_E(I) \cap \mathcal{G}(I) \subset \mathcal{N}(I)$ for all $I \subset R$.

Proof: Suppose $F \in (\Delta_E(I) \cap \mathcal{G}(I)) \setminus \mathcal{N}(I)$ for some interval I. By the hypothesis that $\Delta(I) \cap \mathcal{G}(I) \subset \mathcal{N}(I)$, F cannot be differentiable on all of I. From previous remarks, it follows that F'_E must be unbounded above and unbounded below on I. The Theorem now guarantees the existence of an interval $J \subset I$ on which F' exists and takes both signs. But then $F \in (\Delta(J) \cap \mathcal{G}(J)) \setminus \mathcal{N}(J)$, contradicting the hypothesis.

By choosing $\mathcal{P}$ appropriately, we can obtain a number of special monotonicity results. For example, if $F'_E \geq 0$ a.e. on its set of differentiability, then F is nondecreasing. Similarly, if F is nondecreasing on each interval on which it is differentiable, then F is nondecreasing. (In both these results, we assume, of course, that F is E-differentiable and E is nonporous and satisfies EIC).

It may be interesting to note that in his doctoral research, Taylor [22] has recently established that the M, -M Theorem is valid without EIC if one assumes F continuous and $F'_E \in B_1$. Thus, in particular, it is valid for sequential-congruent derivatives of continuous functions as studied by Petruska and Laczkovich [20], provided the sequential paths are nonporous. For example, if F is continuous and $\lim_{n \to \infty} n[F(x + 1/n) - F(x)]$ and $-\lim_{n \to \infty} n[F(x - 1/n) - F(x)]$ are equal for all x, the M, -M Theorem and its corollaries hold. In particular, if F is monotonic on each interval on which it is differentiable, it is monotonic on all of R.

The path derivative approach can also be useful when dealing with generalized derivatives for which a desirable property is not valid in general. For example, certain kinds of density derivatives have paths that satisfy the EIC. Their primitives will be differentiable a.e. on some dense open set. But if the paths can be chosen nonporous, then the primitive will be differentiable everywhere on a dense open set. This always happens, for example with the approximate derivative, but not with the preponderant derivative, unless, of course, the particular function under consideration allows nonporous paths.

We close this section by mentioning that there are other notions of intersection conditions and of porosity conditions that have also proved useful [4]. Furthermore, the results we stated are not always the best possible [4], [22], [25], [26]. Our aim was merely to give an indication of how simple conditions on a system of paths E can give rise to desirable behavior of E-differentiable functions and of E-derivatives. Observe, that a system satisfying only the two conditions we mentioned gives rise to all the desirable properties we mentioned about approximate differentiation in the Introduction.

Representing Functions by Derivatives

A difficulty in dealing with the class Δ' of derivatives is that this class is not an algebra. Wilkosz [31] showed that while the function $f(x) = \cos \frac{1}{x}$ $(f(0) = 0)$ is in Δ', f^2 is not. Thus Δ' is not closed under multiplication. In fact, for both f and f^2 to be derivatives, severe conditions on f are required [31], [10]. Suppose, for example, that $F' = f$ and $G' = f^2$ on an interval I. Since $G' \geq 0$, f^2 is summable on I, so f must also be summable. Thus, the square of any nonsummable derivative on I cannot be a derivative.

The fact that the product of two derivatives need not be a derivative raises a number of interesting questions. Let Δ denote the class of differentiable functions.

1) For which $F \in \Delta$ is it true that $F'G' \in \Delta'$ for every $G \in \Delta$?
2) Which functions f are products of n derivatives?
3) What other representations of functions by derivatives can be characterized?
4) What functions are in the algebra generated by Δ'?

Question 1) has been studied in great detail. The survey article by Fleissner [8] was up-to-date at the time it was written. Some additional results have recently been obtained by Mařík [13]. We mention only that even for $F,G \in \Delta$, the product FG' need not be in Δ'.

Question 2) has led to some very interesting work. It is easy to verify that the characteristic function of a singleton is a product of two derivatives. One can actually obtain the same result for the characteristic function of any closed set. But the characteristic function of nonempty open set (not all of R) can never be expressed as a product of any number of derivatives. Why does the representation hold for closed sets but not open sets? Mařík and Weil [4] have proved some very interesting theorems related to this question. Here is a corollary to one of their theorems:

Let u be a nonnegative function defined on $[0,1]$ and continuous on $(0,1]$. There are numbers $q_2 > q_3 > \ldots$ such that u is a product of n nonnegative derivatives if and only if $u(0) \geq q_n$. Mařík and Weil determined these numbers q_n precisely. For $u = \chi_{[0,1]}$, u is a product of two derivatives because $u(0) = 1 = q_2$. But for $u = \chi_{[0,1]}$, $u(0) < q_n$ for every n, so u is not a product of any number of derivatives.

A different sort of condition has been obtained in [5]. Every function in the first class of Baire that vanishes a.e. is a product of two derivatives. This fact leads to a very short proof of the existence of differentiable nowhere monotonic functions. Let

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \ q \text{ even} \\ -\frac{1}{q} & \text{if } x = \frac{p}{q}, \ q \text{ odd} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We assume as usual that p and q are relatively prime for rational $x = p/q$. Then $f \in B_1$, and $f = 0$ a.e. Thus there are $g,h \in \Delta$ such that $f = g'h'$. If g' is unisigned on an interval I then h' takes both signs on every subinterval of I so h is the desired function. If g' is not unisigned on any interval, then g is the desired function.

Of major interest to us is question 3), because applications of some of the results shed light on why certain generalized derivatives mimic the behavior of the ordinary derivative.

To date, two types of representations have been studied in detail: representations of the form $f = g' + hk'$ $(g,h,k \in \Delta)$, [1]; and representations of the form

$$f = f_1'^{\beta_1} f_2'^{\beta_2} \cdots f_n'^{\beta_n}$$

$$(f_j \in \Delta, \ f_j' \geq 0, \ \beta_j > 0, \ j = 1,\ldots,n), \qquad [14].$$

We emphasize the former class because of the applications. We observe first that if

$$f = g' + hk' \qquad (g,h,k \in \Delta) \qquad (*)$$

then f can fail to be in Δ' only because the product hk' fails. We have already observed that this is possible. However, it is not difficult to verify that hk' is the derivative of its integral on any interval on which hk' is bounded, in particular on appropriate neighborhoods of points of continuity of hk'. Thus hk' is the derivative of some function on a dense open set. Thus, if f admits the representation (*), then there is a function F de-

fined and differentiable on a dense open set U such that $F' = f$ on U. Actually, the class of functions satisfying (*) can be characterized in a number of ways.

Theorem [1]. Let f be a function on R. The following conditions are equivalent.

(a) There exist differentiable functions g, h and k such that $f = g' + hk'$.

(b) There exists a dense open set U and functions F and G such that F is defined and differentiable on U, G is differentiable on R and $F' = f$ on U, $G' = f$ on $R\backslash U$.

(c) There exist differentiable functions $H_1, H_2, \ldots$ and closed sets $A_1, A_2, \ldots$ such that $R = \bigcup_{j=1}^{\infty} A_j$ and $f = H_j'$ on A_j, $j = 1,2,\ldots$.

(d) To each nonempty closed set P there corresponds an open interval I and a differentiable function F such that $I \cap P \neq \emptyset$ and $F' = f$ on $I \cap P$.

Observe that condition (b) indicates a sense in which there exist two functions which "together have f as a derivative". Of course, the function F is defined only on U and cannot in general be extended to a differentiable function on all of R. On the other hand, condition (c) guarantees the existence of a sequence of functions differentiable on all of R which "together have f as a derivative".

The importance of this class to a study of behavior of generalized derivatives lies in the fact that many classes of functions related to differentiation theory admit the representation (*) [1]. For example, each approximate derivative does. A number of the properties of approximate derivatives and their primitives follow readily.

Various other representations have been investigated. Mařík, for example, observed that many types of functions admit representations of the form $f = g' + h'k'$ $(g,h,k \in \Delta)$. This included all functions of bounded variation as well as functions of the form $\sum_{k=1}^{\infty} \alpha_k f_k$ where each f_k is the characteristic function of a closed

set A_k, the sets A_k are disjoint and $\sum_{k=1}^{\infty} \alpha_k < \infty$. In fact, no Baire 1 functions were discovered that cannot be represented in this form!.

Question (4) now arises naturally: What functions are in Alg Δ', the algebra generated by Δ'? Since $\Delta' \subset B_1$, and B_1 is an algebra it is clear that Alg $\Delta' \subset B_1$. In 1982, Preiss [21] established the fact that Alg $\Delta' = B_1$. In fact, he obtained the remarkable result that every Baire 1 function f admits the representation

$$f = g' + h'k' \qquad (g,h,k \in \Delta).$$

Thus, although a Baire 1 function can be much more pathological than can any derivative, it can still be represented by derivatives in the above form. This also sheds some light on the difficulty of Question (2), because it shows that to each Baire 1 function f, there corresponds a derivative g' such that $f-g'$ is a product of two derivatives. A characterization of the class of functions expressible as a product of derivatives must be very difficult. Perhaps even more difficult than obtaining an answer to Young's problem, mentioned in the Introduction, of characterizing those functions which are derivatives.

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NON-EXISTENCE OF GLOBAL SOLUTIONS OF $\Box u = \frac{\partial}{\partial t} F(u_t)$ IN TWO AND THREE SPACE DIMENSIONS[1]

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This paper deals with solutions $u(x_1,\dots,x_n,t) = u(x,t)$ of certain nonlinear hyperbolic equations of the form

$$u_{tt} - \sum_{i,k=1}^{n} a_{ik}(u')u_{x_i x_k} = 0 \,. \tag{1}$$

(Here u' stands for the set of first partial derivatives of u). Equations or systems of equations of type (1) describe the propagation of waves in a hyperelastic material. Solutions u corresponding to initial conditions

$$u(x,0) = f(x);\ u_t(x,0) = g(x) \quad \text{for} \quad x \in R^n \tag{2}$$

may or may not exist "globally", i.e., for all $t \geq 0$. The "life-span" T of a solution is the largest value such that a C^2-solution of (1), (2) exists for $x \in R^n$, $0 \leq t < T$. Global existence corresponds to $T = \infty$, "blow-up in finite time" to $T < \infty$.

S. Klainerman [1], [2] proved that $T = \infty$ for "sufficiently small" initial data, in case the number n of space dimensions exceeds 3. For initial data of the form

$$u(x,0) = \varepsilon\phi(x), \quad u_t(x,0) = \varepsilon\psi(x) \tag{3}$$

with a constant $\varepsilon > 0$, smallness can be measured conveniently by the size of ε for fixed ϕ,ψ. For $n = 3$ (see Klainerman [2], [3]

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

and John and Klainerman [4]) we only get "almost global" existence of solutions in the sense that $T = T(\varepsilon)$ has a lower bound of the form

$$T > Ae^{B/\varepsilon} \tag{4}$$

with positive constants A,B depending on ϕ,ψ.

This behavior for higher n contrasts with the case $n = 1$. There it is known (see Lax [17], John [18]) that $T < \infty$ for non-trivial sufficiently small data of compact support, provided the equation is "genuinely nonlinear". More precisely T behaves then like A/ε for small ε. By imbedding, this result for $n = 1$ implies that there exist for any n "large" data[1] for which $T < \infty$. That actually $T < \infty$ for $n = 3$ and some arbitrarily small data was shown by F. John [5], at least for some equations. An example is the "model" equation

$$\square u = u_{tt} - \Delta u = F'(u_t)u_{tt} \tag{5}$$

where

$$F(0) = F'(0) = 0;\quad F''(s) > c > 0 \text{ for all } s\ . \tag{6}$$

It is shown in [5] that here $T < \infty$ for data (2) of compact support, provided the data satisfy the inequality

$$K = \int_{R^3} [g(x) - F(g(x))]\,dx \geq 0 \tag{7}$$

(with $dx = dx_1dx_2dx_3$). More precisely for data of compact support of type (3) for which

$$\int_{R^3} \psi(x)dx > 0 \tag{8}$$

one has

$$T < A \exp(B/\varepsilon^4)\ . \tag{9}$$

[1] Conditions for non-existence of global solutions of systems of conservation laws for sufficiently large data and any n are given by Sideris [19].

Similar results for other equations with spherical symmetry were obtained by Sideris [6].

These results for $n = 3$ have certain drawbacks. As a consequence of assumption (6) equation (5) becomes elliptic for $u_t > 1/c$. This raises the question if blow-up is just due to this feature, and if it would also accur in equations that are hyperbolic for all arguments[1]. A second undesirable feature is the inequality restriction (7) imposed on the data, and a third is the fact that the upper bound (9) for T is unrealistically large.

If we restrict ourselves to <u>radial</u> solutions u of (5), (those depending only on $|x|$ and t), the problem becomes essentially one-dimensional and the analysis simplifies, as is shown in [7]. All that matters then for solutions with small initial data is the behavior of $F(s)$ for small s, so that (6) can be replaced by the weaker assumptions

$$F(0) = F'(0) = 0,\quad F''(0) > 0\ . \tag{10}$$

For small initial data of type (3) no inequality restriction on $\phi(x),\psi(x)$ is needed, and blow-up for non-trivial data of compact support occurs at a finite time T with an upper bound of the form

$$T < A^* e^{B^*/\varepsilon} \tag{11}$$

for small ε. This bound is optimal in view of the lower bound (4)[2].

No analogous results for general non-radial solutions u of (5) have been established. The present paper extends the results of [5] to equations (5) that are hyperbolic for all u_t. It proves blow-up in finite time with the optimal bound (11) for T when $n=3$, but only for data that are subject to a slightly generalized inequality (7). The paper also derives results for $n=2$ with an inequality[3].

[1] Equation (5) is hyperbolic iff $F'(u_t) < 1$.

[2] For analogous results for radial elastic waves see [16].

[3] The estimate (12) again is optimal, that is T can be shown to have a lower bound $A\varepsilon^{-2}$. This follows by a slight modification of the arguments used in [8] when $n > 3$. One only has to observe that for $n=2$ the right hand side of formula (98), p. 555 of [8] stays bounded for $T_o = O(\varepsilon^{-2})$.

(12) $$T < A^*/\varepsilon^2$$

taking the place of (11). The essential difference in the proofs is that here we use plane integrals instead of the spherical means used in [5]. This facilitates a unified treatment of the cases $n = 2,3$. The everywhere hyperbolic character of the differential equation somewhat complicates the argument compared with [5], amd makes it necessary to appeal to the energy integral associated with (5).

We assume that our function $F(s)$, is of class $C^3(R)$ and satisfies

(13) $$F(0) = F'(0) = 0;\ F''(s) > 0,\ F'(s) \le q < 1 \quad \text{for all } s$$

so that equation (5) is hyperbolic for all u_t. In what follows n will always have the values 2 or 3. We prescribe initial conditions

(14) $$u(x,0) = f(x) = \varepsilon\phi(x);\ u_t(x,0) = g(x) = \varepsilon\psi(x) \quad \text{for } x \in R^n$$

with $\varepsilon > 0$. The data shall have compact support, say

(15) $$f(x) = g(x) = \phi(x) = \psi(x) = 0 \quad \text{for } |x| > R .$$

We introduce

(16) $$h(x) = g(x) - F(g(x))$$

and set

(17a) $$K = \int_{R^n} h(x)dx$$

(17b) $$L(\xi,s) = \int_{x\cdot\xi>-s} [f(x)+(x\cdot\xi+s)h(x)]\ dx \quad \text{for } \xi \in S^{n-1},\ s \in R$$

(17c) $$\lambda(\xi,s) = \int_{x\cdot\xi>-s} [\phi(x)+(x\cdot\xi+s)\psi(x)]\ dx \quad \text{for } \xi \in S^{n-1},\ s \in R .$$

Here $dx = dx_1 \ldots dx_n$, $x\cdot\xi = x_1\xi_1 + \ldots + x_n\xi_n$, and S^{n-1} denotes the unit sphere in n-space. Under the assumptions (13), (15) we have

(18a) $$L(\xi,s) = 0 \quad \text{for } s < -R$$

(18b) $$L(\xi,s) = \int_{R^n} [f(x) + (x\cdot\xi)h(x)]\ dx + Ks \quad \text{for } s > R$$

(18c) $$L(\xi,s) = \varepsilon\lambda(\xi,s) + O(\varepsilon^2) \quad \text{for fixed } \phi,\psi \text{ and small } \varepsilon .$$

THEOREM. Let $n = 2$ or 3. Let $u(x,t)$ be a C^2-solution of (5) for $x \in R^n$, $0 \le t < T$ with initial data (14) satisfying (15). Then $T < \infty$ if either

(19a) $$L(\xi,s) > 0 \quad \text{for some } \xi \in S^{n-1},\ s \in R$$

or[1]

(19b) $$K \ge 0, \quad u \not\equiv 0 .$$

More precisely, if

(19c) $$\lambda(\xi,s) > 0 \quad \text{for some } \xi \in S^{n-1},\ s \in R$$

then there exist positive A^*,B^* (depending on ϕ,ψ,F) such that for all sufficiently small ε (11) holds when $n = 3$ and (12) holds when $n = 2$.

Corollary. Let $v(x,t)$ for $n = 2,3$ be a nontrivial solution of class C^3 of the nonlinear equation

(20) $$\Box v = v_{tt} - \Delta v = F(v_{tt}) \quad \text{for } x \in R^n,\ 0 \le t < T$$

where F satisfies (13). Let v have initial values $v(x,0)$ and $v_t(x,0)$ of compact support. Then $T < \infty$.

Proof of the Corollary. The function $u = v_t$ is a C^2-solution of (5) for which $u(x,0) = v_t(x,0)$ and

$$h(x) = u_t(x,0) - F(u_t(x,0)) = v_{tt}(x,0) - F(v_{tt}(x,0)) = \Delta v(x,0)$$

have compact support. Then $u_t(x,0)$ also has compact support, since by (13)

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[1] Observe that $K > 0$ implies by (18b) that (19a) holds for all sufficiently large s.

$$|s - F(s)| = \left| \int_0^s (1 - F'(z))dz \right| \geq (1 - q)|s| \neq 0 \quad \text{for} \quad s \neq 0 .$$

Moreover here

$$K = \int_{R^n} h(x)\, dx = \int_{R^n} \Delta v(x,0)\, dx = 0 .$$

Applying the Theorem for $T = \infty$ yields $u(x,t) = v_t(x,t) \equiv 0$. Then $v(x,t) = v(x,0)$ and by (20) $\Delta v(x,0) = 0$, which implies $v(x,0) = 0$, since $v(x,0)$ has compact support. It follows that $v(x,t) \equiv 0$.

Proof of the Theorem. The proof uses the common type of argument that might be called "method of moments". (See e.g., references [10], [11], [12], [13], [14], [15], [19].) A differential inequality is established for a certain "moment" (a functional in integral form formed from the solution). On the basis of this inequality the moment is shown to grow with time in a manner incompatible with continued existence. By this method one proves non-existence of a global solution, without, however, gaining any insight into the process of singularity formation constituting blow-up. The actual blow-up involving possibly only higher derivatives, quite likely, takes place some time before the moments in question show any drastic behavior. The method of moments then just confirms that the solution (in the strict sense) has disintegrated after a finite time, without establishing the cause of death.

First of all $u(x,t)$ is of compact support in x. More precisely for data satisfying (15) we have

(21) $$u(x,t) = 0 \quad \text{for} \quad |x| > R + t$$

(see [5], p. 49). Introducing

(22) $$v(x,t) = \int_0^t u(x,s)\, ds \quad \text{for} \quad x \in R^n, \quad 0 \leq t < T$$

we have $v_t = u$ and

$$v(x,0) = 0, \quad v_t(x,0) = f(x)$$

(23) $$v(x,t) = 0 \quad \text{for} \quad |x| > R + t$$

(24) $$\Box v(x,t) = F(v_{tt}(x,t)) + h(x) \quad \text{for} \quad x \in R^n, \quad 0 \leq t < T$$

with $h(x)$ defined by (16). We associate with the function $v(x,t)$ the "plane integral"

(25) $$v^*(r,t) = \int_{x_1=r} v(x,t)\, dS \quad \text{for} \quad r \quad R, \quad 0 \leq t < T$$

Where $dS = dx_1 \ldots dx_n/dx_1$, that is $dS = dx_2$ when $n = 2$ and $dS = dx_2 dx_3$ when $n = 3$. Then

(26a) $$v^*(r,t) = 0 \quad \text{for} \quad |r| > R + t$$

$$v^*(r,0) = 0$$

(26b) $$v_t^*(r,0) = f^*(r) = \int_{x_1=r} f(x)\, dS$$

(26c) $$v_{tt}^*(r,t) - v_{rr}^*(r,t) = F^*(r,t) + h^*(r)$$

where we define

(26d) $$F^*(r,t) = \int_{x_1=r} F(v_{tt}(x,t))\, dS$$

(26e) $$h^*(r) = \int_{x_1=r} h(x)\, dS .$$

It follows that

(27a) $$v^*(r,t) = v_o^*(r,t) + \frac{1}{2} \int_{T_{r,t}} F^*(\rho,\tau)\, d\rho d\tau \quad \text{for} \quad r \in R, \; 0 \leq t < T$$

where $T_{r,t}$ is the "characteristic" triangle with vertices $(r,t), (r - t, 0), (r + t, 0)$ and

(27b) $$v_o^*(r,t) = \frac{1}{2} \int_{r-t}^{r+t} [f^*(\rho) + (t - |r - \rho|)h^*(\rho)]\, d\rho .$$

Since by (15), (16)

(28) $$f^*(\rho) = h^*(\rho) = 0 \quad \text{for} \quad |\rho| \geq R$$

we have

(29) $$v_0^*(r,t) = \frac{1}{2} \int_{r-t}^{R} [f^*(\rho) + (t - r + \rho)h^*(\rho)]\, d\rho = \frac{1}{2}M(t - r)$$

for $r \geq R$, $t \geq 0$, where in the notation of (17b)

$$M(z) = \int_{-z}^{R} [f^*(\rho) + (z + \rho)h^*(\rho)]\, d\rho$$

(30) $$= \int_{x_1 > -z} [f(x) + (z + x_1)h(x)]\, dx = L(\xi^1, z)$$

with ξ^1 denoting the unit vector in the x_1-direction:

(31) $$\xi^1 = (1,0,\ldots,0) .$$

Our assumptions (15) on F imply that

(32a) $$F(z) > 0 \quad \text{for} \quad z \neq 0$$

and that there exist positive constants a,b such that

(32b) $$F(z) \geq az^2 \quad \text{for} \quad |z| \leq b .$$

Then by (15)

(32c) $$\frac{F(z)}{z} \geq \frac{F(b)}{b} \geq ab \quad \text{for} \quad z \geq b$$

(32d) $$ab \leq q < 1 .$$

Since $F(z)$ is convex and $v_{tt}(x,t)$ has its support in $|x| \leq R + t$ we have from Jensen's inequality applied to (26d)

$$\frac{F^*(r,t)}{c(r,t)} \geq F\left(\frac{1}{c(r,t)} \int_{x_1 = r} v_{tt}(x,t)\, dS\right)$$

(33a) $$= F\left(\frac{v^*_{tt}(r,t)}{c(r,t)}\right) \geq 0$$

for $r < t + R$. Here

(33b) $$c(r,t) = \int_{\substack{x_1 = r \\ |x| < t+r}} dS = \gamma((t + R)^2 - r^2)$$

with $\gamma(z)$ defined by

(33c) $\gamma(z) = 0$ for $z < 0$, $\gamma(z) = 2z^{1/2}$ for $z \geq 0$ when $n = 2$

(33d) $\gamma(z) = 0$ for $z < 0$, $\gamma(z) = \pi z$ for $z \geq 0$ when $n = 3$.

Blow-up will be established by deriving an integral inequality for $v^*(r,t)$ along lines $t - r = \text{const.} = z$. In what follows let z be a fixed number with

(34a) $$z > -R .$$

Define

(34b) $$P(r) = v^*(r, z + r) \quad \text{for} \quad R \leq r < T - z .$$

By (27a), (29)

(34c) $$P(r) = M(z) + \frac{1}{2} \int_{T_{r,z+r}} F^*(\rho,\tau)\, d\rho d\tau \quad \text{for} \quad R \leq r < T - z .$$

For $R \leq r_1 \leq r < t - z$ we have $T_{r_1, z+r_1} \subset T_{r,z+r}$ and

(34d) $$P(r) = P(r_1) + \frac{1}{2} \int_{\Gamma} F^*(\rho,\tau)\, d\rho d\tau$$

with $\Gamma = T_{r,z+r} \setminus T_{r_1,z+r_1}$. Set

(34e) $$r_2 = z + 2r_1 .$$

Then for $r_2 < r$ the region Γ contains the parallelogram

(34f) $$r_2 < \rho < r, \quad \rho - R < \tau < \rho + z$$

and it follows from (34d), (33a) that

(34g) $$P(r) > P(r_1) + \frac{1}{2} \int_{r_2}^{r} d\rho \int_{\rho-R}^{\rho+z} c(\rho,\tau) F\left(\frac{v^*_{tt}(\rho,\tau)}{c(\rho,\tau)}\right) d\tau$$

for $r_2 < r < T - z$.

Since by (26a) $v^*(\rho,\tau) = 0$ for $\tau < \rho - R$, we have for $R < \rho < T - z$

(35a) $$P(\rho) = v^*(\rho, z + \rho) = \int_{\rho-R}^{\rho+z} (\rho + z - \tau) v^*_{tt}(\rho,\tau)\, d\tau .$$

Then by Jensen's inequality and the convexity of F

(35b) $$C(\rho) F\left(\frac{P(\rho)}{C(\rho)}\right) \leq \int_{\rho-R}^{\rho+z} (\rho + z - \tau) c(\rho,\tau) F\left(\frac{v^*_{tt}(\rho,\tau)}{c(\rho,\tau)}\right) d\tau$$

where

$$C(\rho) = \int_{\rho-R}^{\rho+z} (\rho + z - \tau) c(\rho,\tau)\, d\tau .$$

(35b) yields

(35d) $$C(\rho) F\left(\frac{P(\rho)}{C(\rho)}\right) \leq (z + R) \int_{\rho-R}^{\rho+z} c(\rho,\tau) F\left(\frac{v^*_{tt}(\rho,\tau)}{c(\rho,\tau)}\right) d\tau$$

Substituting into (34g) gives the desired inequality for P:

(35e) $$P(r) \geq P(r_1) + \frac{1}{2} \int_{r_2}^{r} \frac{C(\rho)}{z + R} F\left(\frac{P(\rho)}{C(\rho)}\right) d\rho \quad \text{for} \quad r_2 < r < T - r .$$

Lemma 1. Let there exist r_1, t_1, k with

(36) $$r_1 \geq R, \quad 0 < t_1 < T, \quad v^*(r_1,t_1) = k > 0 .$$

Then $T < \infty$.

Proof of Lemma 1. Set $z = t_1 - r_1$. Then $z > -R$ by (26a). Define P, r_2 as in (34b), (34e). Then

(37a) $$P(r_1) = k > 0 .$$

We compare $P(r)$ with the solution $p(r)$ of the integral equation

(37b) $$p(r) = P(r_1) + \frac{1}{2} \int_{r_2}^{r} \frac{C(\rho)}{z + R} F\left(\frac{p(\rho)}{C(\rho)}\right) d\rho \quad \text{for} \quad r \geq r_2$$

that is with the solution of the differential equation problem

(37c) $$p'(r) = \frac{C(r)}{2(z + R)} F\left(\frac{p(r)}{C(r)}\right), \quad p(r_2) = P(r_1) .$$

Since $p(r) > 0$ by (37a), (32a), and F is increasing for positive arguments by (13), we have

(37d) $$P(r) > p(r) \quad \text{for} \quad r_2 < r < T - z$$

by Gronwall's lemma.

The inequalities (32b,c) furnish different lower bounds for $F(p(\rho)/C(\rho))$ as $p/C > b$ or $< b$. Let $\rho \geq r_2$ be a value for which

(38a) $$\frac{p(\rho)}{C(\rho)} \geq b .$$

Then by (35c), (32c)

$$\frac{d\,\frac{p(\rho)}{C(\rho)}}{d\rho} = \frac{1}{2(z + R)} F\left(\frac{p(\rho)}{C(\rho)}\right) - \frac{p(\rho) C'(\rho)}{C^2(\rho)}$$

$$\geq \frac{p(\rho)}{C(\rho)} \left(\frac{ab}{2(z + R)} - \frac{C'(\rho)}{C(\rho)}\right) .$$

Here by (35c), (33b)

$$C(\rho) = \int_{\rho-R}^{\rho+z} (\rho + z - \tau)\gamma((\tau + R)^2 - \rho^2)\, d\tau$$

$$= \int_0^{z+R} (z + R - \sigma)\gamma(\sigma(\sigma + 2\rho))\, d\sigma \tag{38b}$$

$$C'(\rho) = \int_0^{z+R} (z + R - \sigma)\gamma(\sigma(\sigma + 2\rho)) \frac{2\sigma\gamma'(\sigma(\sigma + 2\rho))}{\gamma(\sigma(\sigma + 2\rho))}\, d\sigma < \frac{C(\rho)}{\rho}$$

since by (33c,d)

$$\frac{\gamma'(\sigma(\sigma + 2\rho))}{\gamma(\sigma(\sigma + 2\rho))} = \frac{n - 1}{2\sigma(\sigma + 2\rho)} < \frac{1}{2\sigma\rho} .$$

Thus

$$\frac{dp(\rho)/C(\rho)}{d\rho} > 0 \quad \text{for} \quad \rho \geq \frac{2(z + R)}{ab} .$$

Set

$$r_3 = \text{Max}(r_2, \frac{2(z + R)}{ab}) \tag{38c}$$

Then

$$\frac{p(\rho)}{C(\rho)} \geq b \quad \text{for} \quad \rho \geq r_3 \tag{38d}$$

implies that

$$\frac{p(\rho)}{C(\rho)} \geq b \quad \text{for} \quad r > \rho . \tag{38e}$$

Let now r be such that

$$r_3 \leq r < T - z, \quad p(\rho) \leq bC(\rho) \quad \text{for} \quad r_3 < \rho < r . \tag{39a}$$

Then by (32b), (37c), (39a)

$$p'(\rho) \geq \frac{ap^2(\rho)}{2(z + R)C(\rho)} \quad \text{for} \quad r_3 < \rho < r \tag{39b}$$

$$\frac{1}{k} = \frac{1}{p(r_2)} \geq \frac{1}{p(r_3)} \geq \frac{1}{p(r_3)} - \frac{1}{p(r)} \geq \frac{a}{2(z + R)} \int_{r_3}^{r} \frac{d\rho}{C(\rho)} .$$

Here by (38), (33c,d)

$$C(\rho) \leq (z+R)^2\gamma((z+R)(z+R+2\rho)) \leq \pi(z+R)^{(n+3)/2}(3\rho)^{(n-1)/2} \tag{39c}$$

since by (38c), (32d)

$$z + R + 2\rho \leq \frac{1}{2} abr_3 + 2\rho \leq 3\rho \quad \text{for} \quad r_3 \leq \rho ,$$

$$\int_{r_3}^{r} \frac{d\rho}{C(\rho)} > \frac{1}{3\pi} (z + R)^{-(n+3)/2} \int_{r_3}^{r} \rho^{(1-n)/2}\, d\rho . \tag{39d}$$

We define r_4 by

$$\frac{1}{k} = \frac{a}{2(z + R)} \int_{r_3}^{r_4} \frac{d\rho}{C(\rho)} . \tag{39e}$$

Then

$$P(r) \geq p(r) \geq bC(r) \quad \text{for} \quad r_4 < r < T - z . \tag{39f}$$

We have by (35a)

$$P(r) = \int_{x_1=r} dS \int_{r-R}^{r+z} (z + r - t)v_{tt}(x,t)\, dt .$$

Hence

$$p^2(r) < J \int_{\substack{x_1=r \\ r-R<t<r+z}} v_{tt}^2(x,t)\, dSdt$$

where

$$J = \int_{\substack{x_1=r,\ |x|<t+R \\ r-R<\ <r+z}} (z + r - t)^2\, dSdt$$

$$\leq (z + R) \int_{r-R}^{r+z} (z + r - t)c(r,t)\, dt = (z + R)C(r) .$$

It follows from (39f) that for $r_4 < r < T - z$

$$(39g)\quad b^2 C(r) < \frac{p^2(r)}{C(r)} < (z + r) \int_{\substack{x_1 = r \\ r-R<t<r+z}} v_{tt}^2(x,t)\, dSdt .$$

Here by (38b)

$$C(r) \geq \int_0^{z+R} (z + R - \sigma)\gamma(2r\sigma)\, d\sigma$$

$$\geq \int_0^{z+R} (z + R - \sigma)(2\sigma)^{(n-1)/2} r^{(n-1)/2}\, d\sigma$$

$$(39h)\quad \geq \frac{2}{3} (z + R)^{(n+3)/2} r^{(n-1)/2} .$$

Let now ρ be a number with $r_4 < \rho < T - 2z - R$ so that

$$r_4 < r < T - z \quad \text{for} \quad \rho < r < \rho + z + R .$$

Integrating (39g) with respect to r from ρ to $\rho + z + R$ and using (39h) we find that

$$\frac{2}{3} b^2 (z + R)^{(n+3)/2} \rho^{(n-2)/2} \leq \int_{\substack{\rho<x_1<\rho+z+R \\ x_1-R<t<x_1+z}} v_{tt}^2(x,t)\, dxdt$$

$$(39i)\quad \leq \int_{\rho-R<t<\rho+2z+R} v_{tt}^2(x,t)\, dxdt .$$

Introducing

$$(40a)\quad G(s) = 2 \int_0^s zF'(z)\, dz$$

we have associated with (5) the "conservation of energy" relation

$$E = \frac{1}{2} \int [u_t^2 - G(u_t) + (\nabla u)^2]\, dx = \text{const.}$$

$$(40b)\quad = \frac{1}{2} \int [g^2 - G(g) + \nabla f)^2]\, dx .$$

Here, because of assumptions (13)

$$(40c)\quad G(s) \leq qs^2 \leq s^2 \quad \text{for all} \quad s \in R .$$

Thus

$$(40d)\quad \int v_{tt}^2(x,t)\, dx = \int u_t^2(x,t)\, dx \leq \frac{2E}{1 - q} \quad \text{for } 0 \leq t < T .$$

It follows from (39i) that

$$(40e)\quad \frac{1 - q}{6} b^2 (z+R)^{(n+1)/2} \rho^{(n-1)/2} \leq E \quad \text{for } r_4 < \rho < T - 2z - R .$$

Consequently either

$$(41a)\quad T - 2z - R < r_4$$

or

$$(41b)\quad \frac{1 - q}{6} b^2 (z + r)^{(n-1)/2} (T - 2z - R)^{(n-1)/2} < E$$

holds. In either case $T < \infty$, proving the lemma.

Lemma 2. Let there exist $\xi \in S^{n-1}$, $s \in R$ such that

$$(42a)\quad k = 2L(\xi,s) > 0 .$$

Then $T < \infty$. More precisely there exist positive constants α,β only depending on the choice of F such that

$$(42b)\quad T < \alpha(s + R)\, \text{Max}\left[\exp\left(\frac{\beta(s + R)^4}{k}\right), \frac{E}{(s + R)^3}\right] \text{ when } n = 3$$

$$(42c)\quad T < \alpha(s + R)\text{Max}\left[1 + \frac{(s + R)^6}{K^2}, \frac{E^2}{(s + R)^4}\right] \text{ when } n = 2 .$$

Corollary. For initial data of type (14) and $\lambda(\xi,s) > 0$ there

exist positive constants A^*, B^* depending on ϕ, ψ, F such that for all sufficiently small $\varepsilon > 0$ relation (11) holds when $n = 3$ and (12) when $n = 2$.

Proof of the Corollary. For fixed ϕ, ψ we have using (13) that $k = L(\xi, s) = \varepsilon\lambda(\xi, s) + o(\varepsilon^2)$, $E = o(\varepsilon^2)$ for small $\varepsilon > 0$. Then (11), (12) follow immediately from (42b,c).

Proof of Lemma 2. By (18a) $s > -R$. Since equation (5) is invariant under rigid motions, we can bring about that the ξ in (42a) is the unit vector ξ^1 defined in (31). Then (29), (30) yield

$$v_o^*(R, s + R) = \frac{1}{2} M(s) = k > 0 . \tag{43a}$$

Thus (36) holds with

$$r_1 = R, \quad t_1 = s + R . \tag{43b}$$

It follows from (34e), (38c), (32d) that here

$$r_2 = s + 2R, \quad r_3 = \frac{2(s + R)}{ab} . \tag{43c}$$

Using the estimate (39c) for $C(\rho)$ we find from (39e) that

$$r_4 \leq \frac{2(s + R)}{ab} \exp\left[\frac{6\pi(s + R)^4}{ak}\right] \quad \text{when} \quad n = 3 \tag{43d}$$

$$r_4 \leq \frac{4(s + R)}{ab} + \frac{6\pi^2(s + R)^7}{a^2k^2} \quad \text{when} \quad n = 2 . \tag{43e}$$

Thus (41a) implies that

$$T \leq \frac{4(s + R)}{ab} \exp\left[\frac{6\pi(s + R)^4}{ak}\right] \quad \text{when} \quad n = 3 \tag{43f}$$

$$T \leq \frac{6(s + R)}{ab} + \frac{6\pi^2(s + R)^7}{a^2k^2} \quad \text{when} \quad n = 2 . \tag{43g}$$

On the other hand (41b) leads to

$$T < 2(s + R) + \frac{6E}{(1 - q)b^2(s + R)^2} \quad \text{when} \quad n = 3 \tag{44a}$$

$$T < 2(s + R) + \frac{36E^2}{(1 - q)^2b^4(z + R)^3} \quad \text{when} \quad n = 2 . \tag{44b}$$

This establishes (42b,c) and proves the lemma.

Lemma 3. $K = 0$ implies that either $T < \infty$ or $u \equiv 0$.

Corollary. Completion of the proof of the Theorem.

Proof of Lemma 3. Assume that $T = \infty$. For any fixed $z > -R$ define $P(r)$ by (34b) for $r \geq r_1 = R$. Then

$$P(r) \leq 0 \tag{45a}$$

by Lemma 1. The representation (34c) for P combined with $F^* > 0$ shows that $P(r)$ is non-decreasing in r for $r \geq R$. Hence

$$\delta = \lim_{r \to \infty} \int_{T_{r,z+r}} F^*(\rho, \tau)\, d\rho d\tau = -M(z) + \lim_{r \to \infty} P(r) \tag{45b}$$

exists, and

$$0 \leq \delta \leq -M(z) . \tag{45c}$$

If here

$$\delta + M(z) = -m < 0$$

it would follow that

$$P(\rho) \leq -m < 0 \quad \text{for} \quad \rho \geq R . \tag{45d}$$

There exists a $\rho^* \geq r_2 = z + 2R$, $-b < \frac{-m}{C(\rho)} < 0$ for $\rho > \rho^*$.

By (32b) then

$$F\left(\frac{P(\rho)}{C(\rho)}\right) \geq F\left(\frac{-m}{C(\rho)}\right) > \frac{am^2}{C^2(\rho)} \quad \text{for} \quad \rho > \rho^* . \tag{45e}$$

Using (35e), (39c) we find that

$$P(r) \geq P(R) + \frac{1}{2} \int_{\rho^*}^{r} \frac{am^2}{C(\rho)} \frac{d\rho}{z+R}$$

$$\geq P(R) + \frac{3^{(1-n)/2} am^2}{2\pi} (z+R)^{-(n+5)/2} \int_{\rho^*}^{r} \rho^{(1-n)/2} d\rho .$$

But this implies $P(r) > 0$ for all sufficiently large r, contrary to (45a). Hence[1]

$$\delta = \frac{1}{2} \int_{0<\tau<z+\rho} F^*(\rho,\tau)\, d\rho d\tau = -M(z) \quad \text{for} \quad z > -R .$$

It follows from (30), (18b) and $K = 0$ that

$$\int_{z_1+\rho<\tau<z_2+\rho} F^*(\rho,\tau)\, d\rho d\tau = 0 \quad \text{for} \quad R < z_1 < z_2 .$$

Consequently

$$F^*(\rho,\tau) = 0 \quad \text{for} \quad \tau > \rho + R, \quad \tau > 0$$

and thus by (26d), (32a)

(46a) $$u_t(x,t) = v_{tt}(x,t) = 0 \quad \text{for} \quad t > x_1 + R, \quad t > 0 .$$

Using the spherical symmetry of equation (5) and of K we deduce from (46a) that more generally

$$u_t(x,t) = 0 \quad \text{for} \quad t > x \cdot \xi + R \quad \text{with any} \quad \xi \in S^{n-1}, \quad t > 0 .$$

---

[1] This identity holds whenever $T = \infty$, regardless of the value of K.

But then

$$u_t(x,t) = 0 \quad \text{for} \quad t > R - |x|, \quad t > 0$$

and in particular

$$u_t(x,t) = 0 \quad \text{for} \quad t > R, \quad \text{and all } x \in R^n .$$

It follows from (5) that

$$\Delta u(x,t) = 0 \quad \text{for} \quad t > R, \quad x \in R^n$$

and then from (21) that

$$u(x,t) = 0 \quad \text{for} \quad t > R, \quad x \in R^n .$$

The uniqueness theorem for equation (5), (see [5], p. 49) then yields that also

$$u(x,t) = 0 \quad \text{for} \quad 0 \leq t < T, \quad x \in R^n ,$$

Completing the proof of Lemma 3.

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CARATTERIZZAZIONE GRAFICA DI VARIETA' ALGEBRICHE RIGATE

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1. Introduzione

Sia (S, L) uno spazi parziale di rette, cioé un insieme S di elementi da dirsi punti, in cui si distingua una famiglia L di parti, i cui elementi chiameremo rette, tale che L sia un ricoprimento di S, ogni retta abbia almeno due punti e per due punti distinti passi al più una retta.

Ogni varietà algebrica rigata di uno spazio affine o proiettivo, qualora si fissi l'attenzione sulle sue rette, può esser di fatto riguardata come uno spazio parziale di rette, godente di opportune date proprietà di incidenza. Si presenta quindi il problema di caratterizzare le varietà algebriche rigate come spazi parziali di rette soddisfacenti ad opportuni assiomi di incidenza riflettenti la struttura della varietà stessa.

Nella presente esposizione tratteremo appunto dei recenti risultati conseguiti dalla Scuola italiana nel suddetto ambito.

Un esame attento delle proprietà geometriche di cui gode la varietà in esame suggerisce il tipo di assiomi da imporre allo spazio parziale di rette. Il punto di vista della presente trattazione è appunto questo.

Di fatto questo approccio porta a caratterizzare le varietà di Grassmann (nn. 3 e 4), le varietà prodotto di Segre (n. 5), le varietà di Veronese (n. 6) e le varietà di Schubert (n. 7).

Inoltre, poiché un grafo è uno spazio parziale di rette, le suddette caratterizzazioni forniscono anche caratterizzazioni dal punto di vista della

teoria dei grafi.

2. Spazi parziali di rette: definizioni e notazioni

Uno spazio parziale di rette; in breve un PLS (partial line space), è una coppia $(S, \mathcal{L})$, ove S è un insieme non vuoto e $\mathcal{L}$ è una famiglia propria di parti di S, tali che:

(2.1) per ogni $\ell \in \mathcal{L}$, $|\ell| \geq 2$;

(2.2) $\mathcal{L}$ è un ricoprimento di S;

(2.3) $\ell, \ell' \in \mathcal{L}, \ell \neq \ell' \Rightarrow |\ell \cap \ell'| \leq 1$.

Gli elementi di S saranno detti punti, quelli di $\mathcal{L}$ rette. Per la (2.3) per due punti qualsiasi x ed y passa al più una retta; x ed y sono detti congiungibili, $x \sim y$, se esiste la retta xy per essi; altrimenti x,y sono detti incongiungibili, se $x \not\sim y$. Se due qualsiasi punti di $(S, \mathcal{L})$ sono congiungibili, allora $(S, \mathcal{L})$ si dirà spazio di rette, altrimenti $(S, \mathcal{L})$ si dirà uno spazio parziale di rette proprio, PPLS. Uno spazio parziale di rette $(S, \mathcal{L})$ si dirà irriducibile se ogni retta contiene almeno tre punti.

Un isomorfismo fra due spazi parziali di rette $(S, \mathcal{L})$ e $(S', \mathcal{L}')$ è una biezione $f : S \to S'$ che muta rette in rette insieme con la sua inversa. Lo studio dei PLS viene fatto a meno di isomorfismi, cioè due spazi isomorfi verranno identificati. Il gruppo degli automorfismi di $(S, \mathcal{L})$ in sé verrà denotato $\mathrm{Aut}(S, \mathcal{L})$.

Sia $(S, \mathcal{L})$ un PLS. Un sottoinsieme T di S è detto un sottospazio di $(S, \mathcal{L})$ se

(2.4) $x, y \in T$, $x \neq y \Rightarrow x \sim y$ ed $xy \subset T$.

Se $|T| \geq 2$, denotiamo con $\mathcal{L}_T$ l'insieme delle rette di $\mathcal{L}$ contenute in T, allora $(T, \mathcal{L}_T)$ è uno spazio di rette. Sia $\mathcal{C}_T$ la famiglia dei sottospazi di $(S, \mathcal{L})$ contenuti in T, cioè la famiglia di sottospazi di $(T, \mathcal{L}_T)$. Allora si ha:

$$(2.5) \quad \begin{cases} \emptyset \in \mathcal{C}_T \ ; \ \forall x \in T, \{x\} \in \mathcal{C}_T \ ; \ \mathcal{L}_T \subset \mathcal{C}_T, \\ \mathcal{C}_T \text{ è un sistema di chiusura per } T. \end{cases}$$

Perciò $\mathcal{C}_T$ definisce un operatore di chiusura in T. Ne consegue che in T rimangono definiti gli insiemi indipendenti, i generatori e le basi. Precisamente X è un sottoinsieme indipendente di T, se per ogni x di X, $x \notin \overline{X - \{x\}}$ X è un generatore, se $\overline{X} = T$; X è una base di T, se è un generatore indipendente.

$(S, \mathcal{L})$ si dirà di rango finito, r, se per ogni base B di qualunque suo sottospazio si ha $|B| \leq r$ ed esiste un sottospazio T avente una base con r elementi.

Se per un qualunque sottospazio T di $(S, \mathcal{L})$ vale il seguente assioma di scambio:

(2.6) $X \subset S$, $x, y \in S$; $y \notin \overline{X}$, $y \in \overline{X \cup \{x\}} \Rightarrow x \in \overline{X \cup \{y\}}$,

allora $(S, \mathcal{L})$ si dirà un PLS combinatorio.

Si prova che, se $(S, \mathcal{L})$ è un PLS combinatorio di rango finito, tutte le basi di un qualunque sottospazio T hanno la stessa cardinalità, il rango di T. Inoltre se T_1 e T_2 sono sottospazi di T, si ha:

(2.7) rango T_1 + rango $T_2 \geq$ rango $(T_1 \cap T_2)$ + rango $(\overline{T_1 \cup T_2})$.

Sia $(S, \mathcal{L})$ un PLS proprio. Un sottospazio M è un sottospazio massimale, se non esiste alcun sottospazio in cui M sia contenuto propriamente. Per mezzo del lemma di Zorn, si prova che ogni sottospazio è contenuto in almeno un sottospazio massimale; perciò i sottospazi massimali formano un ricoprimento di S. Se un qualsiasi sottospazio massimale è uno spazio proiettivo, il che accade se, e soltanto se, vale l'assioma di Veblen (cioè se dati due qualsiasi rette sghembe ℓ ed ℓ' ed un qualsiasi punto P non su di esse esiste al più una retta per P che incontra ℓ ed ℓ'), allora $(S, \mathcal{L})$ si dice un PLS proiettivo. In tal caso nella (2.7) vale il segno di uguaglianza.

Una poligonale in $(S, \mathcal{L})$ è una m.pla di rette distinte $(\ell_1, \ell_2, \ldots, \ell_m)$ tale che $\ell_i \cap \ell_{i+1} \neq \phi$, $i = 1, 2, \ldots, m-1$. La poligonale sarà chiusa, ovvero aperta, a seconda che $\ell_1 \cap \ell_m \neq \phi$, ovvero $\ell_1 \cap \ell_m = \phi$. $(S, \mathcal{L})$ si dirà connesso, se per ogni $x, y \in S$ esiste una poligonale per x ad y. Se $(S, \mathcal{L})$ non è connesso, la relazione ρ definita da:

(2.8) $x, y \in S$, $x \rho y \Leftrightarrow$ esiste una poligonale per x ad y,

è una relazione d'equivalenza in S. Data una qualsiasi classe di equivalenza S_i, $i \in I$, sia $\mathcal{L}_i$ l'insieme delle rette di $\mathcal{L}$ contenute in S_i; poiché $\mathcal{L} = \cup (\mathcal{L}_i : i \in I)$ e $\mathcal{L}_i \cap \mathcal{L}_j = \phi$, $i \neq j$, $(S_i, \mathcal{L}_i)$ è un PLS connesso, $i \in I$, che sarà detto componente connessa di $(S, \mathcal{L})$. Perciò ogni $(S, \mathcal{L})$ o è connesso, ovvero può esser data una sua partizione in componenti connesse, pertanto lo studio dei PLS può esser svolto limitandosi a quelli connessi.

Nel seguito saranno usate le seguenti notazioni per un PLS $(S, \mathcal{L})$. Per

$p \in S$ ed $\ell \in L$,

(2.9) $p \sim \ell$ (cioè p congiungibile con ℓ) ogni punto di ℓ è congiungibile con p;

(2.10) $p \not\sim \ell$ (negazione di $p \sim \ell$) esiste qualche punto di ℓ non congiungibile con p:

(2.11) $p \nsim \ell$ (cioè p incongiungibile con ℓ) non esistono punti di ℓ congiungibili con p:

(2.12) $p \dashv \ell$ esiste uno ed un sol punto q su ℓ tale che $p \sim q$;

(2.13) $p,q,t \in S$, $q \tilde{p} t$ se, e solo se, o $q \sim p \sim t$ ovvero $q \not\sim p \not\sim t$.

(2.14) $\ell, \ell' \in L$, $\ell \sim \ell'$ se, e solo se, per ogni $p \in \ell$ e $p' \in \ell'$ risulta $p \sim p'$.

3. Una caratterizzazione della varietà di Grassmann rappresentante le rette di uno spazio proiettivo

E' ben noto che le rette di $\mathbb{P}_{n,K}$ spazio proiettivo n-dimensionale sopra il campo K, possono essere rappresentate iniettivamente come punti di $\mathbb{P}_{N,K}$ ove $N = n(n+1)/2 - 1$. Inoltre tali punti costituiscono una varietà algebrica G, intersezione di opportune quadriche, detta la varietà di Grassmann rappresentativa delle rette di $\mathbb{P}_{n,K}$. Denotato con R la famiglia delle rette di $\mathbb{P}_{n,K}$ sia $\rho : R \to G$ la suddetta rappresentazione. Sia S l'insieme dei punti di G e L l'insieme delle rette di $\mathbb{P}_{N,K}$ appartenenti a G; la coppia $G = (S, L)$ è un PLS e ρ induce una biezione fra i fasci di rette di $\mathbb{P}_{n,K}$ e le rette di L. Inoltre i sottospazi di G sono spazi proiettivi e i sottospazi massimali possono esser ripartiti in due famiglie $\mathcal{S}$ e $\mathcal{P}$. Gli elementi di $\mathcal{S}$ sono spazi (n-1)-dimensionali provenienti mediante ρ dalle stelle di rette di $\mathbb{P}_{n,K}$ (una stella di rette essendo l'insieme di tutte le rette per un punto). Gli elementi di $\mathcal{P}$ sono piani proiettivi provenienti mediante ρ dai piani rigati, cioè i piani considerati come insiemi delle loro rette in $\mathbb{P}_{n,K}$ E' facile verificare che $\mathcal{S}$ e $\mathcal{P}$ soddisfano alle seguenti condizioni:

(α) Ogni terna di punti due a due congiungibili appartengono ad uno stesso sottospazio.

(i) $T, T' \in \mathcal{S}$, $T \neq T'$ implicano $|T \cap T'| = 1$.

(ii) $T \in \mathcal{S}$, $\pi \in \mathcal{P}$ implicano o $T \cap \pi = \emptyset$, ovvero $T \cap \pi \in L$.

(iii) Per ogni $\ell \in L$, esistono un unico $T \in \mathcal{S}$ e un unico $\pi \in \mathcal{P}$ tali che $\ell \subset T \cap \pi$.

Osservazione. Sia $\mathbb{P}$ uno spazio proiettivo di dimensione maggiore di due e supponiamo che $\mathbb{P}$ non sia coordinatizzabile sopra un campo. Ancora a $\mathbb{P}$ si può associare lo spazio parziale di rette $G(\mathbb{P}) = (S(\mathbb{P}), L(\mathbb{P}))$ i cui punti sono le rette di $\mathbb{P}$ e le cui rette sono i fasci di rette in $\mathbb{P}$. E' facile provare che i sottospazi massimali di $G(\mathbb{P})$ sono ancora ripartiti in due famiglie $\mathcal{S}(\mathbb{P})$ e $\mathcal{P}(\mathbb{P})$ soddisfacenti (α), (i), (ii) e (iii). Ovviamente, se $\mathbb{P}$ può coordinatizzarsi sopra un campo, tale costruzione fornisce la struttura d'incidenza di punti e rette sulla varietà di Grassmann delle rette di $\mathbb{P}$.

In [22] G. Tallini ha dimostrato che (α), (i), (ii) e (iii) caratterizzano i PLS del tipo $G(\mathbb{P})$. Cioè precisamente egli ha provato il seguente teorema.

I. Sia (S, L) un PLS per cui valgono le proprietà seguenti:

P.1. Ogni terna di punti due a due congiungibili appartiene a un sottospazio.

P.2. Nessuna retta è un sottospazio massimale e l'insieme dei sottospazi massimali di (S, L) può essere ripartito in due famiglie $\mathcal{S}$ e $\mathcal{P}$ tali che:

(i) $T, T' \in \mathcal{S}$, $T \neq T'$ implicano $|T \cap T'| = 1$;

(ii) $T \in \mathcal{S}$, $\pi \in \mathcal{P}$ implicano o $T \cap \pi = \emptyset$, ovvero $T \cap \pi \in L$;

(iii) per ogni $\ell \in L$, esistono un unico $T \in \mathcal{S}$ e un unico $\pi \in \mathcal{P}$ tali che $\ell \subset T \cap \pi$.

Sotto queste ipotesi, esiste uno spazio proiettivo $\mathbb{P}$ di dimensione almeno tre, tale che (S, L) è isomorfo a $G(\mathbb{P}) = (S(\mathbb{P}), L(\mathbb{P}))$.

In particolare, se (S, L) è un PLS finito e irriducibile, esso è isomorfo alla varietà di Grassmann rappresentante le rette di uno spazio di Galois PG(n,q). Se (S, L) è un grafo nel qual caso la P.1 vale sempre, allora il PLS è isomorfo al "line graph" di un grafo completo.

Proveremo ora il suddetto teorema. Per ogni $p \in S$, poiché L è un ricoprimento di S, esiste una retta $r \in L$ per p. Ma allora per (iii), esiste un $\pi \in \mathcal{P}$ per r e quindi per p; sarà $\pi \neq r$ (perché r non è massimale), onde esistono due rette distinti per p, di π e quindi per la (iii) esistono due elementi di $\mathcal{S}$ per p, necessariamente distinti per la (ii). Pertanto per ogni $p \in S$ passano almeno due elementi distinti di $\mathcal{S}$. Sia ℓ_p l'insieme degli elementi di $\mathcal{S}$ passanti per p; sarà $|\ell_p| \geq 2$. Nell'insieme $\mathcal{S}$ si consideri la famiglia di parti

(3.1) $\mathcal{R} = \{\ell_p\}_{p \in S}$:

essa è propria, per (iii). La coppia $(\mathcal{S},\mathcal{R})$ è uno spazio di rette, in forza di (i).

Proviamo che:

(3.2) ogni $\pi \in \mathcal{P}$ è un piano proiettivo.

cioè che due rette di π si incontrano in un punto. Ragionando per assurdo, supponiamo che in π esistano due rette r_1 ed r_2, tali che $r_1 \cap r_2 = \phi$. Siano T_1 e T_2 gli elementi di $\mathcal{S}$ contenenti rispetivamente r_1 ed r_2 (esistenti per la (iii)). Per (i), T_1 e T_2 si incontrano in un punto p, che non può appartenere a π (altrimenti $r_1 \cap r_2 = \{p\}$). Siano $p_1 \in r_1$ e $p_2 \in r_2$; i punti p_1, p_2 e p sono distinti e due a due congiungibili (in quanto $p_1, p_2 \in \pi$; $p_1, p \in T_1$; $p_2, p \in T_2$), quindi (cfr P.1) esiste un sottospazio contenente p_1, p_2, p. Sia τ un massimale contenente tale sottospazio; esso (cfr P.2) deve appartenere o a $\mathcal{S}$ oppure a $\mathcal{P}$ ma non può appartenere a $\mathcal{S}$ in quanto ha in comune con $T_1 \in \mathcal{S}$ la retta pp_1 e quindi, se $\tau \in \mathcal{S}$ per (i), si avrebbe $\tau = T_1$, onde $p_2 \in T_1$ e, dato che $T_1 \cap \pi = r_1$, cfr. (ii), si avrebbe $p_2 \in r_1 \cap r_2$, mentre è $r_1 \cap r_2 = \phi$); pertanto $\tau \in \mathcal{P}$. Ma allora per la retta p_1p_2 passerebbero due elementi di $\mathcal{P}$ distinti (dati da π e τ) e ciò contrasta con (iii). L'assurdo prova l'asserto.

Siano ora T_1, T_2, T_3 tre elementi distinti ed indipendenti di $(\mathcal{S},\mathcal{R})$, cioè tre elementi di $\mathcal{S}$ che si incontrino due a due in punti distinti. Siano $p_1 = T_2 \cap T_3$, $p_2 = T_3 \cap T_1$, $p_3 = T_1 \cap T_2$ (cfr (i)). I punti p_1, p_2 (appartenendo a T_3) sono congiungibili, esiste cioè la retta p_1p_2 in (S,L); analogamente, esistono le rette p_2p_3 e p_3p_1. In forza dell'assioma P 1, i punti p_1, p_2, p_3 appartengono ad un sottospazio di (S,L) e quindi ad un massimale τ, il quale non può appartenere a $\mathcal{S}$ (in quanto, essendo $p_3 \in \tau$ e $p_3 \notin T_3$, è $\tau \neq T_3$ e τ e T_3 hanno la retta p_1p_3 in comune, mentre due elementi distinti di $\mathcal{S}$ si incontrano solamente in un punto per (i)), onde $\tau \in \mathcal{P}$ e quindi esso è un piano proiettivo per la (3.2). Si consideri l'insieme α costituito dagli elementi di $\mathcal{S}$ che incontrano τ ciascuno in una retta; esso contiene T_1, T_2, T_3; inoltre è un piano proiettivo in $(\mathcal{S},\mathcal{R})$ (infatti due elementi distinti T, T' di α appartengono ad un solo elemento ℓ_p ($\in \mathcal{R}$) tutto contenuto in α, ove p è il punto di intersezione delle rette $r = T \cap \tau$ ed $r' = T' \cap \tau$, in forza di (ii); inoltre se ℓ_p ed $\ell_{p'}$ sono elementi distinti di $\mathcal{R}$ contenuti in α, si ha che $p, p' \in \tau$, onde per la retta pp' di τ passa uno ed uno solo elemento di α comune ad ℓ_p ed $\ell_{p'}$, per (iii)). Ne segue che in $(\mathcal{S},\mathcal{R})$ lo spazio congiungente T_1, T_2, T_3 è un piano proiettivo (dato da α). Pertanto $(\mathcal{S},\mathcal{R})$ è uno spazio proiettivo.

L'applicazione $p \in S \to \ell_p \in \mathcal{R}$ è biettiva; sia

(3.3) $\rho: \ell_p \in \mathcal{R} \to p \in S$

la biezione inversa di $p \in S \to \ell_p \in \mathcal{R}$. Evidentemente ρ muta ogni retta della stella di centro $T \in \mathcal{S}$ in $(\mathcal{S},\mathcal{R})$ in un punto dello spazio T di (S,L), ponendo una biezione fra le rette della stella ed i punti di T, e ciò per ogni T di $\mathcal{S}$.

Proviamo che:

(3.4) ρ muta le rette di un piano rigato di $(\mathcal{S},\mathcal{R})$ nei punti di un piano $\pi \in \mathcal{P}$, inducendo un isomorfismo fra essi. Viceversa, per ogni $\pi \in \mathcal{P}$, si ha che $\rho^{-1}(\pi)$ è un piano rigato di $(\mathcal{S},\mathcal{R})$.

Sia α un piano di $(\mathcal{S},\mathcal{R})$ e siano T_1, T_2, T_3 tre suoi punti indipendenti. Essi sono tre sottospazi di (S,L) (elementi di $\mathcal{S}$) a due a due incidenti in punti distinti ed indipendenti $p_1 = T_2 \cap T_3$, $p_2 = T_3 \cap T_1$, $p_3 = T_1 \cap T_2$.

Poiché i punti p_1, p_2, p_3 sono a due a due congiungibili, esiste in (S,L) un sottospazio massimale per essi, π (cfr P.1), che è necessariamente un elemento di $\mathcal{P}$ (cfr. (i)). L'insieme dei sottospazi $T \in \mathcal{S}$ che intersecano π in rette risulta in $(\mathcal{S},\mathcal{R})$ un piano contenente T_1, T_2, T_3, cioè coincide con il piano α. Ogni retta ℓ_p di α è dunque costituita da tutti gli elementi di $\mathcal{S}$ che incontrano π nelle rette del fascio di centro nel punto p di π. Pertanto ρ muta ogni retta ℓ_p di α nel punto p di π, e viceversa. Ne segue l'asserto.

Dato che ρ trasforma una stella dirette in un sottospazio $T \in \mathcal{S}$ ed un piano rigato in un sottospazio $\pi \in \mathcal{P}$, si ha che:

(3.5) ρ muta un fascio di rette di $(\mathcal{S},\mathcal{R})$ in una retta di (S,L). Viceversa, per ogni $r \in L$, $\rho^{-1}(r)$ è un fascio di rette di $(\mathcal{S},\mathcal{R})$.

Ne segue che ρ induce un isomorfismo tra una stella di rette di $(\mathcal{S},\mathcal{R})$ di centro $T \in \mathcal{S}$, strutturata come spazio proiettivo mediante i fasci di rette, ed il sottospazio T di (S,L), quindi tali sottospazi sono spazi proiettivi. Osserviamo infine che se (S,L) è irriducibile (cioè ogni sua retta ha almeno

tre punti), anche $(\mathcal{S},\mathcal{R})$ è irriducibile, e viceversa.

Concludendo si ha che:

II. Sia $\Gamma(S,L)$ uno spazio parziale di rette proprio, soddisfacente agli assiomi P.1 e P.2. Allora esistono uno spazio proiettivo $(\mathcal{S},\mathcal{R})$ ed una biezione $\rho: \mathcal{R} \to S$ che muta fasci di rette di $(\mathcal{S},\mathcal{R})$ in rette di (S,L), con $\rho^{-1}: S \to \mathcal{R}$ che muta $T \in \mathcal{S}$ e $\pi \in \mathcal{P}$ in stelle di rette e piani rigati. Infine $(\mathcal{S},\mathcal{R})$ è irriducibile se, e soltanto se, lo è (S,L). Ne segue che, se $(\mathcal{S},\mathcal{R})$ è irriducibile, finitamente generato e pascaliano, allora (S,L) è isomorfo alla varietà grassmanniana delle rette di $(\mathcal{S},\mathcal{R})$.

Ciò prova il teorema I. Poichè ogni spazio proeittivo irriducibile finito è finitamente generato e pascaliano, dalla proposizione I otteniamo la seguente caratterizzazione della varietà grassmanniana rappresentativa delle rette di uno spazio di Galois.

III. Ogni spazio parziale di rette proprio, irriducibile, finito, soddisfacente agli assiomi P.1, P.2, è isomorfo alla varietà grassmanniana delle rette di uno spazio di Galois.

Il teorema ora dimostrato è stato il punto di partenza di varie investigazioni aventi lo scopo di studiare le strutture di incidenza delle varietà di Grassmann di indice qualunque, di generalizzare i risultati agli spazi affini e di considerare altre varietà algebriche notevoli da questo punto di vista. Nei numeri seguenti esporremo appunto tali questioni.

4. Spazi di Grassmann

Sia Σ uno spazio affine, o proiettivo, finito, o infinito. Per ogni intero h, $1 \leq h \leq \dim \Sigma - 1$, sia $G^h(\Sigma)$ la famiglia degli spazi h-dimensionali di Σ.

Lo h-mo spazio di Grassmann associato a Σ, ovvero lo spazio di Grassmann dei sottospazi h-dimensionali di Σ, è definito come lo spazio parziale di rette $\Gamma^h(\Sigma) = (G^h(\Sigma), F^h(\Sigma))$, dove $F^h(\Sigma)$ denota la famiglia di tutti gli h-fasci di Σ (un h-fascio è l'insieme di tutti i sottospazi h-dimensionali passanti per uno spazio (h-1)-dimensionale ed appartenenti ad un sottospazio (h+1)-dimensionale). E' evidente che $\Gamma^h(\Sigma)$ è un PLS proprio e connesso.

Osserviamo che quando $\Sigma = \mathbf{P}_{n,K}$ è lo spazio proiettivo n-dimensionale sopra il campo K, $\Gamma^h(\Sigma)$ è isomorfo alla varietà di Grassmann rappresentativa dei sottospazi h-dimensionali di Σ.

Denotiamo con $S(\Sigma)$ e $T(\Sigma)$ le famiglie di sottospazi massimali di $\Gamma^h(\Sigma)$ definite come segue:

$S \in S(\Sigma)$ ⇔ S è una h-stella completa di Σ, cioè l'insieme di tutti i sottospazi h-dimensionali di Σ per un sottospazio (h-1)-dimensionale, il centro di S, eventualmente all'infinito quando Σ è uno spazio affine.

$T \in T(\Sigma)$ ⇔ T è la famiglia di tutti i sottospazi h-dimensionali di Σ appartenenti ad un sottospazio (h+1)-dimensionale.

E' facile controllare che $S(\Sigma)$ e $T(\Sigma)$ ripartiscono l'insieme $M(\Sigma)$ si tutti i sottospazi massimali di $\Gamma^h(\Sigma)$. Ovviamente, quando Σ è uno spazio affine, si distinguerà una partizione in $S_2(\Sigma)$ e $S_1(\Sigma)$ a seconda che il centro della stella sia, o meno, all'infinito.

Lo spazio $\Gamma^h(\Sigma)$ soddisfa le seguenti proprietà di incidenza:

A1. Tre qualunque punti a due a due congiungibili appartengono ad un sottospazio.

A2. Nessuna retta è sottospazio massimale. La famiglia $M(\Sigma)$ dei sottospazi massimali può essere suddivisa in tre classi S_1, S_2 e T, tali che, posto $S = S_1 \cup S_2$, valgano le seguenti condizioni:

(i) $S,S' \in S_2$, $S \neq S' \Rightarrow S \cap S' = \phi$;

(ii) $S \in S$, $T \in T \Rightarrow S \cap T = \phi$ ovvero $S \cap T$ è una retta;

(iii) per ogni $f \in F^h(\Sigma)$, esistono un unico $S \in S$ e $T \in T$, tali che $f \subset S \cap T$;

(iv) (assioma di Veblen in S) dati $S_1, S_2 \in S$, o incontrantisi in un punto ovvero appartenenti ad S_2, se $S'_1, S'_2 \in S$ sono distinti ed ambedue incontrano S_1 ed S_2 in punti distinti, allora o S'_1 e S'_2 hanno intersezione non vuota, ovvero appartengono ad S_2.

A3. Se $S_2 \neq \phi$; allora ogni punto appartiene almeno ad un sottospazio massimale di S_2.

Ovviamente o $S_2 \neq \phi$, ovvero $S_2 = \phi$, a seconda che Σ è uno spazio affine o proiettivo. Inoltre, quando $S_2 = \phi$, le proprietà (i) e A3 sono vuote.

E' stato dimostrato che A1, A2, A3 caratterizzano gli spazi di Grassmann associati agli spazi affini o proiettivi in un senso che verrà precisato in seguito.

Un PLS, $\Gamma = (G,F)$, proprio e connesso è detto uno spazio di Grassmann affine (proiettivo) se valgono A1, A2, A3 e $S_2 \neq \phi$ ($S_2 = \phi$). Uno spazio di

Grassmann affine o proiettivo si dice di indice finito h se contiene una catena C, satura di lunghezza h+1 di sottospazi il cui elemento inferiore è una retta e il cui elemento superiore è un sottospazio di $\mathcal{T}$. Questa nozione è ben posta, poiché se tale catena C esiste, allora ogni altra catena della stessa specie si dimostra che ha la stessa lunghezza h.

Il teorema seguente caratterizza gli spazi di Grassmann:

TEOREMA 1. - Se Γ è uno spazio di Grassman proiettivo (affine) di indice h esiste uno spazio proiettivo $\mathbb{P}$ (uno spazio affine A) tale che Γ e $\Gamma^h(\mathbb{P})$ (Γ e $\Gamma^h(A)$) sono isomorfi. Se h = 1, (iv), cioè l'assioma di Veblen in S, è una conseguenza degli altri assiomi che definiscono uno spazio di Grassmann.

Il teorema 1 è stato dimostrato per h = 1 da G. Tallini (vedi n. 3) nel caso proiettivo, [22], e da A. Bichara e F. Mazzocca nel caso affine [1], [4]. La dimostrazione del caso generale (h > 1) è basata su un ragionamento per induzione che utilizza i risultati contenuti nei citati lavori ed è stata data da A. Bichara e G. Tallini nel caso proiettivo [10], [11] e da A. Bichara e F. Mazzocca nel caso affine [3], [6].

La natura ed il numero degli assiomi atti a definire uno spazio di Grassmann suggerirono di investigare una loro possibile dipendenza. Una risposta negativa in tal senso è stata data da A. Bichara ed F. Mazzocca [2], [5] con il seguente

TEOREMA 2. - Gli assiomi definenti gli spazi di Grassmann sia affini che proiettivi sono indipendenti.

Il teorema 1 caratterizza gli spazi $\Gamma^h(\Sigma)$ per mezzo delle proprietà di incidenza delle due famiglie $\mathcal{S}(\Sigma)$ e $\mathcal{T}(\Sigma)$ di sottospazi massimali. Si presenta allora la questione se sia sufficiente poggiare su una sola di queste famiglie, precisamente su $\mathcal{S}(\Sigma)$. Di fatto, le sole proprietà di incidenza della famiglia $\mathcal{T}(\Sigma)$ sono attualmente troppo generali per fornire la caratterizzazione cercata.

Siffatto problema è lungi dall'esser completamente risolto. Comunque, sotto l'ipotesi in cui Σ sia uno spazio proiettivo e sia h = 1, N. Melone e D. Olanda hanno provato il teorema che segue [19], [21].

TEOREMA 3. - Sia $\Gamma = (G,F)$ un PLS proprio nessuna delle cui rette sia un sottospazio massimale. Γ è isomorfo allo spazio di Grassmann rappresentativo delle rette di uno spazio proiettivo se, e soltanto se, esiste un ricoprimento $\mathcal{S}$ di Γ dato da spazi massimali tale che:

Se $S \in \mathcal{S}$ e $p \in G - S$, ogni S' di $\mathcal{S}$ per p incontra S in un unico punto e l'insieme di tutti siffatti punti è una retta i cui punti sono precisamente i punti di S congiungibili con p.

La dimostrazione di tale teorema è conseguita costruendo un'altra famiglia $\mathcal{T}$ di sottospazi massimali di Γ tale' che per $\mathcal{S}$ e $\mathcal{T}$ valgano gli assiomi A1 e A2 del Teorema 1, sotto l'ipotesi h = 1 e $\mathcal{S}_2 = \Phi$.

Un'altra possibile caratterizzazione degli spazi $\Gamma^h(\Sigma)$ fa uso di proprietà di incidenza coinvolgenti soltanto le rette, cioè non è richiesta alcuna proprietà per i sottospazi massimali. Attualmente non si ha ancora un analogo risultato per h > 1. Il teorema che segue fornisce una risposta nel caso h = 1.

TEOREMA 4. - Un PLS (G,F) irriducibile è isomorfo allo spazio di Grassmann rappresentativo delle rette di uno spazio proiettivo se, e soltanto se, valgono le condizioni seguenti:

(4.1) $\ell \in F$, $p \in G - \ell \Rightarrow p \not\sim \ell$, o $p \dashv \ell$, ovvero $p \sim \ell$

(4.2) $\ell \in F$, $p,q,t \in G - \ell$: $p \sim \ell$, $q \sim \ell$, $t \sim \ell$, $q \not\sim t \Rightarrow q \sim t$.

(4.3) $\ell \in F \Rightarrow$ esistono p e q in G tali che $p \not\sim q$, $p \sim \ell$ e $q \sim \ell$

(4.4) $\ell \in F$, $p \in G - \ell$, $p \not\sim \ell \Rightarrow \hat{\ell}_p = \{x \in G : x \sim p,\ x \sim \ell\} \in F$.

(4.5) $\ell, \ell' \in F$, $p \in G$: $\ell \cap \ell' = \phi$, $\ell \sim \ell'$, $p \not\sim \ell$, $p \not\sim \ell' \Rightarrow \hat{\ell}_p = \hat{\ell}'_p$

Tale teorema fu dapprima dimostrato da P.M. Lo Re e D. Olanda [12], [13] per un PLS finito; essi hanno provato che la famiglia di sottospazi massimali di (G,F) può essere ripartita in due famiglie, $\mathcal{S}$ e $\mathcal{T}$, per le quali valgono gli assiomi A1 e A2 del teorema 1 in cui si faccia h = 1 e $\mathcal{S}_2 = \Phi$. N. Melone e D. Olanda, [19], [21], hanno esteso questo risultato al caso infinito, dimostrando che sotto l'ipotesi del Teorema 4, (G,F) possiede una famiglia di sottospazi massimali soddisfacenti le condizioni del Teorema 3.

Risultati analoghi nel caso affine sono stati stabiliti da F. Mazzocca e D. Olanda [14], [15]. Precisamente è stato provato che:

TEOREMA 5. - Sia (G,F) un PLS le cui rette possono venir ripartite in due famiglie F_1 ed F_2. Allora (G,F) è isomorfo allo spazio di Grassmann rappresentativo delle rette di uno spazio affine se, e soltanto se, valgono le condizioni seguenti:

(4.6) $\ell \in F$, $p \in G - \ell \Rightarrow p \not\sim \ell$ ovvero $p \dashv \ell$ ovvero $p \sim \ell$.

(4.7) Per ogni ℓ in F, esistono p e q in $G - \ell$, $p \neq q$, tali che $p \not\sim q$

$p \sim \ell$ e $q \sim \ell$.

(4.8) Per ogni retta $\ell \in F_1$ ed ogni punto $p \in G - \ell$, $p \neq \ell$, l'insieme di tutti i punti q tali che $q \sim p$ e $q \sim \ell$ è una retta $\ell(p)$ di F_1. Inoltre per ogni punto $t \in \ell$ ed ogni $q \in \ell(p)$, la retta per t e q appartiene a F_1.

(4.9) Siano p,q,t tre qualsiasi punti distinti. Se p,q e q,t appartengono a rette di F_2, i punti p e t sono congiungibili e la retta per essi appartiene ad F_2.

(4.10) Sia ℓ una retta e p,q due punti congiungibili con ogni punto di ℓ. Scriveremo $p \gamma \ell$ (rispettivamente $p \gamma_2 \ell$) se ogni retta (p,x), $x \in \ell$ appartiene ad F_1 (rispettivamente a F_2) e $p \not\gamma \ell$ negli altri casi; diremo poi che p,q sono dello stesso tipo rispetto ad ℓ quando $p \gamma \ell$ e $q \gamma \ell$, ovvero $p \gamma_2 \ell$ e $q \gamma_2 \ell$, ovvero $p \not\gamma_2 \ell$ e $q \not\gamma_2 \ell$. Allora p e q sono dello stesso tipo rispetto ad ℓ se, e soltanto se, essi sono congiungibili.

La dimostrazione del teorema è conseguita utilizzando il Teorema 1. Precisamente si dimostra che la famiglia dei sottospazi massimali si può ripartire in tre classi verificanti le ipotesi del Teorema 1, per $h = 1$.

5. Spazi pseudoprodotto e varietà di C. Segre

Siano $\mathbf{P}_n$ e $\mathbf{P}'_m$ due spazi proiettivi di dimensioni n ed m. Poniamo $S = \mathbf{P}_n \times \mathbf{P}'_m$ e denotiamo con L l'insieme delle parti di S date dai sottoinsiemi $\{p\} \times \ell'$ e $\ell \times \{q'\}$, dove $p \in \mathbf{P}_n$ e $q' \in \mathbf{P}'_m$ ed ℓ, ℓ' sono rette di $\mathbf{P}_n$ e $\mathbf{P}'_m$ rispettivamente. La coppia (S, L) è un PLS che sarà detto spazio prodotto di C. Segre di tipo $\{n,m\}$ e denotato con $S_{n,m}$.

OSSERVAZIONE. Se $\mathbf{P}_n$ e $\mathbf{P}'_m$ sono coordinatizzabili sopra lo stesso campo K, $S_{n,m}$ è isomorfo alla varietà di C. Segre, modello proiettivo di $\mathbf{P}_n \times \mathbf{P}'_m$.

Lo spazio $S_{n,m} = (S,L)$ soddisfa alle condizioni seguenti:

S.1. $\ell \in L$, $p \in S - \ell$, $p \not\sim \ell$ implica che esiste un unico punto $t_{p,\ell}$ in S tale che $p \sim t_{p,\ell}$ e $t_{p,\ell} \sim \ell$;

S.2 $p,q \in S$, $p \not\sim q$ implica che esistono esattamente due punti, $\hat{p}$ e $\hat{q}$, tali che $p \sim \hat{p} \sim q$ e $p \sim \hat{q} \sim q$.

La definizione di spazio pseudoprodotto si presenta spontaneamente come quella di un PLS (S, L) soddisfacente all'assioma di Veblen e agli assiomi S.1 ed S.2.

N. Melone e D. Olanda hanno studiato gli spazi pseudoprodotto, brevemente indicati con PPS, [18], [20] e riassumeremo ora i risultati da essi ottenuti.

Si dimostra che un PPS irriducibile contiene punti singolari, cioè punti che sono congiungibili con ogni punto, se, e soltanto se, esso è uno spazio proiettivo. Perciò basta limitarsi a considerare PPS senza punti singolari. Siano p e q due qualsiasi punti non congiungibili in tale spazio (S, L): si definisce quadrangolo rispetto a p e q insieme dato dai quattro punti p, q, $\hat{p}$, $\hat{q}$ e dalle quattro rette $(p,\hat{p})$, $(p,\hat{q})$, $(q,\hat{p})$, $(q,\hat{q})$: esso sarà denotato con $\langle p,q,\hat{p},\hat{q}\rangle$. Se L_p denota l'insieme delle rette di (S, L) per p, allora è

(5.1) $\qquad |L_p| \geq 2.$

PROPOSIZIONE (5.2). - In un PPS (S, L) irriducibile le seguenti condizioni sono equivalenti.

(i) Per ogni $p \in S$, $|L_p| = 2$.

(ii) (S, L) è isomorfo ad uno spazio prodotto di C. Segre $S_{1,1}$

(iii) S ha rango due.

PROPOSIZIONE (5.3). - In un PPS (S, L) irriducibile sono equivalenti le seguenti condizioni:

(i) Esiste almeno un punto p di S tale che $|L_p| \geq 3$.

(ii) Esistono coppie punto-retta non congiungibili.

(iii) S ha rango maggiore di due.

Per la Proposizione (5.2) possiamo limitarci a studiare i PPS irriducibili di rango maggiore di due. Sotto queste ipotesi, sia (p, ℓ) una coppia punto-retta non congiungibile di (S, L) e per ogni punto x di ℓ, sia $\langle p, x, t_{p,\ell}, \hat{x}\rangle$ il quadrangolo relativo a p,x. (S, L) sarà detto regolare se per ogni punto p ed ogni retta ℓ non congiungibile con p, il punto $\hat{x}$ descrive una retta mentre x varia su ℓ.

Dati due punti distinti congiungibili p,q di (S, L), sia

(5.4) $\qquad V(p,q) = \{x \in S : x \sim p \text{ e } x \sim q\}.$

PROPOSIZIONE (5.5). - Sia (S, L) un PPS irriducibile di rango maggiore di due. Allora si ha:

(i) Per ogni coppia (p,q) di punti distinti congiungibili, $V(p,q)$ è un sottospazio massimale di (S, L).

(ii) Un qualsiasi sottospazio massimale di (S, L) è un $V(p,q)$.

(iii) Due qualunque sottospazi massimali si intersecano in al più un punto.

(iv) Ogni punto p di S appartiene esattamente a due distinti sottospazi massimali ed ogni retta per p appartiene ad uno almeno di questi spazi

(v) L'insieme dei sottospazi massimali di $(S, \mathcal{L})$ è ripartito in due famiglie Σ e T tali che sottospazi della stessa famiglia siano due a due sghembi e due sottospazi di famiglie distinte si incontrino esattamente in un punto.

Siano ora V,W due sottospazi massimali sghembi e sia $\omega_{V,W}: V \to W$ l'applicazione definita come segue. Per ogni $x \in V$, $\omega_{V,W}(x)$ è il punto in cui W incontra il sottospazio massimale per x distinto da V. Si ha la seguente:

PROPOSIZIONE (5.6). - Un PPS irriducibile è regolare se, e soltanto se, per ogni coppia di sottospazi massimali sghembi V,W l'applicazione $\omega_{V,W}$ è una collineazione.

Tenendo conto della Proposizione suddetta, gli spazi prodotto di C. Segre possono venire caratterizzati come segue.

TEOREMA - Un PPS iriducibile è isomorfo ad uno spazio prodotto di C. Segre se, e soltanto se, esso è regolare.

Daremo ora un cenno della dimostrazione, mettendo in luce che, valendo l'assioma di Veblen, ogni sottospazio di $(S, \mathcal{L})$ è uno spazio proiettivo. Denotiamo con n,m le dimensioni dei sottospazi massimali appartenenti a Σ e T rispettivamente e prendiamo $V \in \Sigma$ e $W \in T$; sia $S_{n,m}$ lo spazio prodotto di C. Segre in $V \times W$. Per ogni $(p,q) \in V \times W$, i sottospazi massimali τ_p e σ_q per p e q rispettivamente, distinti da V e da W si incontrano in un punto. L'applicazione:

$$f: (p,q) \in V \times W \to \tau_p \cap \sigma_q \in S$$

risulta un isomorfismo e ne segue il teorema.

6. Spazi di Veronese

Sia $\mathbf{P}_n$ uno spazio proiettivo finito di dimensione $n \geq 3$ e $V(\mathbf{P}_n) = (S, \mathcal{L})$ sia il PLS i cui punti sono le coppie non ordinate di iperpiani di $\mathbf{P}_n$ e le cui rette sono i sottoinsiemi di coppie con un elemento fisso, l'altro descrivendo gli iperpiani di un fascio in $\mathbf{P}_n$. Lo spazio $V(\mathbf{P}_n)$ sarà detto spazio di Veronese associato allo spazio proiettivo $\mathbf{P}_n$.

OSSERVAZIONE. Se $\mathbf{P}_n$ è coordinato sopra un campo di caratteristica diversa da due, allora $V(\mathbf{P}_n)$ è isomorfo alla struttura d'incidenza punto-retta

fornita dalla varietà degli spazi tangenti alla varietà di Veronese di indici (n,2).

E' facile provare che $V(\mathbf{P}_n) = (S, \mathcal{L})$ verifica le seguenti proprietà:

V1. Per ogni $\ell, \ell' \in \mathcal{L}$, $\ell \neq \ell'$ esiste un sol punto p di S tale che $p \sim \ell$ e $p \sim \ell'$

V2. Se $\ell \in \mathcal{L}$, $p,q \in S$, $p \sim \ell$, $q \sim \ell$, allora $p \sim q$.

V3. Per ogni $\ell \in \mathcal{L}$, $p \in S$, $p \not\sim \ell$, $|\{x \in S : p \sim x \sim \ell\}| \leq 2$.

V4. Per ogni $\ell \in \mathcal{L}$, esiste un sol $d_\ell \in S$ tale che $d_\ell \sim \ell$ e $d_\ell \not\sim x$, per ogni $x \in S$ e $x \not\sim \ell$.

Queste proprietà suggeriscono di chiamare spazio di Veronese ogni PLS $(S, \mathcal{L})$ proprio le cui rette non siano sottospazi massimali e per il quale valgano le condizioni V1, V2, V3, V4.

Riassumeremo ora i risultati ottenuti al riguardo da N. Melone [16], [17].

Uno spazio di Veronese $(S, \mathcal{L})$ si dirà proiettivo, se soddisfa all'assioma di Veblen. Per ogni retta $\ell \in \mathcal{L}$, sia

(6.1) $\qquad T(\ell) = \{x \in S : x \sim \ell\}$.

PROPOSIZIONE (6.2). - Per uno spazio di Veronese $(S, \mathcal{L})$ valgono le seguenti condizioni:

(i) Per ogni retta ℓ, $T(\ell)$ è l'unico sottospazio massimale a cui ℓ appartiene

(ii) Ogni sottospazio massimale di $(S, \mathcal{L})$ è un $T(\ell)$, $\ell \in \mathcal{L}$; perciò questi sottospazi ricoprono S.

(iii) Due qualunque sottospazi massimali distinti si incontrano in un unico punto.

(iv) Ogni punto appartiene al più a due sottospazi massimali.

Si dirà punto doppio un punto $p \in S$ per il quale passi soltanto un sottospazio massimale; V denoterà l'insieme di tutti i punti doppi.

PROPOSIZIONE (6.3). - Per ogni retta $\ell \in \mathcal{L}$, d_ℓ è l'unico punto doppio nel sottospazio $T(\ell)$. Conseguentemente ogni sottospazio massimale contiene un unico punto doppio.

Siano T,T' due qualunque sottospazi distinti massimali e siano d_T, $d_{T'}$ i loro punti doppi; per ogni $x \in T - \{d_T\}$, denotiamo con T_x il sottospazio massimale per x distinto da T. Definiamo l'applicazione $\omega_{TT'}: T \to T'$ nel modo seguente:

$$(6.4)\quad \omega_{TT'}(x) = \begin{cases} d_{T'}, & \text{se } x = T \cap T' ; \\ T \cap T' & \text{se } x = d_T ; \\ T' \cap T_x & \text{se } x \in T - \{d_T , T \cap T'\} \end{cases}$$

$\Sigma(S,L)$ sarà chiamato spazio di Veronese regolare, se $\omega_{TT'}$ è una collineazione per ogni coppia di sottospazi distinti massimali T,T'.

Sia ora ℓ una retta qualunque, T il sottospazio massimale per ℓ; l'insieme

$$(6.5)\quad \alpha(\ell) = \begin{cases} \{ d_{T_x} \in V : x \in \ell \} , & \text{se } d_T \notin \ell , \\ \{ d_{T_x} \in V : x \in \ell - d_T \} \cup \{ d_T \} , & \text{se } d_T \in \ell , \end{cases}$$

sarà detto arco associato ad ℓ. Denotiamo con A l'insieme di tutti gli archi.

PROPOSIZIONE (6.6). - La coppia (V,A) è uno spazio di rette, più precisamente uno spazio proiettivo.

La precedente proposizione porta al risultato finale.

TEOREMA - Sia (S,L) uno spazio di Veronese, V l'insieme dei suoi punti doppi ed A la famiglia dei suoi archi. Se (S,L) è proiettivo, regolare e di rango finito, allora esso è isomorfo allo spazio di Veronese associato ad uno spazio proiettivo, cioè il duale di (V,A).

7. Spazi di Schubert

Sia $\mathbb{P}$ uno spazio proiettivo di dimensione $n > 1$ e denotiamo con $\mathcal{P}_h$, $0 \le h \le n-1$, la famiglia dei sottospazi h-dimensionali di $\mathbb{P}$. Una n-pla $(S_0, S_1, \ldots, S_{n-1})$ di sottospazi di $\mathbb{P}$ sarà detta una bandiera, se per ogni $i = 0,1,\ldots,n-2$, $S_i \subset S_{i+1}$ e $\dim S_j = j$ per tutti gli interi $j = 0,1,\ldots,n-1$. Sia $\mathcal{S}(\mathbb{P})$ la famiglia di tutte le bandiere di $\mathbb{P}$.

Sia $h \in \{0,1,\ldots,n-1\}$; per ogni catena di sottospazi $\bar{S}_0 \subset \bar{S}_1 \subset \ldots \subset \bar{S}_{h-1} \subset \bar{S}_{h+1} \subset \ldots \subset \bar{S}_{n-1}$ tale che $\dim \bar{S}_i = i$, denotiamo con $f_h(\bar{S}_0, \ldots, \bar{S}_{h-1}, \bar{S}_{h+1}, \ldots, \bar{S}_{n-1})$ l'insieme delle bandiere $\{(\bar{S}_0, \ldots, \bar{S}_{h-1}, S_h, \bar{S}_{h+1}, \ldots, \bar{S}_{n-1}) : S_h \in \mathcal{P}_h\}$ e con $\mathcal{F}_h = \mathcal{F}_h(\mathbb{P})$ la famiglia dei sottoinsiemi di $\mathcal{S}(\mathbb{P})$ ognuno dei quali è un $f_h(\bar{S}_0, \ldots, \bar{S}_{h-1}, \bar{S}_{h+1}, \ldots, \bar{S}_{n-1})$. Poniamo

$$\mathcal{F} = \mathcal{F}(\mathbb{P}) = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{n-1}) \text{ e}$$
$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\mathbb{P}) = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{n-1}$$

La coppia $\Sigma(\mathbb{P}) = (\mathcal{S}(\mathbb{P}), \mathcal{F}(\mathbb{P}))$ sarà detta spazio delle bandiere di $\mathbb{P}$ ed è facile verificare che $(\mathcal{S}(\mathbb{P}), \mathcal{F}(\mathbb{P}))$ è un PLS proprio e connesso. Inoltre $\Sigma(\mathbb{P})$ soddisfa alle seguenti condizioni:

B1. Ogni poligonale contenente esattamente una retta di $\mathcal{F}_i(\mathbb{P})$, $i=0,1,\ldots,n-1$, è aperta.

B2. Una poligonale di lunghezza quattro è chiusa se, e soltanto se, le sue rette appartengono a $\mathcal{F}_i \cup \mathcal{F}_j$, $i < j$, $j \neq i+1$, $i=0,\ldots,n-3$, $j=2,\ldots,n-1$.

B3. Ogni poligonale di lunghezza cinque le cui rette appartengono a $\mathcal{F}_i \cup \mathcal{F}_{i+1}$, $i=0,1,\ldots,n-2$, è contenuta in una poligonale chiusa di lunghezza sei le cui rette appartengono a $\mathcal{F}_i \cup \mathcal{F}_{i+1}$.

OSSERVAZIONE. Se $\mathbb{P}$ è coordinatizzato sopra un campo, allora lo spazio $(\mathcal{S}(\mathbb{P}), \mathcal{F}(\mathbb{P}))$ è isomorfo alla struttura di incidenza di punti e rette fornita dalla varietà di Schubert rappresentativa delle bandiere di $\mathbb{P}$.

Le argomentazioni precedenti suggeriscono la seguente definizione. Sia $\mathcal{S}$ un insieme non vuoto e $\mathcal{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_{n-1})$, $n \ge 2$, una n-pla di partizioni di $\mathcal{S}$ tali che due a due non si intersechino in un blocco. Poniamo $\tilde{\mathcal{F}} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{n-1}$. La coppia $\Sigma = (\mathcal{S}, \mathcal{F})$ sarà detta spazio di Schubert di indice finito n se la coppia $(\mathcal{S}, \mathcal{F})$ è un PLS proprio e connesso soddisfacente alle condizioni B1, B2, B3.

Gli spazi di Schubert sono stati definiti e studiati da A. Bichara e C. Somma che hanno provato il seguente risultato [7], [8]:

TEOREMA (7.1). - Ogni spazio di Schubert di indice finito n è isomorfo allo spazio delle bandiere di uno spazio proiettivo n-dimensionale.

Recentemente A. Bichara e C. Somma hanno esteso agli spazi affini i risultati esposti in questo numero, cioè essi hanno definito e caratterizzato completamente lo spazio delle bandiere di uno spazio affine, [9].

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STABILITY RESULTS FOR VARIATIONAL INEQUALITIES AND APPLICATIONS TO TRAFFIC EQUILIBRIUM PROBLEM (*)

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In the works [1], [2], [3] we presented a computational procedure to calculate solutions of variational inequalities that express the conditions for the traffic equilibrium. The procedure is founded on a technique that allows us to decrease the dimension of the problem as long as (if certain condition are always violated) solution must coincide with a vertex of the polyhedron where the variational inequality is defined.

In this paper, assuming the cost function to be linear, we give some conditions for the stability of solution of the variational inequality under changes in the travel demand.

In [3] references on traffic equilibrium problems and variational inequalities may be found.

1. CONDITIONS FOR THE STABILITY

Let us consider the V.I. (for the sake of brevity this abbreviation will stand for variational inequality) which express the traffic equilibrium problem

(1.1) *Find* $H \in K$ *such that* $C(H)(F-H) \geq 0 \quad \forall F \in K$ *where*

$$K = \left\{ F \in R^m : F_r \geq 0 \sum_{R_r \in \mathcal{R}(P_i,P_j)} F_r = \rho(P_i,P_j) \right\} \quad (1)$$

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(1) For the meaning of $\rho(P_i,P_j)$, R_r, $\mathcal{R}(P_i,P_j)$, ρ_i, $\mathcal{R}_i$, $\tilde{H}$ see [2].

and $C(F)$ *is a strictly monotone and continuous application from* K *in* R^m .

In the following in fact we will assume $C(F)$ to be linear, that is

$$C(F) = AF$$

where $A = \{a_{rs}\}$ $r,s = 1,\dots,m$.

If we set

$$\varphi_{ir} = \begin{cases} 1 & \text{if } R_r \in \mathcal{R}_i \\ 0 & \text{if } R_r \notin \mathcal{R}_i \end{cases}$$

we can write

$$K = \left\{ F \in R^m \,,\ F_r \geq 0 \quad r = 1,\dots,m \,,\ \sum_{r=1}^{m} \varphi_{ir} F_r \leq \rho_i \quad i = 1,\dots,n^2 \right\} .$$

Now we observe that it is possible to derive the values of n^2 variables, for example those of F_i $i = 1,\dots,n^2$, in the following way

$$F_i = \rho_i - \sum_{r=n^2+1}^{m} \varphi_{ir} F_r \qquad i = 1,\dots,n^2 \,. \tag{1.2}$$

So that using (1.2), we can transform (1.1) into the following

(1.3) *Find* $\tilde{H} \in \tilde{K}$ *such that* $\Gamma(\tilde{H})(\tilde{F}-\tilde{H}) \geq 0 \quad \forall\, \tilde{F} \in \tilde{K}$ *where*

$$\tilde{K} = \left\{ \tilde{F} \in R^{m-n^2} \,,\ \tilde{F} \geq 0 \quad \varphi\tilde{F} \leq \rho \right\} \quad (^2) \tag{1.4}$$

and

$$\Gamma(\tilde{F}) = \alpha\tilde{F} - \beta\rho$$

(²) For the sake of brevity, we have denoted by $\tilde{F}$ and ρ the vectors column whose components are F_r and ρ_i respectively and by φ the matrix (φ_{ir}) $i = 1,\dots,n^2$ $r = n^2+1,\dots,m$.

with

$$\alpha = (\alpha_{rj}) \quad r,j = n^2+1,\dots,m \qquad \beta = (\beta_{rs}) \quad r = n^2+1,\dots,m \quad s = 1,\dots,n^2$$

and

$$\alpha_{rj} = \left\{ \sum_{k=1}^{n^2} \left[\left(\sum_{s=1}^{n^2} a_{ks}\varphi_{sj} \right) - a_{kj} \right] \varphi_{kr} - \left(\sum_{s=1}^{n^2} a_{rs}\varphi_{sj} - a_{rj} \right) \right\} ,$$

$$\beta_{rs} = -\left(a_{rs} - \sum_{k=1}^{n^2} a_{ks}\varphi_{kr} \right) .$$

We are concerned with the stability of the solution of V.I. (1.3) with respect to ρ_i and we meet this problem using the results of [2]. Following [2] let us consider the solution of the system

$$\Gamma(\tilde{H}_o) = \alpha\tilde{H}_o - \beta\rho = 0 \quad ; \tag{1.6}$$

the strictly monotonicity of A , and hence of α , allows us to say that A and α are not degenerate; then we have

$$\tilde{H}_o = \alpha^{-1}(\beta\rho) = (\alpha^{-1}\beta)\rho \tag{1.7}$$

and $\tilde{H}_o$ coincides with $\tilde{H}$ if $\tilde{H}_o \in \tilde{K}$; this means

$$(\alpha^{-1}\beta)\rho \geq 0 \tag{1.8}$$

and

$$\varphi[(\alpha^{-1}\beta)\rho] = (\varphi\alpha^{-1}\beta)\rho \leq \rho \,. \tag{1.9}$$

Now suppose that (1.8) and (1.9) are fulfilled and that the travel demand assumes a new value ρ' ; then in order that solution of V.I. (1.3) is given by $(\alpha^{-1}\beta)\rho'$ and does not change, it must result

$$(\alpha^{-1}\beta)\rho = (\alpha^{-1}\beta)\rho' \tag{1.10}$$

$$(\alpha^{-1}\beta)\rho' \geq 0 \tag{1.11}$$

(1.12) $$(\varphi\alpha^{-1}\beta)\rho' \leq \rho' .$$

From (1.10) we deduce

$$\alpha^{-1}\beta(\rho-\rho') = 0 .$$

Hence, if we suppose that there exist vectors $\rho^* \neq 0$ such that

$$\alpha^{-1}\beta\rho^* = 0$$

i.e. such that

(1.13) $$\beta\rho^* = 0 ,$$

every vector ρ' of the form

$$\rho' = \rho + \lambda\rho^* \qquad \lambda \in R$$

verifies (1.10); moreover we have

$$(\alpha^{-1}\beta)\rho' = (\alpha^{-1}\beta)\rho + \lambda(\alpha^{-1}\beta)\rho^* = (\alpha^{-1}\beta)\rho \geq 0$$

and hence (1.11) holds; with regard to (1.12) we have

$$(\varphi\alpha^{-1}\beta)\rho' = (\varphi\alpha^{-1}\beta)\rho + \lambda(\varphi\alpha^{-1}\beta)\rho^* = (\varphi\alpha^{-1}\beta)\rho ;$$

since

$$(\varphi\alpha^{-1}\beta)\rho \leq \rho$$

if we suppose that

$$\rho \leq \rho' = \rho + \lambda\rho^*$$

and hence

(1.14) $$\lambda\rho^* \geq 0 ,$$

the following theorem holds:

Theorem 1.1. *If* ρ^* *and* λ *verify* (1.13) *and* (1.14) *with travel demand given by* $\rho' = \rho + \lambda\rho^*$, *the solution of V.I.* (1.3) *does not change.*

Now let us suppose that (1.8) and (1.9) are not fulfilled; then $\dot{\tilde{H}}$ belongs to some face of the polyhedron $\tilde{K}$; following the notations of [2], let us consider the face

$$\tilde{K}(S^h, J^k) = \left\{ \tilde{F}(S^h, J^k) \in R^{m-n^2-(h+k)} \, , \, \tilde{F}(S^h, J^k) \geq 0 \, , \, \varphi^{(k)}\tilde{F}(S^h, J^k) \leq \rho^{(k)} \, , \right.$$

$$\left. \varphi^{(E)}\tilde{F}(S^h, J^k) \leq \rho^{(E)} \right\}$$

where

$$S^h = (s_1, \ldots, s_h) \, , \quad J^k = (j_1, \ldots, j_k) \, , \quad L = I - \{l_1, \ldots, l_k\}$$

with [3]

$$I = \{n^2+1, \ldots, m\} - \{s_1, \ldots, s_k\}$$

$$F_{l_1} = \rho_{j_i} - \sum_{\substack{r \in I \\ r \neq l_i}} \varphi_{j_i r} F_r \qquad l_i \in I ,$$

$$E = \{1, \ldots, n^2\} - \{j_1, \ldots, j_k\}$$

$$\varphi^{(k)} = (\varphi_{j_i r}) \qquad j_i \in J^k \, , \quad r \in L$$

$$\varphi^{(E)} = (\varphi_{ir}) \qquad i \in E \, , \quad r \in L$$

and $\rho^{(k)}$, $\rho^{(E)}$, $\tilde{F}(S^h, J^k)$ are the column vectors whose components are ρ_{j_i} , $j_i \in J^k$, ρ_i $i \in E$, F_r $r \in L$ respectively.

([3]) In [2] can be found the method by which the face $\tilde{K}(S^h, J^k)$ has been defined.

If we consider V.I. on the face $\tilde{K}^{(S^h,J^k)}$ we obtain

$$\Gamma^{(S^h,J^k)}(\tilde{H}^{(S^h,J^k)})(\tilde{F}^{(S^h,J^k)} - \tilde{H}^{(S^h,J^k)}) \geq 0 \quad \forall\, \tilde{F}^{(S^h,J^k)} \in \tilde{K}^{(S^h,J^k)}$$

where

$$\Gamma^{(S^h,J^k)}(\tilde{H}^{(S^h,J^k)}) = E\tilde{H}^{(S^h,J^k)} - \Lambda\rho$$

with

$$E = (\varepsilon_{rj}) \qquad r,j \in L$$

$$\varepsilon_{rj} = \left[\left(\alpha_{rj} - \sum_{i=1}^{k} \varphi_{j_i r}\alpha_{l_i j}\right) - \sum_{i=1}^{k}\left(\alpha_{rl_i} - \sum_{n=1}^{k} \varphi_{j_n r}\alpha_{l_n l_i}\right)\varphi_{j_i j}\right]$$

and

$$\Lambda = (\lambda_{rs}) \qquad r \in L \qquad s = 1,\dots,n^2$$

$$\lambda_{rs} = \begin{cases} \eta_{rs} + \theta_{ri} & \text{if } s = j_i \\ \eta_{rs} & \text{if } s \in E \end{cases}$$

$$\eta_{rs} = \beta_{rs}\left(\sum_{i=1}^{k} \varphi_{j_i r} - 1\right) \qquad \theta_{ri} = \left(\sum_{n=1}^{k} \varphi_{j_n r}\alpha_{l_n l_i} - \alpha_{rl_i}\right)$$

Then the system

$$E\tilde{H}_o^{(S^h,J^k)} = \Lambda\rho$$

admits the solution

(1.15) $$\tilde{H}_o = E^{-1}(\Lambda\rho) = (E^{-1}\Lambda)\rho$$

and, in order that $\tilde{H}_o$ coincides with the solution $\tilde{H}$ of V.I.

(1.3), $\tilde{H}_o$ must belong to $\tilde{K}^{(S^h,J^k)}$ and hence $\tilde{H}_o$ must verify the conditions

(1.16) $$(E^{-1}\Lambda)\rho \geq 0$$

(1.17) $$\varphi^{(k)}[(E^{-1}\Lambda)\rho] = (\varphi^{(K)}E^{-1}\Lambda)\rho \leq \rho^{(k)}$$

(1.18) $$\varphi^{(E)}[(E^{-1}\Lambda)\rho] = (\varphi^{(E)}E^{-1}\Lambda)\rho \leq \rho^{(E)}$$

and moreover

i) if $\varphi_{j_i s_q} = 0 \quad i = 1,\dots,k \quad q = 1,\dots,h$ $\tilde{H}_o$ must verify

$$\Gamma_r((E^{-1}\Lambda)\rho) \geq 0 \qquad r \in S^h,$$

$$\Gamma_{l_i}((E^{-1}\Lambda)\rho) \leq 0 \qquad i = 1,\dots,k;$$

ii) or, if there exist a subset J_p of J^k constituted by p indices and p subsets S_{j_i}, $j_i \in J_p$, of S^h such that

$$\varphi_{j_i s_q} = 1 \qquad j_i \in J_p \qquad s_q \in S_{j_i},$$

$\tilde{H}_o$ must verify

$$\Gamma_r((E^{-1}\Lambda)\rho) \geq 0 \quad r \in S^h - \bigcup_{j_i \in J_p} S_{j_i}, \quad \Gamma_{l_i}((E^{-1}\Lambda)\rho) \leq 0 \quad i = 1,\dots,k$$

$$\Gamma_{s_q}((E^{-1}\Lambda)\rho) - \Gamma_{l_i}((E^{-1}\Lambda)\rho) \geq 0 \qquad s_q \in S_{j_i},\ j_i \in J_p$$

Let us examine the case i): being

$$\Gamma_r(\tilde{F}^{(S^h,J^k)}) = \sum_{j \in L} \gamma_{rj}F_j + \sum_{i=1}^{k} \alpha_{rl_i}\rho_{j_i} - \sum_{s=1}^{n^2} \beta_{rs}\rho_s$$

where

(1.19) $$\gamma_{rj} = \alpha_{rj} - \sum_{i=1}^{k} \alpha_{rl_i} \varphi_{j_i j} \qquad r \in S^h \ , \ j \in L \ ,$$

if we denote by $\Gamma^{(S^h)}$ the column vector whose components are Γ_r $r \in S^h$, the conditions

$$\Gamma_r((E^{-1}\Lambda)\rho) \geq 0 \qquad r \in S^h$$

becomes

(1.20) $$\Gamma^{(S^h)}((E^{-1}\Lambda)\rho) = \gamma^{(h)}[(E^{-1}\Lambda)\rho] + \alpha^{(h,k)}\rho^{(k)} - \beta^{(h)}\rho =$$
$$= [(\gamma^{(h)}E^{-1}\Lambda) - \beta^{(h)}]\rho + \alpha^{(h,k)}\rho^{(k)} \geq 0$$

where

$$\gamma^{(h)} = (\gamma_{rj}) \qquad r \in S^h \ , \ j \in L \ , \qquad \alpha^{(h,k)} = (\alpha_{rl_i}) \qquad r \in S^h \ , \ i = 1,\dots,k$$

$$\beta^{(h)} = (\beta_{rs}) \qquad r \in S^h \ , \ s = 1,\dots,n^2 \ .$$

In an analogous way, denoting by $\Gamma^{(J^k)}$ the column vector whose components are $\Gamma_{l_1},\dots,\Gamma_{l_k}$, we obtain

(1.21) $$\Gamma^{(J^k)}((E^{-1}\Lambda)\rho) = [(\gamma^{(k)}E^{-1}\Lambda) - \beta^{(k)}]\rho + \alpha^{(k,k)}\rho^{(k)} \leq 0$$

where

$$\gamma^{(k)} = (\gamma_{l_i j}) \qquad i = 1,\dots,k \qquad j \in L$$

$$\alpha^{(k,k)} = (\alpha_{l_n l_i}) \qquad i,n = 1,\dots,k$$

$$\beta^{(k)} = (\beta_{l_n s}) \qquad n = 1,\dots,k \ , \ s = 1,\dots,n^2 \ .$$

Now let us suppose that the travel demand is ρ' and let us

look for some conditions under which the solution of V.I. (1.3) does not change and is given by solution of the system

$$E\tilde{H}_o^{(S^h, J^k)} = \Lambda\rho'$$

To obtain this it is necessary and sufficient that

(1.22) $$(E^{-1}\Lambda)\rho = (E^{-1}\Lambda)\rho'$$

(1.23) $$(E^{-1}\Lambda)\rho' \geq 0$$

(1.24) $$(\varphi^{(k)}E^{-1}\Lambda)\rho' \leq \rho'^{,(k)} \ ; \ (\varphi^{(E)}E^{-1}\Lambda)\rho' \leq \rho'^{,(E)}$$

(1.25) $$[(\gamma^{(h)}E^{-1}\Lambda) - \beta^{(h)}]\rho' + \alpha^{(h,k)}\rho'^{,(k)} \geq 0$$

(1.26) $$[(\gamma^{(k)}E^{-1}\Lambda) - \beta^{(k)}]\rho' + \alpha^{(k,k)}\rho'^{,(k)} \leq 0$$

If we suppose that there exist nonzero vectors ρ^* such that

$$(E^{-1}\Lambda)\rho^* = 0 \ ,$$

and hence such that

(1.27) $$\Lambda\rho^* = 0$$

every vector ρ' of the type

(1.28) $$\rho' = \rho + \lambda\rho^* \qquad \lambda \in R$$

verifies the conditions (1.22) and (1.23);moreover it results

$$(\varphi^{(k)}E^{-1}\Lambda)\rho' = (\varphi^{(k)}E^{-1}\Lambda)\rho + \lambda\varphi^{(k)}(E^{-1}\Lambda\rho^*) = (\varphi^{(k)}E^{-1}\Lambda)\rho \leq \rho^{(k)}$$

$$(\varphi^{(E)}E^{-1}\Lambda)\rho' = (\varphi^{(E)}E^{-1}\Lambda)\rho + \lambda(\varphi^{(E)}E^{-1}\Lambda\rho^*) = (\varphi^{(E)}E^{-1}\Lambda)\rho \leq \rho^{(E)} \ .$$

Then if

$$\rho^{(k)} \leq \rho'^{(k)} = \rho^{(k)} + \lambda\rho^{*(k)} \qquad \rho^{(E)} \leq \rho'^{(E)} = \rho^{(E)} + \lambda\rho^{*(E)}$$

that is

$$\lambda\rho^{*(k)} \geq 0 \qquad \lambda\rho^{*(E)} \geq 0 \tag{1.29}$$

conditions (1.24) are fulfilled; to verify (1.25), we notice that (1.25) can be written

$$[\gamma^{(h)}E^{-1}\Lambda - \beta^{(h)}]\rho + \alpha^{(h,k)}\rho^{(k)} \geq \lambda[\beta^{(h)}\rho^{*} - \alpha^{(h,k)}\rho^{*(k)}] .$$

So that, assuming

$$\lambda[\beta^{(h)}\rho^{*} - \alpha^{(h,k)}\rho^{*(k)}] \leq 0 . \tag{1.30}$$

because (1.20) holds, condition (1.25) is verified. Analogously, being

$$[\gamma^{k}E^{-1}\Lambda - \beta^{(k)}]\rho + \alpha^{(k,k)}\rho^{k} \leq \lambda[\beta^{(k)}\rho^{*} - \alpha^{(k,k)}\rho^{*(k)}] ,$$

assuming that

$$\lambda[\beta^{(k)}\rho^{*} - \alpha^{(k,k)}\rho^{*(k)}] \geq 0 , \tag{1.31}$$

since (1.21) holds, also conditions (1.26) is fulfilled. Then we have the following:

Theorem 1.2. *Let us suppose that condition* i) *holds and that* ρ^{*} *and* λ *verify conditions* (1.27), (1.29), (1.30) *and* (1.31); *then if the travel demand is* $\rho' = \rho + \lambda\rho^{*}$, *the solution of V.I.* (1.3) *dos not change.*

Now let us consider the case ii); in this case conditions (1.26) and (1.31) do not change while the conditions (1.25) and (1.30) must be replaced by

$$[\gamma^{(h,p)}E^{-1}\Lambda - \beta^{(h,p)}]\rho + \alpha^{(h,k,p)}\rho^{(k)} \geq 0 \tag{1.32}$$

$$\lambda[\beta^{(h,p)}\rho^{*} - \alpha^{(h,k,p)}\rho^{*(k)}] \leq 0 \tag{1.33}$$

where

$$\gamma^{(h,p)} = (\gamma_{r,j}) \qquad r \in S^{h} - \bigcup_{j_i \in J_p} S_{j_i} \quad , \quad j \in L$$

$$\alpha^{(h,k,p)} = (\alpha_{rl_i}) \qquad r \in S^{h} - \bigcup_{j_i \in J_p} S_{j_i} \quad , \quad i = 1,\dots,k$$

$$\beta^{(h,p)} = (\beta_{rs}) \qquad r \in S^{h} - \bigcup_{j_i \in J_p} S_{j_i} \quad , \quad s = 1,\dots,n^{2} .$$

Moreover, as an easy calculation shows that

$$\Gamma_{sq}((E^{-1}\Lambda)\rho) - \Gamma_{l_i}((E^{-1}\Lambda)\rho) = \sum_{j}\sum_{L}(\gamma_{s_q j} - \gamma_{l_i j})H_j + \sum_{n=1}^{k}(\alpha_{s_q l_n} - \alpha_{l_i l_n})\rho_{j_n} - \sum_{s=1}^{n^2}(\beta_{s_q s} - \beta_{l_i s})\rho_{s} ,$$

if we set

$$\gamma^{(S_{j_i})} = (\gamma_{s_q j} - \gamma_{l_i j}) \qquad s_q \in S_{j_i} \quad , \quad j \in L ,$$

$$\alpha^{(S_{j_i})} = (\alpha_{s_q l_n} - \alpha_{l_i l_n}) \qquad s_q \in S_{j_i} \quad , \quad n = 1,\dots,k$$

$$\beta^{(S_{j_i})} = (\beta_{s_q s} - \beta_{l_i s}) \qquad s_q \in S_{j_i} \quad , \quad s = 1,\dots,n^{2} ,$$

we achieve the conditions

$$\Gamma_{sq}((E^{-1}\Lambda)\rho) - \Gamma_{l_i}((E^{-1}\Lambda)\rho) =$$

$$= \gamma^{(S_{j_i})}[(E^{-1}\Lambda)\rho] + \alpha^{(S_{j_i})}\rho^{(k)} - \beta^{(S_{j_i})}\rho =$$

$$= \left[\gamma^{(S_{j_i})}E^{-1}\Lambda) - \beta^{(S_{j_i})}\right]\rho + \alpha^{(S_{j_i})}\rho^{(k)} \geq 0 \qquad j_i \in J_p .$$

Then, in order to have

$$\left[(\gamma^{(S_{j_i})} E^{-1}\Lambda) - \beta^{(S_{j_i})} \right](\rho + \lambda\rho^*) + \alpha^{(S_{j_i})} (\rho^{(k)} + \lambda\rho^{*(k)}) \geq 0 ,$$

it is sufficient that

$$\lambda\left[\beta^{(S_{j_i})} \rho^* - \alpha^{(S_{j_i})} \rho^{*(k)}\right] \leq 0 \qquad j_i \in J_p . \tag{1.34}$$

Then we have shown the following

Theorem 1.3. *If* ρ^* *and* λ *verify conditions* (1.27), (1.29), (1.31), (1.33) *and* (1.34) *and the travel demand is* $\rho' = \rho + \lambda\rho^*$, *the solution of V.I.* (1.3) *does not change.*

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A NEW PROOF OF THE GIERING THEOREM (°)

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Let $K \subset \mathbb{R}^2$ be a convex body, i.e. a convex compact set with non empty interior, and let S be a direction in $\mathbb{R}^2$. We identify S with an equivalence class in the set of all lines in the plane, with respect to the relation of parallelism.

DEFINITION

The Steiner symmetral of a convex body $K \subset \mathbb{R}^2$, centered on a fixed line u orthogonal to S, is the union of all closed segments on lines $s \in S$ such that $s \cap K \neq \emptyset$, whose midpoint is on u and whose length is equal to the one-dimensional measure of $s \cap K$.

It is well known that the Steiner symmetral of a convex body is again a convex body having the same measure ([7], Theorem 12.11).

GIERING'S RESULT

Giering proved in [4] the following result: Given a plane convex body K , there exist three directions S,S' and T, depending on K, such that the Steiner symmetrals with respect to them distinguish K from any other plane convex body.

The aim of this paper is to provide a more transparent proof of that theorem and also to permit a larger choice of the triplets of directions.Our proof is inspired by the original one, but uses also a result ([3], Lemma 4) due to Gardner and McMullen.

A clear and short proof of the same theorem has been given recently by R.J. Gardner [2], but the choice of the triplets of directions is more res-

(°) This paper is in final form and no version of it will be submitted for publication elsewhere.

tricted than in the original paper.

A recent paper of K.J. Falconer [1] contains an analogous uniqueness theorem corresponding to the case when instead of taking chords on parallel lines, one considers chords on lines issuing from points. Problems of this kind have been asked in [5].

Using the concept of one-sided cross point (Definition 1.9) we show that the presence of faces in the boundary of K does not affect the choice of the first two directions, unlike [2] and [4]. We believe that this will help to answer Problem 2 of 2.5, asking if the triplet of directions can be selected in such a way, that it depends continuously on K.

Section 1 contains preliminary results, definitions and notations. In section 2 we give a new proof of the Giering theorem. At the end we state some open questions.

1. Let us begin with some definitions and notations.

1.1 DEFINITION

Given two directions S and S', a parallelogram P is said to be an (S,S')-parallelogram, if its sides are parallel to S and S'.

1.2 NOTATIONS

Given a polygon (in particular a parallelogram) Q, we shall denote with $\operatorname{vert} Q$ the set of its vertices.

Given a set $A \subset \mathbb{R}^2$, we shall denote with ∂A its boundary.

1.3 DEFINITION

Given a plane convex body K, we shall say that a polygon Q is inscribed in K, if $\operatorname{vert} Q \subset \partial K$.

Using a continuity argument it is easy to see that the following lemma holds (for similar results, see [6], §3)

1.4 LEMMA

Given a convex plane body K and a direction S, there exists a direction S' and a non degenerate (S,S')-parallelogram P, inscribed in K.

We have moreover the following result. For its proof, see [4], Satz 9 or also [2], Lemma 2.2.

1.5 LEMMA

Let P be an (S,S')-parallelogram inscribed in a plane convex body K. If H is any plane convex body having the same Steiner symmetrals with respect to S and S', then P is inscribed in H, too.

1.6 DEFINITION

A line l is said to support (in z) a convex plane body K, if $l \cap K \neq \emptyset$ and $l \cap \operatorname{int} K = \emptyset$ (and $z \in l \cap K$).

The following definition is taken from [3].

1.7 DEFINITION

Given a set of directions $\mathcal{D}$, a polygon Q is said to be a $\mathcal{D}$-polygon, if for each $S \in \mathcal{D}$, whenever z is a vertex of Q and $z \in s$, $s \in S$, then s either supports Q in z alone, or contains another vertex w of Q.

1.8 DEFINITION

Given two convex bodies K and H, we define the set of cross points (denoted by $\operatorname{cross}(K,H)$) as follows: $z \in \operatorname{cross}(K,H)$ if $z \in \partial K \cap \partial H$ and if there exists an open connected component R of $\operatorname{int}(K \triangle H)$ such that $z \in \bar{R}$.

It follows immediately from the definition that $\operatorname{cross}(K,H) \subset \partial K \cap \partial H$ and therefore, by convexity, that for every line l, $l \cap \operatorname{cross}(K,H)$ has at most two points.

1.9 DEFINITION

A point $z \in \operatorname{cross}(K,H)$ is said to be a two-sided cross point, if it belongs to the closure of two distinct open connected components of $\operatorname{int}(K \triangle H)$. A cross point, which is not two-sided, is said to be one-sided.

1.10 NOTATIONS

Given two distinct points z and w or a point z and a direction S, we shall denote with (z,w) and (S,z) the line through z and w and, respectively, the line through z having direction S.

We shall denote with zw the segment joining the points z and w.

If $z \in \operatorname{cross}(K,H)$ and K and H have the same Steiner symmetrals with respect to S, we shall denote with $S(z)$ (if it exists) the other cross point on (S,z). The meaning of $S_1S_2(z)$ is also clear.

The proof of the following lemma follows immediately from [3], Lemma 4.

1.11 LEMMA

Let $\mathcal{D} = \{S_1, S_2, \ldots, S_n\}$ be a set of directions and let K and H be two distinct convex plane bodies having the same Steiner symmetrals. Suppose that z_o is a cross point. Then the set of cross points $\{S_{i_1} S_{i_2} \ldots S_{i_k}(z_o) \; ; \; 1 \leq i_j \leq n\}$ is finite.

1.12 LEMMA

Suppose that K and H have the same Steiner symmetrals with respect to all the directions of $\mathcal{D}$ and that $z \in \text{cross}(K,H) \cap s$ with $s \in S$ and $S \in \mathcal{D}$. Let moreover $s \cap \text{int } K \cap \text{int } H \neq \emptyset$. Then the line s contains two cross points.

Proof: As z is a common endpoint of the two segments $s \cap K$ and $s \cap H$ having the same length, two possibilities may occur: either $s \cap K$ and $s \cap H$ are consecutive segments on s or they coincide. But by assumption they have common interior points, so they coincide and therefore the other endpoint of that segment is again a cross point.

1.13 LEMMA

If K and H are two distinct plane convex bodies having the same Steiner symmetrals with respect to $S_1, \ldots, S_n$ $(n \geq 2)$, and if $z, w \in \text{cross}(K,H) \cap s$, $s \in S_1$, then z is a two-sided cross point if and only if w is a two-sided cross point.

Proof: By definition, a two sided cross point z belongs to the boundary of two distinct open connected components R_1 and R_2 of $\text{int}(K \triangle H)$ and R_1 and R_2 give rise to other two connected components R_1' and R_2' having w in the boundary.

1.14 LEMMA

If K and H are two distinct plane convex bodies having the same Steiner symmetrals with respect to $S_1, \ldots, S_n$ $(n \geq 2)$ and if (S_1, z) supports K in z alone, then z is a two-sided cross point.

Proof: It is clear that if $s \in S_1$ and $s \cap \text{cross}(K,H) \neq \emptyset$, then s intersects the closure of at least two connected components, R_1 and R_2, say, of $\text{int}(K \triangle H)$. As $s=(S,z)$ intersects K only in z, $z \in \bar{R}_1 \cap \bar{R}_2$, so z is a two-sided cross point.

1.15 LEMMA

If K and H are two plane convex bodies such that $\text{int } K \cap \text{int } H \neq \emptyset$ having the same Steiner symmetrals with respect to S, if (S,z) supports K in

z and z is a one-sided cross-point, then there exists $w \neq z$ such that $w \in l(S,z) \cap \text{cross}(K,H)$. Moreover w is a one-sided cross-point and $zw \subset \partial K \cap \partial H$.

Proof: As z belongs to the closure of only one connected component R of $(\text{int } K) \setminus H$, say, if we consider the set

$$R' = \bigcup \{((\text{int } H) \setminus K) \cap s; \; s \cap R \neq \emptyset\},$$

there is only one point $w \in l(S,z) \cap \bar{R}'$. It is $z \neq w$, because clearly $z \notin \bar{R}'$. Obviously $w \in \text{cross}(K,H)$ and w is one-sided by Lemma 1.13. From $z, w \subset \partial K \cap \partial H$, from the fact that $l(z,w)$ supports K and H and from the convexity of K and H it follows finally that $zw \subset \partial K \cap \partial H$.

2. We are now in position to prove the main result of the paper.

2.1 THEOREM

If K is a convex plane body, then there exist three appropriate directions such that the Steiner symmetrals with respect to them distinguish K from any other plane convex body.

Proof: The proof will be achieved in several steps.

I) Determination of the first two directions:

Take any direction S. By Lemma 1.4 there exists another direction S' such that there is a non-degenerate (S,S')-parallelogram inscribed in K. These are the first two directions.

II) Determination of the third direction:

After we have taken S and S', two possibilities may occur.

i) Either K is union of all the (S,S')-parallelograms inscribed in K,

ii) or this is not the case.

In case i), K is obviously determined already by the Steiner symmetrals in directions S and S', because the vertices of the (S,S')-parallelograms inscribed in K cover ∂K and are, by Lemma 1.5, common to every convex body having the same Steiner symmetrals with respect to S and S'. In this case a third direction T can be taken arbitrarily.

In case ii), consider the set $M = \bigcup s$, the union being taken over all the lines $s \in S$, such that $s \cap K \neq \emptyset$. This set is obviously closed. For a line $s \subset M$ either

a) $s \cap K$ contains a side of an inscribed (S,S')-parallelogram, or

b) $s \cap K$ does not contain a side of an inscribed (S,S')-parallelogram.

Let us remark that in case a), if $s \cap \operatorname{int} K \neq \emptyset$, then $s \cap K$ is in fact the side of an inscribed (S,S')-parallelogram. It is convenient to include in case a) also the degenerate case.

The union M_0 of all the lines of type b) is an open set relative to M. As we are now in case ii), it is $M_0 \neq \emptyset$. There exists therefore a collection (which is at most countable) of disjoint strips M_n $(n=1,2,\ldots)$, open relative to M, such that $M_0 = \bigcup_n M_n$.

The same construction can be repeated for the direction S'.

For every line s bounding one of the strips M_n (except possibly the lines supporting K), $s \cap K$ is a side of an inscribed (S,S')-parallelogram. If on the contrary s supports K, and there exist more (S,S')-parallelograms having a side on s, we associate to s the parallelogram having its side on s of maximal length.

In this way we have identified an at most countable collection $\mathcal{P} = \{P_n ; n \in N' \subset \mathbb{N}\}$ of (S,S')-parallelograms inscribed in K. To simplify the notations, we shall introduce a cartesian coordinate system such that $\{(x,y): y = 0\} \in S$ and $\{(x,y): x = 0\} \in S'$. We can therefore represent the parallelograms of $\mathcal{P}$:

$P_n = [a_n,b_n] \; [c_n,d_n]$, $n \in N' \subset \mathbb{N}$.

Let us now consider the at most countable set of directions $\mathcal{D}$ corresponding to the lines $((a_n,c_n),(b_m,d_m))$ and $((a_n,d_n),(b_m,c_m))$ $n,m \in \mathbb{N}'$ and take a direction T such that $T \notin \mathcal{D} \cup \{S,S'\}$.

We shall now prove that the Steiner symmetrals in directions S,S' and T distinguish K from any other plane convex body.

The proof will be achieved proving the following statement:

(1) Assume that H is a convex body having the same Steiner symmetrals as K with respect to the directions S,S' and T. Then

$$\operatorname{cross}(K,H) = \bigcup_{n \in N_0} \operatorname{vert} P_n \qquad \text{with } N_0 \subset N'$$

Moreover, if $K \neq H$, at least one of the P_n's $(n \in N_0)$ is non degenerate.

Proof of the theorem assuming (1):

Let us suppose that there exists a convex body $H \neq K$ having the same Steiner symmetrals as K with respect to the directions S, S' and T. By (1) there exists a non degenerate $P_n \in \mathcal{P}$ such that $\{z_1,z_2,z_3,z_4\} = \operatorname{vert} P_n \subset \operatorname{cross}(K,H)$; as T is not parallel to any line (z_i,z_j) $(i \neq j)$, there is a

point z_i such that $(T,z_i) \cap \operatorname{int} K \neq \emptyset$. Obviuosly it is also $(T,z_i) \cap \operatorname{int} H \cap \operatorname{int} K \neq \emptyset$. There exist therefore, by Lemma 1.12, two cross-points on (T,z_i), z_i and w, say. By (1) there exists $m \in N_0$ such that $w \in \operatorname{vert} P_m$, so $(T,z_i) = (w,z_i)$. But this contradicts the choice of T and the theorem is proved.

We shall prove (1) through the following steps:

(2) $\operatorname{cross}(K,H) \cap M_n = \emptyset$ for every n;

(3) $\operatorname{cross}(K,H) \cap s = \emptyset$ for every line s accumulating on both sides lines bounding the open strips M_n;

(4) $\operatorname{cross}(K,H) \subset \bigcup_{n \in N'} \operatorname{vert} P_n$;

(5) if $\operatorname{cross}(K,H) \cap \operatorname{vert} P_n \neq \emptyset$, then $\operatorname{vert} P_n \subset \operatorname{cross}(K,H)$;

(6) if $\operatorname{cross}(K,H) \neq \emptyset$, then there exists a non degenerate parallelogram P_n such that $\operatorname{vert} P_n \cap \operatorname{cross}(K,H) \neq \emptyset$.

Proof of (2):

Suppose $z \in \operatorname{cross}(K,H) \cap M_n$, and denote with (x,y) the coordinates of z. There exists an open strip M'_k (in direction S') such that $z \in M'_k$. It is easily seen that M_n and M'_k cannot be both bounded by lines supporting K: let in fact $P = [a,b] \times [c,d]$ be any non degenerate (S,S')-parallelogram inscribed in K. If M_n and M'_k were both bounded by lines supporting K, then $M_n \cap P = \emptyset$ and $M'_k \cap P = \emptyset$, so we should have $x < a$ or $x > b$ and $y < c$ or $y > d$, but this contradicts the convexity of K.

We see then that the following cases may occur:

a) neither M_n nor M'_k are bounded by a line supporting K;

b) M_n, say, is bounded by a line supporting K and (S,z) supports K in z;

c) M_n, say,is bounded by a line supporting K and (S',z) supports K in z;

d) M_n (or M'_k) is bounded by a line supporting K and (S,z) and (S',z) both support K in z.

Case a): Let $M_n = \{(x,y): d_1 < y < d_2\}$ and $M'_k = \{(x,y): a_1 < x < a_2\}$. The lines $\{y = d_i\}$ and $\{x = a_i\}$ support unique (S,S')-parallelograms inscribed in K. We may suppose without restriction that $P_i = [a_i,b_i] \times [c_i,d_i]$ $(i=1,2)$ are inscribed (S,S')-parallelograms.

Put $z = z_0$. As $(S,z_0) \cap \operatorname{int} K \cap \operatorname{int} H \neq \emptyset$, from Lemma 1.12 we see that there exists $z_1 \neq z_0$ such that $z_1 \in \operatorname{cross}(K,H) \cap (S,z_0)$. From the convexity of K, we have $b_2 < x_1 < b_1$ (and $y_1 = y_0$). We can now repeat the procedure and consider $z_2 \neq z_1$ such that $z_2 \in (S',z_1) \cap \operatorname{cross}(K,H)$. From the convexity of K, we have $c_2 < y_2 < c_1$

and $x_2=x_1$. Applying once more the same argument, we prove the existence of $z_3\neq z_2$ such that $z_3\in(S,z_2)\cap \mathrm{cross}(K,H)$. From the convexity of K it follows that $a_1<x_3<a_2$ and $y_3=y_2$.

If we had $x_3=x_\circ$, then the points $z_\circ,z_1,z_2$ and z_3 would define an (S,S')-parallelogram inscribed in K and this would be against the assumption $z_\circ\in M_n$. Therefore $x_3\neq x_\circ$. If $x_3<x_\circ$ $(x_3>x_\circ)$ the convexity of K implies that we can construct iteratively a sequence of points $\{z_k\}$ on the boundary of K such that for every $k\geq 1$

$$y_{4k}<y_{4k-4}\ ,\ x_{4k+1}>x_{4k-3}\ ,\ y_{4k+2}>y_{4k-2}\ ,\ x_{4k+3}<x_{4k-1}$$

$$(y_{4k}>y_{4k-4}\ ,\ x_{4k+1}<x_{4k-3}\ ,\ y_{4k+2}<y_{4k-2}\ ,\ x_{4k+3}>x_{4k-1})$$

and moreover

$$x_{4k},x_{4k+3}\in[a_1,a_2]\ ,\ x_{4k+1},x_{4k+2}\in[b_2,b_1]$$

$$y_{4k},y_{4k+1}\in[d_1,d_2]\ ,\ y_{4k+2},y_{4k+3}\in[c_2,c_1]\ .$$

Therefore $z_n\neq z_m$ if $n\neq m$. By Lemma 1.12 $z_n\in\mathrm{cross}(K,H)$ for every n, and this contradicts Lemma 1.11.

In cases b), c) and d) at least one of the strips is bounded by a line supporting K. We may assume, without loss of generality, that $P=[a,b]\times[c,d]$ is an (S,S')-parallelogram inscribed in K, having its S'-side of maximal length and that $z_\circ\in\mathrm{cross}(K,H)$, $z_\circ=(x_\circ,y_\circ)$ is such that $y_\circ\geq d$ ($y_\circ>d$ in cases b) and c), $y_\circ\geq d$ in case d)).

Let us first exclude case d).

Suppose that $(S,z_\circ)$ and $(S',z_\circ)$ support K in $z_\circ$. From the convexity of K it follows that $(S',z_\circ)=\{x=a\}$ (or $(S',z_\circ)=\{x=b\}$). Therefore $(x_\circ,c)$, $(x_\circ,d)$ and $(x_\circ,y_\circ)$ are three points of $\partial K\cap\partial H$, hence $(x_\circ,c)z_\circ\subset\partial K\cap\partial H$. As K and H have the same Steiner symmetrals with respect to S, for every line $s=\{y=t\}$ with $t\in[c,y_\circ]$, we have $K\cap s=H\cap s$, therefore $z_\circ\notin\mathrm{cross}(K,H)$, a contradiction. So case d) is excluded.

Case b) splits in three cases:

b1) $z_\circ$ is the only cross-point on $(S,z_\circ)$;

b2) there exists $z_{-1}\neq z_\circ$ such that $z_{-1}\in(S,z_\circ)\cap\mathrm{cross}(K,H)$ and $z_\circ$ and z_{-1} are both one-sided cross-points;

b3) there exists $z_{-1}\neq z_\circ$ such that $z_{-1}\in(S,z_\circ)\cap\mathrm{cross}(K,H)$ and $z_\circ$ and z_{-1} are both two-sided cross-points.

In cases b2) and b3) we may assume without restriction that $x_\circ<x_{-1}$.

By Lemma 1.12 there exists a point $z_1\neq z_\circ$ such that $z_1\in(S',z_\circ)\cap\mathrm{cross}(K,H)$. It is $y_1\leq c$ and $x_1=x_\circ$. In fact it is $y_1<c$, because $y_1=c$ would imply that (a,c), $(x_\circ,c)$ and (b,c) belong to $\partial K\cap\partial H$ and hence $z_1\notin\mathrm{cross}(K,H)$.

We have $(S,z_1)\cap\mathrm{int}\,K\cap\mathrm{int}\,H\neq\emptyset$. If we suppose the contrary, the segment $z_\circ z_1$ is a degenerate (S,S')-parallelogram inscribed in K and H and this contradicts the assumption on $z_\circ$.

Applying Lemma 1.12 again, there exists $z_2\neq z_1$ such that $z_2\in(S,z_1)\cap\cap\mathrm{cross}(K,H)$. From the convexity of K we have $a\leq x_2\leq b$. Suppose $x_2>x_1$ (in the case $x_2<x_1$ the proof goes the same way; we shall indicate in square brackets the alternative inequalities and arguments, when different).

Let us first prove that $x_2\neq b$ $[x_2\neq a]$. If $x_2=b$ $[x_2=a]$, then the points (b,c), (b,d) and z_2 $[(a,c)$, (a,d) and $z_2]$ belong to $\partial K\cap\partial H$ and from the convexity of K it follows that $z_2(b,d)$ $[z_2(a,d)]$ is contained in $\partial K\cap\partial H$. This implies that z_2 is a one-sided cross point and so is therefore $z_\circ$. This shows a contradiction in cases b1) and b3), and also in case b2) with some additional argument: Let us remark that it is

P(2) $\quad K\cap\{(x,y):\ y\in[y_2,c]\}=H\cap\{(x,y):\ y\in[y_2,c]\}$.

As there is no inscribed (S,S')-parallelogram having its S'-side longer than $d-c$, we have $m(K\cap s)<b-a$, where $s=\{y=y'\}$, with $y'\in[y_2,c[$. Therefore $K\cap s'=H\cap s'$ for every s' of the kind $s'=\{x=x'\}$ with $x'\in[a,x_\circ]$ $[x'\in[x_{-1},b]]$. This implies, as $z_\circ$ $[z_{-1}]$ is a one-sided cross point, that $K\cap s'\neq H\cap s'$ for $s'=\{x=x'\}$ with x' in a right [left] neighborhood of $x_\circ$ $[x_{-1}]$, but this is not possible, because the segment $z_\circ z_{-1}$ is contained in $\partial K\cap\partial H$.

Therefore $x_2\neq b$ $[x_2\neq a]$ and as $x_1<x_2$, $x_2\neq a$ $[x_1>x_2$, $x_2\neq b]$. We can now apply again Lemma 1.12 to obtain a point $z_3\neq z_2$ such that $z_3\in(S',z_2)\cap\mathrm{cross}(K,H)$. The convexity of K implies that $y_3>d$. Of course $x_3=x_2$. Equality $y_3=y_\circ$ would imply that the points $z_\circ$, z_1, z_2 and z_3 define an (S,S')-parallelogram inscribed in K, against our assumption. Therefore $y_3<y_\circ$.

Again by Lemma 1.12 we can construct a point $z_4\neq z_3$ such that $z_4\in(S,z_3)\cap\cap\mathrm{cross}(K,H)$. From the convexity of K it follows $a\leq x_4<x_\circ$ $[x_\circ<x_4\leq b]$. Let us prove that in fact $x_4>a$ $[x_4<b]$. From $x_4=a$ $[x_4=b]$ we deduce that (a,c), (a,d) and z_4 belong to $\partial K\cap\partial H$ [(b,c), (b,d) and z_4 belong to $\partial K\cap\partial H$] and therefore $(a,c)z_4\subset\partial K\cap\partial H$ [$(b,c)z_4\subset\partial K\cap\partial H$] This implies that

P(4) $\quad K\cap\{(x,y):\ y\in[c,y_4]\}=H\cap\{(x,y):\ y\in[c,y_4]\}$,

therefore

$$K \cap \{(x,y):\ x \in [x_3,b]\} = H \cap \{(x,y):\ x \in [x_3,b]\}$$

$$[K \cap \{(x,y):\ x \in [a,x_3]\} = H \cap \{(x,y):\ x \in [a,x_3]\}]$$

and from this identity we deduce in the same way the equality

$$K \cap \{(x,y):\ y \in [y_2,c]\} = H \cap \{(x,y):\ y \in [y_2,c]\}$$

which coincides with P(2), and therefore brings to a contradiction.

Suppose now that we have constructed, applying repeatedly Lemma 1.12, the cross points $z_0, z_1, z_2, \ldots, z_{2n}$. The construction of z_r with r even never gives problems, because the convexity of K implies

$$x_{4k-3} < x_{4k+1} < c \qquad (y_{4k+1} = y_{4k})$$

$$x_{4k-1} < x_{4k+3} < d \qquad (y_{4k+3} = y_{4k+2})$$

therefore Lemma 1.12 applies. The only feature which may stop the construction of the sequence $\{z_n\}$ is therefore one of the identities

P' $\quad x_{4k}=a \qquad [x_{4k}=b]$

P" $\quad x_{4k+2}=b \qquad [x_{4k+2}=a]$

We shall now prove that from P' or P" follows P(2) and therefore a contradiction. This will end the proof of (2) in case b), because we note as in case a) that $\{z_n\}$ has to be a finite set.

Let $2n=4k$ and suppose P' holds. Then z_{4k}, (a,c) and (a,d) [z_{4k}, (b,c) and (b,d)] belong to $\partial K \cap \partial H$, therefore $z_{4k}(a,d) \subset \partial K \cap \partial H$ [$z_{4k}(b,d) \subset \partial K \cap \partial H$] and therefore

P(4k) $\quad K \cap \{(x,y):\ y \in [d,y_{4k}]\} = H \cap \{(x,y):\ y \in [d,y_{4k}]\}$;

from this we deduce

$$K \cap \{(x,y):\ x \in [x_{4k-2},b]\} = H \cap \{(x,y):\ x \in [x_{4k-2},b]\}$$

$$[K \cap \{(x,y):\ x \in [a,x_{4k-2}]\} = H \cap \{(x,y):\ x \in [a,x_{4k-2}]\}]$$

and in the same way we prove that

P(4k-2) $\quad K \cap \{(x,y):\ y \in [y_{4k-2},c]\} = H \cap \{(x,y):\ y \in [y_{4k-2},c]\}$.

Suppose now $2n=4k+2$ and P". We have then that z_{4k+2}, (b,c) and (b,d) [z_{4k+2}, (a,c) and (a,d)] belong to $\partial K \cap \partial H$, therefore $z_{4k+2}(b,c) \subset \partial K \cap \partial H$ [$z_{4k+2}(a,c) \subset \partial K \cap \partial H$], hence we have

P(4k+2) $\quad K \cap \{(x,y):\ y \in [y_{4k+2},c]\} = H \cap \{(x,y):\ y \in [y_{4k+2},c]\}$.

This now implies that

$$K \cap \{(x,y):\ x \in [a,x_{4k+1}]\} = H \cap \{(x,y):\ x \in [a,x_{4k+1}]\}$$

$$[K \cap \{(x,y):\ x \in [x_{4k+1},b]\} = H \cap \{(x,y):\ x \in [x_{4k+1},b]\}]$$

and this implies that

$$K \cap \{(x,y):\ y \in [d,y_{4k}]\} = H \cap \{(x,y):\ y \in [d,y_{4k}]\}$$

i.e. P(4k). We have therefore proved that P' implies P(4k), which implies P(4k-2) and that P" implies P(4k+2), which implies P(4k) for every $k \in \mathbf{N}$. Therefore from P' or P" follows P(2) and the proof of (2) in case b) is completed.

Case c)

We may suppose without restriction that $z \in \mathrm{cross}(K,H)$ and $P=[a,b]\times[c,d]$ are such that $z=(a,y)$ [$z=(b,y)$] with $y>d$ and P is the (S,S')-parallelogram inscribed in K having the length of its S'-side maximal.

As $z(a,c) \subset \partial K \cap \partial H$, z is a one sided cross-point and by Lemma 1.15 there is another cross point z_0 on (S,z). Put $z_0=(x_0,y_0)$. We have $y_0=y$ and $x_0<b$, because $x_0=b$ would imply that z, (a,c), (b,c) and z_0 define an (S,S')-parallelogram inscribed in K, stictly including P. We can now apply to the point z_0 the arguments used in case b), and so (2) is proved.

Let us prove (3).

Suppose $z_0=(x_0,y_0) \in \mathrm{cross}(K,H)$. Then there exists an open connected component R of $\mathrm{int}(K \triangle H)$ such that $z_0 \in \bar{R}$. Suppose $R \subset \{(x,y):\ y>y_0\}$ and put $A=\{y_1:\ R \cap \{y=y_1\} \neq \emptyset\}$. Then A is open and for every $y' \in A$, we have that $K \cap \{y=y'\} \neq H \cap \{y=y'\}$. But if $s'=\{y=y'\}$ bounds some M_n, we have $K \cap \{y=y'\} = H \cap \{y=y'\}$, contradicting the assumption on $\{y=y_0\}$.

Let us prove (4).

From (2) and (3) we have that $\mathrm{cross}(K,H) \cap s = \emptyset$ for every $s \in S$ such that $K \cap s$ does not contain a side of one of the P_m's. On the other hand, if $K \cap s$ contains a side of P_m and $K \cap s$ is not contained in ∂K, then obviously $\mathrm{cross}(K,H) \cap s \subset \mathrm{vert}\ P_m$. The same inclusion holds also when s supports K : It is clear that if $P=[a,b]\times[c,d]$ is an (S,S')-parallelogram inscribed in K such that, for instance, $\{y=d\} \cap K \subset \partial K$, then no point (x,d) with $x \in]a,b[$ can belong to $\mathrm{cross}(K,H)$. We have to prove moreover that if $z_0=(x_0,d)$ does not belong to any (S,S')-parallelogram inscribed in K, then $z_0 \notin \mathrm{cross}(K,H)$. But if z_0 is such a point, $z_0 \in M'_k$ for some k and we can now apply (2) with respect to S'.

Let us prove (5).

Suppose first that $P_m=P$ is not degenerate. Let $P=[a,b]\times[c,d]$ be an (S,S')-parallelogram inscribed in K and suppose, for instance, that $z_1=(a,c) \in \mathrm{cross}(K,H)$. If $(S,z_1) \cap \mathrm{int}\ K \cap \mathrm{int}\ H \neq \emptyset$ and $(S',z_1) \cap \mathrm{int}\ K \cap \mathrm{int}\ H \neq \emptyset$, then $\mathrm{vert}\ P \subset \mathrm{cross}(K,H)$ follows immediately from Lemma 1.11. Both identities

(7) $(S,z_1) \cap \operatorname{int} K = \emptyset$ and $(S',z_1) \cap \operatorname{int} K = \emptyset$

cannot occur simultaneously, as we have seen excluding case d) in proving (2). It remains to check what happens if one of the equalities (7), suppose the first one, holds. The point z_1 is in this case a one sided cross point, because $(a,c)(b,c) \subset \partial K \cap \partial H$. From Lemma 1.15 it follows that there exists $z_2 \neq z_1$ such that $(x_2,y_1) = z_2 \in (S,z_1) \cap \operatorname{cross}(K,H)$. By the same Lemma, z_2 is a one sided cross point. From $z_1 z_2 \subset \partial K \cap \partial H$ follows immediately that $x_2 \geq b$; P is the parallelogram having its S-side on (S,z_1) of maximal length, so $x_1 > b$ would imply $z_2 \in M_k'$ for some k and by (2) $z_2 \notin \operatorname{cross}(K,H)$, a contradiction. Therefore $z_2 = (b,c)$.

As the identities (7) cannot occur simultaneously, and as $(S,z_2) \cap \operatorname{int} K = \emptyset$ and also $(S,z_2) \cap \operatorname{int} H = \emptyset$, we have that $(S',z_2) \cap \operatorname{int} K \cap \operatorname{int} H \neq \emptyset$. We can therefore apply Lemma 1.12; there exists a point $z_3 = (b,y_3) \neq z_2$ such that it belongs to $(S',z_2) \cap \operatorname{cross}(K,H)$. By the same argument, $(S',z_1) \cap \operatorname{int} K \cap \operatorname{int} H \neq \emptyset$ and so by Lemma 1.11 there exists $z_o = (a,y_o) \neq z_1$ such that $z_o \in (S',z_1) \cap \operatorname{cross}(K,H)$. As $(S',z_i) \cap \partial K$ (i=1,2) has exactly two points, we have that $z_o = (a,d)$ and $z_3 = (b,d)$, so $\operatorname{vert} P_m \subset \operatorname{cross}(K,H)$.

In case P_m is degenerate, we may suppose $P_m = P = \{ (x,y): x=a, y \in [c,d] \}$ Suppose $(a,c) = z_o$ is a cross point. Then also $z_1 = (a,d)$ is a cross point because z_o belongs by definition to the closure of some open connected component of $(\operatorname{int} K) \setminus H$, say, and therefore $z_1 \in \bar{R}'$, with

$R' = \bigcup ((\operatorname{int} H) \setminus K)\ s';\ s' \in S',\ s' \cap R \neq \emptyset \}$

Let us finally prove (6).

Suppose that $K \neq H$. Then $\operatorname{cross}(K,H) \neq \emptyset$. Suppose that every point of $\operatorname{cross}(K,H)$ belongs to the set of vertices of some degenerate (S,S')-parallelogram of the collection $\mathfrak{P}$.

Let $P = [a,b] \times [c,d]$ be any non degenerate (S,S')-parallelogram inscribed in K and let $(x_1,y_1) = z_1$ be a cross point and $P' = z_1 z_2$ be a degenerate (S,S')-parallelogram inscribed in K, with $z_2 = (x_1,y_2)$ and $y_1 < y_2$. From the convexity of K we deduce immediately that $y_1 \leq c < d \leq y_2$ and $x_1 \in [a,b]$. Suppose first $x_1 = a$, $y_1 \leq c$. Then $z_1(a,d) \subset \partial K \cap \partial H$ and therefore $K \cap \{(x,y): y \in [y_1,d] \} = H \cap \{(x,y): y \in [y_1,d] \}$. As (S,z_1) supports K in z_1, it follows easily that that z_1 is not a cross point, a contradiction. In the same way we exclude the case $x_1 = b$. It is therefore $x_1 \in]a,b[$. From the convexity of K and H we deduce also that $y_1 < c$ and $d < y_2$. In fact, if $y_1 = c$, say, then (a,c), (x_1,c) and (b,c) belong to $\partial K \cap \partial H$, therefore $(a,c)(b,c) \subset \partial K \cap \partial H$, hence z_1 is not

a cross point. As $P' \in \mathfrak{P}$ either (S,z_1) or (S,z_2) support K only in z_1 or, respectively, in z_2. Suppose for instance $(S,z_1) \cap \partial K = \{z_1\}$. By Lemma 1.14 z_1 is a two-sided cross point. Let R_1, R_2 be the two open connected components of $\operatorname{int}(K \triangle H)$ such that $z_1 \in \bar{R}_1 \cap \bar{R}_2$. As $K \cap \{y=c\} = H \cap \{y=c\}$, we have that $R_i \subset \subset \{(x,y): y < c\}$ for i=1,2. Therefore there exist other cross points $z_i = (x_i,y_i)$ for i=3,4 suc that $z_i \in \bar{R}_i$ with $x_i \in [a,b]$ and $y_1 < y_i < c$ for i=3,4. From $x_3, x_4 \in [a,b]$ and the convexity of K it follows now that z_3 and z_4 are not the vertices of a degenerate (S,S')-parallelogram inscribed in K. So (6) is proved.

2.2 REMARK

If we assume (as in [2], p. 240) that the boundary of K does not contain any segment parallel to S or S', then the proof can be made considerably shorter, because cases b2) and b3) considered in proving (2) cannot occur, and also the proofs of (4) and (6) become simpler.

2.3 REMARK

It is not difficult to see that we do not require in fact the convexity of H, but just that $H \cap l$ is a segment for every line having one of the three directions (remarked also in [4], pp. 236-237).

2.4 REMARK

Recently R.J. Gardner gave a short proof of the Giering theorem. The disadvantage of it is that the allowed triplets of directions are in general a proper subset of the triplets considered by Giering. Our aim was the opposite and our triplets include, in general strictly, the triplets considered by Giering. It should be mentioned that from this point of view our result is not optimal. In fact a direct computation shows that the triangle $\{(x,y): y \geq 0,\ y \leq 2x+2,\ y \leq -\frac{2}{3}(x-3) \}$ is uniquely determined by the triplet of directions corresponding to the lines $y=x$, $y=-x$ and $y=\frac{1}{2}x$, while according to 2.1 (and [4]) S or S' has to be the direction of one of its sides (see Problem 4 below).

2.5 OPEN PROBLEMS

1. By the theorem of Gardner and McMullen, four directions distinguish between all the convex bodies in $\mathbb{R}^n$. Giering's result shows that in $\mathbb{R}^2$ we need only

three directions to distinguish a given convex body from all the others. It is an open question if this is true also in higher dimensions.

2. Theorem 2.1 defines on the collection $\mathcal{K}$ of all convex bodies a set-valued mapping F, assigning to each convex body K the triplets of directions (S,S',T). Let us give to $\mathcal{K}$ the topology induced by the Hausdorff metric and to the triplets of directions the natural topology on C^3, where C is the unit circle in $\mathbb{R}^2$. Professor D. Kölzow asked the following questions: Does F have any continuity property? Does there exist a continuous selection?

The answer to the second question would be relevant for a well-posedness problem asked in [8].

3. Characterize those convex bodies, which are determined by the Steiner symmetrals with respect to two appropriate directions ([4], footnote 3).

4. Find a proof of the Giering theorem providing, for each given convex body K, all the triplets of directions which distinguish K from any other convex body.

5. Find an algorithm to reconstruct a convex body from the Steiner symmetrals in the two settings (Gardner-McMullen's and Giering's). It has to be noted that the proof of Theorem 2.1 is to a certain extent constructive and show how to obtain from the vertices of some inscribed parallelograms a subset of ∂K. This rizes the question whether is it possible to combine the proof 2.1 with some other techniques to find a dense subset of ∂K.

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WIJSMAN CONVERGENCE IN THE HYPERSPACE OF A METRIC SPACE

A. Lechicki and S. Levi

Let (X,d) be a metric space; 2^X will denote the power set of X and 2^X_o the subfamily of non-empty subsets.

A net (A_i) in 2^X is Wijsman convergent (w-convergent) to $A \in 2^X$ if $d(y,A) = \lim_i d(y,A_i)$ for every y in X, where $d(y,\emptyset) = +\infty$.

R.A. Wijsman introduced this definition in [3] and applied it to the study of sequences of convex subsets of Euclidean spaces.

Wijsman convergence is topological and induces a (non T_1) completely regular topology.

In Euclidean spaces w-convergence agrees with Kuratowski convergence on closed sets (theorem 2.6 of [1]). In general Wijsman convergence is weaker than Vietoris topology and stronger than Kuratowski convergence and appears to preserve the general properties of the underlying space more faithfully than the above mentioned convergences, although it depends on the metric d: equivalent metrics on X need not generate the same w-convergence in the hyperspace.

Wijsman convergence turns out to be a useful tool for the study of multifunctions (see [1], §4 and 7).

It can be verified that Wijsman convergence is admissible, that is the mapping $j: X \to 2^X$ defined by $j(x) = \{x\}$ is a homeomorphism.

Let $y \in X$, $\delta > 0$ and $0 < \varepsilon < \alpha$; put

$$W_{y,\delta} = \{(A,B) \in 2^X_o \times 2^X_o : |d(y,A) - d(y,B)| < \delta\}, \text{ and}$$

$$L_{y,\varepsilon,\alpha} = \{(A,B) \in 2^X \times 2^X : A \cap B_\varepsilon(y) \neq \emptyset \Rightarrow B \cap B_\alpha(y) \neq \emptyset \text{ and } A \cap B_\alpha(y) = \emptyset \Rightarrow B \cap B_\varepsilon(y) = \emptyset\}$$

where $B_\varepsilon(y)$ is the open ball of radius ε centered at y.

The family $\{ W_{y,\delta} : y \in X, \delta > 0 \}$ (resp. $\{ L_{y,\varepsilon,\alpha} : y \in X, 0 < \varepsilon < \alpha \}$) is a subbase for the uniformity $\mathcal{W}$ (resp. $\mathcal{L}$) on 2^X_o (resp. 2^X) which is compatible with Wijsman convergence on 2^X_o (resp. 2^X).

Let us define, for $A,B \in 2^X_o$, $y \in X$

$$\alpha_y(A,B) = \frac{| d(y,A) - d(y,B) |}{\sqrt{1+d(y,A)^2}\sqrt{1+d(y,B)^2}}$$

and $\alpha_y(A,\emptyset) = \alpha_y(\emptyset,A) = \frac{1}{\sqrt{1+d(y,A)^2}}$

Each α_y is a pseudometric and we have the

Theorem 1: The family $\{\alpha_y : y \in X\}$ generates the uniformity $\mathcal{L}$ on 2^X.

Theorem 2 : The uniform space $(2^X, \mathcal{L})$ is totally bounded; $(2^X_o, \mathcal{W})$ is totally bounded if and only if d is bounded.

A metric space (X,d) is boundedly compact if every closed ball for d is compact. Euclidean spaces are boundedly compact.

Theorem 3: Let d be unbounded. Then $(j(X) \cup \{\emptyset\}, w)$ is the Alexandrov compactification of X if and only if (X,d) is boundedly compact.

Let c(X) denote the hyperspace of X, that is the family of closed subsets of X and $c_o(X)$ the subfamily of non-empty subsets.

Corollary 4: X is compact if and only if $c_o(X)$ is compact; c(X) is compact if and only if (X,d) is boundedly compact.

If (X,d) is separable, then w-convergence on c(X) is metrizable and theorem 1 can be used to deduce an explicit formula for a metric. This answers a question raised by G. Salinetti and R. Wets in §5 of [2].

As was pointed out, Wijsman convergence depends on the metric on X. For every M>0, let d_M denote the metric min (d,M); d_M is uniformly equivalent to d and the following result holds:

Theorem 5 : For every M>0 $w_d | 2^X_o = w_{d_M} | 2^X_o$ if and only if every closed ball ($\neq X$) in (X,d) is totally bounded.

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ON MULTIFUNCTIONS WITH CONVEX GRAPH

Biagio Ricceri

The purpose of this talk is to present a characterization of the lower semicontinuity of multifunctions with convex graph as well as some of its consequences.

Before stating our results, let us recall some basic definitions on multifunctions. Let X, Y be two topological spaces and F a multifunction from X into Y. For every $A \subseteq X$, $\Omega \subseteq Y$, we put $F(A) = \bigcup_{x \in A} F(x)$ and $F^-(\Omega) = \{x \in X : F(x) \cap \Omega \neq \emptyset\}$. We say that F is *lower* (resp. *upper*) *semicontinuous* if the set $F^-(\Omega)$ is open (resp. closed) in X for every open (resp. closed) set $\Omega \subseteq Y$. We say that F is *open* (resp. *closed*) if the set $F(A)$ is open (resp. closed) in Y for every open (resp. closed) set $A \subseteq X$. The set $\{(x,y) \in X \times Y : y \in F(x)\}$ is the *graph of* F.

From now on, X is a non-empty open convex subset of a real topological vector space E, Y is a real locally convex topological vector space and F is a multifunction from X into Y, with non-empty values and convex graph.

Our main result is the following.

THEOREM 1. - *The following are equivalent:*

(1) *F is lower semicontinuous.*

(2) *F is lower semicontinuous and the set $\{(x,y) \in X \times Y : y \in \overline{F(x)}\}$ is closed in $X \times Y$.*

(3) *There exist a saturated family $\{p_i\}_{i \in I}$ of semi-norms on Y, generating the topology of Y, and a family $\{\Omega_i\}_{i \in I}$ of sub-*

sets of Y such that, for every $i \in I$, the set $p_i(\Omega_i)$ is bounded and $\mathrm{int}(F^-(\Omega_i)) \neq \emptyset$.

A first consequence of Theorem 1 is the following

THEOREM 2. - *If $\dim(E) < +\infty$, then F is lower semicontinuous.*

Another consequence of Theorem 1 is the following

THEOREM 3. - *Let $\dim(E) < +\infty$. Let S be a convex subset of $X \times Y$ and let Ω be a subset of Y containing the projection of S on Y. If, for each $x \in X$, the set $\{y \in \Omega : (x,y) \in S\}$ is non-empty and closed in Ω, then the set S is closed in $X \times \Omega$.*

Here is an application of Theorem 3.

THEOREM 4. - *Let $\dim(E) < +\infty$. Let Ω be a non-empty convex subset of Y and let f be a quasi-convex real function on $X \times \Omega$. If, for each $x \in X$, the function $f(x,\cdot)$ is lower semicontinuous and unbounded below on Ω, then f is lower semicontinuous on $X \times \Omega$.*

Furthermore, applying jointly Theorem 1 and Theorem 4.2 of [4], we obtain the following

THEOREM 5. - *Let $\dim(E) < +\infty$ and $X = E$. Assume that $F(x)$ is closed for all $x \in X$ and that there exists $y_o \in F(X)$ such that the set $\{x \in X : x + F^-(y_o) \subseteq F^-(y_o)\}$ is a linear subspace of X. Then, $F(X)$ is closed.*

The next result relates the two types of semicontinuity for multifunctions with convex graph.

THEOREM 6. - *The following are equivalent:*

(1) *Any lower semicontinuous convex real function on X is continuous.*

(2) *For every locally convex topological vector space Σ, any upper semicontinuous multifunction from X into Σ, with convex graph, is lower semicontinuous.*

Now, we present the dual versions of Theorems 1, 2 and 6, in terms of open and closed multifunctions.

In the sequel, T is a non-empty convex subset of Y and G is a multifunction from T onto X, with convex graph.

THEOREM 7. - *The following are equivalent:*

(1) *G is open.*

(2) *G is open and the set $\{(x,y) \in X \times Y : y \in \overline{G^-(x)}\}$ is closed in $X \times Y$.*

(3) *There exist a saturated family $\{p_i\}_{i \in I}$ of semi-norms on Y, generating the topology of Y, and a family $\{\Omega_i\}_{i \in I}$ of subsets of T such that, for every $i \in I$, the set $p_i(\Omega_i)$ is bounded and $\mathrm{int}(G(\Omega_i)) \neq \emptyset$.*

THEOREM 8. - *If $\dim(E) < +\infty$, then G is open.*

THEOREM 9. - *The following are equivalent:*

(1) *Any lower semicontinuous convex real function on X is continuous.*

(2) *For every locally convex topological vector space Σ and every non-empty convex set $V \subseteq \Sigma$, any closed multifunction from V onto X, with convex graph, is open.*

The following final result is a consequence of Theorem 7.

THEOREM 10. - *Let E be Hausdorff and T be σ-compact. Moreover, suppose that G is upper semicontinuous and that $G(y)$ is compact for all $y \in T$. Then, for any vector topology τ on E, stronger than the original one, such that (E,τ) is a Baire space, G is open with respect to τ.*

The proofs of the above results can be found in [6]. For other re-

sults on multifunctions with convex graph and their applications, see, for instance, [1], [2], [3], [4], [5], [7], [8].

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TRUNCATION AND DENSITY FOR NON-NEGATIVE SEQUENCES

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1. Introduction The results to be discussed here, suggested by ways in which (C,1) limits arise in ergodic theory, are of two kinds. One kind comes from the fact that the orbit of a general point, under an ergodic measure-preserving transformation T of the unit interval I, is equidistributed, and determines the integrals of L^1 functions by $\int f\,d\lambda = \lim_1 \langle f(T^n(x)) : n \in N \rangle$ a.e., where $N = \{1, 2, 3, \ldots\}$ and $\lim_1$ denotes the (C,1) limit. The other kind arises from the sequence of Radon-Nikodym derivatives $\omega_n(x)$, $n \in N$, of a (not necessarily ergodic) measurable transformation T; the criterion for the existence of an invariant measure can be expressed in terms of a certain integral (see [2] and [3]) involving relative asymptotic densities determined by the sequence $\langle \omega_n(x) : n \in N \rangle$. We shall show that this integral can be expressed, in more accessible form, by a (C,1) limiting process, involving also a process of "truncation".

2. Integrability criteria For a fixed positive integer d, let S be a bounded Lebesgue measurable subset of Euclidean space R^d, of positive Lebesgue measure $\lambda(S)$. A sequence $\xi = \langle \xi(1), \xi(2), \ldots \rangle$ of points of S is said to be uniformly distributed (on S) provided, for each "box"

$$B = \{ x \in I^d : a_1 \le x_1 \le b_1, \ \ldots, \ a_d \le x_d \le b_d \}$$

(where $a_i < b_i$ for $i = 1, 2, \ldots, n$), the asymptotic density $\delta(M)$ of the set M of integers n for which $\xi(n) \in B$, exists and equals $\lambda(S \cap B)/\lambda(S)$. We refer to the set $S \cap B$ as "B relative to S". Two sequences ξ, η of points of S are said to be "equivalent" ($\xi \sim \eta$) provided the Euclidean distance $\| \xi(n) - \eta(n) \| \to 0$ as $n \to \infty$. It is easy to see that if $\xi \sim \eta$ and ξ is uniformly distributed on S, then so is η.

A real-valued function f on S is said to be Riemann

integrable on S provided it is bounded and the upper and lower Darboux sums (formed with respect to finite partitions of S into relative boxes by hyperplanes parallel to the coordinate axes) converge to a common value (which then, of course, is defined to be the Riemann integral of f over S); equivalently, f is bounded and the set of points of discontinuity of the restriction $f|S$ has Lebesgue measure 0.

The following theorem is an improvement, for which I am indebted to Prof. J.H.B. Kemperman, of an earlier version of mine, in which I had assumed the function f to be bounded.

Theorem 1. Suppose ξ is a uniformly distributed sequence on S, a bounded Lebesgue measurable subset of R^d having positive measure. Suppose f is a real-valued function on S such that, for every sequence η on S equivalent to ξ, $\lim_1 f(\eta)$ exists and is finite. Then f is Riemann integrable on S, and its integral equals $\lim_1 f(\xi)/\lambda(S)$.

The point of this theorem is that f is Riemann integrable; the equality of the integral and the (C,1) limit then follows from well-known classical results, at least if $S = I^d$. The converse of Theorem 1 is also true; in fact the relation between functions and sequences here can be used to characterize both uniform distribution and Riemann integrability. The theorem extends to more general topological measure spaces, but the hypotheses become more complicated, and we do not go into that here. In fact, the special case $S = I^d$, the d-dimensional unit cube, is probably the case of most interest.

Theorem 1 can be proved by direct calculation, as Professor Kemperman does. An alternative proof depends on a special case of the following Lemma, stated here in full generality because it is useful for some other results later on.

Main lemma. Suppose $I^d \supset U_1 \supset U_2 \supset \ldots$, where each U_n is open relative to I^d and where $\lim_{n\to\infty} \lambda(U_n) = 0$. Let ξ be a uniformly distributed sequence in I^d. Then there exists a sequence η in I^d, equivalent to ξ, such that the asymptotic density $\delta(\eta^{-1}(U_n))$ exists for all $n \in N$, and such that η is uniformly distributed relative to each U_n.

One can also arrange that η consists of distinct points (that is, is one-to-one) all of whose co-ordinates are irrational.

The proof of the Main Lemma requires several preliminary lemmas, of which we mention one as of some intrinsic interest:

Sublemma Suppose $U \subset I^d$ is open relative to I^d, and ξ is uniformly distributed in I^d, then there is a set M of positive integers such that (i) $\xi(M) \subset U$, (ii) $\delta(M) = \lambda(U)$, (iii) the distance from ξ_n to $I^d \setminus U$ tends to 0 when $n \to \infty$ in M.

It is natural to look for characterizations of Lebesgue measurability and summability analogous to Theorem 1. We consider only the simpler case $S = I^d$. First, some notation: $\chi(E)$ denotes the characteristic function of a set E; its value at a point x is written $\chi(E, x)$ (= 1 if $x \in E$, 0 otherwise).

Theorem 2 Let f be a real-valued function on I^d. The following statements are equivalent:

(1) f is Lebesgue measurable.

(2) For each uniformly distributed sequence ξ in I^d, there is an equivalent sequence $\eta \sim \xi$ in I^d, and an increasing sequence $M_1 \supset M_2 \supset \ldots$ of sets of positive integers such that

(a) the upper asymptotic density $\delta^*(M_n) \to 1$ as $n \to \infty$,

(b) the restriction of f to the closure $Cl(\eta(M_n))$ is continuous for each $n \in N$.

(2') Same as (2), except that (a) is replaced by the stronger condition

(a') $\delta(M_n)$ exists and $\to 1$ as $n \to \infty$; and $U\{M_n : n \in N\} = N$.

(3) For each uniformly distributed sequence ξ in I^d, there exist: a sequence $\eta \sim \xi$ in I^d, and a partition of N into subsets M_n $(n \in N)$ such that

(i) the sets $\eta(M_n)$, $n \in N$, have pairwise disjoint closures;

(ii) for each n, $\delta(M_n)$ exists and equals $\lambda(Cl(\eta(M_n)))$

(iii) $\Sigma\{\delta(M_n) : n \in N\} = 1$; and

(iv) If $\zeta \sim \eta$ and the closures $Cl(\zeta(M_n))$, $Cl(\eta(M_n))$ are equal $(n \in N)$, then, for each n,

$$\lim_{k\to\infty 1} f(\zeta(k))\chi(M_n, k) \text{ exists and is finite.}$$

(3') Same as (3), except that (iv) is replaced by the stronger condition

(iv') If ζ is uniformly distributed in I^d and $\delta(\zeta^{-1}(Cl(\eta(M_n))))$ exists and equals $\lambda(Cl(\eta(M_n)))$ for some $n \in N$, then (for that n) $\lim_{k\to\infty 1} f(\zeta(k))\chi(M_n, k)$ exists and is finite.

That these conditions follow from the Lebesgue measurability of f, comes essentially from translating the Lusin criterion for measurability, using the Main Lemma to manufacture the appropriate η. In the converse direction, we show from (3) that f is Riemann integrable on each set $Cl(\eta(M_n))$, and then apply Theorem 1.

Now suppose $f: I^d \to R$ is Lebesgue measurable; what further conditions will make f summable (that is, in L^1)? It is sufficient, and convenient, to consider f^+, f^- separately; thus we assume $f \geq 0$. The following is then a straightforward deduction from Theorem 2.

Theorem 3 Suppose f is a non-negative real-valued function on I^d. For f to be summable, it is necessary and sufficient that f satisfies condition (3) above with the extra requirement:

(v) In (iv), $\lim_{n\to\infty} \lim_{k\to\infty} f(\gamma(k))\chi(M_n, k)$ exists and is finite. Further, this limit is then $\int_{I^d} f\, d\lambda$.

In a similar way we can characterize the bounded measurable real functions f on I^d: it is necessary and sufficient that the five conditions (3) (i) - (v) hold and in addition

(vi) $\lim_{k\to\infty} f(\gamma(k))$ exists and equals the limit in (v).

3. Integrals and truncation sequences It is a well-known result (and an easy consequence of Fubini's theorem) that, if m is a countably additive non-negative σ-finite measure on a measure space X, and if f is a non-negative real-valued measurable function on X, then $\int_X f\, dm = \int_0^\infty \lambda(f^{-1}[\alpha,\infty))d\alpha$. This has an extension to finitely additive measures, in terms of a familiar generalization of Riemann integration, as follows. Let $\mathcal{E}$ be a (finitely additive) field of subsets of a set X, and let m be a finitely additive measure, non-negative and finite, defined on $\mathcal{E}$. Suppose f is a bounded real-valued function on X. For each finite partition τ of X into (pairwise disjoint) sets $E_1, E_2, \dots, E_n \in \mathcal{E}$, put $S^*(f,\tau) = \Sigma\{\sup(f|E_i)m(E_i): 1 \leq i \leq n\}$, $S_*(f,\tau) = \Sigma\{\inf(f|E_i)m(E_i): 1 \leq i \leq n\}$. We say f is Riemann integrable providing $S^*(f,\tau)$ and $S_*(f,\tau)$ have equal limits under refinement, as τ varies; the common value of these limits is then, by definition, the Riemann integral $\int_X f\, dm$. Of course, this process includes as a special case the notion of Riemann integral used in Theorem 1 (§2), and it can be extended further (to include "improper Riemann integrals"), though we shall not be concerned with this.

Let $\mathcal{R}(\mathcal{E},m)$ denote the family of all Riemann integrable functions on X; this is a (real) linear space (in fact an algebra). Let $\mathcal{J} = \mathcal{J}(\mathcal{E},m)$ denote the family of all "Jordan measurable" subsets of X, consisting of all $J \subset X$ such that $\chi(J) \in \mathcal{R}(\mathcal{E},m)$. The measure m extends to a finitely additive measure (also denoted by m) on $\mathcal{J}$, by defining $m(J) = \int_X \chi(J)\, dm$.

One easily sees that this extension leads to the same Riemann integral, so that $\mathcal{R}(\mathcal{J},m) = \mathcal{R}(\mathcal{E},m)$. A theorem, due essentially to Ridder and Frink (see [5] and [1]), can be stated as follows:

Theorem 4 With the set-up described above, the following statements about a bounded real-valued function f are equivalent:

(1) $f \in \mathcal{R}(\mathcal{E},m)$,

(2) $f^{-1}[\alpha,\infty)$ is Jordan measurable, for all but (at most) a countable set of values of $\alpha \in R$.

(3) $f^{-1}[\alpha,\infty)$ is Jordan measurable for a dense set of values of $\alpha \in R$.

An addendum to this theorem (which should surely be known, but which I have not seen in the literature) then asserts:

If further $f \geq 0$, and $f \in \mathcal{R}(\mathcal{E},m)$, then $\int_X f\, dm = \int_0^\infty \lambda(f^{-1}[\alpha,\infty))d\alpha$.

Remark The exceptional set of α's, in Theorem 3(2), can be infinite, even when (X,m) is the unit interval with Lebesgue measure and f is continuous.

Our object in this section is an analog (Theorem 6) of Theorem 4 for sequences. Throughout what follows, ξ denotes a sequence of non-negative real numbers. We use the following notation. For each $c > 0$, ξ^c is the sequence $\min(\xi,c)$; that is, $\xi^c(n) = \min(\xi(n),c)$, for all $n \in N$. For each $M \subset N$, ξ_M denotes the sequence $\xi\chi(M)$; that is, $\xi_M(n) = \xi(n)$ if $n \in M$, 0 otherwise. For $a > 0$, put $M(a) = \{n \in N : \xi(n) \geq a\}$. We abbreviate $\xi_{M(a)}$ to ξ_a; thus $\xi_a(n) = 0$ if $\xi(n) < a$, and $\xi_a(n) = \xi(n)$ otherwise.

As a preparation for formulating the sesired analog, we state:

Theorem 5 The following assertions (about a sequence ξ of non-negative reals) are equivalent.

(1) For each $c > 0$, $\lim_1 \xi^c$ exists.

(2) For a dense set of values of c in $(0,\infty)$, $\lim_1 \xi^c$ exists.

(3) For a dense set of values of t in $(0,\infty)$, the asymptotic density $\delta(\xi^{-1}[t,\infty))$ exists.

(4) For all but countably many values of t in $(0,\infty)$, $\delta(\xi^{-1}[t,\infty))$ exists.

The proof goes by showing (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2) $\Rightarrow$ (1); of course, (4) $\Rightarrow$ (3) is trivial.

We shall say that a sequence ξ satisfying the conditions of Theorem 5 is a truncation sequence. Note that, for example,

every uniformly distributed sequence in $[0,1]$ is a truncation sequence; on the other hand, if M is a subset of N that does not have an asymptotic density, then $\chi(M)$ is not a truncation sequence.

The following further assertions are also equivalent to those in Theorem 5:

(5) For all $a > 0$, ξ_a is a truncation sequence.

(6) For some sequence $\langle a(n) : n \in N \rangle$ of positive numbers with limit 0, each $\xi_{a(n)}$ is a truncation sequence.

(One shows (6) $\Rightarrow$ (1) $\Rightarrow$ (5).)

The analog is now:

Theorem 6 Let ξ be a truncation sequence. Then both $\lim_{c\to\infty} \lim_1(\xi^c)$ and $\int_0^\infty \delta(\xi^{-1}[t,\infty))dt$ exist (perhaps infinite), and they are equal.

The proof proceeds in stages; first when ξ takes only finitely many different values, then when ξ is merely bounded, and finally the general case.

Remark If further ξ is a bounded truncation sequence, Theorem 6 gives $\int_0^\infty \delta(\xi^{-1}[t,\infty))dt = \lim_1 \xi$; but this need not hold for unbounded truncation sequences; the two expressions can be different, even when both are finite. (For example, let $\xi(n) = 0$ if n is not a square, and let $\xi(k^2) = k$.)

The resemblance between Theorems 4 and 6 is not merely superficial. Let us consider the following measure space in Theorem 4 : $X = N$, $\mathcal{E} = \mathcal{P}(N)$, the family of all subsets of N, with a finitely additive measure $\tilde{\delta}$ obtained in the following way. Let $\mathcal{B}$ denote the family of all bounded real sequences η (negative values are allowed here, of course); let $\mathcal{L}$ consist of all $\eta \in \mathcal{B}$ for which $\lim_1 \eta$ exists (necessarily finite). Then $\lim_1$ is a (finitely additive) linear functional on $\mathcal{L}$, and extends (Hahn-Banach style) to a positivity- preserving linear functional $\tilde{\ell} : \mathcal{B} \to R$. Define $\tilde{\delta}$ by $\tilde{\delta}(E) = \tilde{\ell}(\chi(E))$, for all $E \subset N$. (Thus $\tilde{\delta}$ extends δ.)

One sees that each $\eta \in \mathcal{B}$ is Riemann integrable (with respect to $\tilde{\delta}$), in the sense of §2, with integral $\tilde{\ell}(\eta)$. The family $\mathcal{J}$ of Jordan measurable sets is here $\mathcal{P}(N)$. And the addendum to Theorem 4 gives $\tilde{\ell}(\eta) = \int_0^\infty \tilde{\delta}(\eta^{-1}[t,\infty))dt$.

Suppose now that ξ is a bounded truncation sequence. Then $\tilde{\ell}(\xi) = \lim_1 \xi$, and (from condition (4) of Theorem 5) $\tilde{\delta}(\xi^{-1}[t,\infty)) = \delta(\xi^{-1}[t,\infty))$ for all but countably many values of t. Thus $\lim_1 \xi = \int_0^\infty \delta(\xi^{-1}[t,\infty))dt$, which is precisely

Theorem 6 for bounded truncation sequences. In other words, Theorem 6 constitutes an extension of Theorem 4, in the present special case, to unbounded functions.

We remark that (from Theorems 6 and 5(3)) the (non-negative) sequence ξ is a truncation sequence if and only if $\int_0^\infty \delta(\xi^{-1}[t,\infty))dt$ exists (as a Lebesgue or improper Riemann integral), perhaps infinite. It can also be shown that the non-negative sequence ξ is a truncation sequence if, and only if, $\tilde{\ell}(\xi^c)$ is, for each $c > 0$, independent of the choice of the the extension $\tilde{\ell}$ of $\lim_1$, in the foregoing construction.

4. Stable sequences A sequence ξ (of non-negative real numbers is said to be stable provided, for every subset M of N with $\delta(M) = 1$, ξ and ξ_M ($= \xi\chi(M)$) have the same (C,1) lim sup and the same (C,1) lim inf. All bounded sequences ξ are stable, and so are the sequences arising from the Birkhoff ergodic theorem (almost everywhere; see [2] for instance). It is easy to give examples of non-stable sequences; for instance, let $\xi(n) = 0$ unless $n = 2^k$, and $\xi(2^k) = 2^k$.

The following lemma (and generalizations of it) will be useful in what follows.

Truncation Lemma Given a truncation sequence ξ, there is a subset M of N with $\delta(M) = 1$, such that ξ_M is a stable truncation sequence, and moreover $\lim_1(\xi_M)$ exists (perhaps infinite) and equals both $\lim_{c\to\infty} \lim_1 \xi^c$ and $\lim_{c\to\infty} \lim_1(\xi_M)^c$.

In view of this lemma, Theorem 6 now gives:

Corollary Let ξ be a stable truncation sequence. Then both $\lim_1 \xi$ and $\int_0^\infty \delta(\xi^{-1}[t,\infty))dt$ exist (possibly infinite), and they are equal.

In applications (such as in the next section) the sequences involved will often be zero except on subsets of N of asymptotic density 0, so we need notions of "relative asymptotic density" and "relative truncation sequence". Suppose E is an infinite subset of N. We define the "relative (C,1)limit", $\lim_1(\xi/E)$, to mean $\lim_{n\to\infty} \left(\sum_{j=1}^n \xi(j)\right) / \sum_{j=1}^n \chi(E,j)$ (assuming, of course, that this limit exists, possibly infinite), and the "relative asymptotic density" $\delta(F/E)$, where also $F \subset N$, to mean $\lim_1(\chi(F)/E)$.

The corresponding notion of "relative truncation sequence" is provided by the following theorem, patterned after Theorem 5.

Theorem 7 The following statements about a sequence ξ (of non-

negative numbers) and an infinite subset E of N are equivalent.

(1) For all but countably many positive real numbers a, $\lim_1((\xi_a)^c/E)$ exists and is finite for all $c > 0$.

(2) The conclusion of (1) holds for a dense set of values of a in $(0,\infty)$.

(3) For a dense set of values of t in $(0,\infty)$, the relative asymptotic density $\delta(\xi^{-1}[t,\infty)/E))$ exists and is finite.

(4) The conclusion of (3) holds for all but countably many values of t in $(0,\infty)$.

When these equivalent conditions hold, we say that ξ is a "truncation sequence relative to E". Note that the condition (2) of Theorem 7, unlike the corresponding condition (2) of Theorem 5, allows no exceptional values of c; such exceptions could probably be allowed, but I have not checked this.

The corresponding "relativised" version of Theorem 6 is now:

Theorem 8 Let ξ be a truncation sequence relative to E. Then

$$\int_0^\infty \delta(\xi^{-1}[t,\infty)/E)dt = \lim_{a\to 0+}\ \lim_{c\to\infty}\ \lim_{n\to\infty} \frac{\Sigma\{\xi^c(j):\ 1\le j\le n,\ \xi(j)\ge a\}}{|\{j:\ 1\le j\le n,\ j\in E\}|}$$

(where the notation $|F|$ means the number of elements of the set F). Here both sides may be infinite.

Note that the double limit implied by $a\to 0+$, $c\to\infty$, is just the least upper bound (for a and c positive).

The sequence ξ is said to be stable relative to E, an infinite subset of N, provided that, for all $F\subset N$ with $\delta(F/E) = 0$, we have $\overline{\lim}_1(\xi/E) = \overline{\lim}_1(\xi_{N\setminus F}/E)$ and $\underline{\lim}_1(\xi/E) = \underline{\lim}_1(\xi_{N\setminus F}/E)$.

Two further definitions are needed. First, ξ is said to be a "special truncation sequence" relative to E if it is a truncation sequence relative to E (satisfying the conditions of Theorem 7) with the further property: there exists some positive $\beta\in D$, the co-countable set of values of t to which Theorem 7(4) applies, such that for all (positive) $\alpha\in D$, $\delta(\xi^{-1}[\alpha,\beta)/\xi^{-1}[\beta,\infty))$ is finite. The least upper bound of the set of all such numbers β is denoted by $\beta^*(\xi)$; it may be infinite. It is easy to see that if $\gamma\in D\cap(0,\beta^*(\xi))$ then $0 < \delta(\xi^{-1}[\gamma,\infty)/E) < \infty$, so that the expression $\delta(\xi^{-1}[\alpha,\infty)/\xi^{-1}[\beta,\infty))$ used here makes sense.

A special truncation sequence ξ relative to E is called strongly stable relative to E provided there is some $\beta\in D\cap(0,\beta^*(\xi))$, for which the sequence ξ_β is stable relative to E. It follows easily that ξ_β is then stable relative to E for all $\beta\in D\cap(0,\beta^*(\xi))$.

For such sequences, the conclusion of Theorem 8 can be simplified; that is:

Theorem 9 Let ξ be a strongly stable special truncation sequence relative to E. Then

$$\int_0^\infty \delta(\xi^{-1}[t,\infty)/E)dt = \lim_{a\to 0}\ \lim_{n\to\infty} \frac{\Sigma\{\xi(j):\ 1\le j\le n,\ \xi(j)\ge a\}}{|\{j:1\le j\le n,\ j\in E\}|}$$

The proof uses a relativised truncation lemma, generalized so as to be applicable to all the relevant sequences ξ_a simultaneously.

5. An application Let m be a probability measure (hence finite, non-negative and countably additive) on a measure space X, and suppose T is a one-to-one transformation of X onto itself for which both T and T^{-1} are measurable and nonsingular. For each $n\in N$ let ω_n denote the Radon-Nikodym derivative of T^n, defined (almost everywhere) by

$$m(T^nA) = \int_A \omega_n(y)\ dm(y)$$

for all measurable $A\subset X$. There is no real loss of generality in assuming that T is incompressible, because the behavior of T on the (maximal invariant) set where T is compressible is well understood. So, for simplicity, we assume T incompressible.

We denote the sequence $\langle\omega_1(x),\ \omega_2(x),\ \ldots\rangle$ by $\Omega(x)$, and the set $\{n\in N:\ \omega_n(x)\ge\alpha\}$, where $\alpha>0$, by $E_\alpha(x)$. Then, after omitting a suitable fixed invariant null set from X, we have:

Theorem 10 For all $x\in X$ and $\alpha>0$, the sequence $\Omega(x)$ is a strongly stable special truncation sequence relative to $E_\alpha(x)$, with $\beta^*(\Omega(x)) = \limsup_{n\to\infty}\omega_n(x)$.

In particular, writing $\limsup_{n\to\infty}\omega_n(x) = \omega^*(x)$, for short, we have that $\delta(E_\alpha(x)/E_\beta(x))$ is positive and finite for all α,β in $(0,\beta^*(x))$; it is, of course, infinite if $0<\alpha<\omega^*(x)<\beta$, and 0 if $0<\beta<\omega^*(x)<\alpha$.

We write $I_\beta(x)$ for $\int_0^\infty \delta(E_\alpha(x)/E_\beta(x))d\alpha$ (possibly infinite) and have, from Theorem 9,

Corollary If $0<\beta<\omega^*(x)$, then

$$I_\beta(x) = \lim_{a\to 0+}\ \lim_{n\to\infty} \frac{\Sigma\{\omega_j(x):1\le j\le n,\ \omega_j(x)\ge a\}}{|\{j:1\le j\le n,\ \omega_j(x)\ge\beta\}|}$$

It should be remarked that (when $0<\beta<\omega^*(x)$) $I_\beta(x)$ is finite

precisely on the largest subset of X (modulo null sets) admitting a σ-finite measure, invariant under T, and equivalent to m (see [2]). Further, the invariant measure in question is obtainable from $\int (1/I_\beta(x))d\beta$; see [3] . Thus this Corollary is of some interest, since it may give a grip on $I_\beta(x)$, which is rather elusive in practice.

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SOME REMARKS ON SEQUENCES HAVING A CORRELATION*

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We denote by $\mathbf{C}$ the set of all complex numbers and by $\mathbf{C}_1$ the unit circle

$$\mathbf{C}_1 = \{z \in \mathbf{C} \mid |z| = 1\} .$$

We now consider the space $\mathbf{S}$ of complex sequences $\underline{a} = (a_k)$ such that

$$\gamma_{\underline{a}}(k) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_{j+k}\bar{a}_j$$

exists for each integer $k \in \mathbf{N}$. This space was introduced by N. Wiener in [3], Chap. IV. The sequence $\gamma_{\underline{a}} = (\gamma_{\underline{a}}(k))$ is called the correlation of $\underline{a}$. It can be extended to the negative integers by setting

$$\gamma_{\underline{a}}(-k) = \overline{\gamma_{\underline{a}}(k)} .$$

It is well known that $\gamma_{\underline{a}}$ is positive definite on $\mathbf{Z}$ and thus by the Herglotz-Bochner theorem we have the representation

$$\gamma_{\underline{a}}(n) = \int_{\mathbf{C}_1} z^n d\lambda_{\underline{a}}(z) \quad , \quad n \in \mathbf{Z}$$

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where the positive Borel measure $\lambda_{\underline{a}}$ is uniquely determined by $\underline{a}$; the measure $\lambda_{\underline{a}}$ is called the spectral measure corresponding to the sequence $\underline{a}$.

We first construct a class of sequences having as spectral measure, Lebesgue measure on $\mathbb{C}_1$.

Lemma. Let (x_n) be a sequence of real numbers such that for every $h \in \mathbb{N}$, $h > 0$,

(i) The sequence $(x_{n+h}-x_n)$ is uniformly distributed (mod 1). Let

$$a_n = e^{2\pi i x_n} , \quad n \in \mathbb{N} .$$

Then the sequence $\underline{a} = (a_n)$ belongs to $\mathcal{S}$ and its spectral measure $\lambda_{\underline{a}}$ = Lebesgue measure on $\mathbb{C}_1$.

Proof: Let $k \in \mathbb{N}$, $k > 0$. Then, by H. Weyl's criterion for uniform distribution [2],

$$\frac{1}{n} \sum_{j=0}^{n-1} a_{j+k}\bar{a}_j = \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i(x_{j+k}-x_j)} \longrightarrow 0 .$$

For $k = 0$,

$$\frac{1}{n} \sum_{j=0}^{n-1} a_j\bar{a}_j = \frac{1}{n} \sum_{j=0}^{n-1} |a_j|^2 = 1 \longrightarrow 1 .$$

Thus $\underline{a} = (a_n)$ has a correlation and $\gamma_{\underline{a}}(k) = 0$ if $k \in \mathbb{N}$, $k > 0$. $\gamma_{\underline{a}}(0) = 1$; hence $\lambda_{\underline{a}}$ = Lebesgue measure on $\mathbb{C}_1$.

Examples. 1) Let $P(x) = \alpha_0 x^q + \alpha_1 x^{q-1} + \ldots + \alpha_q$, $q \geq 2$ be a polynomial with real coefficients and α_0 irrational. Then the sequence

$$(x_n) = (P(n))$$

satisfies assumption (i) of the previous lemma.

In fact, fix $h \in \mathbb{N}$, $h > 0$. Then

$$P_h(x) = P(x+h) - P(x)$$

is a polynomial of degree $q - 1$ and the coefficient of x^{q-1} is $\alpha_0 \binom{q}{1} h$, hence irrational. Thus by a classical theorem of Weyl [2],

$$(P_h(n)) = (P(n+h)-P(n)) \text{ is u.d. (mod 1) .}$$

2) Let (n_k) be a strictly increasing sequence of positive integers such that for each $h \in \mathbb{N}$, $h > 0$, $p_h(k) = n_{k+h} - n_k$ is strictly increasing in k . Then by a classical theorem in the theory of uniform distribution [2], for (Lebesgue) a.e. $x \in (0,1)$ and for each $h \in \mathbb{N}$, $h > 0$,

$$(p_h(k)x) = (n_{k+h}x-n_kx) \text{ is u.d. (mod 1) .}$$

Fix such an irrational x for example; then

$$(x_k) = (n_kx)$$

satisfies assumption (i) of the previous lemma.

As concrete examples, one may take for instance

2.1) (m_k) defined by $m_0 = 1$ and $m_k = 2^k + [\log_2 k]$ for $k \geq 1$ (here $[y]$ denotes of course the integral part of y).

2.2) $n_k = 2^k$ for $k \in \mathbb{N}$.

Theorem. There are sequences $\underline{a} \in \mathcal{S}$, $\underline{b} \in \mathcal{S}$ with $\lambda_{\underline{a}} = \lambda_{\underline{b}}$ = Lebesgue measure on $\mathbb{C}_1$, such that the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} a_j\bar{b}_j$$

does not converge to a limit as $n \to \infty$.

Proof: Let (m_k) and (n_k) be as in Example 2.1), 2.2). Fix now an irrational $\theta \in (0,1)$ such that both sequences $(m_k\theta)$ and $(n_k\theta)$ satisfy assumption (i) of the previous lemma. Let $\underline{a} = (a_k)$, $\underline{b} = (b_k)$ be given by

$$a_k = e^{2\pi i\theta m_k}, \quad b_k = e^{2\pi i\theta n_k}.$$

Then

$$\frac{1}{n}\sum_{j=0}^{n-1} a_j\bar{b}_j = \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} e^{2\pi i\theta[\log_2 j]}\right) = \frac{1}{n} + \frac{1}{n}\sum_{j=1}^{n-1} e^{2\pi i\theta[\log_2 j]}.$$

Now $[\log_2 k]$ is constant on the successive blocks $[2^j, 2^{j+1}-1]$ of length 2^j. Thus we have

$$T_p = \frac{1}{2^p}\sum_{k=0}^{2^p-1} a_k\bar{b}_k = \frac{1}{2^p} + \frac{1}{2^p}(1+2e^{2\pi i\theta}+2^2e^{2\pi i2\theta}+\ldots+2^{(p-1)}e^{2\pi i(p-1)\theta})$$

and hence

$$|T_n - T_{n+1}| = \left|\frac{1}{2}T_n - \frac{e^{2\pi in\theta}}{2}\right| \tag{1}$$

Suppose now that $\frac{1}{n}\sum_{j=0}^{n-1} a_j\bar{b}_j$ tends to a limit c, say. Then

$$T_p \longrightarrow c \quad \text{as} \quad p \longrightarrow \infty \tag{2}$$

and

$$|T_n - T_{n+1}| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3}$$

But in formula (1), as $n \to \infty$,

$$\frac{1}{2}T_n \longrightarrow \frac{c}{2},$$

This would imply that $\frac{e^{2\pi in\theta}}{2}$ must converge to $\frac{c}{2}$; but since $(n\theta)$ is u.d. (mod 1), hence dense in $[0,1)$, this is impossible. Thus we reached a contradiction.

Remarks. 1) Note that for sequences $\underline{a}$ and $\underline{b}$ in the proof of the theorem we have

$$\underline{a} + \underline{b} \notin \mathcal{S},$$

since the correlation cannot exist. In fact

$$\frac{1}{n}\sum_{j=0}^{n-1}(a_j+b_j)(\bar{a}_j+\bar{b}_j) = \frac{1}{n}\sum_{j=0}^{n-1}|a_j|^2 + \frac{1}{n}\sum_{j=0}^{n-1}|b_j|^2 + \frac{1}{n}\sum_{j=0}^{n-1}(a_j\bar{b}_j+b_j\bar{a}_j)$$

$$= 1 + 1 + 2\mathrm{Re}\left(\frac{1}{n}\sum_{j=0}^{n-1} a_j\bar{b}_j\right)$$

and the last term cannot converge to a limit (the argument is entirely similar to that in the proof of the theorem: take real parts in formula (1)).

2) It was shown in [1] that if $\underline{a} = (a_n)$ is a bounded complex sequence having a mean, a correlation and such that its spectral measure $\lambda_{\underline{a}}$ is absolutely continuous with respect to Lebesgue measure, then $\underline{a}$ is a good universal weight relative to the Individual Ergodic Theorem. The above theorem shows that such a sequence need not be a "good universal weight" with respect to the Wiener space $\mathcal{S}$ (not even with respect to the class of sequences

in S having for their spectral measure Lebesgue measure). This answers a question raised by B. Weiss.

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CLOSED TILINGS OF EUCLIDEAN SPACES [1]

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Introduction A "tiling" of a topological space X is a covering $\mathcal{T}$ of X by regular closed sets T (that is, each is the closure of its interior) with pairwise disjoint interiors; the sets $T \in \mathcal{T}$ are the "tiles". As usual, the "order" $r(\mathcal{T})$ of $\mathcal{T}$ is the greatest number of members of $\mathcal{T}$ with a common point (or ∞ if there is no greatest number). Analogously, the "density" $\ell(\mathcal{T})$ is defined to be the least upper bound of the number of "neighbours" of T_o as T_o varies in $\mathcal{T}$, where two members T, T' of $\mathcal{T}$ are "neighbours" if $T \neq T'$ and $T \cap T' \neq \emptyset$. (See [1, 6].) In this communication we are concerned only with the case $X = R^n$, n-dimensional Euclidean space. A tiling $\mathcal{T}$ of R^n is said to be convex if its members are convex sets. In a recent letter V. Klee asked the author what can be said about the order $r(\mathcal{T})$ of a convex tiling of R^n by uniformly bounded sets. Here we consider this question and also the analogous question about the density $\ell(\mathcal{T})$.

Locally finite case If the uniformly bounded tiling $\mathcal{T}$ of R^n is locally finite, but not necessarily convex, information is provided by some well known topological theorems. For $\mathcal{T}$ yields a finite closed covering of an arbitrarily large n-cube; a homothetic transformation converts this to an arbitrarily fine finite closed covering of the standard unit cube I^n, which by a theorem of Lebesgue must have order at least $n + 1$ (see for example [2, p.•42]). Thus $r(\mathcal{T}) \geq n + 1$.

This result is "best possible" even when the tiles are convex, as is shown by the standard example (covering I^n by staggered rows of congruent bricks).

The results for the density, $\ell(\mathcal{T})$, are much less precise. It has been conjectured [4] that $\ell(\mathcal{T}) \geq 2^{n+1} - 2$ (the value achieved by the staggered rows of bricks), but this has been

proved only for $n = 1$ (trivial) and $n = 2$ ([5]; see also [1] and [6]). When $n = 3$ it can be shown that $\ell(\mathcal{J}) \geq 7$; even this weak result requires some trouble. For large n, only the trivial estimate $\ell(\mathcal{J}) \geq r(\mathcal{J}) - 1$ seems to be known.

It should be remarked that Klee has proved that the condition of local finiteness can (for tilings) be replaced by the formally weaker (but actually equivalent) condition that the tiling be point-finite and "protective" (that is, each point is interior to its star in the covering), and that for convex tilings of R^n one can even omit "point-finite" here (see [3]).

General case Now let us drop the condition that $\mathcal{J}$ be locally finite. Without affecting the results we may (and do) assume that $\emptyset \notin \mathcal{J}$; thus (from the disjointness of the interiors) $\mathcal{J}$ is necessarily countable. The applicable results from topology are of course much weaker now. First,

$$r(\mathcal{J}) \geq 2$$

(unless $\mathcal{J}$ has only one member). For, as before, $\mathcal{J}$ induces a countable cover of a sufficiently large cube by two or more compact non-empty sets; since the cube is connected, a theorem of Sierpinski (see for example [7, p. 16]) shows that these sets cannot be pairwise disjoint.

It follows that

$$\ell(\mathcal{J}) \geq 2$$

(unless $\mathcal{J}$ has at most two members), because otherwise the different sets $St(T,\mathcal{J})$, $T \in \mathcal{J}$ (where $St(T,\mathcal{J}) = \cup \{ T' \in \mathcal{J} : T' \cap T \neq \emptyset \}$) form a countable closed cover $\mathcal{J}^*$ of R^n by pairwise disjoint sets, contradicting $r(\mathcal{J}^*) \geq 2$.

An example Now restrict attention to uniformly bounded, convex tilings of R^n. The surprising thing is that the foregoing crude estimates remain sharp. We prove this by exhibiting a uniformly bounded convex tiling $\mathcal{J}$ of R^n for which $r(\mathcal{J}) = 2 = \ell(\mathcal{J})$.

It is convenient to introduce some terminology and notation. A "closed rectangular box", or simply "box" for short, is a set of the form $I_1 \times I_2 \times \cdots \times I_n$, where each I_i is a closed interval $[a_i, b_i]$ with $a_i < b_i$ $(i = 1, 2, \ldots, n)$. If one particular I_j is replaced here by a half-open interval $J_j = (a_j, b_j]$, we call the resulting product set $J_1 \times J_2 \times \cdots \times J_n$, where $J_i = I_i$ if $i \neq j$, a "lidless box".

The construction depends on two facts. Fact 1: Each lidless box is the union of a sequence $\mathcal{S}$ of boxes, in such a way that $r(\mathcal{S}) = 2 = \ell(\mathcal{S})$.

To see this, suppose the lidless box is $J_1 \times J_2 \times \cdots \times J_n$ where $J_j = (a_j, b_j]$ and $J_i = [a_i, b_i]$ if $i \neq j$ (and it is always understood that $a_1 < b_1$, $a_2 < b_2$, ..., $a_n < b_n$). Take a sequence of numbers $c_1, c_2, \ldots$, converging to a_j, with $b_j > c_1 > c_2 > \cdots > a_j$, and let $\mathcal{S}$ consist of the boxes S_k $(k = 1, 2, \ldots)$ where $S_k = I_{k1} \times I_{k2} \times \cdots \times I_{kn}$ where $I_{kj} = [c_k, b_j]$ and $I_{ki} = [a_i, b_i] = J_i$ if $i \neq j$.

Fact 2: the half-open unit cube $J^n = [0, 1) \times [0, 1) \times \cdots \times [0, 1)$ is expressible as the union of a pairwise disjoint system of boxes, all but one of them lidless.

Proof: Fix a sequence $d_1, d_2, \ldots$ converging to 1, with $0 < d_1 < d_2 < \cdots < d_k < \cdots < 1$. Consider the set

$$[0, d_2]^n - [0, d_1]^n$$

(where $[a, b]^n$ is the product of n intervals $[a, b]$). This is easily seen to be the union of the n pairwise disjoint lidless boxes T_{1j} $(1 \leq j \leq n)$ where $T_{1j} = A_{1j} \times A_{2j} \times \cdots \times A_{nj}$ and $A_{ij} = [0, d_1]$ if $i < j$, $A_{ij} = [0, d_2]$ if $i > j$, and $A_{jj} = (d_1, d_2]$. Put $\mathcal{J}_1 = \{T_{1j} : 1 \leq j \leq n\}$.

In exactly the same way, we obtain (for each $k = 2, 3, \ldots$) a family $\mathcal{J}_k = \{T_{kj} : 1 \leq j \leq n\}$ of pairwise disjoint lidless boxes with union $[0, d_{k+1}]^n - [0, d_k]^n$. Then $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \cdots \cup \mathcal{J}_k$ forms a pairwise disjoint covering of $[0, d_{k+1}]^n - [0, d_1]^n$, and so $\cup \{\mathcal{J}_k : k = 1, 2, \ldots\}$ is a pairwise disjoint family of lidless boxes with union $[0, 1)^n - [0, d_1]^n$.

Because of Fact 1, each $T_{ij} \in \mathcal{J}_i$ is the union of a sequence $\mathcal{S}_{ij}$ of boxes B_{ijk}, with $r(\mathcal{S}_{ij}) = 2 = \ell(\mathcal{S}_{ij})$. Adjoining the box $[0, d_1]^n$ to these boxes B_{ijk}, we obtain a covering $\mathcal{B}$ of $[0, 1)^n$ by boxes, for which $r(\mathcal{B}) = 2 = \ell(\mathcal{B})$.

Finally, the translates of $\mathcal{B}$ by the integer lattice (of all points of R^n with integer coordinates) constitute a covering $\mathcal{J}$ of R^n by uniformly bounded boxes, with $r(\mathcal{J}) = 2 = \ell(\mathcal{J})$, as promised.

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ON UNIFORM MONOTONICITY OF NORMS AND ERGODIC THEOREMS IN FUNCTION SPACES*

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1. Introduction Let E be a Banach Lattice; the elements of E are denoted by f,g,h etc. The norm on a Banach Lattice is monotone in the following sense: $0 \le f \le g$ implies $\|f\| \le \|g\|$. (Complete definitions are given in Section 2). It is, however, often necessary to assume that the norm is strictly monotone, perhaps in some uniform sense. The oldest such assumption seems to be the uniform monotonicity or the UMB condition, introduced by G. Birkhoff ([3], p.371). In recent papers [1],[2] the authors considered other conditions on the norm involving various degrees of monotonicity. These are, in the increasing order stringency, the conditions C_1, C and a uniform form of C. It is shown below that this uniform form of C is in fact equivalent to UMB. The conditions C and C_1 are in general different from UMB and from each other. Another monotonicity condition, called condition δ, was used by Y. Katznelson and L. Tzafriri to prove a "zero-two" theorem ([6], Theorem 6). We observe below that also δ is equivalent with UMB.

A Banach Lattice is said to be order continuous if every decreasing sequence of positive elements converges in norm. A Banach Lattice is said to satisfy the condition B if every norm-bounded increasing sequence converges in norm. (An equivalent assumption is the weak sequential completeness of the Banach Lattice).

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An important class of Banach Lattices consists of Orlicz function spaces $L_M = L_M(X,F,\mu)$ where M is an Orlicz function and (X,F,μ) is a non-atomic σ-finite measure space. We prove that in an Orlicz function space the conditions UMB, C, C_1, order continuity and B are all equivalent and that they are also equivalent with the well known Δ_2 condition, which will be stated in Section 3 in a slightly non standard form.

In addition to Orlicz spaces L_M we also consider H_M spaces, the closure in the L_M norm of integrable simple functions. We will observe that H_M is always order continuous but it satisfies the condition B if and only if $H_M = L_M$ or, which will be shown to be equivalent, if and only if Δ_2 holds. If $M(x) = x(\log^+ x)^k$, $k = 1,2,\ldots$, then H_M is the R_k space, a natural setting for the multiparameter Dunford-Schwartz theorem, as shown by N. Fava [4] (see also [5]).

In [1] and [2] we have considered Banach Lattices E satisfying the conditions B and C and also the following condition A :

(A) E has a weak unit, i.e. an element $u \in E^+$ such that $|f| \wedge u = 0$ implies $f = 0$.

We have proved ([1],[2]) that under A, B and C, the Cesaro averages $(T + \ldots + T^n)/n = A_n$ of iterates of a positive contraction operator T, applied to $f \in E$, have "truncated limits" (This is usually equivalent with convergence of $A_n f$ in measure on sets of finite measure). Since separable Banach Lattices have a weak unit and since the condition Δ_2 implies separability, all the three conditions A, B, C of our theorem hold in Orlicz spaces with the Δ_2 condition. This provides a large class of non-reflexive Banach Lattices in which the theorem is valid. (In reflexive Banach Lattices this result is an immediate consequence of the mean ergodic theorem). In the spaces H_M, if they are different from L_M, our theorem is not applicable, because the condition B fails. In this case also Birkhoff's condition UMB (which implies B) fails in H_M.

2. The L_M and H_M Spaces Let E be a Banach Lattice, i.e. a real Banach Space with a partial order $\leq$ satisfying the following conditions

(i) $f \leq g$ implies $f+h \leq g+h$,

(ii) $0 \leq \alpha f$ whenever $0 \leq f$ and α is a non negative real number,

(iii) for every f, g there is a least upper bound $f \vee g$ and a greatest lower bound $f \wedge g$; the absolute value of f is $|f| = f \vee (-f)$. If $f_n \leq g$ then f_n's have a least upper bound.

(iv) if $|f| \leq |g|$ then $\|f\| \leq \|g\|$.

The set (cone) of positive elements f $(0 \leq f)$ of E is denoted by E^+. We define E^{++} as the set of all non zero elements in E^+. If f_n is an increasing sequence and if $f = \sup f_n$, the least upper bound of f_n's, then we write $f_n \uparrow f$. Similarly one defines $f_n \downarrow f$. This is convergence for order; it need not be monotone, because there are obvious definitions of $\limsup f_n$ and $\liminf f_n$ and if these two elements exist and are equal to each other the common value is called the limit for order of f_n. The Banach Lattices we consider are in fact function spaces over a measure space, with the usual partial ordering; in this case convergence for order corresponds to the almost everywhere convergence (Many Banach Lattices admit a representation as function spaces; see [8], p.25).

Let (X,F,μ) be a σ-finite non-atomic measure space. Let M be an Orlicz function, i.e. a continuous convex non decreasing function in $[0,\infty)$, satisfying $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. It will be convenient to write $\underline{M}(f)$ for the integral $\int M(f)d\mu$. The Orlicz space L_M is the class of all (equivalence classes) of functions f such that $\underline{M}(f/\rho) \leq 1$ for some $\rho > 0$; the infimum of all such ρ is the norm $\|f\|$ $(= \|f\|_M)$ of f in L_M. (Equivalently, $f \in L_M$ if and only if $\underline{M}(f/\rho) < \infty$ for some $\rho > 0$). The space H_M consists of all f such that $\underline{M}(f/\rho) < \infty$ for all $\rho > 0$. Clearly, H_M is a linear subspace of L_M. For the sake of completeness we show the following known fact.

2.1 Lemma H_M is the closure in the L_M norm of all simple integrable functions.

Proof These functions are clearly in H_M. We show that H_M is closed. Let $f_n \in H_M$, $\|f_n - f\| \to 0$. Let $\rho > 0$. We show that $\underline{M}(f/\rho) < \infty$. Choose n such that $\|f_n - f\| < \rho/2$. This means that $\underline{M}(2(f_n - f)/\rho) < \infty$. Now $|f| \leq |f - f_n| + |f_n|$ implies, since M is non decreasing and convex, $2M(|f|/\rho) \leq 2M((|f-f_n|/\rho) + (|f_n|/\rho))$ $\leq M(2|f_n - f|/\rho) + M(2|f_n|/\rho)$. Hence $2\underline{M}(f/\rho) \leq \underline{M}(2(f-f_n)/\rho)$ $+ \underline{M}(f_n/\rho) < \infty$. It remains to show that integrable simple functions are dense in H_M. It suffices to consider elements of H_M^+. Let $f \in H_M^+$. Let f_n be a sequence of integrable simple functions such that $f_n \uparrow f$ a.e. Hence $M((f-f_n)/\rho) \downarrow 0$ a.e. for each $\rho > 0$. Since $\underline{M}((f-f_n)/\rho) < \infty$, the monotone convergence theorem implies

that $\underline{M}((f-f_n)/\rho) \downarrow 0$ as $n \to \infty$. Hence $\underline{M}((f-f_n)/\rho) \le 1$ and consequently $\|f-f_n\| \le \rho$ for all sufficiently large n. It follows that $\|f-f_n\| \to 0$ as $n \to \infty$. □

The last argument can be applied to an arbitrary sequence $g_n \downarrow 0$, instead of $f - f_n$, and gives the following lemma.

2.2 Lemma The space H_M has an order continuous norm; any decreasing non negative sequence in H_M converges in norm.

3. The Δ_2 Condition We state this condition in the following form.

(Δ_2) There are (finite) numbers k and ℓ such that $M(2t) \le kM(t)$ whenever $\ell \le t\mu(X)$.

Note that if $\mu(X) < \infty$ then Δ_2 is equivalent to the condition that $\lim \sup_{t\to\infty}(M(2t)/M(t)) < \infty$. This condition is sometimes referred to as the "Δ_2 condition at infinity". If $\mu(X) = \infty$ then Δ_2 is equivalent to the condition that $\sup_{t>0}(M(2t)/M(t)) < \infty$, which can also be expressed as the condition that $\lim \sup(M(2t)/M(t)) < \infty$ both as $t \to \infty$ and as $t \to 0$ $(t > 0)$. This condition is sometimes referred to as the "Δ_2 condition at infinity and at zero".

3.1 Lemma The Δ_2 condition is satisfied if and only if $\underline{M}(2f) < \infty$ whenever $\underline{M}(f) < \infty$.

Proof If Δ_2 is not satisfied then there is a sequence $t_n > 0$ such that $M(2t_n) > nM(t_n)$. We now consider the cases of $\mu(X) < \infty$ and $\mu(X) = \infty$ separately.

(a) If $\mu(X) < \infty$ then we also have that $t_n \to \infty$. Hence we will assume that if $\alpha_n = n^{-2}(M(t_n))^{-1}$ then $\sum_{n=1}^{\infty}\alpha_n < \mu(X)$. We now let A_n be a sequence of pairwise disjoint sets in X such that $\mu(A_n) = \alpha_n$ and define f as t_n on A_n and as 0 on the complement of $\bigcup_{n=1}^{\infty} A_n$. Then $\underline{M}(f) = \sum_{n=1}^{\infty}M(t_n)\alpha_n = \sum_{n=1}^{\infty}n^{-2} < \infty$ but $\underline{M}(2f) = \sum_{n=1}^{\infty}M(2t_n)\alpha_n \ge \sum_{n=1}^{\infty}n^{-1} = \infty$.

(b) If $\mu(X) = \infty$ and if $M(t_n) > 0$ for all n then we find a sequence of pairwise disjoint sets A_n such that $\mu(A_n) = n^{-2}(M(t_n))^{-1}$ and define f as before. If $M(t_n) = 0$ for some n then we may let $f(x) = t_n$ for all $x \in X$. We then see that $\underline{M}(f) < \infty$ but $\underline{M}(2f) = \infty$.

Conversely, assume that Δ_2 is satisfied and let $\underline{M}(f) < \infty$.

We will show that $\underline{M}(2f) < \infty$. If $\mu(X) = \infty$ then $M(2t) \le kM(t)$ for all $t \ge 0$. Hence $\underline{M}(2f) \le k\underline{M}(f) < \infty$. If $\mu(X) < \infty$ then there is a $t_1 > 0$ such that $M(2t) \le kM(t)$ whenever $t \ge t_1$. Hence $M(2t) \le kM(t) + M(2t_1)$ for all $t \ge 0$. This means that $\underline{M}(2f) \le k\underline{M}(f) + M(2t_1)\mu(X) < \infty$. □

3.2 Theorem For an Orlicz space L_M the following conditions are equivalent.

(3.3) The condition Δ_2 is satisfied,

(3.4) $L_M = H_M$,

(3.5) L_M is order continuous,

(3.6) L_M satisfies the condition B; any norm-bounded increasing sequence in L_M converges in norm,

(3.7) H_M satisfies the condition B.

Proof (3.3) => (3.4) Let $\underline{M}(f) < \infty$. Since Δ_2 is satisfied, Lemma 3.1 gives that $\underline{M}(2f) < \infty$. By induction we see that $\underline{M}(2^n f) < \infty$ for any $n \ge 1$. Hence $\underline{M}(f/\rho) < \infty$ for any $\rho > 0$. Therefore $H_M = L_M$.

(3.4) => (3.5) Since H_M is always order continuous, if $L_M = H_M$ then L_M is also order continuous.

(3.5) => (3.6) Let f_n be a sequence in L_M such that $0 \le f_n \le f_{n+1}$ and such that $\sup_{n\ge1}\|f_n\| = \alpha < \infty$. We will show that f_n converges in the norm of L_M. Without loss of generality assume that $\alpha \le \frac{1}{2}$. Then $\underline{M}(f_n) \le 1$ for all $n \ge 1$. If $f = \sup_{n\ge1} f_n$, i.e. if f is the pointwise limit of increasing sequence of functions f_n then $M(f_n) \uparrow M(f)$ and, consequently, $\underline{M}(f) = \lim_{n\to\infty}\underline{M}(f_n) \le 1$. Therefore $f \in L_M$. Then $f - f_n$ is a positive decreasing sequence in L_M and hence converges in L_M norm, by the order continuity of L_M.

(3.6) => (3.7) This part is trivial.

(3.7) => (3.3) If Δ_2 is not satisfied then there is a non negative function g such that $\underline{M}(g) < \infty$ but $M(2g) = \infty$, by Lemma 3.1. Letting $f = 2g$ we see that there is a function $f \in L_M$ such that $\underline{M}(f) = \infty$. Hence $f \in L_M - H_M$. Let f_n be a sequence of integrable non negative simple functions such that $f_n \uparrow f$ (i.e. $f_n \uparrow f$ a.e.). Since $0 \le f_n \le f$, $\|f_n\| \le \|f\|$, i.e. f_n is a norm-bounded sequence in H_M. Now f_n can not converge to f in norm, since f is not in H_M, the closure of integrable simple functions. Since f_n converges to f a.e. it is easy to see that f_n can not converge to any other function in norm. Hence f_n can

not converge in norm. This violates (3.7). □

We will now give an equivalent formulation of Δ_2, which will be used in the next section. Although this second formulation is also very well known, we will prove the equivalence of these two formulations for completeness, assuming only the elementary fact that M can be represented as $M(t) = \int_0^t p(\alpha)d\alpha$, $t \geq 0$, where p is a non decreasing right continuous function with $p(0) = 0$. The existence of this representation depends only on the fact that M is a non decreasing convex function on $[0,\infty)$ with $M(0) = 0$ (see, e.g. [7], p.5).

3.8 Lemma Let $t_0 = \sup\{t \mid t \geq 0, M(t) = 0\}$. For each $s > t_0$ let $K(s) = \sup_{t \geq s} \frac{tp(t)}{M(t)} \leq \infty$. If $\mu(X) < \infty$ then Δ_2 is satisfied if and only if $K(s) < \infty$ for all $s > t_0$. If $\mu(X) = \infty$ then Δ_2 is satisfied if and only if $t_0 = 0$ and $K = \sup_{s>0} K(s) < \infty$. Furthermore, in any case, $M(\lambda t) \leq \lambda^{K(s)} M(t)$ whenever $t \geq s$ and $\lambda \geq 1$.

Proof The integration of $\frac{p(t)}{M(t)} \leq \frac{K(s)}{t}$, $t \geq s$, gives that $\log \frac{M(\lambda t)}{M(t)} \leq K(s) \log \frac{\lambda t}{t}$, $t \geq s$, $\lambda \geq 1$. Hence $M(\lambda t) \leq \lambda^{K(s)} M(t)$ whenever $t \geq s$ and $\lambda \geq 1$. If $K(s) < \infty$ for some $s > t_0$ and if $\mu(X) < \infty$ then $M(2t) \leq 2^{K(s)} M(t)$ whenever $t \geq s$. Hence Δ_2 is satisfied. If $t_0 = 0$ and $K(s) \leq K < \infty$ for all $s > 0$ then $M(2t) \leq 2^K M(t)$ for all $t \geq 0$. Hence Δ_2 is again satisfied, even if $\mu(X) = \infty$.

Conversely, if Δ_2 is satisfied then there are two numbers $t_1 \geq 0$ and $k > 0$ such that $M(2t) \leq kM(t)$ whenever $t \geq t_1$. Then $kM(t) \geq M(2t) = \int_0^{2t} p(\alpha)d\alpha \geq \int_0^{2t} p(\alpha)d\alpha \geq tp(t)$ whenever $t \geq t_1$. Hence $\limsup_{t\to\infty} \frac{tp(t)}{M(t)} \leq k$. This means that $\sup_{t \geq s} \frac{tp(t)}{M(t)} = K(s) < \infty$ whenever $s > t_0$. If $\mu(X) = \infty$ then Δ_2 gives that $M(2t) \leq kM(t)$ for all $t \geq 0$. Hence $t_0 > 0$ gives the contradiction that $0 < M(2t_0) \leq kM(t_0) = 0$. Therefore $t_0 = 0$ and $kM(t) \geq tp(t)$ for all $t \geq 0$. Hence $\sup_{t>0} \frac{tp(t)}{M(t)} = k < \infty$. □

4. Uniform Monotonicity of Norms We now introduce the following conditions that specify various degrees of uniform monotonicity on the norm in a Banach Lattice E.

(C_1) If $f \in E^{++}$ and $g \in E^+$ then $\|f+g\| > \|g\|$.

(C) For each $f' \in E^+$ and for each number $\alpha > 0$ there is a number $\beta = \beta(f',\alpha) > 0$ such that if $f \in E^+$, $g \in E^+$ and $\|g\| \leq 1$, $f \leq f'$, $\|f\| \geq \alpha$ then $\|f+g\| \geq \|g\| + \beta$.

(UC) (A uniform form of C) For each $\alpha > 0$ there is a $\beta = \beta(\alpha) > 0$ such that $f \in E^+$, $g \in E^+$, $\|f\| \geq \alpha$, $\|g\| \leq 1$ implies $\|f+g\| \geq \|g\| + \beta$.

(δ) There exists a strictly increasing continuous function δ on $[0,1]$ with $\delta(0) = 0$ so that if $f \in E^+$, $g \in E^+$ with $1 = \|g\| \geq \|f\|$ then $\|f+g\| \geq 1 + \delta(\|f\|)$.

(UMB) Given $\xi > 0$ there is an $\eta = \eta(\xi) > 0$, such that if $f \in E^+$, $g \in E^+$ with $1 = \|g\|$ and $\|f\| \geq \xi$ then $\|f+g\| \geq 1 + \eta$.

We remark that the original formulation of UMB in [3] was the following condition: "Given $\varepsilon > 0$ there is a $\delta > 0$ such that if $f \in E^+$, $g \in E^+$ with $1 = \|g\|$ and $\|f+g\| \leq 1 + \delta$ then $\|f\| \leq \varepsilon$". It is clear, however, that these two formulations give equivalent conditions.

4.1 Lemma The conditions UC, δ and UMB are all equivalent. Also, UC implies C and C implies C_1.

Proof It is clear that UC implies C and C implies C_1. It is also clear that UC => UMB. To see that UMB => UC, given $\alpha > 0$, find an $\eta > 0$ from UMB corresponding to $\xi = \alpha$. Let $\beta = \min\{\frac{\alpha}{2}, \eta\frac{\alpha}{2}\}$. Let $f,g \in E^+$, $\|f\| \geq \alpha$ and $\|g\| \leq 1$. If $\|g\| \leq \frac{\alpha}{2}$ then $\|f+g\| \geq \alpha \geq \|g\| + \frac{\alpha}{2} \geq \|g\| + \beta$. If $\frac{\alpha}{2} < \|g\| = t \leq 1$ then $\|f+g\| = t\|\frac{1}{t}f + \frac{1}{t}g\| \geq t(1+\eta) = t + t\eta \geq \|g\| + \frac{\alpha}{2}\eta \geq \|g\| + \beta$, where the first inequality follows from the fact that $\|\frac{1}{t}f\| \geq \|f\| \geq \alpha = \xi$ and $\|\frac{1}{t}g\| = 1$. Hence we see that UC holds.

It is also clear that δ => UMB. We will now show that UMB => δ. For each $\xi \geq 0$ let $\delta(\xi) = \inf\{\|f+g\| - 1 \mid f \in E^+, g \in E^+, \|f\| = \xi, \|g\| = 1\}$. We see that $\delta(\xi) \geq \eta(\xi) > 0$ for each $\xi > 0$. If $0 \leq \xi < \xi' = \xi + s$ $(s > 0)$ then any $f' \in E^+$ with $\|f'\| = \xi'$ can be written as $f' = f + h$ where $f \in E^+$, $h \in E^+$, $\|f\| = \xi$, $\|h\| = s$ (i.e. let, $f = \frac{\xi}{\xi'}f'$, $h = \frac{s}{\xi'}f'$). Then $\|f' + g\| - 1 = \|f + g + h\| - 1 \geq \|f + g\| - 1 \geq \delta(\xi)$. Hence $\delta(\xi') \geq \delta(\xi)$. Furthermore we also have, with $t = \|g+f\|$ $(1 \leq t \leq 1+\xi)$, $\|f'+g\| - 1 = \|f+g+h\| - 1 = t\|\frac{1}{t}(f+g) + \frac{1}{t}h\| - 1 \geq t\left(1 + \delta(\frac{s}{t})\right) - 1$ $= \|f+g\| - 1 + \delta(\frac{s}{t}) \geq \delta(\xi) + \delta(\frac{s}{1+\xi})$. Hence $\delta(\xi') > \delta(\xi)$ whenever $\xi' > \xi$. Finally to see that δ is continuous, given $\varepsilon > 0$ find $g,f \in E^+$, $\|g\| = 1$, $\|f\| = \xi$ such that $\|f+g\| - 1 \leq \delta(\xi) + \varepsilon$.

Hence, with $f' = f + \frac{s}{\xi}f$, $\delta(\xi+s) \le \|f'+g\| - 1 \le \|f+g+\frac{s}{\xi}f\| - 1 \le \|f+g\| + s - 1 \le \delta(\xi) + \varepsilon + s$. Therefore $\delta(\xi) \le \delta(\xi+s) \le \delta(\xi) + s$. □

We are now going to show that in an Orlicz space L_M all these conditions on the norm are equivalent to each other and they hold if and only if Δ_2 holds. We start with the following observations (made in our earlier paper [2]).

4.2 Lemma If Δ_2 is satisfied then for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $f \ge 0$ with $\underline{M}(f) \le 1$ then $\underline{M}((1+\delta)f) \le 1 + \varepsilon$.

Proof We apply Lemma (3.8). If $\mu(X) < \infty$ then first choose $s > t_0$ such that $0 < M(2s)\mu(X) < \frac{\varepsilon}{2}$. Then choose δ , $0 < \delta < 1$ such that $(1+\delta)^{K(s)} < 1 + \frac{\varepsilon}{2}$. Hence, if $A = \{x | f(x) \ge s\}$ and $B = \{x | f(x) < s\}$ and $1 \le \lambda \le 1 + \delta$ then $\underline{M}(\lambda f) = \int M(\lambda f) = \int_A M(\lambda f) + \int_B M(\lambda f) \le \lambda^{K(s)} \int_A M(f) + M(2s)\mu(X) \le (1+\delta)^{K(s)} + \frac{\varepsilon}{2} \le 1 + \varepsilon$. If $\mu(X) = \infty$ then, since $\sup_{s>0} K(s) = K < \infty$, $M(\lambda t) \le \lambda^K M(t)$ for all $t \ge 0$ and for all $\lambda \ge 1$. Hence $\underline{M}(\lambda f) \le \lambda^K \underline{M}(f) \le \lambda^K$. Therefore, given $\varepsilon > 0$ it is enough to choose $\delta > 0$ such that $(1+\delta)^K < 1 + \varepsilon$. □

4.3 Lemma If Δ_2 is satisfied then for each $\lambda \ge 1$ there is an $\varepsilon > 0$ such that if $f \ge 0$ and if $\underline{M}(\lambda f) \ge 1$ then $\underline{M}(f) \ge \varepsilon$.

Proof If $\mu(X) = \infty$ then $\underline{M}(\lambda f) \le \lambda^K \underline{M}(f)$ shows that $\underline{M}(f) \ge \lambda^{-K} \underline{M}(\lambda f) \ge \lambda^{-K} = \varepsilon$. If $\mu(X) < \infty$ then find $s > t_0$ such that $M(\lambda s)\mu(X) \le \frac{1}{2}$. Then, defining A and B as before we have $1 \le \underline{M}(\lambda f) = \int_A M(\lambda f) + \int_B M(\lambda f) \le \lambda^{K(s)} \int_A M(f) + \frac{1}{2} \le \lambda^{K(s)} \underline{M}(f) + \frac{1}{2}$. Hence we can let $\varepsilon = \frac{1}{2}\lambda^{-K(s)}$. □

4.4 Theorem In an Orlicz Space L_M all the conditions C_1 , C and UMB are equivalent to each other and to Δ_2 .

Proof Assume that Δ_2 is satisfied. Note that in this case $\underline{M}(\lambda h)$ is a continuous function of λ , for each fixed $h \in L_M$. Hence, in particular, $\underline{M}\{\frac{1}{|h|}h\} = 1$ for any $h \in L_M$. We will now verify that UMB is satisfied. Let $\xi > 0$ be given. We will assume, without loss of generality that $\xi < 1$. Let $f,g \in E^+$, $\|f\| \ge \xi$, $\|g\| = 1$. Hence $\underline{M}(\frac{1}{\xi}f) \ge 1$ and $\underline{M}(g) = 1$. Find $\varepsilon > 0$ from Lemma 4.3 corresponding to $\lambda = \frac{1}{\xi}$. Then $\underline{M}(f+g) \ge \underline{M}(f) + \underline{M}(g) \ge 1 + \varepsilon$, where the first inequality follows from the convexity of M . Then Lemma 4.2 shows that there is an $\eta > 0$, depending only on $\varepsilon > 0$ (consequently depending only on $\xi > 0$) such that $\underline{M}(\frac{1}{1+\eta}(f+g)) \ge 1$. Hence $\|f+g\| \ge 1 + \eta$. Therefore UMB is satisfied. Hence C and C_1 are also satisfied.

We now show that C_1 implies Δ_2 . In fact, if Δ_2 is not satisfied, then we are going to show there are two non-negative functions f and g with disjoint support such that $\underline{M}(f) = \underline{M}(g) \le \frac{1}{2}$ but $\underline{M}(\lambda f) = \underline{M}(\lambda g) = \infty$ whenever $\lambda > 1$. This would mean that $\|f\| = \|g\| = 1$ and also that $\|f+g\| = 1$, since $\underline{M}(f+g) = \underline{M}(f) + \underline{M}(g) \le 1$. This will show that C_1 is not satisfied.

The construction of f and g is very similar to the construction given in the proof of Lemma 3.1. We will use the formulation of Δ_2 given by Lemma 3.8 and consider the cases of $\mu(X) < \infty$ and $\mu(X) = \infty$ separately.

(a) If $\mu(X) < \infty$ and if Δ_2 is not satisfied then there is a sequence $t_n > 0$ such that $t_n \to \infty$ and such that $t_n p(t_n) \ge nM(t_n)$ for all $n = 1,2,\ldots$ Choose a sequence $\alpha_n > 0$ such that $\sum_{n=1}^{\infty} \alpha_n \le \frac{1}{2}$, $\sum_{n=1}^{\infty} n\alpha_n = \infty$. By passing to a subsequence, if necessary, we will assume that $\sum_{n=1}^{\infty} \frac{\alpha_n}{M(t_n)} < \frac{1}{2}\mu(X)$. Hence we can find a family A_n , B_n , $n \ge 1$, of pairwise disjoint sets such that $\mu(A_n) = \mu(B_n) = \alpha_n [M(t_n)]^{-1}$. We then let $f = \sum_{n=1}^{\infty} t_n \chi_{A_n}$, $g = \sum_{n=1}^{\infty} t_n \chi_{B_n}$. It is clear that $\underline{M}(f) = \underline{M}(g) = \sum_{n=1}^{\infty} \alpha_n = \frac{1}{2}$. Since $M\big((1+\varepsilon)t_n\big) \ge M(t_n) + \varepsilon t_n p(t_n) \ge (1+\varepsilon n)M(t_n)$ for each $\varepsilon > 0$, we also see that $\underline{M}\big((1+\varepsilon)f\big) = \underline{M}\big((1+\varepsilon)g\big) \ge \sum_{n=1}^{\infty} (1+\varepsilon n)\alpha_n = \infty$. Hence f and g have the required properties.

(b) Let $\mu(X) = \infty$. If $t_0 = \sup\{t | t \ge 0 , M(t)=0\} > 0$ then $M(t_0) = 0$ but $M(\lambda t_0) > 0$ for all $\lambda > 1$. Then we may let $f = t_0 \chi_A$, $g = t_0 \chi_B$ where A and B are two disjoint sets with $\mu(A) = \mu(B) = \infty$. Since $\underline{M}(f) = \underline{M}(g) = \underline{M}(f+g) = 0$ and $\underline{M}(\lambda f) = \underline{M}(\lambda g) = \infty$ whenever $\lambda > 1$ we see that $\|f\| = \|g\| = \|f+g\| = 1$. If $t_0 = 0$ but $\sup_{t>0} \frac{tp(t)}{M(t)} = \infty$ then we again find a sequence $t_n > 0$ such that $t_n p(t_n) \ge nM(t_n)$ for all $n \ge 1$. If $\alpha_n > 0$ is a sequence as in part (a) we may find disjoint A_n and B_n such that $\mu(A_n) = \mu(B_n) = \mu_n [M(t_n)]^{-1}$. Since $\mu(X) = \infty$ there is no restriction on $\sum_{n=1}^{\infty} \alpha_n [M(t_n)]^{-1}$. We then define f and g as in part (a). □

5. An Application to Ergodic Theory The following theorem has been obtained in [2], which we state in a self contained form.

5.1 Theorem Let E be a Banach Lattice satisfying the following conditions

(A) E has a weak unit, i.e. an element $u \in E^+$ such that $|f| \wedge u = 0$ implies $f = 0$.

(B) Any norm-bounded increasing sequence in E converges in norm.

(C) For each $\alpha > 0$ and $f' \in E^+$ there is a $\beta = \beta(f',\alpha)$ such that $\|f+g\| \geq \|g\| + \beta$ whenever $f,g \in E^+$, $f \leq f'$, $\|f\| \geq \alpha$, $\|g\| \leq 1$.

Let $T : E \to E$ be a positive contraction. Let $A_n = \frac{1}{n}(1 + T + \ldots + T^{n-1})$. Then for each $f \in E$ there is a $\phi \in E$ such that $T\phi = \phi$ and such that $|A_n f - \phi| \wedge u$ converges to zero in norm.

The arguments given in the present note show that this theorem is applicable to an Orlicz function space L_M if Δ_2 is satisfied. If Δ_2 is not satisfied then this theorem is not applicable to L_M or even to H_M , since in this case B fails.

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QUELQUES RESULTATS D'EXISTENCE POUR LE PROBLEME DE LA COURBURE DANS R^n

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INTRODUCTION.

On considère l'équation de la courbure scalaire dans R^n pour $n > 2$.

(1) $$\Delta_g u(x) - k(x)u(x) + K(x)\, u(x)^{n+2/n-2} = 0$$

g est une métrique complète donnée sur R^n ; on cherche u, facteur de conformité, tel que $\bar{g} = u^{4/n-2}\, g$ soit une métrique complète avec :

$$K(x) = \frac{(n-2)}{4(n-1)}\ R(\bar{g})$$

$$k(x) = \frac{(n-2)}{4(n-1)}\ R(g)$$

où K est une fonction donnée (cf[2]).

Remarque 1. Une condition suffisante pour que la métrique $\bar{g}$ soit complète est que le facteur de conformité soit positif. Dans le cadre de cette étude, on ne cherchera que des solutions positives.

L'équation (1) sera étudiée dans le cadre des espaces de Sobolev à poids (cf.[3],[4]), et l'on cherchera des solutions bornées, tendant vers 1 à l'infini ; le cas n=2, a été étudié dans[6].

Notations.

- $M^{p,m}_{\beta}(R^n)$ est la fermeture de $C^{\infty}_{0}(R^n)$ pour la norme suivante :

$$\| u \|_{M^{p,m}_{\beta}} = \sum_{|\alpha| \leq m} \| \sigma^{\beta+|\alpha|} D^{\alpha} u \|_{L^p} \qquad 1 < p < +\infty$$

- $C^{k}_{\beta}(R^n) = \{ f \in C^{k}(R^n) \ / \ \sup_{\substack{|\alpha| \leq k \\ x \in R^n}} | \sigma^{\beta+|\alpha|} D^{\alpha} f(x) | \leq C \}$

où $\sigma(x) = (1+ |x|^2)^{1/2}$

Pour les propriétés de ces espaces voir [3] , [4] , [5] .

ETUDE DE L'EQUATION (1).

Soit δ la métrique canonique sur R^n. On fait les hypothèses suivantes :

(1-1) $\quad -n/p < \beta < -2+n/p, \qquad (1/p + 1/p') = 1$

$\quad m > 2+n/p \qquad m \in N$

(1-2) $\quad g_{ij} - \delta_{ij} \in M^{p,\ell}_{\beta}(R^n)$ avec $\ell > m+n/p \qquad \forall i,j \qquad 1 \leqslant i \leqslant j \leqslant n.$

(1-3) $\quad k \in M^{p,\ell-2}_{\beta+2}(R^n),\ K \in M^{p,m-2}_{\beta+2}(R^n)$

On peut énoncer le :

Théorème 1.

On se place sous les hypothèses (1-1)-(1-3). Alors il existe une solution $u > o$, telle que $1-u \in M^{p,m}_{\beta}(R^n)$ si :

(i) $k(x) \geqslant 0$, $K(x) \leqslant 0$ et $n = 3$ ou $n = 4$

(ii) pour tout n, avec $k(x) \leqslant 0$, $K(x) \leqslant 0$

(iii) pour tout n, avec $k(x) > 0$, $K(x) > 0$ et $\inf_{x \in R^n} \frac{k(x)}{K(x)} > \frac{n+2}{n-2}$

De plus dans le cas (i) la solution est unique (cf [7]).

La démonstration se fera en plusieurs étapes en utilisant la méthode de LERAY-SCHAUDER (cf [7]).

Si u est solution de (1), alors : $v = 1 - u$, vérifie :

$$\Delta_g v = -k(1-v) + K(1-v)^{n+2/n-2}$$

A cette équation on associera l'équation linéaire :

$$\Delta_g v = -k(1-\omega) + K(1-\omega)^{n+2/n-2}$$

où ω sera choisi dans un ouvert Ω, à fixer.

Alors on peut trouver Ω tel que l'application :

$$\omega \mapsto v(\omega)$$

ait un point fixe dans Ω .

Lemme 1. Pour $m > k + n/p$, $\beta' < \beta + n/p$ on a l'inclusion :

$$M^{p,m}_{\beta}(R^n) \subset C^k_{\beta'}(R^n) \text{ pour tout } \beta \text{ , et } p \in [1,+\infty[$$

Preuve :

Ce lemme généralise un résultat de [8]

On considère le difféomorphisme $\Phi_\varepsilon : \begin{cases} R^n \to R^n \\ x \to \frac{x}{\sigma(x)^{1-\varepsilon}} \end{cases}$, $\varepsilon \in]0,1]$

Soit T_ε l'application $f \to f \circ \Phi_\varepsilon^{-1}$ où $f : R^n \to R$ (cf [9])

On pose $\nu_\varepsilon = \beta/\varepsilon + n/p\left(\frac{\varepsilon - 1}{\varepsilon}\right)$

Alors T_ε est un isomorphisme de $M^{p,m}_{\beta}(R^n)$ sur $M^{p,m}_{\nu_\varepsilon}(R^n)$ pour tout $\varepsilon \in]0,1]$. En effet :

$$\int \sigma(x)^{p\nu_\varepsilon} [f(\Phi_\varepsilon^{-1}(x))]^p dx = \int J_\varepsilon(t) \sigma(\Phi_\varepsilon(t))^{p\nu_\varepsilon} |f(t)|^p dt$$

or $J_\varepsilon(t)$ est de l'ordre de $\frac{1}{\sigma(t)^{n(1+\varepsilon)}}$ et $\sigma(\Phi_\varepsilon(t))^{p\nu_\varepsilon}$ de l'ordre de

$\sigma(t)^{p\beta} \sigma(t)^{n(1-\varepsilon)}$. Puis on remarque que T_ε envoie $C^m_{\beta}(R^n)$ dans $C^m_{\beta/\varepsilon}(R^n)$ et que :

$$u \in M^{p,m}_{\beta}(R^n) \Rightarrow \sigma^{\beta+k} D^k u \in W^{m-k,p}(R^n) \quad \text{pour tout } k \leqslant |\alpha|$$

avec $\alpha = m-1-E(n/p)$ (E : partie entière)

Il suffit alors d'utiliser les théorèmes d'injection des espaces $W^{n,p}(R^n)$ dans les espaces de fonctions continues (cf [9]) pour conclure.

Lemme 2. Pour tout $w \in M^{p,m-1}_{\gamma}(R^n)$ avec $-n/p < \gamma < \beta$, et $w > 0$, on définit les fonctions ;

$$\Phi : w \to \Phi(w) = kw - Kw^{n+2/n-2} \text{ et } f \text{ telle que } f(w) = -\Phi(1-w).$$

Alors, sous les hypothèses (1-1)-(1-3), $f(w)$ et $\Phi(w)$ appartiennent à $M^{p,m-2}_{\beta+2}(R^n)$

Preuve :

$k \in M^{p,\ell-2}_{\beta+2}(R^n)$ donc $k \in M^{p,m-2}_{\beta+2}(R^n)$ avec $\ell > m$

Or avec le lemme 1, $w \in C^{m'}_{\gamma'}$, $\gamma' < \gamma + n/p$

et $0 < m' < m-1-n/p$

On peut donc choisir $\gamma' > 0$ car : $\gamma + n/p > 0$ donc : $\sup_{x \in R^n} |w(x)| \leq \frac{c}{\sigma^{\gamma'}}$

donc w est borné. On en conclut avec (1-3) que $\Phi(w)$ et $f(w) \in M^{p,m-2}_{\beta+2}(R^n)$

Lemme 3. Avec les hypothèses (1-1) et (1-2), Δ_g est un isomorphisme de $M^{p,m}_{\beta}(R^n)$ sur $M^{p,m-2}_{\beta+2}(R^n)$

Preuve :

Il suffit de remarquer avec (1-2) et le lemme 1 que g est asymptotiquement euclidienne jusqu'au rang (m-1). Le résultat suit avec [4] (Th. 6-2).

Considérons maintenant la famille d'équations non linéaires suivante :

(2) $\Delta_g u = t\,\Phi(u) \qquad t \in [0,1]$

Lemme 4. On suppose, que pour tout $x \in R^n$, $k(x) \geq 0$, $K(x) \leq 0$, et $n = 3$ ou 4. Si u est solution de (2), telle que : $1-u \in M^{p,m}_{\beta}(R^n)$, alors : $u > 0$; d'autre part, pour toute valeur n, il existe une constante $M > 1$, telle que $u < M$.

Preuve :

Pour $n = 3,4$ $\frac{n+2}{n-2}$ est impair. Supposons que u prenne des valeurs négatives ou nulles ; alors il existerait x_o tel que $u(x_o)$ soit un minimum négatif. Avec (1-1) et le lemme 1, $M^{p,m}_{\beta}(R^n) \subset C^2_o(R^n)$. On peut donc appliquer le principe du maximum à Δ_g. En x_o, on aurait : $\Delta_g u(x_o) \geq 0$ et $\Phi(u)(x_o) \leq 0$. Ce qui est impossible.

En outre, pour $u > o$ la fonction Φ est croissante, avec $\Phi(o) = o$. Si u prenait des valeurs strictement supérieures à 1, alors elle atteindrait son maximum $M > 1$, en un point x_1 de R^n ; on aurait $\Delta_g u(x_1) \leq o$, et $t\Phi(M) \geq o$. Le résultat suit.

Soit R une constante positive à fixer. On pose :

(1-4) $\Omega^1_R = \{\omega, \|\omega\|_{M^{p,m-1}_{\gamma}} < R, \quad 0 < 1 - \omega(x) < M, \text{pour tout } x \in R^n\} ; -\frac{n}{p} < \gamma < \beta$

Lemme 5. On suppose : $k(x) \leq 0$ et $K(x) \leq 0$ pour tout $x \in R^n$. Alors si u est solution de (2), avec $(1-u) \in M^{p,m}_{\beta}(R^n)$, on peut trouver deux constantes ℓ et M, avec : $0 < \ell < 1 < M$, telles que : $\ell \leq u \leq M$ entraîne $\ell < u < M$.

Preuve :

L'étude de la fonction Φ montre que l'on peut trouver ℓ et M, telles que : $\Phi(\ell) < 0$ et $\Phi(M) > 0$. Si l'on considère l'ensemble des solutions u de (2), telles que : $(1-u) \in M^{p,m}_{\beta}(R^n)$, et $\ell \leq u \leq M$, alors nécessairement : $\ell < u < M$, d'après le principe du maximum.

On pose :

(1-5) $\Omega^2_R = \{\omega, \|\omega\|_{M^{p,m-1}_{\gamma}} < R, \ell < 1 - \omega < M\}$, $(-\frac{n}{p} < \gamma < \beta)$

Lemme 6. On suppose $k(x) > 0$ et $K(x) > 0$, pour tout $x \in R^n$. De plus, on suppose que : $\inf_{x \in R^n} \frac{k(x)}{K(x)} > \frac{n+2}{n-2}$. Si u est solution de (2), telle que $u > 0$ et : $1-u \in M^{p,m}_{\beta}(R^n)$ alors il existe : $0 < u_o < 1$, tel que $1 - u_o \in M^{p,m}_{\beta}(R^n)$, et une constante $M > 1$, vérifiant : $u_o < u < M$.

Preuve :

Soit l'équation

(3) $\Delta_g u - ku = 0$

avec $k > 0$; on vérifie que $\Delta_g - k$ est un isomorphisme de $M^{p,m}_{\beta}(R^n)$ sur $M^{p,m-2}_{\beta+2}(R^n)$.

Il existe donc une solution de (3), u_o, telle que $1-u_o \in M^{p,m}_{\beta}(R^n)$

En appliquant le principe du maximum, on vérifie que : $0 < u_o < 1$

Si u est solution de (2), alors $u - u_o$ vérifie :

$$\Delta_g(u-u_o) - k(u-u_o) = -K\,u^{n+2/n-2}$$

D'autre part si $\inf_{x\in R^n} \frac{k(x)}{K(x)} > \frac{n+2}{n-2}$, l'étude de la fonction Φ montre que l'on peut trouver $M > 1$ tel que $\Phi(M) > 0$.

On conclut avec des arguments analogues à ceux des lemmes précédents.

On pose :

$$\Omega_R^3 = \{ \omega,\ \| \omega \|_{M_\gamma^{p,m-1}} < R,\ u_o < 1 - \omega < M \},\ (-n/p < \gamma < \beta)$$

Revenons à la démonstration du théorème 1.

On a posé : $v = 1-u$.

Si u est solution de (1), alors v vérifie :

(4) $$\Delta_g v = f(v)$$

avec, rappelons-le, $f(v) = -\Phi(1-v)$

A (4), associons l'équation linéaire :

(4') $\Delta_g v = f(\omega)$, où ω est élément de Ω_R^i $i = 1,2$ ou 3 suivant les hypothèses.

D'après le lemme 2, $f(\omega) \in M_{\beta+2}^{p,m-2}(R^n)$.

On en déduit qu'il existe une unique solution v à (4'), avec $v \in M_\beta^{p,m}(R^n)$.

On définit alors une application compacte T de Ω_R^i dans $M_\beta^{p,m-1}(R^n)$

$(\omega \to v(\omega)$, v solution de (4'). On a en effet la décomposition suivante de T :

$$\Omega_R^i \subset M_\gamma^{p,m-1}(R^n) \xrightarrow{f} M_{\beta+2}^{p,m-2}(R^n) \xrightarrow{\Delta_g^{-1}} M_\beta^{p,m}(R^n) \xrightarrow{i} M_\gamma^{p,m-1}(R^n)$$

Et T est la composée d'applications continues avec i qui est compacte $(\gamma < \beta)$

L'existence d'une solution de l'équation (1) découle alors du :

Lemme 7. Il existe $R > 0$, tel que l'application T ait un point fixe dans Ω_R^i, pour tout $i = 1,2,3$.

Preuve :

L'application T_o : $\omega \longrightarrow 0$ a un point fixe dans Ω_R^i (remarquons, que $0 \in \Omega_R^i$ pour tout i ; en particulier pour $i = 3$, on a : $u_o < 1$).

Pour tout $t \in [0,1]$ définissons une famille d'applications compactes :

$$T_t = t\,T + (1-t)T_o,$$ par résolution du problème :

(5) $$\begin{cases} \Delta_g v = t\, f(\omega),\ \omega \in \Omega_R^i \\ v \in M_\gamma^{p,m-1}(R^n) \end{cases}$$

Alors :

$$\| v(\omega) \|_{M_\gamma^{p,m-1}} \leq C_1 \| v(\omega) \|_{M_\beta^{p,m}} \leq C_2 \| \Delta_g v(\omega) \|_{M_{\beta+2}^{p,m-2}} = C_2 \| f(\omega) \|_{M_{\beta+2}^{p,m-2}}$$

or :

$$\| f(\omega) \|_{M_{\beta+2}^{p,m-2}} \leq \| k(1-\omega) + K(1-\omega)^{n+2/n-2} \|_{M_{\beta+2}^{p,m-2}}$$

Par définition des Ω_R^i, il existe une constante :

$C = C(n, k, K, M)$ telle que

$$\| f(\omega) \|_{M_{\beta+2}^{p,m-2}} \leq C$$

Il suffit alors de choisir $R > C_2 \times C$

D'après les lemmes 4, 5, 6, pour tout $t \in [0,1]$, T_t n'a pas de point fixe sur le bord de Ω_R^i. Le résultat découle alors du théorème de LERAY-SCHAUDER.

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ON AN APPROXIMATION PROBLEM BY TRIGONOMETRIC POLYNOMIALS (*)

L.Gotusso - D.Roux

In this paper we deal with a problem about the approximation of a function by suitable trigonometric polynomials; this problem concernes both Numerical and Harmonic Analysis.

1. We all know that if we consider a generally continuous real funtion $f \in L^2$, its Fourier polynomials give a good approximation of f in L^2 but in general a really bad representation of f from a geometrical point of wiew.
Indeed, also if we disregard the Gibbs phenomenon which arises near to the discontinuity points, the graphics of Fourier polynomials have an oscillatory character which often is not in the nature of the approximated function f. In particular, these oscillations are very emphasized when the polynomial's degree is small. There are many thecniques of smoothing the Fourier polynomials in order to avoid this trouble (we remember, for instance, the Fejer or Weierstrass coefficients). In 1981, a smoothing method was suggested in [2] for functions of one real variable. This method, which seems to be very natural in modern approximation theory, tries to realize a good approximation of f in the least squares sense with trigonometric polynomials having a small mean curvature. The smoothing process involves a parameter which in some sense measures the importance given to both the aspects of the problem.
We can formalize the problem as follows. Let be T the one-dimensional torus and f a function of the Lebesque space $L^2(T)$. We want to minimize in the class of trigonometric polynomials of a fixed degree m

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This work is in its final form and it will not appear elsewhere in this form.

$$P(t) = \sum_{|n| \leq m} a_n e^{2\pi int} \quad , t \in T$$

the functional

$$J(P) = \int_T |f-P|^2 + \sigma \int_T \left| \frac{d^2P}{dt^2} \right|^2$$

where $\sigma > 0$ is the (arbitrary) smoothing coefficient.
The properties of trigonometric functions allow to obtain a very simple result. The polynomial $P_{m,\sigma}$ which realizes the minimum of J is

$$P_{m,\sigma}(t) = \sum_{|n| \leq m} \frac{\hat{f}(n)}{1+\sigma n^4} e^{2\pi int}$$

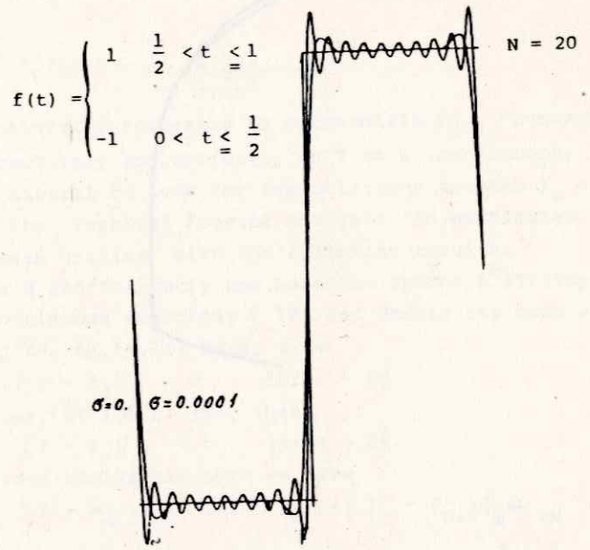
where $\{\hat{f}(n)\}_{n=-\infty}^{n=+\infty}$ is the sequence of Fourier coefficients of f.
So, the smoothing problem is solved for every m simply by multiplying the coefficients of the Fourier polynomial of f

$$P_m(t) = \sum_{|n| \leq m} \hat{f}(n) e^{2\pi int}$$

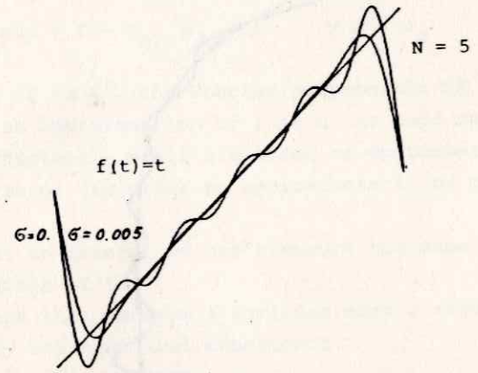
by the smoothing factors $\frac{1}{1 + \sigma n^4}$.

We include some pictures (see tab. 1,2 and 3) obtained by a numerical experience at Politecnico of Milano. They show how the polynomials $P_{m,\sigma}$, expecially for small values of m, represent f better than its Fourier polynomials, because they appear to be more regular and more suitable for applications.
The analysis of numerical experience pointed out a very interesting phenomenon. For all the considered functions f and values of m it is possible to find values of σ such that the polynomials $P_{m,\sigma}$ turn out to give a better approximation of f in L^1 than that one given by the Fourier polynomials P_m.
Moreover, this approximation is also better than that ones given by other smoothing thecniques (in particular, that ones of Fejer and of sigma factors [5]).
This lead us to study the problem from a theoretical point of wiew, also in order to find the motivation of the experimentally observed phenomenons.
We begin to remark that if $f \in L^1$ the Fourier series

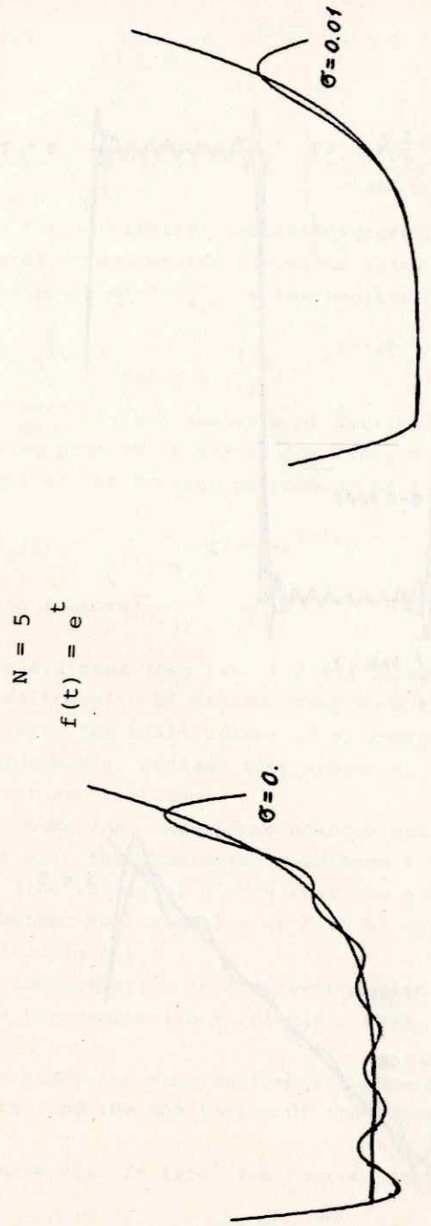
$$f(t) \sim \sum_{-\infty}^{+\infty} \hat{f}(n) e^{2\pi i n t}$$



TAB. 1



TAB. 2



TAB. 3

doesn't in general converge in any classical sense; on the contrary, the series

$$f_\sigma(t) = \sum_{-\infty}^{+\infty} \frac{\hat{f}(n)}{1+\sigma n^4} e^{2\pi i n t}$$

(which is naturally suggested by polynomials $P_{m,\sigma}$) converges for every $\sigma > 0$ absolutely and uniformly on T to a (continuous) function f_σ. Then it is natural to look for the relations between f_σ and f. The methods of the classical Fourier Analysis (in particular the theory of approximate unities) give the following results.

Let us name B indifferently the Lebesgue spaces $L^p(T)$, $1<p<+\infty$ or the space of continuous functions C (T) and denote its norm with $\| \ \|_B$.

THEOREM 1 ([4], th.1). *If* $f \in B$, *then*

$$\| f - f_\sigma \|_B \to 0 \qquad \text{for } \sigma \to 0+.$$

In particular, if $f \in L^1(T)$, then

$$\| f - f_\sigma \|_1 \to 0 \qquad \text{for } \sigma \to 0+.$$

Since in every admissible norm we have

$$(1) \qquad \| f - P_{m,\sigma} \| < \| f - f_\sigma \| + \| f_\sigma - P_{m,\sigma} \|$$

and for every $\sigma > 0$, whichever is B, we have

$$(2) \qquad \| f_\sigma - P_{m,\sigma} \|_B \to 0 \qquad \text{if } m \to +\infty,$$

the following Corollary holds.

COROLLARY 1. *If* $f \in L^1(T)$, *then for every* $\varepsilon > 0$ *there exists* $\sigma_o = \sigma_o(\varepsilon)$ *such that*

$$\limsup_{m \to +\infty} \| f - P_{m,\sigma} \|_1 < \varepsilon \qquad \forall\, \sigma < \sigma_o$$

This means that if $f \in L^1$, the Fourier polynomials of f_σ, which are our $P_{m,\sigma}$, give an approximation of f in L^1 as good as we want when we choose σ sufficiently small (in order to approximate f by f_σ) and m sufficiently large (in order to approximate f_σ by its Fourier polynomials).

We remember that in general we can't ensure the same result by using Fourier polynomials of f.

Let us now remark that Theorem 1 includes also a result which was not suggested by the numerical experience.

Indeed, if $f \in C(T)$, we have

$$\| f - f_\sigma \|_\infty \to 0 \quad \text{when} \quad \sigma \to 0+.$$

Consequently (1) and (2) give the

COROLLARY 2. *If* $f \in C(T)$ *then for every* $\varepsilon > 0$ *there exists* $\sigma_o = \sigma_o(\varepsilon)$

such that

$$\limsup_{m \to +\infty} \| f - P_{m,\sigma} \|_\infty < \varepsilon \qquad \forall 0<\sigma<\sigma_o .$$

Then, if $f \in C(T)$, the polynomials $P_{m,\sigma}$ approximate f as well as we want in every L^p, $1 \leq p \leq +\infty$. It is well known that also this result doesn't hold in general for the Fourier polynomials of f.
Finally, if $f \in C^{(k)}(T)$ $(k=1,2....)$, all the derivatives of f of order $j \leq k$ can be approximated uniformly on T by the corresponding derivatives of polynomials $P_{m,\sigma}$ as well as we want; indeed

$$f \in C^{(k)}(T) \Longrightarrow \| f^{(j)} - P_{m,\sigma}^{(j)} \|_\infty < \varepsilon$$

for every $j=0,1... k$ if $0<\sigma\leq\sigma_o(\varepsilon)$ and $m \geq m_o(\sigma)$.

We can also give, for every $\varepsilon > 0$, some evaluation about the values of σ and m which ensures the asked approximation ([4], th.2).

2. Let us now consider the general N-dimensional case. If $N > 1$ the problem appears to be more complicated.
Indeed, the functional we have to minimize, in analogy to the case N=1, in order to obtain smooth polynomials which approach, in the sense of least squares, a generally continuous function $f \in L^2(T^N)$, may be choosen in different ways.
We can for instance minimize, in the class of trigonometric polynomials P of fixed "degree", the functional

$$J(P) = \int_{T^N} |f - P|^2 + \sigma \int_{T^N} \sum_{j,k=1}^{N} a_{j,k} \frac{\partial^2 P}{\partial t_j^2} \frac{\partial^2 P}{\partial t_k^2}$$

where $\sigma < 0$ is the (arbitrary) smoothing factor and $\sum_{j,k=1}^{N} a_{j,k} t_j t_k$ is a definite positive quadratic form.
The choice

$$a_{j,k} = \delta_{j,k}$$

(where $\delta_{j,k}$ is the Kronecker's symbol) may appear to be more natural then the other ones; but a different choice allows us to weight the smooth differently in the different directions.
We remark also that we can't obtain a more wide class of functionals if we consider that one given by

$$\int_{T^N} |f - P|^2 + \sigma \int_{T^N} \sum \beta_{j,r,k,s} \frac{\partial^2 P}{\partial t_j \partial t_r} \frac{\partial^2 P}{\partial t_k \partial t_s}$$

(where $\beta_{j\,r\,k\,s}$ $(j,r,k,s = 1....N)$ satisfy obvious conditions), which seems to be more general. Indeed

$$\int_{T^N} \frac{\partial^2 P}{\partial t_j \partial t_r} \frac{\partial^2 P}{\partial t_k \partial t_s} = \int_{T^N} \frac{\partial^2 P}{\partial t_j^2} \frac{\partial^2 P}{\partial t_k^2} .$$

It is easy to verify that the polynomial which achieves the asked minimum of J(P) is obtained from the corrisponding Fourier polynomial of f by multiplying its coefficients by the smoothing factors

$$(3) \qquad \left(1 + \sum_{j,k=1}^{N} a_{jk} n_j^2 n_k^2\right)^{-1} .$$

The numerical experiments in [3] concerne only the case N=2 with some particular choices of parameters $a_{j,k}$. The graphics of tab. 4 and 5 show the efficacy of this smoothing method.
Obviously, if N=2, it is hard to give a numerical evaluation of $\| f - P \|_1$ with a great accuracy; then, it is better to study the problem if the obtained polynomials approximate f in L^1 as well as we want from a theoretical point of wiew.
First of all, we remark that the smoothing factors (3) have the form

$$(1 + \sigma P(n))^{-1}$$

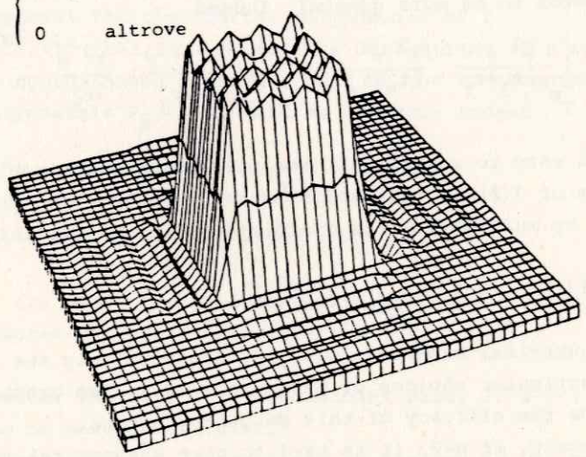
where P is a particular, homogeneous polynomial of $x \in R^2$, of degree four, which is positive if $x \neq 0$.
This leads to consider, if $f \in L^1(T^N)$ $(N=1,2,...)$ the series

$$\sum_{n \in Z^N} \frac{\hat{f}(n)}{1+\sigma P(n)} e^{2\pi i n t} , \qquad t \in T^N$$

where Z^N is the lattice of integers $n \in R^N$ and $P: R^N \to R$, here and in the following, is a homogeneous polynomial of even degree $k > 0$, which is positive out of the point zero.
If $N \neq 1$ this series in general doesn't converge pointwise on T^N and moreover we don't know "a priori" if this series converges at least in some L^p; so the study of the general case is more difficult than the case N=1.
Then, let us consider for every $\sigma > 0$ the distribution

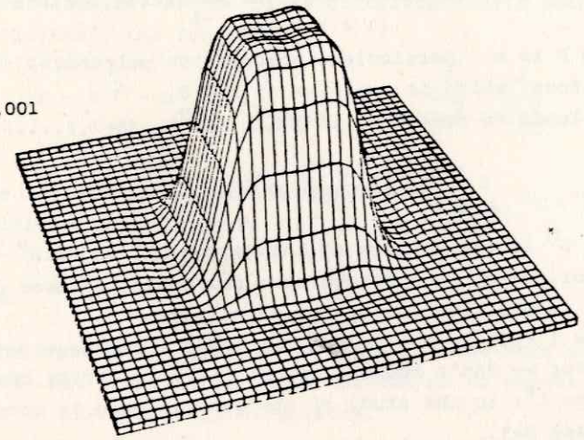
$$f(y_1,x_2) = \begin{cases} 1 & \frac{1}{3} < x_1 \leq \frac{2}{3} \\ 0 & \text{altrove} \end{cases}$$

$\sigma = 0$



N = 10

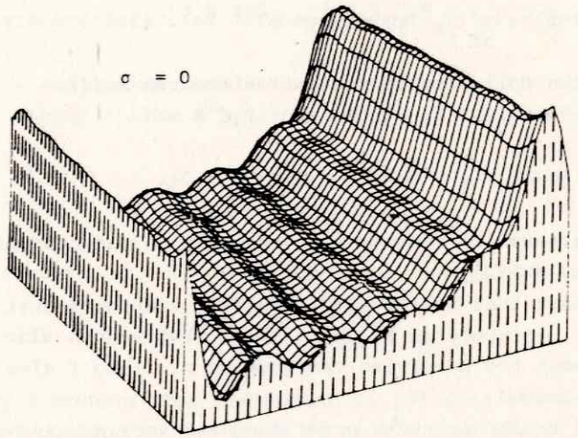
$\sigma = 0{,}001$



TAB. 4

$f(x_1,x_2) = e^{x_1 + x_2}$

$\sigma = 0$

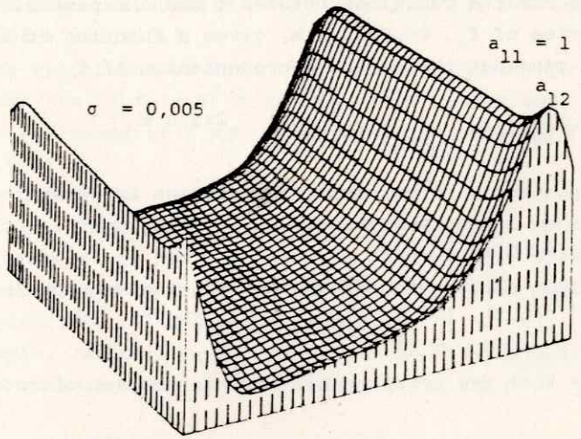


N = 10

$a_{11} = 1 \quad a_{22} = 2$

$a_{12} = a_{21} = 0$

$\sigma = 0{,}005$



TAB. 5

$$K_\sigma \sim \sum_{n \in Z^N} \frac{1}{1+\sigma P(n)} e^{2\pi i n t} \qquad t \in T^N$$

and, for every $f \in L^1(T^N)$, the convolution

$$f_\sigma = K_\sigma * f \sim \sum_{n \in Z^N} \frac{\hat{f}(n)}{1+\sigma P(n)} e^{2\pi i n t}.$$

Fist of all, the following approximation theorem holds.

THEOREM 2. ([1] th.1). *If $f \in B$, then $f_\sigma \in B$ and*

$$\| f - f_\sigma \|_B \to 0 \qquad \text{when } \sigma \to 0+.$$

This theorem generalize Theorem 1 in the case N=1 and extends the result to every N>1.

Let us now remark that if $f \in C(T^N)$, Theorem 2 ensures that $f_\sigma(t)$ converges to f(t) uniformly on T^N when $\sigma \to 0+$. But we can also give some information about the pointwise convergence of f_σ to f also in the case $B \neq C(T^N)$. Indeed

THEOREM 3 ([1] th.2). *In every point t of the Lebesgue set of f (*)*

$$f_\sigma(t) \to f(t) \qquad \text{when } \sigma \to 0+.$$

Then f_σ converges to f almost everywhere on T^N when $\sigma \to 0+$; in particular $f_\sigma(t) \to f(t)$ in every point of continuity of f.

Let us now look for the relations between f and the partial sums of the Fourier series of f_σ. To this aim, given a function $f \in L^p(T^N)$, $1 \leq p \leq +\infty$, let us consider the series representation of f_σ

$$(4) \qquad f_\sigma(t) \sim \sum_{n \in Z^N} \frac{\hat{f}(n)}{1+\sigma P(n)} e^{2\pi i n t}$$

regarding the pointwise convergence. This series in general doesn't converge, not even almost everywhere; but, if the degree k of P is sufficiently large with respect to N and p, then it is possible to ensure its uniform convergence in every summation method. Indeed

THEOREM 4. ([1], th.3). *Let be $f \in L^p(T^N)$, $1 \leq p \leq +\infty$, and*

$$p_o = \min(p,2).$$

(*) We remember that the Lebesgue set of f is the set of points $t \in T^N$ such that

$$\int_{|h|<r} |\hat{f}(t+h) - \hat{f}(t)| \, d h = o(r^N)$$

where $\hat{f}$ is the function obtained by prolonging f to R^N with periodicity law.

Then, if $k p_o > N$,

$$\sum_{n \in Z^N} \frac{\hat{f}(n)}{1+\sigma P(n)} e^{2\pi i n t}, \qquad t \in T^N$$

converges absolutely (and uniformly) on T^N to the (continuous) function f_σ.

Theorems 2 and 4 give immediately the following Corollary.

Let be C a body (2) in R^N and for every $\tau > 0$ let us set

$$S_\tau^C f = \sum_{n \in Z^N \cap \tau C} \hat{f}(n) e^{2\pi i n t}, \qquad t \in T^N.$$

COROLLARY 3. *Let be $f \in B$ and $k p_o > N$.*
Then, for every $\varepsilon > 0$, there exists $\sigma_o(\varepsilon)$ such that, if $0 < \sigma < \sigma_o$

$$(5) \qquad \limsup_{\tau \to +\infty} \| f - S_\tau^C f_\sigma \|_B < \varepsilon.$$

Corollary 3 ensures that the method in [3] (and its generalization (3)), when applied to functions $f \in L^1(T^N)$ with N=2,3, gives the some results about the approximation of f by the Fourier polynomials of f_σ previously obtained in the case N=1. This Corollary also ensures that, if $f \in C(T^N)$, it is possible to approximate f uniformly on T^N by the Fourier polynomials of f_σ, not only for N=1 (see Corollary 2), but also for N=2,3...7.

Under the hypotheses of Theorem 4 we also suggest how to choose σ and τ in order that $\| f_\sigma - S_\tau^C f_\sigma \|$ is as small as we want in every admissible norm. Indeed if $k p_o > N$ we have for every $x \in T^N$ (see [1], formula (5,6)):

$$(6) \qquad |f_\sigma(x) - S_\tau^C f_\sigma(x)| \leq \frac{1}{\sigma} \| f \|_{p_o} \left(\sum_{n \in \tau C} p(n)^{-p_o} \right)^{\frac{1}{p_o}}$$

and we can always approximate the series on the right side by a suitable integral.

For example, let us suppose that C is the unit ball of R^N and denote by I the minimun of P on the boundary of S. Then

$$\sum_{n \in \tau S} P(n)^{-p_o} \leq \frac{1}{I} \int_{R^N - \tau S} |x|^{-k p_o} \, dx \; (1+\sigma(1))$$

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(2) i.e. a closed bounded neighbourhood of the point 0 in R^N.

$$= \frac{\alpha}{I} \int_{\tau}^{+\infty} r^{-k p_o + N - 1} \, dr \quad (1 + o(1))$$

$$= \frac{\alpha}{(kp_o - N) I} \tau^{-kp_o + N} \quad (1 + o(1))$$

where α is the surface area of S.
Let us also remark that when C is a cube we can give an evaluation of the right side of (6) which holds for every τ ([1], theorem 4). If $kp_o \leq N$, Theorem 4 and Corollary 3 don't hold in general. Nevertheless, the approximation result (5) still holds if f and C are such that

$$\| f - S_\tau^C f \|_B \to 0 \qquad \text{when } \tau \to +\infty,$$

i.e. if Fourier series of f converges to f in B-norm in the considered summation method.
In particular if C is a cube, (5) holds for every $f \in L^p(T^N)$, $1 < p \leq \infty$ in every space L^1 with $1 \leq s \leq p$; if C is a sphere, (5) holds for every $f \in L^2(T^N)$ in every L^1 with $1 \leq s \leq 2$.

3. The key of proofs of theorems 2,3 and 4 is the following one ([1], th.5). The function $g: R^N \to R$ given by

$$g(x) = (1+P(x))^{-\alpha}, \qquad \alpha > 0$$

is the Fourier transform of a function $G \in L^1(R^N) \cap C^{(\infty)}(R^N/\{0\})$ which goes to zero when $|x| \to \infty$ as fast as we want and doesn't go to infinity too fast when $|x| \to 0$.
This allows to apply the theory of multipliers in L^p; this theory and some other tricks of Harmonic Analysis give the theorems.
It is interesting to remark that if we weaken just one of the hypoteses on P, g doesn't have, in general, the above properties.
Thus, the obtained results are, in some sense, the best possible results.

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A MEASURE THEORETICAL MAX-FLOW-PROBLEM

Konrad Jacobs in Erlangen

The following generalization of Ford-Fulkerson's max-flow problem in finite networks is discussed. Let X be a compact metric space and Ω a compact metric space of paths in X. Let P be a measure kernel from Ω to X such that for every $0 \leq f \in C(X, \mathbb{R})$ the function Pf is lower semicontinuous on Ω. A measure μ in Ω is called a flow, a measure c in X is called a capacity (constraint) and P is called the feeding kernel. We pose the following

Max-Flow Problem. Maximize the strength $\mu 1 = \mu(\Omega)$ of the flow μ under the constraint

$$\mu P \leq c$$

Interpretation: an animal migrating through X on path $\omega \in \Omega$ eats $P(\omega,.)$ of the food available according to c.

The dual problem is:

Minimize cf by choice of $0 \leq f \in C(X, \mathbb{R})$ under the constraint

$$1 \leq Pf$$

The following inequalities are obvious

$$\max_{\mu P \leq c} \mu 1 \leq \max_{\mu P \leq c} \inf_{1 \leq Pf} \mu Pf$$

$$\leq \inf_{1 \leq Pf} \max_{\mu P \leq c} \mu Pf$$

$$\leq \inf_{1 \leq Pf} cf.$$

An application of the Hahn-Banach theorem shows that equality

holds here in all three places (letter of H.Kellerer (München) to the author, dated 16.11.1984). If we interpret the f with $1 \leq Pf$ as cuts, the equality

$$\max_{\mu P \leq c} \mu 1 = \inf_{1 \leq Pf} cf$$

is the counterpart of Ford-Fulkerson's max-flow-min-cut theorem in the present context. In a special case, the author has obtained an approximation theorem which allows to find approximative solutions to the above max-flow problem by application of the finite-network theory (Jacobs [1984]). The question of how to obtain "min" instead of "inf" in Kellerer's result, plus the search for an analogon of the marking algorithm in the measure-theoretical case, plus the search for more general versions of the mentioned approximation theorem etc. is the objective of current research in Erlangen

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STABILITY THEOREMS FOR THE FEYNMAN INTEGRAL[1]

by G.W. Johnson[2] and D.L. Skoug[3]

1. Introduction. The Feynman integral was introduced by Feynman in 1948 [19] as an alternate approach to quantum mechanics which restores, in altered form of course, many concepts of classical mechanics and makes clearer, at least heuristically, the relationship between classical and quantum mechanics. Physicists have found the subject intuitively appealing since its inception, and, in the last several years especially, rather heavy use has been made of the integral for several topics of current physical interest. See [37 or 38] for example.

Despite considerable progress in recent years it is still fair to say that the Feynman integral has proven difficult to deal with in a mathematically rigorous way. An abundance of different definitions of "the" Feynman integral have been given usually with no clear indication of how they are related to one another. Pleasant analytic properties have been difficult to establish. One of the shortcomings that has been pointed to, the lack of satisfactory stability theorems, is the subject of this paper. The physical world seems to be stable under small perturbations of the potential and the initial state, but it has not been clear that the mathematical model provided by the Feynman integral has such stability properties.

The main result in a recent paper by the 1st author [23] was, as far as we know, the 1st reasonably satisfactory stability theorem. [23] dealt with complex-valued (or $\mathbb{C}$-valued), essentially bounded, time-independent potentials and depended in an essential way on an earlier paper [27] of both authors. The setting of [23] was intentionally restricted in order to exhibit clearly the basic simplicity of the result, but it was conjectured [23,p.1326] that the results could be extended to $\mathbb{C}$-valued, time-dependent potentials with certain

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2. Presenter.
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kinds of singularities as in [27]. We carry this out in the present paper. The outline of our proof follows [23] rather closely, but filling in the outline is definitely more involved. Most of the additional complications are handled by adapting techniques from [27]. We include most of these adaptations since some of them are rather complicated, since they involve techniques from scattered parts of a long paper, and since these techniques are somewhat obscured, for our present purposes, by the more general framework of [27].

The fact that we are able to treat potentials which are time-dependent and complex-valued is of interest physically. In particular, complex-valued potentials are of interest in connection with "open quantum systems" [18].

There were stability results in the literature before [23] that we are aware of, but they are not very satisfactory. Most of the work in the mathematical theory of the Feynman integral in recent years has involved one or another "transform approach". (See [1,2,9-17,22,28-30,32,33,39] for example.) When these theories are applied to specific quantum mechanical problems, the assumption is made that the potential and the initial state involved are Fourier transforms of bounded Borel measures on the "configuration space" $\mathbb{R}^\nu$. (In the setting of Albeverio and Høegh-Krohn's beautiful theory of the Fresnel integral [1,2], the authors and Kun Chang have recently shown [14] that this assumption on the initial state is actually necessary. We have some further related partial results.) Stability of the Fresnel (or Feynman) integral with respect to convergence in the total variation norm on the measures is built into the theory. Further, it seems certain that at least somewhat more can be done in the direction of stability theorems. However there seem to be severe limitations on how far this can be carried. It is often difficult to tell if a function is a transform or not, but if it fails to be bounded or uniformly continuous, it is definitely not a transform. Also the space of transforms is not closed under pointwise or even uniform convergence.

We close this introduction by calling the reader's attention to a nice stability theorem [35] which was very recently proved by Michel Lapidus for his "modified Feynman integral". The modified Feynman integral is a variation of Nelson's approach to the Feynman integral via the Trotter product formula [36]. See [34] for the most recent information on the modified Feynman integral. Lapidus' work [35] was motivated by a preprint of [23] and by discussions with the 1st author.

2. Background. Our setting is essentially an approach to the Feynman integral introduced by Cameron and Storvick [4] and studied further by them [5-8] and others including Haugsby [20] and the authors [25-27]. The approach is closely related to the 2nd approach in Nelson's beautiful paper [36].

For our present purposes, $t > 0$ will be fixed. $C[0,t]$ will denote the space of $\mathbb{R}$-valued continuous functions on $[0,t]$. $C_0[0,t]$ denotes the space of functions x in $C[0,t]$ such that $x(0) = 0$. $C^\nu[0,t]$ and $C_0^\nu[0,t]$ will denote the product of ν copies of $C[0,t]$ and $C_0[0,t]$ respectively. m will be referred to as Wiener measure and will denote the product of ν copies of 1-dimensional Wiener measure.

Let $F : C^\nu[0,t] \to \mathbb{C}$. We mention that functions F of the form

$$F(Y) = \exp\{\int_0^t \theta(t-s, Y(s))ds\}$$

where θ is the potential function involved in the problem are of primary interest in quantum mechanics and will be our concern in the stability theorem.

Let $1 < p \le 2$ be given and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Given $\lambda > 0$, ψ in $L_p(\mathbb{R}^\nu)$, and ξ in $\mathbb{R}^\nu$, let

$$(I_\lambda(F)\psi)(\xi) = \int_{C_0^\nu[0,t]} F(\lambda^{-\frac{1}{2}}X+\xi)\psi(\lambda^{-\frac{1}{2}}X(t)+\xi)dm(X).$$

This formula may define, for each $\lambda > 0$, a bounded linear operator $I_\lambda(F)$ in $\mathcal{L} = \mathcal{L}(L_p(\mathbb{R}^\nu), L_{p'}(\mathbb{R}^\nu))$, the space of bounded linear operators from $L_p(\mathbb{R}^\nu)$ to $L_{p'}(\mathbb{R}^\nu)$. If this is so, and if the operator-valued function $\lambda \to I_\lambda(F)$ has an analytic continuation to $\mathbb{C}^+ := \{\lambda \text{ in } \mathbb{C} : \text{Re } \lambda > 0\}$, we denote this analytic continuation $I_\lambda^{an}(F)$. Finally given a real parameter q, $q \neq 0$, the operator-valued, analytic Feynman integral $J_q^{an}(F)$ is defined by

$$J(F) = J_q^{an}(F) := \lim_{\lambda \to -iq} I_\lambda^{an}(F)$$

where the limit is taken in the strong operator topology and where λ approaches $-iq$ through $\mathbb{C}^+$.

Except for one remark, we will be considering only the analytic, operator-valued Feynman integral. Also, except for Corollary 2 of Theorem 2, q will be arbitrary but fixed. Hence we will usually simply write $J(F)$ rather than $J_q^{an}(F)$.

That the Wiener integral is somehow related to Feynman's integral was already hinted at in Feynman's original paper [19]. In [31] Kac established a precise tie, the now famous "Feynman-Kac formula". The first rigorous paper using analytic continuation to define the Feynman integral seems to have been Cameron's 1960 paper [3].

3. The Stability Theorem. We assume throughout unless we specifically mention otherwise that the numbers p, γ, ν and r satisfy the following restrictions:

(a) $1 < p \le 2$.

(b) $\gamma = \frac{p}{2-p}$. Note that $\gamma = \gamma(p)$ is an increasing function taking the interval $(1,2]$ onto the interval $(1,+\infty]$. We remark that γ is chosen so that multiplication by a function in $L_\gamma(\mathbb{R}^\nu)$ is a bounded linear operator from $L_{p'}(\mathbb{R}^\nu)$ to $L_p(\mathbb{R}^\nu)$.

(c) ν is a positive integer satisfying $1 \le \nu < 2\gamma$.

(d) $\frac{2\gamma}{2\gamma-\nu} < r \le +\infty$ except that we allow $1 \le r \le +\infty$ when $p = 2$ and $\gamma = +\infty$.

The mixed norm space $L_{\gamma r} = L_{\gamma r}([0,t]\times\mathbb{R}^\nu)$ will play a key role throughout. A complex-valued Lebesque measurable function g on $[0,t]\times\mathbb{R}^\nu$ is said to be in $L_{\gamma r}$ if and only if $g(s,\cdot)$ is in $L_\gamma(\mathbb{R}^\nu)$ for almost every (a.e.) s in $[0,t]$ and $||g(s,\cdot)||_\gamma$ is in $L_r([0,t])$. Note that $L_{\gamma r_2} \subseteq L_{\gamma r_1}$ if $1 \le r_1 \le r_2 \le +\infty$.

The following existence theorem is essentially contained in [27,Theorem 5.1, p.126].

Theorem 1. Let θ belong to $L_{\gamma r}([0,t]\times\mathbb{R}^\nu)$. Let $F : C^\nu[0,t] \to \mathbb{C}$ be given by

$$(1)\quad F(X) = \exp\{\int_0^t \theta(t-s,X(s))ds\}.$$

Then $J(F)$ exists as an element of $\mathcal{L} = \mathcal{L}(L_p(\mathbb{R}^\nu), L_{p'}(\mathbb{R}^\nu))$ and, for any ψ in $L_p(\mathbb{R}^\nu)$, it is given by

$$\begin{aligned}(2)\quad (J(F)\psi)(\xi) = \sum_{n=0}^{\infty} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} (-iq)^{\nu/2}\,[2\pi(t-s_n)]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \theta(s_n,V_n)\exp[\frac{iq||V_n-\xi||^2}{2(t-s_n)}](-iq)^{\nu/2}[2\pi(s_n-s_{n-1})]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \theta(s_{n-1},V_{n-1})\exp[\frac{iq||V_{n-1}-V_n||^2}{2(s_n-s_{n-1})}]\times\cdots\times(-iq)^{\nu/2}[2\pi(s_2-s_1)]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \theta(s_1,V_1)\exp[\frac{iq||V_1-V_2||^2}{2(s_2-s_1)}](-iq)^{\nu/2}[2\pi s_1]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \psi(V_0)\exp[\frac{iq||V_0-V_1||^2}{2s_1}]dV_0dV_1\cdots dV_n ds_1\cdots ds_n\end{aligned}$$

where the integrals over $\mathbb{R}^\nu$ are interpreted in the mean (as in the L_p-Fourier transform theory) and the integral over

$$\Delta_n := \{(s_1,\ldots,s_n): 0<s_1<\ldots<s_n<t\}$$

may be interpreted either as a Bochner integral [21,p.71-89] of an $L_{p'}(\mathbb{R}^\nu)$-valued function or as a Lebesgue integral.

(2) is the "Dyson series" or "perturbation series" associated with the system. We will need some additional notation and further information about the terms involved in (2). Let

$$\begin{aligned}(3)\quad (B_n(F)(s_1,\ldots,s_n)\psi)(\xi) := (-iq)^{\nu/2}[2\pi(t-s_n)]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \theta(s_n,V_n)\exp[\frac{iq||V_n-\xi||^2}{2(t-s_n)}](-iq)^{\nu/2}[2\pi(s_n-s_{n-1})]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \theta(s_{n-1},V_{n-1})\exp[\frac{iq||V_{n-1}-V_n||^2}{2(s_n-s_{n-1})}]\times\cdots\times(-iq)^{\nu/2}[2\pi(s_2-s_1)]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \theta(s_1,V_1)\exp[\frac{iq||V_1-V_2||^2}{2(s_2-s_1)}](-iq)^{\nu/2}[2\pi s_1]^{-\nu/2} \\ \times \int_{\mathbb{R}^\nu} \psi(V_0)\exp[\frac{iq||V_0-V_1||^2}{2s_1}]dV_0dV_1\cdots dV_n\end{aligned}$$

and let

$$(4)\quad (A_n(F)\psi)(\xi) = \int_{\Delta_n} (B_n(F)(s_1,\ldots,s_n)\psi)(\xi)dS$$

where $S = (s_1,\ldots,s_n)$. Note that $(A_n(F)\psi)(\xi)$ is just the nth term of the series (2).

It is helpful to think of $B_n(F)(s_1,\ldots,s_n)$ as the composition of a succession of alternating convolution and multiplication operators. Let

$$(5)\quad e_s(U) := (-iq)^{\nu/2}[2\pi s]^{-\nu/2}\exp[iq||U||^2/2s]$$

and let C_s be the operator of convolution by e_s; that is,

$$(6)\quad (C_s\psi)(\xi) := (-iq)^{\nu/2}[2\pi s]^{-\nu/2} \int_{\mathbb{R}^\nu} \psi(U)e_s(\xi-U)dU$$

where the integral is interpreted in the mean. It is known (See [27,Lemma 1.1, p.98] for example.) that C_s is in $\mathcal{L}(L_p(\mathbb{R}^\nu),L_{p'}(\mathbb{R}^\nu))$ with

(7) $||C_s|| \le (\frac{|q|}{2\pi s})^{\nu/2\gamma}$. In particular,

(8) $||C_s|| \le 1$ when $p = 2$ and $\gamma = +\infty$.

(In fact when $p = 2$, C_s is well-known to be a unitary operator on L_2 [27,p.99].)

Let $M_{\theta(s,\cdot)}$ denote the operator of multiplication by $\theta(s,\cdot)$. Since $\theta(s,\cdot)$ is in $L_\gamma(\mathbb{R}^\nu)$ for a.e. s, it is an easy consequence of Hölder's inequality that for a.e. s, $M_{\theta(s,\cdot)}$ is in $\mathcal{L}(L_{p'}(\mathbb{R}^\nu), L_p(\mathbb{R}^\nu))$ [27,p.102] and

(9) $||M_{\theta(s,\cdot)}|| \le ||\theta(s,\cdot)||_\gamma$.

Now close inspection of (3) shows that

(10) $B_n(F)(s_1,\ldots,s_n) = C_{t-s_n} \circ M_{\theta(s_n,\cdot)} \circ C_{s_n-s_{n-1}} \circ \cdots \circ C_{s_2-s_1} \circ M_{\theta(s_1,\cdot)} \circ C_{s_1}$

and so from (10), (9), and (7),

(11) $||B_n(F)(s_1,\ldots,s_n)|| \le [\prod_{i=1}^{n} ||\theta(s_i,\cdot)||_\gamma][\prod_{i=1}^{n+1} \frac{|q|}{2\pi(s_i-s_{i-1})}]^{\nu/2\gamma}$

where we have let $s_0 = 0$ and $s_{n+1} = t$.

We begin the proof of our stability theorem with a simple lemma.

Lemma. Let $h(U)$ belong to $L_\gamma(\mathbb{R}^\nu)$. Let $\{\theta_m(U)\}$ be a sequence from $L_\gamma(\mathbb{R}^\nu)$ such that $\theta_m(U) \to \theta(U)$ a.e. and such that, for every m, $|\theta_m(U)| \le h(U)$ for a.e. U. Then θ is in $L_\gamma(\mathbb{R}^\nu)$ and $M_{\theta_m} \to M_\theta$ in the strong operator topology on $\mathcal{L}(L_{p'}(\mathbb{R}^\nu), L_p(\mathbb{R}^\nu))$.

Proof. It is clear that θ is in $L_\gamma(\mathbb{R}^\nu)$ and satisfies $|\theta(U)| \le h(U)$ a.e. . Now

$$|\theta_m(U)-\theta(U)| \le |\theta_m(U)| + |\theta(U)| \le 2h(U)$$

and hence $|\theta_m(U)-\theta(U)|^p \le 2^p h^p(U)$. Let ϕ be in $L_{p'}(\mathbb{R}^\nu)$. Then

(12) $|\phi(U)[\theta_m(U)-\theta(U)]|^p \le 2^p|\phi(U)|^p h^p(U)$.

But, by (9), ϕ in $L_{p'}$ and h in L_γ implies that ϕh is in L_p and so the right-hand side of (12) is integrable. The left-hand side of (12) converges to zero for a.e. U since $\theta_m \to \theta$ a.e. . It now follows from the Dominated Convergence Theorem that

$$||(M_{\theta_m}-M_\theta)\phi||_p^p = \int_{\mathbb{R}^\nu} |\phi(U)[\theta_m(U)-\theta(U)]|^p dU \to 0$$

as $m \to +\infty$ as desired.

Given θ and θ_m, m=1,2,... from $L_{\gamma r}([o,t]\times\mathbb{R}^\nu)$, let F and F_m, m=1,2,... be defined on $C^\nu[0,t]$ by

(13) $F(X) = \exp\{\int_0^t \theta(t-s,X(s))ds\}$,

(14) $F_m(X) = \exp\{\int_0^t \theta_m(t-s,X(s))ds\}$.

Theorem 2. (A Stability Theorem for the Feynman Integral.) Let h be in $L_{\gamma r}$. Let $\{\theta_m\}$ be a sequence from $L_{\gamma r}$ such that $\theta_m(s,U) \to \theta(s,U)$ for a.e. (s,U) and such that, for every m, $|\theta_m(s,U)| \le h(s,U)$ for a.e. (s,U). Then, of course, θ is also in $L_{\gamma r}$ and so $J(F)$ and $J(F_m)$, m=1,2,... all exist by Theorem 1 and, as $m \to +\infty$,

(15) $J(F_m) \to J(F)$ in the strong operator topology on $\mathcal{L}(L_p(\mathbb{R}^\nu), L_{p'}(\mathbb{R}^\nu))$.

Remarks. 1. If the potentials $\{\theta_m\}$ are time-independent, the domination assumption in the theorem reduces to the existence of a function h in $L_\gamma(\mathbb{R}^\nu)$ such that, for every m, $|\theta_m(U)| \le h(U)$ for a.e. U.

2. The stability theorem in [23] is the special case of the above where $p = p' = 2$, $\gamma = +\infty$, and the θ_m's are time-independent.

Proof of Theorem 2. Fix ψ in $L_p(\mathbb{R}^\nu)$. Also, for now, we fix n. We first claim that for a.e. $(s_1,\ldots,s_n)$ in Δ_n

(16) $||B_n(F_m)(s_1,\ldots,s_n)\psi - B_n(F)(s_1,\ldots,s_n)\psi||_{p'} \to 0$ as $m \to \infty$.

Temporarily freeze $(s_1,\ldots,s_n)$ in Δ_n such that

(i) $h(s_i,\cdot)$ is in $L_\gamma(\mathbb{R}^\nu)$, i = 1,...,n;

(ii) $|\theta_m(s_i,U)| \le h(s_i,U)$ for a.e. U, i = 1,...,n, m = 1,2,...; and

(iii) $\theta_m(s_i,U) \to \theta(s_i,U)$ for a.e. U, i = 1,...,n.

For such an $(s_1,\ldots,s_n)$ we see from (9) and (ii) above that the family of multiplication operators $\{M_{\theta_m(s_i,\cdot)}: i=1,\ldots,n; m=1,2,\ldots\}$ is bounded in operator norm by $\max\{||h(s_1,\cdot)||_\gamma,\ldots,||h(s_n,\cdot)||_\gamma\}$. Hence (16) follows from formula (10), the Lemma, and the fact that the composition of operators is jointly continuous in the strong operator topology provided one of the operators is restricted to lie

in a bounded subset of the relevant space of operators.

We will omit the proof that $B_n(F_m)(s_1,\ldots,s_n)$ is measurable as a function of $(s_1,\ldots,s_n)$ in the appropriate sense because (i) the proof can be extracted from the proof of Theorem 2.1 of [27] (See especially the paragraph just after formula (2.13),p.107.), and (ii) Lapidus has shown that a much shorter proof can be given using facts and ideas from the theory of semigroups of operators. This shorter proof in a more general setting will be given in [24].

Next we wish to show that

$$(17)\quad \|A_n(F_m)\psi - A_n(F)\psi\|_{p'} \to 0 \text{ as } m \to \infty.$$

We will use the Dominated Convergence Theorem for Bochner integrals [21, Theorem 3.7.9,p.83]. Recalling the definition (4) of A_n, we see that the necessary pointwise convergence has just been established above. They key to establishing (17) will be to find an appropriate dominating function. In the process we will get norm estimates on $A_n(F_m)\psi$ and $A_n(F)\psi$ which will also be the key to finishing the proof.

Inequality (11) and our assumption that, for every m, $|\theta_m(s,U)| \le h(s,U)$ a.e. allows us to write

$$(18)\quad \|B_n(F_m)(s_1,\ldots,s_n)\psi\|_{p'} \le \Big[\prod_{i=1}^{n}\|\theta_m(s_i,\cdot)\|_\gamma\Big]\Big[\prod_{i=1}^{n+1}\frac{|q|}{2\pi(s_i-s_{i-1})}\Big]^{\nu/2\gamma}\|\psi\|_p$$

$$\le \Big[\prod_{i=1}^{n}\|h(s_i,\cdot)\|_\gamma\Big]\Big[\prod_{i=1}^{n+1}\frac{|q|}{2\pi(s_i-s_{i-1})}\Big]^{\nu/2\gamma}\|\psi\|_p.$$

Note that when $p = 2$ and $\gamma = +\infty$, $\nu/2\gamma = 0$ and the inequality in (18) becomes much simpler. Because of this the rest of the proof becomes <u>much</u> simpler in this case even with $r = 1$. We will write out the rest of the proof under the assumption that $1 < p < 2$. After going through this, the reader should return to this point and check the substantial simplications when $p = 2$.

Now we return to the proof itself. To justify the use of the Dominated Convergence Theorem for Bochner integrals, it suffices to show that

$$(19)\quad \ell(s_1,\ldots,s_n) := \Big[\prod_{i=1}^{n}\|h(s_i,\cdot)\|_\gamma\Big]\Big[\prod_{i=1}^{n+1}\frac{|q|}{2\pi(s_i-s_{i-1})}\Big]^{\nu/2\gamma}\|\psi\|_p$$

is integrable over Δ_n. We will do this by finding a number K_n such that $\int_{\Delta_n}\ell(s_1,\ldots,s_n)dS \le K_n$. The estimate on $\|A_n(F_m)\psi\|_{p'}$, which we need later will then follow immediately from a basic inequality for Bochner integrals [21, Theorem 3.7.6,p.82]:

$$(20)\quad \|A_n(F_m)\psi\|_{p'} \le \int_{\Delta_n}\|B_n(F_m)(s_1,\ldots,s_n)\psi\|_{p'}dS$$

$$\le \int_{\Delta_n}\ell(s_1,\ldots,s_n)dS$$

$$\le K_n.$$

Since $\|B_n(F)(s_1,\ldots,s_n)\psi\|_{p'}$ is also dominated by $\ell(s_1,\ldots,s_n)$, we will immediately have as well that

$$(21)\quad \|A_n(F)\psi\|_{p'} \le K_n.$$

Now by Hölder's inequality and what is sometimes referred to as "the simplex trick"

$$(22)\quad \int_{\Delta_n}\ell(s_1,\ldots,s_n)dS = \int_{\Delta_n}\Big[\prod_{i=1}^{n}\|h(s_i,\cdot)\|_\gamma\Big]\Big[\prod_{i=1}^{n+1}\frac{|q|}{2\pi(s_i-s_{i-1})}\Big]^{\nu/2\gamma}\|\psi\|_p dS$$

$$\le \Big\{\int_{\Delta_n}\Big[\prod_{i=1}^{n}\|h(s_i,\cdot)\|_\gamma^r dS\Big\}^{1/r}\Big\{\int_{\Delta_n}\Big[\prod_{i=1}^{n+1}\frac{|q|}{2\pi(s_i-s_{i-1})}\Big]^{\frac{\nu r'}{2\gamma}}dS\Big\}^{1/r'}\|\psi\|_p$$

$$= \Big\{\frac{1}{n!}\int_{[0,t]^n}\Big[\prod_{i=1}^{n}\|h(s_i,\cdot)\|_\gamma^r\Big]dS\Big\}^{1/r}\Big(\frac{|q|}{2\pi}\Big)^{\frac{(n+1)\nu}{2\gamma}}$$

$$\cdot\Big\{\int_{\Delta_n}[s_1(s_2-s_1)\ldots(s_n-s_{n-1})(t-s_n)]^{-\frac{\nu r'}{2\gamma}}dS\Big\}^{1/r'}\|\psi\|_p$$

$$= \Big(\frac{1}{n!}\Big)^{1/r}\Big\{\prod_{i=1}^{n}\int_0^t\|h(s_i,\cdot)\|_\gamma^r\,ds_i\Big\}^{1/r}\Big(\frac{|q|}{2\pi}\Big)^{\frac{(n+1)\nu}{2\gamma}}$$

$$\cdot\Big\{\int_{\Delta_n}[s_1(s_2-s_1)\ldots(s_n-s_{n-1})(t-s_n)]^{-\frac{\nu r'}{2\gamma}}dS\Big\}^{1/r'}\|\psi\|_p$$

$$= \Big(\frac{1}{n!}\Big)^{1/r}\|h\|_{\gamma r}^n\Big(\frac{|q|}{2\pi}\Big)^{\frac{(n+1)\nu}{2\gamma}}\Big\{\int_{\Delta_n}[s_1(s_2-s_1)\ldots(s_n-s_{n-1})(t-s_n)]^{-\frac{\nu r'}{2\gamma}}dS\Big\}^{1/r'}\|\psi\|_p.$$

Next we let $\delta = \frac{\nu}{2\gamma}$ for convenience and we calculate the remaining integral in the last expression in (22); that is, we calculate

$$(23)\quad H_n = \int_0^t\int_0^{s_n}\cdots\int_0^{s_2}[s_1(s_2-s_1)\ldots(s_n-s_{n-1})(t-s_n)]^{-r'\delta}ds_1\ldots ds_n.$$

Beginning with the innermost integral with respect to s_1, we let $u = \frac{s_1}{s_2}$ and obtain

$$\int_0^{s_2} [s_1(s_2-s_1)]^{-r'\delta} ds_1 = s_2 \int_0^1 [s_2 u(s_2-s_2u)]^{-r'\delta} du$$

$$= s_2^{1-2r'\delta} \int_0^1 u^{(1-r'\delta)-1}(1-u)^{(1-r'\delta)-1} du$$

$$= s_2^{1-2r'\delta}\, \beta(1-r'\delta, 1-r'\delta)$$

where β denotes the β-function. At the next stage we make the substitution $u = \frac{s_2}{s_3}$ and obtain

$$\int_0^{s_3} s_2^{1-2r'\delta}(s_3-s_2)^{-r'\delta} ds_2$$

$$= s_3 \int_0^1 (s_3u)^{1-2r'\delta}(s_3-s_3u)^{-r'\delta} du$$

$$= s_3^{2-3r'\delta} \int_0^1 u^{1-2r'\delta}(1-u)^{-r'\delta} du$$

$$= s_3^{2-3r'\delta} \int_0^1 u^{(2-2r'\delta)-1}(1-u)^{(1-r'\delta)-1} du$$

$$= s_3^{2-3r'\delta}\, \beta(2-2r'\delta, 1-r'\delta).$$

We continue in this fashion until at the nth and last stage we make the substitution $u = \frac{s_n}{t}$ and obtain

$$\int_0^t s_n^{(n-1)-nr'\delta}(t-s_n)^{-r'\delta} ds_n$$

$$= t \int_0^1 (tu)^{(n-1)-nr'\delta}(t-tu)^{-r'\delta} du$$

$$= t^{n-(n+1)r'\delta} \int_0^1 u^{(n-1)-nr'\delta}(1-u)^{-r'\delta} du$$

$$= t^{n-(n+1)r'\delta} \beta(n-nr'\delta, 1-r'\delta).$$

Combining the results of these n integrations and using the relationship between the β function and the Γ function we obtain

$$H_n = t^{n-(n+1)r'\delta} \prod_{i=1}^{n} \beta(i-ir'\delta, 1-r'\delta)$$

$$= t^{n-(n+1)r'\delta} \prod_{i=1}^{n} \frac{\Gamma(i-ir'\delta)\Gamma(1-r'\delta)}{\Gamma[(i+1)-(i+1)r'\delta]}$$

$$= t^{n-(n+1)r'\delta} \frac{\Gamma^{n+1}(1-r'\delta)}{\Gamma[(n+1)(1-r'\delta)]}.$$

Returning to (22) we now see that

$$(24)\quad \int_{\Delta_n} \ell(s_1,\ldots,s_n)dS \leq K_n = \left(\frac{1}{n!}\right)^{1/r} ||h||_{\gamma r}^{n} \left(\frac{|q|}{2\pi}\right)^{(n+1)\delta} \frac{t^{n(1-r'\delta)/r'}\{\Gamma(1-r'\delta)\}^{\frac{n+1}{r'}}}{t^{\delta}\{\Gamma(n+1)(1-r'\delta)]\}^{1/r'}} ||\psi||_p.$$

In particular $\ell(s_1,\ldots,s_n)$ is integrable over Δ_n and so the use of the Dominated Convergence Theorem is justified and (17) holds.

We now know that each term of the series $J(F_m)\psi = \sum_{n=0}^{\infty} A_n(F_m)\psi$ converges in $L_{p'}$-norm to the corresponding term of the series $J(F)\psi = \sum_{n=0}^{\infty} A_n(F)\psi$. It remains to show that

$$(25)\quad ||J(F_m)\psi - J(F)\psi||_{p'} \to 0 \text{ as } m \to \infty.$$

As noted in (20) and (21), both $||A_n(F_m)\psi||_{p'}$ and $||A_n(F)\psi||_{p'}$ are bounded by K_n, the right-hand side of (24). We rewrite K_n as:

$$(26)\quad K_n = ||\psi||_p \left(\frac{|q|}{2\pi t}\right)^{\delta} [\Gamma(1-r'\delta)]^{1/r'} \frac{\{||h||_{\gamma r}\left(\frac{|q|}{2\pi}\right)^{\delta} t^{\frac{1-r'\delta}{r'}} [\Gamma(1-r'\delta)]^{1/r'}\}^n}{(n!)^{1/r}\{\Gamma[(n+1)(1-r'\delta)]\}^{1/r'}}$$

It is not hard to see that $\sum_{n=0}^{\infty} K_n < +\infty$.

Now given $\varepsilon > 0$, we take N_0 so large that

$$(27)\quad \sum_{n=N_0+1}^{\infty} K_n < \varepsilon/4.$$

Next using (17), let N be so large that for $m \geq N$,

$$(28)\quad \sum_{n=0}^{N_0} ||A_n(F_m)\psi - A_n(F)\psi||_{p'} < \varepsilon/2 .$$

Now let $m \geq N$. Using (27) and (28), we can write

$$\begin{aligned}
&||J(F_m)\psi - J(F)\psi||_{p'} \\
&= ||\sum_{n=0}^{\infty} A_n(F_m)\psi - \sum_{n=0}^{\infty} A_n(F)\psi||_{p'} \\
&\leq ||\sum_{n=0}^{N_0} [A_n(F_m)\psi - A_n(F)\psi]||_{p'} + \sum_{n=N_0+1}^{\infty} ||A_n(F_m)\psi||_{p'} + \sum_{n=N_0+1}^{\infty} ||A_n(F)\psi||_{p'} \\
&\leq \varepsilon/2 + 2 \sum_{n=N_0+1}^{\infty} K_n \\
&< \varepsilon/2 + 2(\varepsilon/4) = \varepsilon
\end{aligned}$$

as desired which finally concludes the proof of Theorem 2.

Using physical language and speaking a bit loosely, the stability theorem says that a slight perturbation in the potential produces only a slight perturbation in the state $J(F)\psi$ at time t. A simple standard result of functional analysis says that if $\{D_m\}$ is a sequence of operators between two normed linear spaces such that $D_m \to D$ in the strong operator topology and if $||Z_m - Z|| \to 0$, then $||D_m Z_m - DZ|| \to 0$ as $m \to \infty$. As an immediate corollary of this and our stability theorem we see that slight perturbations of both the potential and the initial state produce only a slight perturbation in the state $J(F)\psi$ at time t. We state this precisely.

Corollary 1. Let the hypotheses of Theorem 2 be satisfied and suppose that $||\psi_m - \psi||_p \to 0$ as $m \to \infty$. Then $||J(F_m)\psi_m - J(F)\psi||_{p'} \to 0$ as $m \to \infty$.

Remarks. 1. It was remarked in [23] that the use of the Dominated Convergence Theorem for Bochner integrals was essential to the proof of the stability theorem. The same remark is appropriate here. An attempt to apply the ordinary Dominated Convergence Theorem putting absolute values inside the integrals defining the B_n's simply does not work; the functions need not even be integrable. The use of the Dominated Convergence Theorem for Bochner integrals allows us to take the norm inside the integral with respect to Δ_n but still to take advantage of the cancelling effects of the integrals over $\mathbb{R}^\nu$.

2. We have assumed that $1 < p \leq 2$. We conjecture that a stability theorem can also be proved when $p = 1$. Our experience with [26 and 27] leads us to believe that the proof should be somewhat easier than the proof of Theorem 2 but that the techniques should be rather different.

3. In the case $p = p' = 2$, $\gamma = +\infty$, the "sequential, operator-valued function space integral", $J_q^{seq}(F)$, of Cameron and Storvick [4] is known to exist and equal $J_q^{an}(F)$ [27,Theorem 4.6,p.124]. A stability result for $J_q^{seq}(F)$ in case $p = 2$ is then an immediate corollary of the stability result in this paper. When $1 < p < 2$, it is known [27,p.123-4] that $J_q^{seq}(F)$ need not even exist under the present assumptions on the potential θ.

4. In a paper of the 1st author and Lapidus [24] which is presently in preparation, very general Dyson series are being studied. Although it is not the main point of [24], it appears that an appropriate stability theorem can be established in that setting.

We finish the paper with a 2nd corollary asserting that $J_q(F)\psi$ is stable in the "mass" parameter q. (For a single quantum particle, q may actually be interpreted as the mass of the particle. In a multi-particle system, q is a certain weighted average of the masses of the individual particles.)

Corollary 2. Let θ satisfy the hypotheses of Theorem 1. Let $\{q_m\}$ be a sequence of real numbers which converges to the non-zero real number q. Then $J_{q_m}(F) \to J_q(F)$ in the strong operator topology as $m \to \infty$. In fact, if $\{\theta_m\}$ is a sequence satisfying the hypotheses of Theorem 2, if $||\psi_m - \psi||_p \to 0$, and if $q_m \to q$, then

$$(29) \quad ||J_{q_m}(F_m)\psi_m - J_q(F)\psi||_{p'} \to 0 \quad \text{as } m \to \infty .$$

Proof. This result is a corollary of the technique of proof of Theorem 2 and of Lemma 1.2 on page 100 of [27]. We will sketch the proof.

For present purposes, we denote the convolution operator given by (6) as $C_{s,-iq}$ rather than simply C_s. Note that $C_{s,\lambda}$ also makes sense for λ in $\mathbb{C}^+$. Let ψ be in $L_p(\mathbb{R}^\nu)$. It is shown in Lemma 1.2 of [27] that $||C_{s,\lambda}\psi - C_{s,-iq}\psi||_{p'} \to 0$ as $\lambda \to -iq$ through $\mathbb{C}^+$. It follows easily from this, as we now show , that $||C_{s,-iq_m}\psi - C_{s,-iq}\psi||_{p'} \to 0$ as $m \to \infty$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $||C_{s,\lambda}\psi - C_{s,-iq}\psi||_{p'} < \varepsilon$ whenever λ is in $\mathbb{C}^+$ and $|\lambda + iq| < \delta$. Now choose N such that $m \geq N$ implies that $|q_m - q| < \delta$. Suppose $m \geq N$. Then we can write

$$(30) \quad ||C_{s,-iq_m}\psi - C_{s,-iq}\psi||_{p'} = \lim_{\eta \to -iq_m} ||C_{s,\eta}\psi - C_{s,-iq}\psi||_{p'} < \varepsilon,$$

where $\eta \to -iq_m$ through $\mathbb{C}^+$. It follows from this, the fact that the multiplication operators $M_{\theta(s_j,\cdot)}$ are independent of q, formula (10), and the norm estimate (11), that $B_{n,-iq_m}(F)(s_1,\ldots,s_n) \to B_{n,-iq}(F)(s_1,\ldots,s_n)$ in the strong operator topology on $\mathcal{L}(L_p(\mathbb{R}^\nu), L_{p'}(\mathbb{R}^\nu))$ as $m \to \infty$.

Now since $q_m \to q$ there is a real number Q such that $|q_m| \le Q$ for all m. Replacing $\ell(s_1,\ldots,s_n)$ from (19) with

$$(31)\quad \ell_Q(s_1,\ldots s_n) := \Big[\prod_{i=1}^{n} ||\theta(s_i,\cdot)||_\gamma\Big]\Big[\prod_{i=1}^{n+1}\frac{Q}{2\pi(s_i-s_{i-1})}\Big]^{\nu/2\gamma} ||\psi||_p ,$$

one can argue just as in the proof of Theorem 2 that $||A_{n,-iq_m}(F)\psi - A_{n,-iq}(F)\psi||_{p'} \to 0$ and then that $||J_{q_m}(F)\psi - J_q(F)\psi||_{p'} \to 0$ as $m \to \infty$.

Once the above is understood, it is not hard to see that one can combine ideas from the proof of Theorem 2 and from the argument above and argue that $||J_{q_m}(F_m)\psi - J_q(F)\psi||_{p'} \to 0$ as $m \to \infty$. The key is replacing $\ell_Q(s_1,\ldots,s_n)$ from (31) with

$$\ell_{Q,h}(s_1,\ldots,s_n) := \Big[\prod_{i=1}^{n} ||h(s_i,\cdot)||_\gamma\Big]\Big[\prod_{i=1}^{n+1}\frac{Q}{2\pi(s_i-s_{i-1})}\Big]^{\nu/2\gamma} ||\psi||_p .$$

Finally (29) follows just as Corollary 1 followed from Theorem 2.

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TOPOLOGICAL PROPERTIES OF THE SET OF GLOBAL SOLUTIONS FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS IN A BANACH SPACE

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This paper contains, as main results, the topological characterization, in suitable function spaces, of the set of global solutions for a semilinear Sobolev evolution equation in a real Banach space and for a large class of semilinear equations of evolution in a real or complex Banach space.

1. INTRODUCTION

In this paper we present a generalization of theorem 3 of [16] for a Fréchet space and we deduce some consequences concerning the topological properties of the set of fixed points of nonlinear operators on topological vector function spaces. Using the specialization of the Phillips integral [22] given by Millionščikov [19], we are able to prove that the set of global solutions is an R_δ-set in three important cases of initial value problems : (I) the Cauchy problem in a Fréchet space under the Carathéodory hypotheses, which for existence was considered by Phillips [22]; (II) the abstract Cauchy problem for the semilinear Sobolev evolution equation in a real Banach space studied for the necessary existence theory by Brill [5]; (III) the initial value problem for the class of semilinear equations of evolution in a Banach considered for the existence of mild solutions by Pazy [20].

2. PRELIMINARIES

Consider the class $\mathcal{M}$ of all metrizable spaces. A topological space S is called an *absolute retract for metrizable spaces*, in symbols $S \in AR(\mathcal{M})$, if $S \in \mathcal{M}$ and, for each closed subset F of a metrizable space M, every continuous function $h : F \to S$ admits a continuous extension $\bar{h} : M \to S$ [4, p.87]. It follows from [10, theorem 4.1] that, if S is a convex subset of a locally convex topological vector space, then $S \in AR(\mathcal{M})$. A topological space S is called an *absolute retract*, in symbols $S \in AR$, if S is compact and $S \in AR(\mathcal{M})$ [4 , p.100]. It follows from [4,

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This paper is in final form and no version of it will be submitted for publication elsewhere

property 2.1, p.101] that, if $S \in AR$ and $h : S \to S'$ is a homeomorphism, then $S' \in AR$. A topological space S is called an R_δ-*set* if $S \in \mathcal{M}$ and S is homeomorphic to the intersection of a decreasing sequence in AR [1]. One of the most important properties of this kind of space is the following : Every R_δ-set is acyclic [17, p.110]. In particular, if S is an R_δ-set, then S is non-empty, compact and connected.

Let J and X be topological spaces. The set of all continuous functions from J to X will be denoted by $C(J,X)$. If X is a uniform space, the symbols τ_u, τ_c denote the topology of uniform convergence, the topology of uniform convergence on compacta, respectively, on every non-empty subset of $C(J,X)$. If X is a Hausdorff topological vector space, the symbols $C_b(J,X)$ denote the vector space of all bounded continuous functions from J to X. In this case, it is well-known that $(C(J,X), \tau_c)$ (resp. $(C_b(J,X), \tau_u)$) is a Hausdorff topological vector space whose topology has as local base the collection of all sets of the form $N(K,U) = \{y \in C(J,X) : y[K] \subseteq U\}$ (resp. $N(U) = \{y \in C_b(J,X) : y(J) \subseteq U\}$), where K is a compact subset of J and U is a neighbourhood of 0 in X.

If x is an element of a metric space S and $\varepsilon > 0$, the symbol $B(x,\varepsilon)$ (resp. $\overline{B}((x,\varepsilon))$ denotes the open ball (resp. the closed ball) of center x and radius ε. Let I denote the identity mapping on S.

Recall that a *Fréchet space* is a complete metrizable locally convex topological vector space. It is well-known that its topology can be induced by a complete invariant metric d such that the open balls $B(x,\varepsilon)$ are convex. We note also that if J is a topological space and X is a Fréchet space, then $(C_b(J,X),\tau_u)$ is a Fréchet space and, if further, J is locally compact and countable at infinity, then $(C(J,X), \tau_c)$ is a Fréchet space (see [28, pp. 446-447]).

The following theorem generalizes théorème 3 of [16]:

THEOREM 2.1. *Let E be a Fréchet space, let Ω be an open non-empty subset of E and let $F : \overline{\Omega} \to E$. Assume that F satisfies the following conditions :*

(1) F is continuous.

(2) If (u_n) is a sequence in $\overline{\Omega}$ such that $u_n - F(u_n) \to 0$, then (u_n) has a convergent subsequence.

(3) There exists a sequence (F_n) of continuous mappings from $\overline{\Omega}$ to E such that

(i) $d(F_n(u), F(u)) \leqslant \frac{1}{n}$ *for all* $u \in \overline{\Omega}$ *and all* $n = 1,2,3, \ldots$;

(ii) *for every* $n = 1,2,3,\ldots$, $I - F_n$ *is a homeomorphism of* $R_n = \{u \in \overline{\Omega} : d(u - F_n(u), 0) \leqslant \frac{1}{n}\}$ *onto* $\overline{B}(0, \frac{1}{n})$.

Then the set Fix(F) *of all fixed points of* F *is an* R_δ*-set.*

PROOF. Let $S = \mathrm{Fix}(F)$, let $T = I - F$ and let $T_n = I - F_n$ for every $n = 1,2,3,..$

We will show that S is a non-empty and compact subset of $\overline{\Omega}$. Since every T_n is a surjection from R_n onto $\overline{B}(0,\frac{1}{n})$, there exists $u_n \in R_n$ such that $T_n(u_n) = 0$. So, by (3)(i), $d(u_n - F(u_n), 0) = d(F_n(u_n), F(u_n)) \leqslant \frac{1}{n}$ for all $n = 1,2,3, \ldots$. Therefore, $u_n - F(u_n) \to 0$. Then, by (2), there exists a subsequence (u_{k_n}) of (u_n) and an element $u \in E$ such that $u_{k_n} \to u$. Since $u_n \in \overline{\Omega}$ for every $n = 1,2,3, \ldots$, $u \in \overline{\Omega}$. It follows from (1) that $T(u_{k_n}) \to T(u)$, and therefore $T(u) = 0$. So $u \in S$. Now the compactness of S is an easy consequence of (2).

Since $d(T_n(x),0) \leqslant \frac{1}{n}$ for all $x \in S$ and all $n = 1,2,3,\ldots$, the sets $T_n[S]$ and $Q_n = \overline{co}(T_n[S])$ are contained in the closed ball $\overline{B}(0,\frac{1}{n})$. Since $T_n[S]$ is compact, it follows from theorem 3.25 of [24] that Q_n is compact. So $Q_n \in AR$. Since by (3)(ii), T_n^{-1} is a homeomorphism of $\overline{B}(0,\frac{1}{n})$ onto R_n it follows that $S_n = T_n^{-1}[Q_n] \in AR$.

It is clear that $S \subseteq S_n$ for every $n = 1,2,3, \ldots$. So $S \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} S_k = \lim_n \inf S_n$. Let now $x \in \lim_n \sup S_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} S_k$. Then there exists a subsequence (S_{k_n}) of (S_n) such that $x \in S_{k_n}$ for all $n = 1,2,3, \ldots$. Thus $T_{k_n}(x) \in Q_{k_n}$ and therefore $T_{k_n}(x) \in \overline{B}(0,\frac{1}{k_n})$. Since $d(T_{k_n}(x), T(x)) \leqslant \frac{1}{k_n}$ for every $n = 1,2,3, \ldots$, it follows that $T(x) = 0$. So $x \in S$ and therefore $S = \lim_n S_n$. Let V be a neighbourhood of S in $\overline{\Omega}$. Suppose that there exists a positive integer k such that $n > k$ implies $S_k \not\subseteq V$. For every $n = k+1, k+2, k+3, \ldots$, choose $x_n \in S_n - V$. Let $y_n = x_{n+k}$ for all $n = 1,2,3, \ldots$. Since $d(T(y_n), 0) \leqslant \frac{2}{n+k}$ for all $n = 1,2,3, \ldots$, it follows that (y_n) is a sequence in $\overline{\Omega}$ such that $y_n - F(y_n) \to 0$. Then, by (2), there exists a subsequence (y_{j_n}) of (y_n) and an element $y \in E$ such that $y_{j_n} \to y$. Since $y_{j_n} \in \overline{\Omega}$ for all $n = 1,2,3, \ldots$, $y \in \overline{\Omega}$ and therefore $T(y_{j_n}) \to T(y)$. So $T(y) = 0$. That is $y \in S \subseteq V$. This is a contradiction, because $y_n \notin V$ for all $n = 1,2,3, \ldots$. Consequently, for each neighbourhood V of S in $\overline{\Omega}$ there exists a subsequence of (S_n) contained in V. By lemma 5 of [7] it follows that S is an R_δ-set.

COROLLARY 2.2. *Let J be a compact convex subset of a normed space, let X be a Fréchet space, let $E = (C(J,X), \tau_u)$ and let $F : E \to E$. Assume that F satisfies the following conditions :*

(i) *F is continuous.*

(ii) *There exist* $t_o \in J$ *and* $x_o \in X$ *such that* $F(u)(t_o) = x_o$ *for every* $u \in E$.

(iii) *For every* $\varepsilon > 0$, *the relations* $u,v \in E$ *and* $u|J_\varepsilon = v|J_\varepsilon$ *imply* $F(u)|J_\varepsilon = F(v)|J_\varepsilon$, *where* $J_\varepsilon = J \cap \overline{B}(t_o,\varepsilon)$.

(iv) $F[C(J,X)]$ *is a relatively compact subset of* E.

Then the set Fix(F) *of all fixed points of* F *is an* R_δ*-set.*

PROOF. The condition (iv) implies that $F[C(J,X)]$ is an equicontinuous subset of E, and, since J is compact, $F[C(J,X)]$ is an equiuniformly continuous subset of E. Then, by lemma 1 of [27], there exists a sequence (F_n) of continuous mappings

from E to E such that each $I - F_n$ is a homeomorphism of E onto E and $\lim_{n\to\infty} F_n(u) = F(u)$ uniformly in $u \in E$. Hence we may suppose (by choosing a subsequence of (F_n) if necessary) that $d(F_n(u), F(u)) \leq \frac{1}{n}$ for all $u \in E$ and all $n = 1,2,3, \ldots$. To complete the proof, it suffices to verify the condition (2) of theorem 2.1. Let (u_n) be a sequence in $C(J,X)$ such that $u_n - F(u_n) \to 0$. Since $F(u_n) \in \overline{F[C(J,X)]}$, the condition (iv) implies that there exist a subsequence (u_{k_n}) of (u_n) and an element $u \in C(J,X)$ such that $F(u_{k_n}) \to u$. So $u_{k_n} = u_{k_n} - F(u_{k_n}) + F(u_{k_n}) \to u$.

LEMMA 2.3. *Let J be a bounded and convex subset of a normed space, let X be a Fréchet space, let* $E = (C_b(J,X), \tau_u)$ *and let* $F : E \to E$. *Assume that F satisfies the following conditions :*

(a) *F is continuous,*

(b) *There exist* $t_o \in J$ *and* $x_o \in X$ *such that* $F(u)(t_o) = x_o$ *for every* $u \in E$.

(c) *For every* $\varepsilon > 0$, *the relations* $u, v \in E$ *and* $u|J_\varepsilon = v|J_\varepsilon$ *imply* $F(u)|J_\varepsilon = F(v)|J_\varepsilon$, *where* $J_\varepsilon = J \cap \bar{B}(t_o,\varepsilon)$.

(d) *For every bounded subset V of E, F[V] is an equiuniformly continuous subset of E.*

Then there exists a sequence (F_n) *of continuous mappings from E to E such that each* $I - F_n$ *is a homeomorphism of E onto E and* $\lim_{n\to\infty} F_n(u) = F(u)$ *uniformly on each bounded subset of E.*

PROOF. A trivial modification of the proof of lemma 1 of [27].

Let X and Y be Hausdorff topological vector spaces, let M be a non-empty subset of X and let $F : M \to Y$. The mapping F is said to be *completely continuous* if F is continuous and the image of each bounded subset of M is relatively compact.

THEOREM 2.4. *Let J be a compact convex subset of a normed space, let X be a Banach space, let* $E = (C(J,X), \tau_u)$ *and let* $F : E \to E$. *Assume that F satisfies the following conditions :*

(i) *F is completely continuous.*

(ii) *There exist* $t_o \in J$ *and* $x_o \in X$ *such that* $F(u)(t_o) = x_o$ *for every* $u \in E$.

(iii) *For every* $\varepsilon > 0$, *the relations* $u,v \in E$ *and* $u|J_\varepsilon = v|J_\varepsilon$ *imply* $F(u)|J_\varepsilon = F(v)|J_\varepsilon$, *where* $J_\varepsilon = J \cap \bar{B}(t_o,\varepsilon)$.

If the set Fix(F) *of all fixed points of F is a non-empty and bounded subset of E, then* Fix(F) *is an* R_δ*-set.*

PROOF. We note first that E is a Banach space, and if we take $d(u,v) = \sup_{t\in J} ||u(t) - v(t)||$, where $u,v \in E$ and $||.||$ denotes the norm on X, the notions of boundedness in E and d-boundedness coincide.

Let V be a bounded subset of E. Then, by the condition (i), $\overline{F[V]}$ is compact,

and therefore F[V] is equicontinuous. Since J is compact, F[V] is an equiuniformly continuous subset of E. Then, by lemma 2.3, there exists a sequence (F_n) of continuous mappings from E to E such that each $I - F_n$ is a homeomorphism of E onto E and $\lim_{n\to\infty} F_n(u) = F(u)$ uniformly on each bounded subset V of E. Hence there exists an strictly increasing sequence $(k_n(V))$ of positive integers such that $d(F_{k_n(V)}(u), F(u)) \leq \frac{1}{n}$ for all $u \in V$ and all $n = 1,2,3, \ldots$

Let $S = Fix(F)$. Since S is a non-empty and bounded subset of E, $S \subseteq \bar{B}(0,r)$, where $r = 1 + d(0,S) + \delta(S)$ and $\delta(S)$ denotes the diameter of S. Put $\Omega = B(0,r)$, $G = F|\bar{\Omega}$ and $G_n = F_{k_n(\bar{\Omega})} | \bar{\Omega}$ for $n = 1,2,3,\ldots$ Then G and the sequence (G_n) satisfy trivially the conditions (1) and (3) of theorem 2.1. To verify the condition (2), let (u_n) be a sequence in $\bar{\Omega}$ such that $u_n - G(u_n) \to 0$. So $G(u_n) \in G[\bar{\Omega}]$. Since $\overline{G[\bar{\Omega}]} = \overline{F[\bar{\Omega}]}$ is compact, there exists a subsequence (u_{k_n}) of (u_n) and an element $u \in \overline{G[\bar{\Omega}]}$ such that $G(u_{k_n}) \to u$. Therefore $u_{k_n} = u_{k_u} - G(u_{k_n}) + G(u_{k_n}) \to u$.

Since $S = Fix(G)$, theorem 2.1 implies that S is an R_δ-set.

LEMMA 2.5. *Let* $J = [0,\infty)$ *be a half-line of the real line, let X be a Banach space, let* $f : J \times X \to X$ *be a continuous mapping and let* $(Nu)(t) = f(t,u(t))$ *for all* $u \in C(J,X)$ *and all* $t \in J$. *Then, for every* $u \in C(J,X)$, *every compact subset K of J and every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $||(Nu)(t) - (Nv)(t)|| < \varepsilon$ *for all* $t \in K$ *whenever* $v \in C(J,X)$ *and* $||u(t) - v(t)|| < \delta$ *for all* $t \in K$.

PROOF. Let $u \in C(J,X)$, let K be a compact subset of J and let $\varepsilon > 0$.

Since $f|K \times X$ is continuous, for every $s \in K$ there exist $\delta_s > 0$ and $\eta_s > 0$ such that $||f(t,x) - f(s,u(s))|| < \frac{1}{2}\varepsilon$ whenever $(t,x) \in K \times X$, $|t-s| < \eta_s$ and $||x-u(s)|| < \delta_s$.

Since $u|K$ is continuous, for every $s \in K$, there exists $\omega_s > 0$ such that $\omega_s < \eta_s$ and $t \in K$, $|t-s| < \omega_s$ imply $||u(t)-u(s)|| < \frac{1}{2}\delta_s$.

From the compactness of K it follows that there exists a finite sequence $(s_i)_{1\leq i\leq n}$ in K such that $K \subseteq \bigcup_{i=1}^{n} (s_i-\omega_{s_i}, s_i+\omega_{s_i})$. Let $\delta = \min\{\frac{1}{2}\delta_i : i = 1,2,3,\ldots,n\}$. Then $\delta > 0$. Let $v \in C(J,X)$ be such that $||u(t)-v(t)|| < \delta$ for all $t \in K$. Let $t \in K$. Then there exists $i = 1,2,\ldots,n$ such that $t \in (s_i-\omega_{s_i}, s_i+\omega_{s_i}) \subseteq (s_i-\eta_{s_i}, s_i+\eta_{s_i})$.

Since $||u(t)-u(s_i)|| < \frac{1}{2}\delta_{s_i} < \delta_{s_i}$ and $||u(s_i)-v(t)|| \leq ||u(s_i)-u(t)|| + ||u(t)-v(t)|| < \frac{1}{2}\delta_{s_i} + \delta \leq \frac{1}{2}\delta_{s_i} + \frac{1}{2}\delta_{s_i} = \delta_{s_i}$, we get finally $||(Nu)(t)-(Nv)(t)|| = ||f(t,u(t)) - f(t,v(t))|| \leq ||f(t,u(t)) - f(s_i,u(s_i))|| + ||f(s_i,u(s_i)) - f(t,v(t))|| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

3. THE PHILLIPS INTEGRAL

We present briefly the specialization of the Phillips integral [22] given by Millionščikov [19], and a list of its main properties.

Let X be a complete Hausdorff locally convex topological vector space and let P be a separating family of semi-norms on X generating its topology. Let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$, let μ the Lebesgue measure on $\mathbb{R}$ and let $A \in \mathcal{L}$. A function x from A to X is *countably-valued* if there exists a countable Lebesgue measurable partition (A_i) of A such that, for every i, $x|A_i$ is a constant of value x_i.

A function $y : A \to X$ is said to be *Phillips integrable* if there exists a net $(x_\alpha)_{\alpha \in D}$ of functions from A to X with the following properties :

(i) each x_α is countably-valued,

(ii) for every $p \in P$, $I_p(x_\alpha) = \sum_i p(x_\alpha^i)\mu(A_\alpha^i) < \infty$, where (A_α^i) is the partition of A such that $x_\alpha|A_\alpha^i$ is a constant of value x_α^i,

(iii) for every $p \in P$ and every $\varepsilon > 0$, there exists $\alpha_o \in D$ such that α', $\alpha'' \in D$ and $\alpha', \alpha'' \geqslant \alpha_o$ imply $I_p(x_{\alpha'} - x_{\alpha''}) < \varepsilon$,

(iv) for every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{L}$ such that $A_\varepsilon \subseteq A$, $\mu(A_\varepsilon) < \varepsilon$ and $(x_\alpha)_{\alpha \in D}$ converges uniformly to y on $A - A_\varepsilon$.

Since X is complete and Hausdorff, the properties (i) - (iii) imply that $\sum_i \mu(A_\alpha^i)x_\alpha^i \in X$ for all $\alpha \in D$ and the corresponding net $(\sum_i \mu(A_\alpha^i)x_\alpha^i)_{\alpha \in D}$ converges to a well-defined element $I(x_\alpha)$ of X. Moreover, using the property (iv) it can be shown that, if $(y_\alpha)_{\alpha \in D}$ is another net of functions from A to X satisfying the conditions (i) - (iv), then $I(y_\alpha) = I(x_\alpha)$. By definition, $\lim_\alpha \sum_i \mu(A_\alpha^i)x_\alpha^i$ is called the *Phillips integral* of y on A and it is denoted by $\int_A y(t)dt$.

We indicate some important properties of this integral :

1. If $y : A \to X$ is integrable, then, for all $p \in P$, $\int_A p(y(t))dt < \infty$ and $p(\int_A y(t)dt) \leqslant \int_A p(y(t))dt$.

2. Let $y : A \to X$ be integrable. If $A = \bigcup_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{L}$ and $A_i \cap A_j = \phi$ if $i \neq j$, then $\int_A y(t)dt = \sum_{i=1}^{\infty} \int_{A_i} y(t)dt$.

3. Let $y : A \to X$ be integrable. Then, for every neighbourhood U of 0 in X, there exists $\delta > 0$ such that $A' \in \mathcal{L}$, $A' \subseteq A$ and $\mu(A') < \delta$ imply $\int_{A'} y(t)dt \in U$.

4. Let $y_1, y_2 : A \to X$ be integrable and let a and b be scalars. Then $ay_1 + by_2$ is integrable and $\int_A (ay_1 + by_2)(t)\,dt = a\int_A y_1(t)dt + b\int_A y_2(t)dt$.

5. Let A be compact and let $y : A \to X$ be a function such that $y[A]$ is a bounded subset of X. If, for every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{L}$ such that $A_\varepsilon \subseteq A$, $\mu(A_\varepsilon) < \varepsilon$ and $y|A-A_\varepsilon$ is continuous, then y is integrable and $\int_A y(t)dt \in \mu(A)\overline{co}(y[A])$.

6. If $y : A \to X$ is integrable and continuous at $t_o \in A$, then

$$\lim_{\delta(A') \to 0,\ t_o \in A'} \frac{1}{\mu(A')} \int_{A'} y(t)dt = y(t_o),$$

where $\delta(A')$ denotes the diameter of E'.

7. Let Y be a complete Hausdorff locally convex topological vector space, let $H : X \to Y$ be a linear continuous operator and let $y : A \to X$. If y is integrable, then $H \circ y$ is integrable and $H(\int_A y(t)dt) = \int_A (H \circ y)(t)dt$.

Henceforth, when A is an interval of $\mathbb{R}$ of extremities a and b with $a \leqslant b$, we write $\int_A y(t)dt = \int_a^b y(t)dt$ and $\int_b^a y(t)dt = -\int_a^b y(t)dt$.

Finally we note that this specialization of the Phillips integral substantially generalizes, when X is a Banach space, the Bochner and the Pettis integral with respect to the Lebesgue measure on $\mathbb{R}$. A detailed discussion of these integrals may be found in [9, pp. 41-56] and [12, pp. 71-89].

4. APPLICATIONS TO DIFFERENTIAL EQUATIONS

This section contains, as examples, three applications of the results of Section 2, using the specialization of the Phillips integral presented in Section 3.

(I) Let $J = [0,1]$ be a compact interval of the real line, let X be a Fréchet space, let $u_o \in X$ and let $f : J \times X \to X$.

Consider the Cauchy problem

$$(CP) \qquad u' = f(t,u),\ u(0) = u_o.$$

A *global solution* of (CP) is a function $u : J \to X$ such that

(i) $u(0) = u_o$,

(ii) u is absolutely continuous,

(iii) u is differentiable on (0,1),

(iv) $u'(t) = f(t, u(t))$ for almost all $t \in (0,1)$.

We note that, if the function $s \to f(s, u(s))$ is integrable on each compact interval $[0,t]$ of $\mathbb{R}$ $(t \in J)$ for every $u \in C(J,X)$, then (CP) is equivalent to the integral equation

$$(IE) \qquad u(t) = u_o + \int_o^t f(s, u(s))ds, \quad t \in J.$$

Therefore the set of all global solutions of (CP) is exactly the set of all fixed points of the integral operator F defined by the formula $F(u)(t) = u_o + \int_o^t f(s,u(s))ds$ for all $t \in J$ and all $u \in C(J,X)$.

THEOREM 4.1. *Assume that* $f : J \times X \to X$ *satisfies the following conditions :*

(1) *For every* $x \in X$, $f(.,x)$ *is measurable.*

(2) *For every* $t \in J$, $f(t,.)$ *is continuous.*

(3) $f[J \times X]$ *is a compact subset of* X.

Then the set of global solutions of (CP) *is an* R_δ*-set in* $(C(J,X), \tau_u)$.

PROOF. Let $u \in C(J,X)$. Since u is measurable, it follows from lemma 9.2 of [22] that the function $t \to f(t, u(t))$ is measurable. We will show that this function is integrable. Let (U_n) be a countable local base at 0 in X such that $U_{n+1} \subseteq U_n$ for all n = 1,2,3,... and let $K = f[J \times X]$. Since K is compact, it is totally bounded. Then, for every n = 1,2,3,..., there exists a finite subset K_n of K such that $K \subseteq U_n + K_n$. Put $K_n = \{x_n^1, x_n^2, \ldots, x_n^{k_n}\}$ for all n = 1,2,3,... . Then, for all $x \in K$ and all n = 1,2,3,..., there exists $i \leqslant k_n$ such that $x_n^i - x \in U_n \subseteq U_k$ for

all $k \leq n$. Define $B_n^i = \{t \in J : x_n^i - f(t, u(t)) \in U_n\}$ for all $i = 1,2,\ldots, k_n$. It is clear that each B_n^i is Lebesgue measurable. Put $A_n^i = B_n^i - \bigcup_{j=1}^{i-1} B_n^j$ for all $i = 1,2,3,\ldots, k_n$. Since $(A_n^i)_{1\leq i\leq k_n}$ is a partition of J, we can define the countably-valued function $y_n : J \longrightarrow X$ by the formula $y_n(t) = x_n^i$ on A_n^i. Then (y_n) converges uniformly to the function $t \longrightarrow f(t, u(t))$, and the remaining conditions (i) - (iii) of the definition of integrable functions are trivially satisfied.

Hence we can define the integral operator $F(u)(t) = u_o + \int_o^t f(s,u(s))ds$ for all $t \in J$ and $u \in C(J,X)$. Consequently, the set of global solutions of (CP) is exactly the set Fix(F) of all fixed points of F. It is clear that F satisfies conditions (ii) and (iii) of corollary 2.2.

We will show now that $F[C(J,X)]$ is an equicontinuous subset of C(J,X). Let $t_o \in J$ and let U be a balanced convex neighbourhood of 0 in X. Since $K = f[J \times X]$ is bounded, there exists $\eta > 0$ such that $\eta\ \overline{co}(K) \subseteq U$. Let $t \in J$. Then, for all $u \in C(J,X)$, the property 5 of Section 3 implies that $F(u)(t) - F(u)(t_o) = \int_{t_o}^t f(s, u(s))ds \in (t-t_o)\ \overline{co}(K) \subseteq \frac{t-t_o}{\eta} U \subseteq U$ if $|t - t_o| < \eta$.

To prove the continuity of F, let $u \in C(J,X)$ and let (u_n) be a sequence in $C(J,X)$ τ_u - converging to u. Let $p \in P$. Since $f[J \times X]$ is bounded, there exists a constant $M_p > 0$ such that $p(f(s,x)) \leq M_p$ for all $(s,x) \in J \times X$. On the other hand, the continuity of f(s,.) implies that $\lim_{n\to\infty} f(s,u_n(s)) = f(s,u(s))$, and therefore $\lim_{n\to\infty} p(f(s,u_n(s)) - f(s,u(s))) = 0$ for all $s \in J$. Since $p(f(s,u_n(s)) - f(s,u(s))) \leq 2M_p$ for all $s \in J$ and $n = 1,2,3,\ldots$, the Lebesgue's dominated convergence theorem implies that $\lim_{n\to\infty} \int_o^t p(f(s,u_n(s)) - f(s,u(s)))ds = 0$ for all $t \in J$. Then, by the property 1 of Section 3, we get $\lim_{n\to\infty} p(\int_o^t (f(s,u_n(s)) - f(s,u(s)))ds = 0$, and therefore $\lim_{n\to\infty} F(u_n)(t) = F(u)(t)$ for all $t \in J$. The equicontinuity of $F[C(J,X)]$ and the compactness of J imply the τ_u-convergence of the sequence $(F(u_n))$ to F(u).

To show that $F[C(J,X)]$ is relatively τ_u-compact it remains, by the Ascoli theorem, to show that, for every $t \in J$, the set $\{F(u)(t) : u \in C(J,X)\}$ is a relatively compact subset of X. Let $t \in J$ and let $K = f[J \times X]$. Since K is compact, $u_o + t\ \overline{co}(K)$ is also compact. For every $u \in C(J,X)$ we have $F(u)(t) = u_o + \int_o^t f(s,u(s))ds \in u_o + t\ \overline{co}(K)$ and therefore $\{F(u(t) : u \in C(J,X)\} \subseteq u_o + t\ \overline{co}(K)$. Thus $\{F(u)(t) : u \in C(J,X)\}$ is relatively compact.

The corollary 2.2 implies finally that Fix(F) is a R_δ-set in $(C(J,X), \tau_u)$.

REMARK 4.2. Theorem 4.1 contains the existence theorem of Phillips [22, theorem 9.5] and generalizes a result of Pulvirenti [23]. Moreover, using the extension theorem of Dugundji [10] it is easy to see that theorem 4.1 contains an existence theorem of Knight [15] and generalizes two results of Szufla ([25, theorem 6] and [26]).

(II) Let X and Y be real Banach spaces. Suppose that $A : D(A) \subseteq X \longrightarrow Y$ and $B : D(B) \subseteq X \longrightarrow Y$ are linear operators satisfying the following conditions :

(1) $D(B) \subseteq D(A)$,

(2) A and B are closed operators,

(3) B is bijective,

(4) $B^{-1} : Y \longrightarrow D(B)$ is a compact operator.

The conditions (1) - (3) imply, by the closed graph theorem, that the linear operator $AB^{-1} : Y \longrightarrow Y$ is continuous.

Let $J = [0,a]$, with $a > 0$, be a compact interval of the real line. Suppose that $f : J \times X \longrightarrow Y$ is a function satisfying the following conditions :

(5) f is continuous,

(6) there exists a constant $c > 0$ such that $||f(t,x)|| \leq c(1 + ||x||)$ for all $(t,x) \in J \times X$.

We are concerned here with the semilinear Sobolev evolution equation

(α) $\frac{d}{dt} Bu(t) + Au(t) = f(t,u(t))$.

As was indicated in [5], we note that the equation (α) contains as a special case the pseudo-parabolic partial differential equation

(β) $\frac{\partial}{\partial t} Mu(t,x) + Lu(t,x) = \tilde{f}(t,x,(D^\alpha u(t,x))_{|\alpha|\leq 2m-1})$

where M and L are linear elliptic differential operators of orders 2m and 2n, respectively, with $m \geq n$. The equation (β) appears, for example, in thermodynamics, in the flow of fluid through fissured rocks and in soil mechanics.

If $u_o \in D(B)$ we can consider the Cauchy problem for the equation (α) :

(CP) $$(Bu)' + Au = f(t,u),\ u(0) = u_o.$$

A *global solution* of (CP) is a continuous function $u : J \longrightarrow X$ such that

a) $u(t) \in D(B)$ for all $t \in J$ and $u(0) = u_o$,

b) the continuous function $Bu : J \longrightarrow Y$ is strongly continuously differentiable on $(0,a)$,

c) $\frac{d}{dt} Bu(t) + Au(t) = f(t,u(t))$ for all $t \in (0,a)$.

Let $u \in C(J,X)$ and let $t \in (0,a]$. Since $s \longrightarrow f(s,u(s))$ is continuous on $[0,t]$ and $e^{-(t-s)AB^{-1}}$ is a linear continuous operator from Y to Y for all $s \in \mathbb{R}$, the mapping $s \longrightarrow e^{-(t-s)AB^{-1}} f(s,u(s))$ is continuous on $[0,t]$. Moreover, for all $s \in [0,t]$ we have $||e^{-(t-s)AB^{-1}} f(s,u(s))|| \leq ||e^{-(t-s)AB^{-1}}||.\ ||f(s,u(s))|| \leq e^{(t-s)||AB^{-1}||} c(1+||u(s)||) \leq e^{a||AB^{-1}||} c(1 + \sup_{s\in[0,t]} ||u(s)||) < \infty$. Then the property 5 of Section 3 implies that the function $s \longrightarrow e^{-(t-s)AB^{-1}} f(s,u(s))$ is integrable on $[0,t]$. Hence we can associate to (CP) the integral equation

(IE) $$u(t) = B^{-1}e^{-t\,AB^{-1}} Bu_o + B^{-1} \int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds.$$

LEMMA 4.3. *Assume that the operator* A *and* B *satisfy conditions* (1), (2) *and* (3) *and the function* f *satisfies conditions* (5) *and* (6). *Then* $u \in C(J,X)$ *is a global solution of* (CP) *if and only if* u *satisfies* (IE).

PROOF. Suppose that $u \in C(J,X)$ is a global solution of (CP). Let $t \in (0,a]$ and

let $g(s) = e^{-(t-s)AB^{-1}} Bu(s)$ for all $s \in [0,t]$. Then $g(t) = Bu(t)$, $g(0) = e^{-tAB^{-1}} Bu_o$ and g is strongly continuously differentiable on (0,t). Since

$$g'(s) = e^{-(t-s)AB^{-1}} AB^{-1}Bu(s) + e^{-(t-s)AB^{-1}} \frac{d}{ds} Bu(s)$$
$$= e^{-(t-s)AB^{-1}} Au(s) + e^{-(t-s)AB^{-1}} [-Au(s) + f(s,u(s))]$$
$$= e^{-(t-s)AB^{-1}} f(s,u(s))$$

we get $Bu(t) = e^{-tAB^{-1}} Bu_o + \int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds$, and therefore u satisfies (IE).

Suppose that $u \in C(J,X)$ satisfies (IE). Then $Bu(t) = e^{-tAB^{-1}} Bu_o + \int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds$. Let $g(t) = \int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds$ for all $t \in J$. We will show that g is strongly continuously differentiable on (0,a) and $g'(t) = AB^{-1}g(t) + f(t,u(t))$. Let $t \in (0,a)$ and let $h > 0$ be such that $t + h \leq a$. Since $g(t+h) = \int_t^{t+h} e^{-(t+h-s)AB^{-1}} f(s,u(s))ds + \int_o^t e^{-(t+h-s)AB^{-1}} f(s,u(s))ds$, the property 7 of Section 3 implies that $g(t+h) = \int_t^{t+h} e^{-(t+h-s)AB^{-1}} f(s,u(s))ds + e^{-hAB^{-1}} g(t)$. Hence $h^{-1}(g(t+h) - g(t)) = h^{-1}(e^{-hAB^{-1}} - I)g(t) + e^{-(t+h)AB^{-1}} h^{-1} \int_t^{t+h} e^{sAB^{-1}} f(s,u(s))ds$. It is clear that the first term converges strongly to $-AB^{-1}g(t)$ as $h \downarrow 0$. Since $e^{-(t+h)AB^{-1}}$ converges to $e^{-tAB^{-1}}$ in the uniform operator topology as $h \downarrow 0$ and, by the property 6 of Section 3, $h^{-1}\int_o^{t+h} e^{sAB^{-1}} f(s,u(s))ds$ converges strongly to $e^{tAB^{-1}} f(t,u(t))$ as $h \downarrow 0$, the second term converges strongly to $e^{-tAB^{-1}} e^{tAB^{-1}} f(t,u(t)) = f(t,u(t))$ as $h \downarrow 0$. Then there exists the strong right derivative $g'_+(t)$ and satisfies the equation $g'_+(t) = -AB^{-1}g(t) + f(t,u(t))$. Since g'_+ is continuous at the point t, it follows that the strong derivative $g'(t)$ exists, it is continuous and satisfies the equation $g'(t) = -AB^{-1}g(t) + f(t,u(t))$ (see [14, p.492, footnote]). Then Bu is strongly continuously differentiable on (0,a) and $\frac{d}{dt} Bu(t) = -AB^{-1}e^{-tAB^{-1}} Bu_o - AB^{-1}[Bu(t) - e^{-tAB^{-1}} Bu_o] + f(t,u(t)) = -Au(t) + f(t,u(t))$ for all $t \in (0,a)$. Hence u satisfies (CP).

THEOREM 4.4. *Assume that the operators* A *and* B *satisfy conditions* (1), (2), (3) *and* (4), *and the function* f *satisfies conditions* (5) *and* (6). *Then the set of global solutions of* (CP) *is an* R_δ*-set in* $(C(J,X), \tau_u)$.

PROOF. For every $u \in C(J,X)$ put $F(u)(t) = B^{-1}e^{-tAB^{-1}} Bu_o + B^{-1}\int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds$. From property 3 of Section 3 it follows that F is a mapping from C(J,X) to C(J,X). Moreover, lemma 4.3 implies that the set of global solutions of (CP) is exactly the set Fix(F) of all fixed points of the integral operator F. We will show that F satisfies the conditions of theorem 2.4. It is clear that F satisfies conditions (ii) and (iii).

To show the continuity of F, let $u \in C(J,X)$ and let $\varepsilon > 0$. If $v \in C(J,X)$ it follows from properties 1 and 7 of Section 3 that $||F(u)(t) - F(v)(t)|| \leq ||B^{-1}||e^{a||AB^{-1}||} \int_o^a ||f(s,u(s)) - f(s,v(s))||ds$. By lemma 2 of [27] there exists $\delta > 0$ such that

$$||f(s,u(s)) - f(s,v(s))|| < \frac{\varepsilon}{a||B^{-1}||e^{a||AB^{-1}||}} \text{ for all } s \in J \text{ whenever } v \in C(J,X) \text{ and}$$

$||u(s)-v(s)|| < \delta$ for all $s \in J$. Hence $||F(u)(t)-F(v)(t)|| < \varepsilon$ for all $t \in J$ whenever $v \in C(J,X)$ and $||u(t)-v(t)|| < \delta$ for all $t \in J$. To finish the verification of condition (i), we must show that, if V is a bounded subset of $(C(J,X), \tau_u)$, then $F[V]$ is relatively τ_u-compact. To this it suffices to apply the Ascoli theorem.

To show that $F[V]$ is equicontinuous, let $t_o \in J$ be fixed. For every $u \in V$ and every $t \in J$ we have

$$(*) \quad F(u)(t) - F(u)(t_o) = B^{-1}(e^{-t\,AB^{-1}} - e^{-t_o\,AB^{-1}})Bu_o +$$
$$B^{-1}\int_o^{t_o} [e^{-(t-s)AB^{-1}} - e^{-(t_o-s)AB^{-1}}]f(s,u(s))ds +$$
$$B^{-1}\int_{t_o}^t e^{-(t-s)AB^{-1}} f(s,u(s))ds.$$

Since $e^{-t\,AB^{-1}} - e^{-t_o\,AB^{-1}} \to 0$ and, for all $s \in J$, $e^{-(t-s)AB^{-1}} - e^{-(t_o-s)AB^{-1}} \to 0$ in the uniform operator topology and $\{f(s,u(s)) : s \in [0,t]\}$ is bounded, the first and second term of the expression (*) converges to 0 as $t \to t_o$. Since V is bounded there exists a constant $M > 0$ such that $||v(t)|| \leq M$ for all $v \in V$ and all $t \in J$. Then, for the third term of the expression (*), we get $||B^{-1}\int_{t_o}^t e^{-(t-s)AB^{-1}} f(s,u(s))ds|| \leq ||B^{-1}||e^{a||AB^{-1}||}c(1+M)|t-t_o| \to 0$ as $t \to t_o$. So $\lim_{t \to t_o} F(u)(t) = F(u)(t)$ uniformly in $u \in V$. We will show that, for every $t \in J$, $\{F(u)(t) : u \in V\}$ is a relatively compact subset of Y. Let $t \in J$. Since $||\int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds|| \leq t\,e^{t||AB^{-1}||}\,c(1+M)$ for all $u \in V$ and B^{-1} is a compact operator, $\{B^{-1}\int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds : u \in V\}$ is relatively compact. Hence $\{F(u)(t) : u \in V\} = \{B^{-1}\,e^{-t\,AB^{-1}} Bu_o + B^{-1}\int_o^t e^{-(t-s)AB^{-1}} f(s,u(s))ds : u \in V\}$ is relatively compact.

By theorem 3.4 of [5], Fix(F) is non-empty. To finish the proof, it remains to show that Fix(F) is a bounded subset of $(C(J,X))$ $\tau_u)$. Let $u \in$ Fix(F) and let $t \in J$. Then

$$||u(t)|| = ||B^{-1}e^{-t\,AB^{-1}}Bu_o + B^{-1}\int_o^t e^{-(t-s)AB^{-1}}f(s,u(s))ds||$$

$$\leqslant ||B^{-1}||e^{a||AB^{-1}||}||Bu_o|| + ||B^{-1}||\int_o^t e^{a||AB^{-1}||}c(1+||u(s)||)ds$$

$$= \delta + \int_o^t \eta||u(s)||ds, \text{ where } \delta = ||B^{-1}||e^{a||AB^{-1}||}(||Bu_o|| + ac)$$

and $\eta = c||B^{-1}||e^{a||AB^{-1}||}$.

The Gronwall's inequality implies that $||u(t)|| \leqslant \delta\ e^{\eta t} \leqslant \delta\ e^{\eta a}$, and therefore Fix(F) is bounded.

(III) Let X be a real or complex Banach space and let $\mathcal{B}(X)$ be the vector space of all linear continuous operators on X.

A mapping $S : [0,\infty) \to \mathcal{B}(X)$ is called a C_o-*semigroup on* X if

i) $S(0) = I$,

ii) $S(t+t') = S(t)S(t')$ for all $t, t' \in [0,\infty)$,

iii) for every $x \in X$, $\lim_{t \downarrow 0} ||S(t)x - x|| = 0$.

Let S be a C_o-semigroup on X. We indicate two important properties of S (see [8, p.8]) :

1) for every $x \in X$, the function $t \to S(t)x$ from $[0,\infty)$ to X is strongly continuous;

2) there exist real constants $M > 0$ and ω such that $||S(t)|| \leqslant M e^{\omega t}$ for all $t \in [0,\infty)$.

A C_o-semigroup S on X is called *compact* if, for every $t \in (0,\infty)$, S(t) is a compact operator on X. We note the following result of Lax (see [2, theorem 4.4.1] and [12, theorem 10.2.2]): If S is a compact C_o-semigroup on X, then the function $t \to S(t)$ from $(0,\infty)$ to $\mathcal{B}(X)$ is continuous in the uniform operator topology.

Let S be a C_o-semigroup on X. It is clear that the set

$D = \{x \in X : \lim_{t \downarrow 0} t^{-1}(S(t) - I)x \text{ exists in } X\}$ is a vector subspace of X. By definition the *infinitesimal generator of* S is the linear operator A with domain $D(A) = D$ and defined by the formula $Ax = \lim_{t \downarrow 0} \frac{(S(t)-I)x}{t}$. It can be shown that A is a closed operator, D(A) is dense in X and, for every $x \in D(A)$, the function $t \to S(t)x$ is strongly differentiable and $\frac{d}{dt} S(t)x = S(t)Ax = AS(t)x$ (see [11, pp. 619-621]).

One of the most important results in the theory of linear semigroup in Banach spaces is the following theorem due to Hille, Yosida and Phillips : A necessary and sufficient condition for a closed linear operator with dense domain $A : D(A) \subseteq X \to X$ to be the infinitesimal generator of a C_o-semigroup on X is that there exist real constants $M > 0$ and ω such that, for every $\lambda > \omega$, λ belongs to the resolvent set of A and $||(\lambda I - A)^{-n}|| \leqslant M(\lambda-\omega)^{-n}$ for every $n = 1,2,3,\ldots$. (For a proof see [8, theorem 1.3.6] or [11, theorem 13, p.624]).

We indicate a few examples of C_o-semigroups with the corresponding infinitesimal generator :

1) Let Ω be a non-empty bounded open set in $\mathbb{R}^n$ and let X be the Sobolev space $W_o^{1,2}(\Omega)$. Define $D(A) = \{u \in X : \Delta u \in X\}$ and $A = \Delta$. Then A is the infinitesimal generator of a compact C_o-semigroup on X.

2) Let $X = L^2(0,\infty)$. For every $t \in [0,\infty)$ and every $x \in X$, define $S(t)x = y$ where $y(s) = 0$ if $0 \leqslant s \leqslant t$ and $y(s) = x(s-t)$ if $s \geqslant t$. Therefore S is a non-compact C_o-semigroup on X with infinitesimal generator $A = -\frac{\partial}{\partial s}$ whose domain is $D(A) = \{x \in X : x(0) = 0 \text{ and } x' \in X\}$.

3) Let X be the Banach space of bounded and uniformly continuous complex-valued functions on $[0,\infty)$ with supremum norm. Let $Ax = x'$ where $D(A) = \{x \in X : x' \in X\}$. Then A is the infinitesimal generator of a C_o-semigroup S on X, where $(S(t)x)(s) = x(t+s)$ for all $t \in [0,\infty)$ and all $x \in X$ and such that $||S(t)|| \leqslant 1$.

Let A be the infinitesimal generator of a C_o-semigroup S on X, let J be an interval of the real line containing 0 and let $f : J \times X \to X$ be a continuous function. We can consider the semilinear equation of evolution

$$(\gamma) \qquad u'(t) = Au(t) + f(t, u(t)).$$

For every $u_o \in X$ we can consider the Cauchy problem for the semilinear equation (γ)

$$(CP) \qquad u' = Au + f(t,u),\ u(0) = u_o.$$

As was indicated in [21] the main interest in the problem (CP) is due to the fact that it abstracts many initial value problems involving partial differential equations. In such problems X is usually a Banach space of functions, as for example $C(\Omega)$ or $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, where Ω is a non-empty open subset of $\mathbb{R}^n$, and A is a differential operator in X whose domain D(A) consists of smooth elements of X satisfying some boundary conditions. For more details concerning the equation (γ) we refer to [13].

Since f is continuous, we can associate to the problem (CP) the integral equation

$$(IE) \qquad u(t) = S(t)u_o + \int_o^t S(t-s)f(s,u(s))ds,\ t \in J.$$

Following Browder [6] we say that a continuous function $u : J \to X$ is a *global mild solution* of (CP) if u satisfies the equation (IE). Our first result improves a recent result of Ballotti [3] :

THEOREM 4.5. *Let A be the infinitesimal generator of a compact C_o-semigroup S on X, let $J = [0,a]$ be a compact interval of the real line with $a > 0$, let $u_o \in X$ and let $f : J \times X \to X$ be a continuous function such that $f[J \times X]$ is a bounded subset of X. Then the set of global mild solutions of (CP) is an R_δ-set in $(C(J,X), \tau_u)$.*

PROOF. For every $u \in C(J,X)$ and every $t \in J$ define

$$F(u)(t) = S(t)u_o + \int_o^t S(t-s)f(s,u(s))ds.$$

From property 3 of Section 3 it follows that $F(u) \in C(J,X)$.

We will show that F satisfies the conditions of corollary 2.2. It is clear that F satisfies the conditions (ii) and (iii).

To prove that F is continuous, let $u \in C(J,X)$ and let $\varepsilon > 0$. If $v \in C(J,X)$ and $t \in J$, then the property 1 of Section 3 implies that

$$||F(u)(t) - F(v)(t)|| \leq \int_0^t ||S(t-s)||\,||f(s,u(s)) - f(s,v(s))||ds$$
$$\leq M e^{\omega a} \int_0^a ||f(s,u(s)) - f(s,v(s))||ds.$$

By lemma 2 of [27], there exists $\delta > 0$ such that $||f(s,u(s))-f(s,v(s))|| < \frac{\varepsilon}{a M e^{\omega a}}$ for all $s \in J$ whenever $v \in C(J,X)$ and $||u(s)-v(s)|| < \delta$ for all $s \in J$. Then $||F(u)(t)-F(v)|| < \varepsilon$ for all $t \in J$ whenever $v \in C(J,X)$ and $||u(t)-v(t)|| < \delta$ for all $t \in J$. So the condition (i) is verified.

To verify the condition (iv) it suffices, by the Ascoli theorem, to show that $F[C(J,X)]$ is equicontinuous and, for all $t \in J$, $\{F(u)(t) : u \in C(J,X)\}$ is a relatively compact subset of X.

Since $f[J \times X]$ is a bounded subset of X, there exists a real constant $N > 0$ such that $||f(t,x)|| \leq N$ for every $(t,x) \in J \times X$. To prove the equicontinuity of $F[C(J,X)]$, let $t_o \in J$ and let $\varepsilon > 0$. If $u \in C(J,X)$ and $t \in J$ we have

$$F(u)(t) - F(u)(t_o) = (S(t) - S(t_o))u_o + \int_o^{t_o} (S(t-s)-S(t_o-s))f(s,u(s))ds$$
$$+ \int_{t_o}^t S(t-s)f(s,u(s))ds,$$

and therefore

$$||F(u)(t) - F(u)(t_o)|| \leq ||S(t) - S(t_o)||\,||u_o|| + N\int_o^{t_o} ||S(t-s) - S(t_o-s)||ds$$
$$+ M N e^{\omega a} |t-t_o|.$$

Choose $\eta > 0$ such that $\eta(||u_o|| + Nt_o + MNe^{\omega a}) < \varepsilon$. Since $t \to S(t)$ is continuous in the uniform operator topology and J is compact, $t \to S(t)$ is uniformly continuous in the uniform operator topology. So there exists δ such that $0 < \delta < \eta$ and $t', t'' \in J$ and $|t'-t''| < \delta$ imply $||S(t') - S(t'')|| < \eta$. Hence $t \in J$ and $|t-t_o| < \delta$ imply $||F(u)(t) - F(u)(t_o)|| < \varepsilon$ for all $u \in C(J,X)$.

Now let $t \in J$. To prove that $\{F(u)(t) : u \in C(J,X)\}$ is a relatively compact subset of X, we may suppose $t > 0$. Choose ε such that $0 < \varepsilon < t$ and define, for every $u \in C(J,X)$.

$$F_\varepsilon(u)(t) = S(t)u_o + \int_o^{t-\varepsilon} S(t-s)f(s,u(s))ds.$$

Then

$$F_\varepsilon(u)(t) = S(t)u_o + S(\varepsilon)\int_o^{t-\varepsilon} S(t-s-\varepsilon)f(s,u(s))ds$$

by the property 7 of Section 3. Since $S(\varepsilon)$ is a compact operator and $||\int_o^{t-\varepsilon} S(t-s-\varepsilon)f(s,u(s))ds|| \leq a M N e^{\omega a}$, the set $\{F_\varepsilon(u)(t) : u \in C(J,X)\}$ is relatively compact for every $0 < \varepsilon < t$. Moreover, for all $u \in C(J,X)$, we have

$$||F(u)(t)-F_\varepsilon(u)(t)|| \leq \int_{t-\varepsilon}^t ||S(t-s)||\,||f(s,u(s))||ds \leq \varepsilon M N$$

for every $0 < \varepsilon < t$. Let $\eta > 0$. Choose $\varepsilon > 0$ such that $\varepsilon < \min\{t, \frac{\eta}{1+2MN}\}$. Since $\{F_\varepsilon(u)(t) : u \in C(J,X)\}$ is totally bounded, there exists a finite sequence $(u_i)_{1 \leq i \leq n}$ in $C(J,X)$ such that $\{F_\varepsilon(u)(t) : u \in C(J,X)\} \subseteq \bigcup_{i=1}^n B(F_\varepsilon(u_i),\varepsilon)$. So $\{F(u)(t) : u \in C(J,X)\} \subseteq \bigcup_{i=1}^n B(F(u_i),\eta)$ and therefore $\{F(u)(t) : u \in C(J,X)\}$ is totally bounded, so relatively compact. Thus the condition (iv) is verified.

By corollary 2.2, Fix(F) is an R_δ-set in $(C(J,X), \tau_u)$, and therefore the set of global mild solutions of (CP) is an R_δ-set in $(C(J,X), \tau_u)$.

Our last result is the initial value problem considered, for the necessary existence theory, by Pazy [20]:

THEOREM 4.6. *Let A be the infinitesimal generator of a compact C_o-semigroup S on X, let $J = [0,\infty)$ be a half-line of $\mathbb{R}$, let $u_o \in X$ and let $f : J \times X \to X$ be a continuous function. If f maps bounded sets in $J \times X$ into bounded sets in X and there exist two locally integrable functions k_1 and k_2 from J to J such that $||f(t,x)|| \leq k_1(t)||x|| + k_2(t)$ for all $(t,x) \in J \times X$, then the set of global mild solutions of (CP) is an R_δ-set in $(C(J,X), \tau_c)$.*

PROOF. By property 3 of Section 3 it follows that the integral operator

$$F(u)(t) = S(t)u_o + \int_o^t S(t-s)f(s,u(s))ds, \quad t \in J$$

is a mapping from $C(J,X)$ to $C(J,X)$.

Let $I = [0,1]$. By theorem 7.1 of [18] there exists a homeomorphism

$$\Phi : (C(I,X), \tau_c) \to (C(J,X), \tau_c)$$

satisfying the following conditions :

a) Φ is linear,

b) $\Phi(u)$ is an extension of u for every $u \in C(I,X)$,

c) $\Phi(u)[J] \subseteq co(u[I])$ for all $u \in C(I,X)$.

Since I is compact, $\tau_c = \tau_u$ on $C(I,X)$.

Define $F_o : (C(I,X), \tau_u) \to (C(I,X), \tau_u)$ by the formula $F_o = \Phi^{-1} o F o \Phi$. We will show that F_o satisfies the conditions of theorem 2.4.

Let $u \in C(I,X)$. Since

$$F_o(u)(0) = (\Phi^{-1}(F(\Phi(u))))(0)$$
$$= F(\Phi(u))(0)$$
$$= u_o$$

F_o satisfies the condition (ii).

Let $\varepsilon > 0$ and let $u,v \in C(I,X)$ be such that $u|J_\varepsilon = v|J_\varepsilon$ where $J_\varepsilon = [0,a]$ with $a = \min\{1,\varepsilon\}$. Then $\Phi(u)|J_\varepsilon = \Phi(v)|J_\varepsilon$, and therefore $F(\Phi(u))|J_\varepsilon = F(\Phi(v))|J_\varepsilon$. Thus $F_o(u)|J_\varepsilon = \Phi^{-1}(F(\Phi(u)))|J_\varepsilon = \Phi^{-1}(F(\Phi(v)))|J_\varepsilon = F_o(v)|J_\varepsilon$. So F_o satisfies the condition (iii).

To establish the continuity of F_o it suffices to show that F is continuous. Let $u \in C(J,X)$, let K be a compact subset of J and let $\varepsilon > 0$. For $v \in C(J,X)$ and $t \in K$ we have

$$||F(u)(t) - F(v)(t)|| \leq \int_o^t ||S(t-s)||\,||f(s,u(s))-f(s,v(s))||\,ds$$
$$\leq M e^{\omega b}\int_o^t ||f(s,u(s))-f(s,v(s))||\,ds, \quad \text{where } b = \sup K.$$

By lemma 2.5 there exists $\delta > 0$ such that $||f(s,u(s))-f(s,v(s))|| < \frac{\varepsilon}{Mbe^{\omega b}}$ for every $s \in K$ whenever $v \in C(J,X)$ and $||u(s)-v(s)|| < \delta$ for every $s \in K$. So $||F(u)(t)-F(v)(t)|| < \varepsilon$ for every $t \in K$ whenever $v \in C(J,X)$ and $||u(t)-v(t)|| < \delta$ for every $t \in K$. Now let V be a bounded subset of $(C(I,X), \tau_u)$. Since $\overline{F_o[V]} = \Phi^{-1}[\overline{F[\Phi[V]]}]$, to complete the verification of condition (i), it suffices to show that $\overline{F[\Phi[V]]}$ is τ_c-compact. Because J is a regular locally compact topological space, by the Ascoli theorem it suffices to prove that $F[\Phi[V]]$ is equicontinuous and $\{F(\Phi(u))(t) : u \in V\}$ is relatively compact for every $t \in J$.

Let V be a bounded subset of $(C(I,X), \tau_u)$. Then there exists a real constant $N > 0$ such that $||u(t)|| \leq N$ for every $u \in V$ and every $t \in I$. The property c) of the mapping Φ implies that $||\Phi(u)(t)|| \leq N$ for every $t \in J$ and every $u \in V$. Then, for every compact subset K of J, $f[K \times \{\Phi(u)(t) : u \in V \text{ and } t \in K\}]$ is a bounded subset of X. Then there exists a real constant $M_K > 0$ such that $||f(t,\Phi(u)(t))|| \leq M_K$ for every $u \in V$ and every $t \in K$. To prove the equicontinuity of $F[\Phi[V]]$, let $t_o \in J$. If $u \in V$ and $t \in J$ such that $t > t_o$ we have

$$F(\Phi(u))(t)-F(\Phi(u))(t_o) = (S(t)-S(t_o))u_o + \int_o^{t_o}(S(t-s)-S(t_o-s))f(s,\Phi(u)(s))\,ds + \int_{t_o}^t S(t-s)f(s,\Phi(u)(s))\,ds$$

and therefore

$$||F(\Phi(u)(t)-F(\Phi(u))(t_o)|| \leq ||S(t)-S(t_o)||\,||u_o|| + M_{[0,t_o]}\int_o^{t_o}||S(t-s)-S(t_o-s)||\,ds + Me^{\omega(t-t_o)}[N\int_{t_o}^t k_1(s)\,ds + \int_{t_o}^t k_2(s)\,ds].$$

To finish the proof it suffices to note that the right-hand side of this inequality is independent of $u \in V$ and tends to 0 as $t \downarrow t_o$, because $t \to S(t)$ is continuous in the uniform operator topology.

Let now $t \in J$. To prove that $\{F(\Phi(u))(t) : u \in V\}$ is a relatively compact subset of X, we may suppose $t > 0$. For $0 < \varepsilon < t$, put $F_\varepsilon(\Phi(u))(t) = S(t)u_o + \int_o^{t-\varepsilon} S(t-s)f(s,\Phi(u)(s))\,ds = S(t)u_o + S(\varepsilon)\int_o^{t-\varepsilon} S(t-s-\varepsilon)f(s,\Phi(u)(s))\,ds$ for every

$u \in V$. Since $S(\varepsilon)$ is a compact operator and $||\int_o^{t-\varepsilon} S(t-s-\varepsilon)f(s,\Phi(u)(s))\,ds \leq M e^{\omega(t-\varepsilon)} M_{[0,t-\varepsilon]}(t-\varepsilon)$ for every $u \in V$, it follows that the set $\{F_\varepsilon(\Phi(u))(t) : u \in V\}$ is relatively compact for every $0 < \varepsilon < t$. Moreover, for all $u \in V$ we have $||F(\Phi(u))(t) - F_\varepsilon(\Phi(u))(t)|| \leq \int_{t-\varepsilon}^t ||S(t-s)||\,||f(s,\Phi(u)(s))||\,ds \leq M e^{\omega\varepsilon}[N\int_{t-\varepsilon}^t k_1(s)\,ds + \int_{t-\varepsilon}^t k_2(s)\,ds] = \delta(\varepsilon) > 0$ for every $0 < \varepsilon < t$. Let $\eta > 0$. Since $\delta(\varepsilon)$ tends to 0 as $\varepsilon \downarrow 0$, there exists $0 < \varepsilon < t$ such that $\delta(\varepsilon) < \frac{\eta}{3}$. But $\{F_\varepsilon(\Phi(u))(t) : u \in V\}$ is totally bounded. So there exists a finite sequence $(u_i)_{1\leq i\leq n}$ in V such that

$$\{F_\varepsilon(\Phi(u))(t) : u \in V\} \subseteq \bigcup_{i=1}^n B(F_\varepsilon(\Phi(u_i)),\delta(\varepsilon)).$$

Then $\{F(\Phi(u))(t) : u \in V\} \subseteq \bigcup_{i=1}^n B(F(\Phi(u_i)),\eta)$, and therefore $\{F(\Phi(u)(t) : u \in V\}$ is totally bounded, so relatively compact.

On the other hand, $S = \mathrm{Fix}(F)$ is non-empty by corollary 3.2 of [20]. Since $S_o = \mathrm{Fix}(F_o) = \Phi^{-1}(S)$, S_o is a non-empty subset of $C(I,X)$. We will show that S_o is bounded. Since Φ^{-1} is linear and continuous, it suffices to show that S is a bounded subset of $(C(J,X), \tau_c)$. Let K be a compact subset of J and let $\varepsilon > 0$. For $u \in S$ and $t \in K$ we have $u(t) = S(t)u_o + \int_o^t S(t-s)f(s,u(s))\,ds$. Therefore

$$||u(t)|| \leq ||S(t)||\,||u_o|| + \int_o^t ||S(t-s)||\,||f(s,u(s))||\,ds \leq M e^{\omega b}||u_o|| + Me^{\omega b}[\int_o^t k_1(s)||u(s)||\,ds + \int_o^b k_2(s)\,ds]$$
$$= \alpha + \int_o^t \beta k_1(s)||u(s)||\,ds$$

where $\alpha = M e^{\omega b}[||u_o|| + \int_o^b k_2(s)\,ds]$, $\beta = M e^{\omega b}$ and $b = \sup K$. By the Gronwall's inequality we deduce $||u(t)|| \leq \alpha \exp(\int_o^t \beta k_1(s)\,ds) \leq \alpha \exp(\int_o^b \beta k_1(s)\,ds)$. Let $\eta > 0$ be such that $\eta\alpha \exp(\int_o^b \beta k_1(s)\,ds) < \varepsilon$ and let $U = \{x \in X : ||x|| < \varepsilon\}$. Then $S \subseteq \frac{1}{\eta} N(K,U)$.

By theorem 2.4, S_o is an R_δ-set in $(C(I,X), \tau_u)$. Since $S = \mathrm{Fix}(F) = \Phi[S_o]$ and Φ is a homeomorphism, S is an R_δ-set in $(C(J,X), \tau_c)$.

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ε-APPROXIMATE SOLUTIONS FOR HEREDITARY NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS (*)

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Abstract.

Relative to the problem

$$(j)\quad \begin{cases} z_x(x,y) = f(x,y;\dots,z(\alpha_i(x,y),\beta_i(x,y)),\dots;z_y(x,y)) & \text{on } I \\ z = \phi^{(o)} & \text{on } I_o \end{cases}$$

for which in [3] an existence theorem was established, we prove that a suitable discretization of Baiada's approximation method [1] gives a numerical approximation of the exact solution, in the same range where this method provides the solution.

Having indicated the approximating surface sequence by $(\pi_n)_n$, we demonstrate that it converges uniformly to the solution z and definitely satisfies the relation

$$\left|\frac{\partial}{\partial x}\pi_n(x,y) - f(x,y;\dots,\pi_n(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}\pi_n(x,y))\right| \le \varepsilon.$$

Finally, we estimate the convergence rate of π_n to z as $n\to+\infty$.

1. Introduction.

The nonlinear Cauchy problem

$$(j)\quad \begin{cases} z_x(x,y) = f(x,y;z(\alpha_1(x,y),\beta_1(x,y)),\dots,z(\alpha_r(x,y),\beta_r(x,y));z_y(x,y)) & \text{on } I \\ z = \phi^{(o)} & \text{on } I_o\ ;\ z \in \tilde{C}(I_o \cup I) \end{cases}$$

where $f,\phi^{(o)},\alpha_i,\ \beta_i (i=1,2,\dots,r)$ are given functions,

(*) This research was carried out within the G.N.A.F.A. of the C.N.R.

$I_o = \{(x,y): p_o \le x \le 0,\ y \in \mathbb{R}\}$, $-\infty \le p_o \le 0$, $I = \{(x,y): 0 \le x \le a,\ y \in \mathbb{R}\}$

and $\tilde{C}$ is a suitable subspace of C^1, is considered.

Local existence theorems for the problem (j) were established in [3], [12], [13].

In [3], by means of a technique which then permitted us to also study equations whose functional argument is more general and more abstract than those in problem (j) (see [4]), we introduced a sequence of problems (j_n) for the equation $p = f(x,y,q)$ and obtained the solution z of the given problem as the limit of the sequence of the respective solutions.

The objective of this note is to obtain a numerical approximation of the problem (j) solution.

While investigating whether Baiada's approximation method [1] which was used to establish the solution existence could provide such an approximation, it is obviously advantageous to apply this method directly to problem (j) without introducing the intermediate problems as in [3].

The particular expression of the functional argument α_i, β_i, $i=1,2,\dots,r$, makes this direct study possible.

Fixed a positive number ε, every sufficiently smooth surface $S(x,y)$ such that

(i) $\quad |S(x,y) - \phi^{(o)}(x,y)| \le \varepsilon \quad$ on I_o,

(ii) $\quad |\frac{\partial}{\partial x} S(x,y) - f(x,y;\dots,S(\alpha_i(x,y),\beta_i(x,y)),\dots;S_y(x,y))| \le \varepsilon \quad$ on I,

(iii) $\quad |S(x,y) - z(x,y)| \le \varepsilon \quad$ on I

is called ε-approximate solution of the problem (j).

In section 3, the surfaces S_n are introduced by definition

$$(1)\quad S_n(x,y) = \begin{cases} \phi^{(o)}(x,y) & \text{on } I_o \\ \frac{1}{2} S_n(i\delta, y + \frac{x-i\delta}{2}) + \frac{1}{2} S_n(i\delta, y - \frac{x-i\delta}{2}) + \\ + \int_{y - \frac{x-i\delta}{2}}^{y + \frac{x-i\delta}{2}} f(i\delta,t;\dots,S_n(\alpha_j(i\delta,t),\beta_j(i\delta,t)),\dots;\ \frac{\partial}{\partial y} S_n(i\delta,t))\,dt \\ \text{if } (x,y) \in I_{i+1} - I_i, \quad i=0,1,\dots,n-1 \end{cases}$$

where $I_i = \{(x,y): p_o \le x \le i\delta,\ y \in \mathbb{R}\}$ and $\delta = a/n$.

Let us point out here that the algorithmic procedure makes the construction of the n-th surface S_n independent from the preceding steps.

We show that the sequence $(S_n)_n$ is definitely made up of ε-approximate surfaces and we estimate the convergence rate of S_n to z.

The S_n surfaces, therefore, represent a problem (j) solution approximation; however the n integrals and derivatives present in (1) obstruct numerical results.

In section 4, so as to avoid the numerical quadrature, we apply a discretization in expression (1) referring to the finite difference method.

Having indicated by $(\pi_n)_n$ the surface sequence obtained with this discretization, we prove that it is definitely made up of ε-approximate solutions of the problem (j) and we estimate the convergence rate of π_n to z.

This proves that Baiada's method gives a numerical approximation of the exact solution, in the same range where this method provides the solution.

The application of this result is particularly important in the cases where Baiada's method gives the global existence (see, for example, semilinear equation and systems [5]).

2. Assumptions and definition of the problem.

Denoted by $I = \{(x,y): 0 \le x \le a,\ y \in \mathbb{R}\}$, $I_o = \{(x,y): p_o \le x \le 0,\ y \in \mathbb{R}\}$, $-\infty \le p_o \le 0$, $\Omega = \{(x,y;z_1,z_2,\dots,z_r;q): 0 \le x \le a,\ y, z_1, z_2, \dots, z_r \in \mathbb{R}, |q| \le S\}$, $S \le +\infty$ and given the functions $f : \Omega \to \mathbb{R}$, $\alpha_i, \beta_i : I \to \mathbb{R}$, $(i=1,2,\dots,r)$, $\phi^{(o)} : I_o \to \mathbb{R}$ we assume that:

(i) f is continuous on its domain, Lipschitzian in x of constant H, for every $y, z_1, \dots, z_r \in \mathbb{R}$, $|q| \le S$, and bounded in Ω by a constant G;

(2i) each partial derivative f_y, f_{z_i}, f_q $(i=1,2,\dots,r)$ satisfies in Ω a Lipschitz condition with respect to $y, z=(z_1,z_2,\dots,z_r), q$ with the constant K_w, $w=y,z,q$;

(3i) the partial derivative f_w is bounded in Ω by a constant L_w, with $L_w \le K_w$, $w=y,z,q$;

(4i) the functions $\alpha_i(x,y), \beta_i(x,y)$, $i=1,2,\dots,r$, are of class C^1 in I, there exist constants A,B,C,D such that

$$|\partial\alpha_i/\partial x| \le A \qquad |\partial\alpha_i/\partial y| \le B$$

$$|\partial\beta_i/\partial x| \le C \qquad |\partial\beta_i/\partial y| \le D$$

and the partial derivatives $(\partial/\partial y)\alpha_i(x,y)$, $(\partial/\partial y)\beta_i(x,y)$ are Lipschitzian in I with respect to y with constants B,D respectively, for every $x\in[0,a]$, $i=1,2,\dots,r$;

(5i) the function $\alpha_i(x,y)$, $i=1,2,\dots,r$, satisfies the condition $p_o\le\alpha_i(x,y)\le x$ for every $(x,y)\in I$;

(6i) the function $\phi^{(o)}$ is of class C^1 in I_o, admits a continuous partial derivative of the second order with respect to y and there exist constants N,M such that

$$|\phi_x^{(o)}(x,y)|\le G, \quad |\phi_y^{(o)}(x,y)|\le N, \quad |\phi_{yy}^{(o)}(x,y)|\le M$$

for every $(x,y)\in I_o$, with $N+a\le S$;

(7i) the partial derivatives $\phi_x^{(o)},\phi_y^{(o)}$ are Lipschitzian in I_o with respect to x with the constants P and F; moreover $\phi_x^{(o)}$ is Lipschitzian in y with constant F; we shall suppose that

$$P\ge\frac{1}{4}(4K_q^2M+a) + K_qR + H + rK_z[GA + (N+a)C],$$

$$F\ge\frac{1}{2}(2K_qM+a) + R,$$

where $R = [K_y + rK_z(GB + (N+a)D) + K_q(M+a)]$;

(8i) the following consistency condition is satisfied

$$\phi_x^{(o)}(0,y) = f(0,y;\phi^{(o)}(\alpha_1(0,y),\beta_1(0,y)), \dots$$

$$\dots,\phi^{(o)}(\alpha_r(0,y),\beta_r(0,y));\phi_y^{(o)}(0,y)).$$

Obviously the condition (8i) is omitted if the set I_o is reduced to the line $x=0$.

DEFINITION 1. We shall say that a function z is of class $C^{(1,\ell)}$ on $I_o\cup I$ if it is of class C^1 on $I_o\cup I$ and satisfies the following conditions

(ℓ_1) $|\frac{\partial}{\partial x}z(x,y)|\le G$, $|\frac{\partial}{\partial y}z(x,y)|\le N+a$, for every $(x,y)\in I_o\cup I$,

(ℓ_2) the partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ are Lipschitzian in x of constants P and F, and in y of constants F and $M+a$.

In the present paper we consider the following Cauchy problem

$$(j)\quad \begin{cases} \frac{\partial}{\partial x}z(x,y) = \\ = f(x,y;z(\alpha_1(x,y),\beta_1(x,y)),\dots,z(\alpha_r(x,y),\beta_r(x,y)); \frac{\partial}{\partial y}z(x,y)) \\ \text{on } I, \\ z = \phi^{(o)} \text{ on } I_o; \quad z\in C^{(1,\ell)}(I_o\cup I). \end{cases}$$

In [3] a local existence theorem for this problem (j) was established. We denoted, for brevity's sake, the range of the solution again as $I_o\cup I$. Arbitrarily fixed a function $z_o\in C^{(1,\ell)}(I_o\cup I)$, we considered the following sequence of Cauchy problems

$$(j_{n+1})\quad \begin{cases} \frac{\partial}{\partial x}z(x,y) = f(x,y;\dots,z_n(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}z(x,y)) \text{ on } I \\ z = \phi^{(o)} \text{ on } I_o, \quad n=0,1,\dots . \end{cases}$$

where z_n is the solution of problem (j_n).

Each problem, now without a functional argument, was studied applying Baiada's approximation method [1] .

It was proved that the sequence $(z_n)_n$ of the solutions of the partial problems converges uniformly to the solution z of the problem (j) and the convergence rate of z_n to z as $n\to+\infty$, of order $\frac{1}{n!}$, was estimated (see [4]).

This technique, which has already been used in literature, then allowed us to study differential equations whose functional argument is more general and more abstract than that present in problem (j) (see [4]).

Applying Baiada's approximation method, we established existence theorems for non linear hereditary differential equations.

It is natural to ask whether this method may be used to obtain a numerical approximation of the solution. It is clear that the direct application of the method to the given problem, without introducing the intermediate problems, represents a simple way of reaching our objective.

The particular expression of the operator and the properties of the functions α_i and β_i $(i=1,2,\dots,r)$ have made it possible to study the problem (j) directly.

To obtain this, we shall prove that the surface sequence $(S_n)_n$ defined by

$$(1)\quad S_n(x,y) = \begin{cases} \phi^{(o)}(x,y) & \text{if } (x,y)\in I_o \\ \phi^{(i+1)}(x,y) & \text{if } (x,y)\in I_{i+1}-I_i, \quad i=0,1,\dots,n-1 \end{cases}$$

where $I_i = \{(x,y): p_o \le x \le i\delta,\ y \in \mathbb{R}\}$, $\delta = a/n$ and

$$\phi^{(i+1)}(x,y) = \frac{1}{2}\phi^{(i)}(i\delta, y+\frac{x-i\delta}{2}) + \frac{1}{2}\phi^{(i)}(i\delta, y-\frac{x-i\delta}{2}) +$$

$$+ \int_{y-\frac{x-i\delta}{2}}^{y+\frac{x-i\delta}{2}} f(i\delta,t;\ldots,\phi^{(i)}(\alpha_j(i\delta,t),\beta_j(i\delta,t)),\ldots;\phi^{(i)}_y(i\delta,t))\,dt\ ,$$

converges in I to the solution of the problem.

The dependence of the functions α_i $(i=1,2,\ldots,r)$ on both the variables, in particular on x, does not allow us to proceed as in [1],[2] because the second derivative $\frac{\partial^2 S_n}{\partial y^2}$ may not exist in a subset $P \subset \mathbb{R}^2$ with measure non zero: this is verified if the image of the set P under α_i is contained in one or more vertical straight lines along which the surface S_n does not admit, by definition, the derivative respect to x.

As Baiada observed, it is, however, possible to obtain the result by substituting the property of boundedness of the function $\frac{\partial^2 S_n}{\partial y^2}$ with the Lipschitz condition of $\frac{\partial S_n}{\partial y}$ in y.

3. ε-approximate solutions.

We prove that the surface sequence defined by (1) converges uniformly to the solution z of the problem (j) and we estimate the convergence rate of S_n to z as $n \to +\infty$.

Apart from multiplicative constants, we suppose that the relations (1) of [3] whose validity is made possible, eventually reducing the width of strip I, by the omotetia $x=mX$, $y=\lambda mY$, where λ, m are opportune positive constants with $\lambda m \le 1$, are verified.

For further details see [3].

LEMMA 1. In virtue of the assumed hypotheses, every surface S_n satisfies in $I_o \cup I$ the properties:[1]

(1) Every property valid for $\partial S_n/\partial x$ in the points where this derivative does not exist is understood to be satisfied by the partial right-hand derivative $\partial^+ S_n/\partial x$ and by the left-hand derivative $\partial^- S_n/\partial x$.

(p_1) $|\partial S_n/\partial x| \le \hat{M}\,\delta/4 + G$

(p_2) $|\partial S_n/\partial y| \le \hat{N}$

(p_3) $|\frac{\partial}{\partial y} S_n(x,y_2) - \frac{\partial}{\partial y} S_n(x,y_1)| \le \hat{M}|y_2 - y_1|$

(p_4) $|\frac{\partial}{\partial y} S_n(x_2,y) - \frac{\partial}{\partial y} S_n(x_1,y)| \le (\frac{\hat{M}}{2} + \hat{R})\cdot|x_2 - x_1|$

(p_5) $|\frac{\partial}{\partial x} S_n(x,y_2) - \frac{\partial}{\partial x} S_n(x,y_1)| \le (\frac{\hat{M}}{2} + \hat{R})\cdot|y_2 - y_1|$

(p_6) $|\frac{\partial}{\partial x} S_n(x_2,y) - \frac{\partial}{\partial x} S_n(x_1,y)| \le \{\frac{\hat{M}}{4} + \frac{\hat{R}}{2} + H + rL_z[(\frac{\hat{M}\delta}{4} + G)A + \hat{N}C]\}\cdot$
$\cdot(|x_2 - x_1| + 2\delta)$

where $\hat{N} = N+a$, $\hat{M} = M+a$, $\hat{R} = K_y + rK_z[(\frac{\hat{M}\delta}{4} + G)B + \hat{N}D] + K_q\hat{M}$.

Proof. We prove the property (p_3) because the other proofs are analogous to those shown in [3].

Fixed $n \in \mathbb{N}^+$, let $P_1 = (x,y_1)$, $P_2 = (x,y_2) \in I_{m+1} - I_m$, $0 \le m < n$, be such that $0 < y_2 - y_1 \le \delta/B$.

Supposing that S_n satisfies in I_m the properties (p) with $\hat{M} = M+m\delta$, $\hat{N} = N+m\delta$ we demonstrate by induction that surface $\phi^{(m+1)}$ verifies the condition (p_3) in $I_{m+1} - I_m$ with constant $M+(m+1)\delta$.

Given, for brevity's sake, $x^o = (x-m\delta)/2$ we consider the difference:

$$\Delta\phi^{(m+1)} = \phi^{(m+1)}(x,y_2) - \phi^{(m+1)}(x,y_1) =$$

$$= \frac{1}{2}\frac{\partial}{\partial y} S_n(m\delta, y_2+x^o) + \frac{1}{2}\frac{\partial}{\partial y} S_n(\cdot, y_2-x^o) - \frac{1}{2}\frac{\partial}{\partial y} S_n(\cdot, y_1+x^o) - \frac{1}{2}\frac{\partial}{\partial y} S_n(\cdot, y_1-x^o) +$$

$$+ [f(m\delta, y_2+x^o, \ldots; S_n(\alpha_i(m\delta, y_2+x^o), \beta_i(m\delta, y_2+x^o)), \ldots; \frac{\partial}{\partial y} S_n(m\delta, y_2+x^o)) +$$

$$- f(\cdot, y_1+x^o, \ldots; S_n(\alpha_i(\cdot, y_1+x^o), \beta_i(\cdot, y_1+x^o)), \ldots; \frac{\partial}{\partial y} S_n(\cdot, y_1+x^o))] +$$

$$- [f(\cdot, y_2-x^o, \ldots; S_n(\alpha_i(\cdot, y_2-x^o), \beta_i(\cdot, y_2-x^o)), \ldots; \frac{\partial}{\partial y} S_n(\cdot, y_2-x^o)) +$$

$$- f(\cdot, y_1-x^o, \ldots; S_n(\alpha_i(\cdot, y_1-x^o), \beta_i(\cdot, y_1-x^o)), \ldots; \frac{\partial}{\partial y} S_n(\cdot, y_1-x^o))].$$

Applying the mean value theorem to the function f we obtain

$$\Delta\phi^{(m+1)} = [f_y(\cdot, \gamma^+, \ldots; S_n(\alpha_i(\cdot, y_2+x^o), \beta_i(\cdot, y_2+x^o)), \ldots; \frac{\partial}{\partial y} S_n(\cdot, y_2+x^o)) +$$

$$- f_y(\cdot, \gamma^-, \ldots; S_n(\alpha_i(\cdot, y_2-x^o), \beta_i(\cdot, y_2-x^o)), \ldots; \frac{\partial}{\partial y} S_n(\cdot, y_2-x^o))](y_2 - y_1) +$$

$$+ \sum_{i=1}^{r} \{[f_z(\cdot, y_1+x^o; \ldots, S_n(\alpha_i(\cdot, \omega_i^+), \beta_i(\cdot, y_2+x^o)), \ldots; \frac{\partial}{\partial y} S_n(\cdot, y_2+x^o))\cdot$$

$$\cdot[S_n(\alpha_i(\cdot,y_2+x^o),\cdot)-S_n(\alpha_i(\cdot,\delta_i^+),\cdot)+S_n(\alpha_i(\cdot,\delta_i^+),\cdot)-S_n(\alpha_i(\cdot,y_1+x^o),\cdot)]] +$$

$$-[f_z(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,\omega_i^-),\beta_i(\cdot,y_2-x^o)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o))\cdot$$

$$\cdot[S_n(\alpha_i(\cdot,y_2-x^o),\cdot)-S_n(\alpha_i(\cdot,\delta_i^-),\cdot)+S_n(\alpha_i(\cdot,\delta_i^-),\cdot)-S_n(\alpha_i(\cdot,y_1-x^o),\cdot)]] +$$

$$+[f_z(\cdot,y_1+x^o;\dots,S_n(\alpha_i(\cdot,y_1+x^o),\beta_i(\cdot,\xi_i^+)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2+x^o))\cdot$$

$$\cdot[S_n(\cdot,\beta_i(\cdot,y_2+x^o))-S_n(\cdot,\beta_i(\cdot,y_1+x^o))]] +$$

$$-[f_z(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,y_1-x^o),\beta_i(\cdot,\xi_i^-)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o))\cdot$$

$$\cdot[S_n(\cdot,\beta_i(\cdot,y_2-x^o))-S_n(\cdot,\beta_i(\cdot,y_1-x^o))]]\} +$$

$$+[f_q(\cdot,y_1+x^o;\dots,S_n(\alpha_i(\cdot,y_1+x^o),\beta_i(\cdot,y_1+x^o)),\dots;\frac{\partial}{\partial y}S_n(\cdot,\rho^+))+\frac{1}{2}]\cdot$$

$$\cdot[\frac{\partial}{\partial y}S_n(\cdot,y_2+x^o)-\frac{\partial}{\partial y}S_n(\cdot,y_1+x^o)] +$$

$$-[f_q(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,y_1-x^o),\beta_i(\cdot,y_1-x^o)),\dots;\frac{\partial}{\partial y}S_n(\cdot,\rho^-))-\frac{1}{2}]\cdot$$

$$\cdot[\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o)-\frac{\partial}{\partial y}S_n(\cdot,y_1-x^o)]$$

where $\gamma^\pm,\omega_i^\pm,\xi_i^\pm,\delta_i^\pm,\rho^\pm \in]y_1\pm x^o, y_2\pm x^o[$, $i=1,2,\dots,r$.

Because of the Lipschitz condition of α_i and the inequality $0<y_2-y_1<\delta/B$, we observe that the differences

$$|\alpha_i(m\delta,y_2\pm x^o)-\alpha_i(m\delta,y_1\pm x^o)|$$

do not exceed δ, so that the points $(\alpha_i(m\delta,y_j\pm x^o),\beta_i(\cdot))$, $j=1,2$. belong to the same strip or to two consecutive strips.

Consequently, with a reduction to intervals contained in a single strip by means of the addition and subtraction of the expression $S_n(\alpha_i(m\delta,\delta_i^\pm),\beta_i(m\delta,y_2\pm x^o))$ (with suitable $\delta_i^\pm$), it is possible to apply the mean value theorem to the function S_n in each interval.

Putting T as the term that contains the partial derivatives f_z and applying the above theorem to functions S_n, α_i, β_i we deduce

$$T=\sum_{i=1}^{r}\{[f_z(\cdot,y_1+x^o;\dots,S_n(\alpha_i(\cdot,\omega_i^+),\beta_i(\cdot,y_2+x^o)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2+x^o))+$$

$$-f_z(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,\omega_i^-),\beta_i(\cdot,y_2-x^o)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o))]\cdot$$

$$\cdot[\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{1i}^+),\beta_i(\cdot,y_2+x^o))\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{1i}^+)\cdot(y_2+x^o-\delta_i^+) +$$

$$+\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{2i}^+),\cdot)\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{2i}^+)\cdot[\delta_i^+-(y_1+x^o)]] +$$

$$+f_z(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,\omega_i^-),\beta_i(y_2-x^o)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o))\cdot$$

$$\cdot[[\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{1i}^+),\beta_i(\cdot,y_2+x^o))-\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{1i}^-),\beta_i(\cdot,y_2-x^o))]\cdot$$

$$\cdot(y_2+x^o-\delta_i^+)\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{1i}^+)+\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{1i}^-),\beta_i(\cdot,y_2-x^o))\cdot$$

$$\cdot[(y_2+x^o-\delta_i^+)\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{1i}^+)-(y_2-x^o-\delta_i^-)\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{1i}^-)] +$$

$$+[\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{2i}^+),\beta_i(\cdot,y_2+x^o))-\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{2i}^-),\beta_i(\cdot,y_2-x^o))]\cdot$$

$$\cdot[\delta_i^+-(y_1+x^o)]\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{2i}^+)+\frac{\partial}{\partial x}S_n(\alpha_i(\cdot,h_{2i}^-),\beta_i(\cdot,y_2-x^o))\cdot$$

$$\cdot[[\delta_i^+-(y_1+x^o)]\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{2i}^+)-[\delta_i^--(y_1-x^o)]\cdot\frac{\partial}{\partial y}\alpha_i(\cdot,k_{2i}^-)]] +$$

$$+(y_2-y_1)\cdot[f_z(\cdot,y_1+x^o;\dots,S_n(\alpha_i(\cdot,y_1+x^o),\beta_i(\cdot,\xi_i^+)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2+x^o))+$$

$$-f_z(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,y_1-x^o),\beta_i(\cdot,\xi_i^-)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o))]\cdot$$

$$\cdot\frac{\partial}{\partial y}S_n(\alpha_i(\cdot,y_1+x^o),\beta_i(\cdot,\tau_i^+))\cdot\frac{\partial}{\partial y}\beta_i(\cdot,\theta_i^+) +$$

$$+(y_2-y_1)\cdot f_z(\cdot,y_1-x^o;\dots,S_n(\alpha_i(\cdot,y_1-x^o),\beta_i(\cdot,\xi_i^-)),\dots;\frac{\partial}{\partial y}S_n(\cdot,y_2-x^o))$$

$$\cdot[[\frac{\partial}{\partial y}S_n(\alpha_i(\cdot,y_1+x^o),\beta_i(\cdot,\tau_i^+))-\frac{\partial}{\partial y}S_n(\alpha_i(\cdot,y_1-x^o),\beta_i(\cdot,\tau_i^-))]\cdot$$

$$\cdot\frac{\partial}{\partial y}\beta_i(\cdot,\theta_i^+)+\frac{\partial}{\partial y}S_n(\alpha_i(\cdot,y_1-x^o),\beta_i(\cdot,\tau_i^-))\cdot[\frac{\partial}{\partial y}\beta_i(\cdot,\theta_i^+)-\frac{\partial}{\partial y}\beta_i(\cdot,\theta_i^-)]]\}$$

where $h_{1i}^+,k_{1i}^+\in]\delta_i^+,y_2+x^o[$, $h_{2i}^+,k_{2i}^+\in]y_1+x^o,\delta_i^+[$,
$h_{1i}^-,k_{1i}^-\in]\delta_i^-,y_2-x^o[$, $h_{2i}^-,k_{2i}^-\in]y_1-x^o,\delta_i^-[$,
$\tau_i^+,\theta_i^+\in]y_1+x^o,y_2+x^o[$, $\tau_i^-,\theta_i^-\in]y_1-x^o,y_2-x^o[$, $i=1,2,\dots,r$.

In virtue of the hypotheses (i)-(5i), considering that $|f_q|\le\frac{1}{2}$ and using natural increases, we have

$$(2)\quad |\Delta\phi^{(m+1)}|\le K_y\{2+r[(M+m\delta)\frac{\delta}{4}+G]B+r(N+m\delta)D+(M+m\delta)\}\delta(y_2-y_1) +$$

$$+r\{K_z[1+r[(M+m\delta)\frac{\delta}{4}+G]2B+r(N+m\delta)D+(M+m\delta)]\cdot$$

$$\cdot\left[(M+m\delta)\frac{\delta}{4}+G\right]B\cdot 2\delta(y_2-y_1)+2L_z\left[(2PB+2P+FD)B+\left[(M+m\delta)\frac{\delta}{4}+G\right]\right]\cdot\delta(y_2-y_1)+$$

$$+K_z\left[1+r\left[(M+m\delta)\frac{\delta}{4}+G\right]B+r(N+m\delta)2D+(M+m\delta)\right]\cdot(N+m\delta)D\cdot\delta(y_2-y_1)+$$

$$+L_z\left[FBD+(M+m\delta)2D^2+(N+m\delta)2D\right]\delta(y_2-y_1)\}+(M+m\delta)(y_2-y_1)+$$

$$+(M+m\delta)K_q\left[1+r\left[(M+m\delta)\frac{\delta}{4}+G\right]B+(N+m\delta)D+2(M+m\delta)\right]\delta(y_2-y_1)\leq$$

$$\leq(M+m\delta)(y_2-y_1)+\delta(y_2-y_1)\cdot\{\left[2+2r\left[(M+m\delta)\frac{\delta}{4}+G\right]B+2r(N+m\delta)D+2(M+m\delta)\right]\cdot$$

$$\cdot\left[K_y+2rK_z+\left[(M+m\delta)\frac{\delta}{4}+G\right]B+(N+m\delta)D+(M+m\delta)\right]+$$

$$+rL_z\left[8PB^2+3FDPB+4\left[(M+m\delta)\frac{\delta}{4}+G\right]B+2D(M+m\delta)+2D(N+m\delta)\right]\}\cdot$$

Finally from the relations (1) of [3] it follows that

$$|\phi_y^{(m+1)}(x,y_2)-\phi_y^{(m+1)}(x,y_1)|\leq[M+(m+1)\delta]\cdot(y_2-y_1) \quad \text{in } I_{m+1}-I_m,$$

which proves the property (p_3).

Furthermore, we observe that every surface S_n, $n\in\mathbb{N}$, satisfies in $x=s\delta$, $s=0,1,\dots,n-1$ the given equation, that is

$$\frac{\partial^+}{\partial x}S_n(s\delta,y)=f(s\delta,y;\dots,S_n(\alpha_i(s\delta,y),\beta_i(s\delta,y)),\dots;\frac{\partial}{\partial y}S_n(s\delta,y)).$$

DEFINITION 2. Each function $S(x,y)$ defined and continuous in $I_o\cup I$ is called the ε-approximate solution of the problem (j) if

(s_1) $|S(x,y)-\phi^{(o)}(x,y)|\leq\varepsilon$ for every $(x,y)\in I_o$;

(s_2) $S(x,y)$ belongs to class C^1 in $I_o\cup I$ apart from, at the most, a finite number of points, segments or straight lines where, anyway, the partial right and left-hand derivatives respect to x and y exist;

(s_3) $|\frac{\partial}{\partial x}S(x,y)-f(x,y;\dots,S(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}S(x,y))|\leq\varepsilon$ for every $(x,y)\in I$;

(s_4) $|S(x,y)-z(x,y)|\leq\varepsilon$ for every $(x,y)\in I$.

THEOREM 1. Arbitrarily fixed $\varepsilon>0$, the sequence $(S_n)_n$ is definitely made up of ε-approximate solutions of the problem (j).

Proof. Every surface S_n satisfies, by definition, the properties (s_1) and (s_2).

Fixed a point $(x,y)\in I_s-I_{s-1}$ $(s=1,2,\dots,n)$, it results

$$(3)\quad \left|\frac{\partial}{\partial x}S_n(x,y)-f(x,y;\dots,S_n(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}S_n(x,y))\right|\leq$$

$$\leq\left|\frac{\partial}{\partial x}S_n(x,y)-\frac{\partial^+}{\partial x}S_n(x,y)\right|+$$

$$+\left|f((s-1)\delta,y;\dots,S_n(\alpha_i((s-1)\delta,y),\beta_i((s-1)\delta,y)),\dots;\frac{\partial}{\partial y}S_n((s-1)\delta,y))\right.+$$

$$\left.-f(x,y;\dots,S_n(\alpha_i(x,y)),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}S_n(x,y))\right|\leq$$

$$\leq[(M+a)/4+R/2]\cdot[n-(s-1)\delta]+$$

$$+\{H+rK_z\left[\left[(M+a)\frac{\delta}{4}+G\right]A+(N+a)C\right]+K_q\left[(M+a)/2+R\right]\}\cdot[n-(s-1)\delta]\leq$$

$$\leq\{(M+a)/4+R/2+H+rK_z\left[\left[(M+a)\frac{\delta}{4}+G\right]A+(N+a)C\right]+$$

$$+K_q\left[(M+a)/2+R\right]\}\cdot\delta=\mathcal{U}\cdot\delta$$

where $R=K_y+rK_z\{\left[(M+a)\frac{\delta}{4}+G\right]B+(N+a)D\}+K_q(M+a)$.

The inequality (3) proves that, having fixed $\varepsilon>0$ and having chosen an integer $\bar{n}=\bar{n}(\varepsilon)$ such that $\mathcal{U}\delta=\mathcal{U}\frac{a}{\bar{n}}<\varepsilon$, every surface S_n, $n>\bar{n}$, satisfies the property (s_3).

Finally we demonstrate that the sequence S_n converges to z as $n\to+\infty$.

Fixed the trapezium T_1 bounded by $x=0$, $x=a$, $y=y_1+K_qx$, $y=y_2-K_qx$, with arbitrary $y_1,y_2\in\mathbb{R}$, $y_1<y_2$, from (3) we deduce

$$\left|z_x(x,y)-\frac{\partial}{\partial x}S_n(x,y)\right|\leq\left|f(x,y;\dots,z(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}z(x,y))\right.+$$

$$\left.-f(x,y;\dots,S_n(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\partial}{\partial y}S_n(x,y))\right|+\mathcal{U}\frac{a}{n}\leq$$

$$\leq K_z\sum_{i=1}^{r}|z(\alpha_i(x,y),\beta_i(x,y))-S_n(\alpha_i(x,y),\beta_i(x,y))|+$$

$$+K_q\left|\frac{\partial}{\partial y}z(x,y)-\frac{\partial}{\partial y}S_n(x,y)\right|+\mathcal{U}\frac{a}{n}\leq$$

$$\leq 2rK_zW+\mathcal{U}\frac{a}{n}+K_q\left|\frac{\partial}{\partial y}z(x,y)-\frac{\partial}{\partial y}S_n(x,y)\right|+\rho|z(x,y)-S_n(x,y)|$$

for every $\rho>0$, where $|S_n(x,y)|\leq W$ for every n and for every $(x,y)\in \mathcal{T}(T_1) = \bigcup_{i=1}^{r}\{\alpha_i(T_1)\times\beta_i(T_1)\}$.

Moreover, as $S_n(0,y)-z(0,y)=0$ for every $y\in\mathbb{R}$, in virtue of a suitable formulation of Haar's lemma [6], it follows that

(4) $|z(x,y)-S_n(x,y)| \leq (2rK_z W + \mathcal{U}a/n)a$ for every $(x,y)\in T_1$

Proceeding in a similar way and using (4), we obtain

$$|z(x,y)-S_n(x,y)| \leq [rK_z(2rK_z W + \mathcal{U}a/n)a + \mathcal{U}a/n]a =$$
$$= (1+rK_z a)\mathcal{U}a^2/n + 2W(rK_z a)^2.$$

By iteration on m we have

$$|z(x,y)-S_n(x,y)| \leq (\mathcal{U}a^2/n)[1+rK_z a+(rK_z a)^2+\ldots+(rK_z a)^m+\ldots]+\lim_{m\to+\infty} 2W(rK_z a)^m$$

from which $|z(x,y)-S_n(x,y)|\leq \mathcal{U}a^2/n(1-rK_z a)$ for every $(x,y)\in T_1$ and therefore, for the arbitrary choice of T_1, we prove the thesis.

The immediate consequence is

COROLLARY 1. The sequence $(S_n)_n$ and its partial derivatives $(\partial S_n/\partial x)_n$, $(\partial S_n/\partial y)_n$ converge uniformly to the functions z, z_x, z_y respectively.

4. Approximation of the solution.

The results obtained in n. 3 permit us to use the surfaces defined by (1) to approximate the solution of problem (j). However, the n integrals present in (1) obstruct the use of these surfaces to obtain an approximation of a numerical type.

To overcome the numerical quadrature obstacle, in this section we applied a discretization to the approximation method given by (1), referring to the finite difference method.

Having put

(9i) $|\beta_i(x,y)|\leq Q$ for every $(x,y)\in I$, $i=1,2,\ldots,r$

let[2] $T = \{(x,y): p_o\leq x\leq a,\ -b-\lambda(a-x)\leq y\leq b+\lambda(a-x)\}$, $\lambda\geq|f_q|$,

(2) The requirement $\lambda\geq|f_q|$ is the Courant - Friedrichs - Lewy condition or the domain of dependence condition.

$b\in\mathbb{R}$, be a trapezium containing the set $\bigcup_{i=1}^{r}\{\alpha_i(I)\times\beta_i(I)\}$

Having fixed an integer n and having subdivided the intervals $[p_o,a]$ and $[-b-\lambda(a-p_o), b+\lambda(a-p_o)]$ in parts whose width is respectively $\Delta x = (a-p_o)/n$, $\Delta y = \lambda\Delta x$ such that the point $(0,0)$ belongs to the division, we cover the trapezium by a net of discrete points with coordinates $(m\Delta x, s\Delta y)$. From the net points $(p_o, \pm s\Delta y)$ we then trace segments with angular coefficient $\pm\lambda$ on this net.

Let $\pi_n: T\to R$ be the polyhedric triangular-faced continuous surface, whose vertices $V_{m,s} = (m\Delta x, s\Delta y, \pi_n(m\Delta x, s\Delta y))$ are projected in the plane $z=0$ on the net, defined in the net points as follows

$$(5)\quad \pi_n((m+1)\Delta x, s\Delta y) = \begin{cases} \phi^{(o)}((m+1)\Delta x, s\Delta y)+\rho((m+1)\Delta x, s\Delta y) \\ \text{if } (m+1)\Delta x\leq 0 \\ \frac{1}{2}\pi_n(m\Delta x,(s+1)\Delta y)+\frac{1}{2}\pi_n(m\Delta x,(s-1)\Delta y)+ \\ +\frac{1}{2}\Delta x\cdot f(m\Delta x, s\Delta y; \ldots, \pi_n(\alpha_i(m\Delta x, s\Delta y), \beta_i(m\Delta x, s\Delta y)), \ldots \\ \ldots; \frac{\Delta\pi_n}{\Delta y}(m\Delta x, s\Delta y)) + \\ +\frac{1}{2}\Delta x\cdot f(m\Delta x, \Delta y; \ldots, \pi_n(\alpha_i(m\Delta x, s\Delta y), \beta_i(m\Delta x, s\Delta y)), \ldots \\ \ldots; \frac{\Delta\pi_n}{\Delta y}(m\Delta x,(s-1)\Delta y))+\rho(m\Delta x, s\Delta y)) \\ \text{if } (m+1)\Delta x>0, \quad s=0,\pm1,\pm2,\ldots,\pm h_{m+1} \end{cases}$$

where $|\rho(m\Delta x, s\Delta y)|\leq\rho(\Delta x)$ for every $m=-n_1,-(n_1-1),\ldots,0,1,2,\ldots,n_2$, $(n_1+n_2=n)$, $s=0,\pm1,\pm2,\ldots,\pm h_{m+1}$ and $\rho(h) = \mathcal{O}(h^\alpha)$ (the order α will later be estimated).

For brevity's sake, we put

$$\frac{\Delta\pi_n}{\Delta x}(m\Delta x, s\Delta y) = \frac{\pi_n((m+1)\Delta x, s\Delta y)-\pi_n(m\Delta x, s\Delta y)}{\Delta x}$$
$$\frac{\Delta\pi_n}{\Delta y}(m\Delta x, s\Delta y) = \frac{\pi_n(m\Delta x,(s+1)\Delta y)-\pi_n(m\Delta x, s\Delta y)}{\Delta y}$$

As in section 3, we prove that the sequence $(\pi_n)_n$ converges

uniformly to the solution z of the problem (j).

Working in a similar way to lemma 1, the following is demonstrated

LEMMA 2. Under the hypotheses (i)-(9i), every surface π_n satisfies in T the properties

$$(P_1) \quad \left|\frac{\Delta\pi_n}{\Delta x}\right| \le G + \frac{\lambda^2}{2}\hat{M}\Delta x + \frac{\rho(\Delta x)}{\Delta x}$$

$$(P_2) \quad \left|\frac{\Delta\pi_n}{\Delta y}\right| \le \hat{N}$$

$$(P_3) \quad \left|\frac{\Delta\pi_n}{\Delta y}(m\Delta x,(s+1)\Delta y) - \frac{\Delta\pi_n}{\Delta y}(m\Delta x,s\Delta y)\right| \le \hat{M}\Delta y$$

$$(P_4) \quad \left|\frac{\Delta\pi_n}{\Delta y}((m+1)\Delta x,s\Delta y) - \frac{\Delta\pi_n}{\Delta y}(m\Delta x,s\Delta y)\right| \le (\lambda\hat{M}+\hat{R}+\frac{2}{\lambda}\frac{\rho(\Delta x)}{\Delta x^2})\Delta x$$

$$(P_5) \quad \left|\frac{\Delta\pi_n}{\Delta x}(m\Delta x,(s+1)\Delta y) - \frac{\Delta\pi_n}{\Delta x}(m\Delta x,s\Delta y)\right| \le (\lambda\hat{M}+\hat{R}+\frac{2}{\lambda}\frac{\rho(\Delta x)}{\Delta x^2})\Delta y$$

$$(P_6) \quad \left|\frac{\Delta\pi_n}{\Delta x}((m+1)\Delta x,s\Delta y) - \frac{\Delta\pi_n}{\Delta x}(m\Delta x,s\Delta y)\right| \le \{(\lambda^2+\tfrac{1}{2}\lambda)\hat{M} + (\lambda+\tfrac{1}{2})\hat{R} +$$
$$+H+rL_z[(G+\frac{\lambda^2}{2}\hat{M})\Delta x+\frac{\rho(\Delta x)}{\Delta x})A + \hat{N}C] + (2+\tfrac{1}{\lambda})\rho(\Delta x)/\Delta x^2\}\Delta x$$

where

$$\hat{N} = \hat{N}(n) = N + a + \frac{2}{\lambda}(n+1)\rho(\Delta x)/\Delta x,$$

$$\hat{M} = \hat{M}(n) = 2M + a + \frac{4}{\lambda^2}(n+1)\rho(\Delta x)/\Delta x^2,$$

$$\hat{R} = \hat{R}(n) = K_y + rK_z[(G+\frac{\lambda^2}{2}\hat{M}\Delta x + \frac{\rho(\Delta x)}{\Delta x})B + \hat{N}D] + K_q\hat{M}.$$

In virtue of what was established in Lemma 2, we can prove the following:

THEOREM 2. If $\rho(h) = \mathcal{O}(h^3)$, arbitrarily fixed $\varepsilon>0$ the sequence $(\pi_n)_n$ is definitely made up of ε-approximate solutions of the problem (j).

Proof. Every surface π_n verifies in T the property (s_2) by construction.

Let $(x,y)\in I_o \cap T$ be a point belonging to the triangle with vertices $(m\Delta x,s\Delta y)$, $((m+1)\Delta x,s\Delta y)$, $(m\Delta x,(s+1)\Delta y)$. By definition we have

$$\pi_n(x,y) = \pi_n(m\Delta x,s\Delta y) + \frac{x-m\Delta x}{\Delta x}[\pi_n((m+1)\Delta x,s\Delta y)-\pi_n(m\Delta x,s\Delta y)] +$$

$$+ \frac{y-s\Delta y}{\Delta y}[\pi_n(m\Delta x,(s+1)\Delta y)-\pi_n(m\Delta x,s\Delta y)].$$

Considering the expression of $\hat{N}$ and $\hat{M}$ it results that

$$(6) \quad |\pi_n(x,y)-\phi^{(o)}(x,y)| \le |\phi^{(o)}(m\Delta x,s\Delta y)-\phi^{(o)}(x,y)| +$$
$$+ |x-m\Delta x|\cdot[G+\frac{\lambda^2}{2}\hat{M}\Delta x + \frac{\rho(\Delta x)}{\Delta x}] + |y-s\Delta y|\hat{N} + \rho(\Delta x) \le$$
$$\le G\Delta x + N\Delta y + \Delta x[G+\frac{\lambda^2}{2}\hat{M}\Delta x + \frac{\rho(\Delta x)}{\Delta x}] + \Delta y\hat{N} + \rho(\Delta x) =$$
$$= \Delta x(2G+2\lambda N+\lambda a) + \Delta x^2(\lambda^2\hat{M} + \frac{\lambda^2}{2}a) + (4n+6)\rho(\Delta x) \le \gamma_1\Delta x + \gamma_2\frac{\rho(\Delta x)}{\Delta x}$$

Expression (6) proves that the sequence $(\pi_n)_n$ definitely satisfies the property (s_1).

Let us now verify the property (s_3).

Fixed (x,y) belonging to the triangle with vertices $(m\Delta x,(s+1)\Delta y)$, $((m+1)\Delta x,(s+1)\Delta y)$, $((m+1)\Delta x,s\Delta y)$ contained in $I\cap T$, we have

$$\left|\frac{\Delta\pi_n}{\Delta x}(x,y)-f(x,y;\ldots,\pi_n(\alpha_i(x,y),\beta_i(x,y)),\ldots;\frac{\Delta\pi_n}{\Delta y}(x,y))\right| = \left|\frac{\Delta\pi_n}{\Delta x}(m\Delta x,(s+1)\Delta y) +\right.$$
$$-f(m\Delta x,(s+1)\Delta y;\ldots,\pi_n(\alpha_i(m\Delta x,(s+1)\Delta y),\beta_i(m\Delta x,(s+1)\Delta y)),\ldots;\frac{\Delta\pi_n}{\Delta y}(m\Delta x,(s+1)\Delta y))+$$
$$+f(m\Delta x,(s+1)\Delta y;\ldots,\pi_n(\alpha_i(m\Delta x,(s+1)\Delta y),\beta_i(m\Delta x,(s+1)\Delta y)),\ldots;\frac{\Delta\pi_n}{\Delta y}(m\Delta x,(s+1)\Delta y))+$$
$$\left.-f(x,y;\ldots,\pi(\alpha_i(x,y),\beta_i(x,y),\ldots;\frac{\Delta\pi_n}{\Delta y}(x,y))\right| \le$$
$$\le \left|\frac{\pi_n((m+1)\Delta x,(s+1)\Delta y) - \pi_n(m\Delta x,(s+1)\Delta y)}{\Delta x} +\right.$$
$$\left.-f(m\Delta x,(s+1)\Delta y;\ldots,\pi_n(\alpha_i(m\Delta x,(s+1)\Delta y),\beta_i(m\Delta x,(s+1)\Delta y)),\ldots;\frac{\Delta\pi_n}{\Delta y}(m\Delta x,(s+1)\Delta y))\right| +$$
$$+ H\Delta x + L_y\Delta y + rL_z\{[G+\frac{\lambda^2}{2}\hat{M}\Delta x + \frac{\rho(\Delta x)}{\Delta x}][A\Delta x + B\Delta y] + \hat{N}[C\Delta x + D\Delta y]\} +$$
$$+ L_q\{[\lambda\hat{M} + \hat{R} + \frac{2}{\lambda}\frac{\rho(\Delta x)}{\Delta x^2}]\Delta x + \hat{M}\Delta y\} \le$$
$$\le \frac{1}{\Delta x}\left|\frac{1}{2}\pi_n(m\Delta x,(s+2)\Delta y) + \frac{1}{2}\pi_n(m\Delta x,s\Delta y) - \pi_n(m\Delta x,(s+1)\Delta y)\right| +$$
$$+\left|\frac{1}{2}f(m\Delta x,(s+1)\Delta y;\ldots,\pi_n(\alpha_i(m\Delta x,(s+1)\Delta y),\beta_i(m\Delta x,(s+1)\Delta y)),\ldots;\frac{\Delta\pi_n}{\Delta y}(m\Delta x,s\Delta y))+\right.$$
$$\left.-\frac{1}{2}f(m\Delta x,(s+1)\Delta y;\ldots,\pi_n(\alpha_i(m\Delta x,(s+1)\Delta y),\beta_i(m\Delta x,(s+1)\Delta y)),\ldots;\frac{\Delta\pi_n}{\Delta y}(m\Delta x,(s+1)\Delta y))\right|+$$

$$+ \bar{W}_1 \Delta x + \bar{W}_2 \rho(\Delta x)/\Delta x^2 ,$$

with an obvious meaning of the constants $\bar{W}_1$ and $\bar{W}_2$.

With opportune increases we obtain

$$(7)\quad \left|\frac{\Delta \pi_n}{\Delta x}(x,y) - f\left(x,y;\dots,\pi_n(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\Delta \pi_n}{\Delta y}(x,y)\right)\right| \le$$

$$\le \frac{\Delta y}{2\Delta x}\left|\frac{\Delta \pi_n}{\Delta y}(m\Delta x,(s+1)\Delta y) - \frac{\Delta \pi_n}{\Delta y}(m\Delta x,s\Delta y)\right| +$$

$$+ \frac{\lambda}{2} L_q \hat{M} \Delta x + \bar{W}_1 \Delta x + \bar{W}_2 \rho(\Delta x)/\Delta x^2 \le W_1 \Delta x + W_2 \rho(\Delta x)/\Delta x^2$$

which proves the property (s_3) definitely respect to n.

The proof is analogous if (x,y) belongs to the triangle with vertices $(m\Delta x,s\Delta y)$, $(m\Delta x,(s+1)\Delta y)$, $((m+1)\Delta x,s\Delta y)$.

We then demonstrate [3] that the difference $|\pi_n - z|$ converges to zero as $n \to +\infty$.

From (7) it follows that

$$\left|z_x(x,y) - \frac{\Delta \pi_n}{\Delta x}(x,y)\right| \le \left|f\left(x,y;\dots,z(\alpha_i(x,y),\beta_i(x,y)),\dots;z_y(x,y)\right) +\right.$$

$$\left. - f\left(x,y;\dots,\pi_n(\alpha_i(x,y),\beta_i(x,y)),\dots;\frac{\Delta \pi_n}{\Delta y}(x,y)\right)\right| + W_1 \Delta x + W_2 \cdot \rho(\Delta x)/\Delta x^2 \le$$

$$\le K_z \sum_{i=1}^{r} \left|z(\alpha_i(x,y),\beta_i(x,y)) - \pi_n(\alpha_i(x,y),\beta_i(x,y))\right| +$$

$$+ K_q \left|z_y(x,y) - \frac{\Delta \pi_n}{\Delta y}(x,y)\right| + W_1 \Delta x + W_2 \rho(\Delta x)/\Delta x^2 \le$$

$$\le 2rK_z V + W_1 \Delta x + W_2 \frac{\rho(\Delta x)}{\Delta x^2} + K_q \left|z_y(x,y) - \frac{\Delta \pi_n}{\Delta y}(x,y)\right| + \sigma \left|z(x,y) - \pi_n(x,y)\right|$$

for every $\sigma > 0$, where we put $|\pi_n(x,y)| \le V$ for every n and for every $(x,y) \in T$.

Considering (6) in virtue of Haar's Lemma, we obtain

$$\left|z(x,y) - \pi_n(x,y)\right| \le \varphi_1 \Delta x + \varphi_2 \frac{\rho(\Delta x)}{\Delta x} + (2rK_z V + W_1 \Delta x + W_2 \rho(\Delta x)/\Delta x^2)a$$

for every $(x,y) \in T$.

Proceeding by iteration, we have

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(3) Let us point out that the properties (s_3) and (P) imply the existence of a subsequence $(\pi_{n_k})_k$ converging to the exact solution of the problem.

$$(8)\quad \left|z(x,y) - \pi_n(x,y)\right| \le \left[\varphi_1 \Delta x + \varphi_2 \frac{\rho(\Delta x)}{\Delta x} + (W_1 \Delta x + W_2 \rho(\Delta x)/\Delta x^2)a\right] \cdot$$

$$\cdot \left[1 + rK_z a + (rK_z a)^2 + \dots + (rK_z a)^m + \dots\right] + \lim_{m \to +\infty} 2V(rK_z a)^m =$$

$$= \left[(\varphi_1 + aW_1)\Delta x + \varphi_2 \frac{\rho(\Delta x)}{\Delta x} + W_2 a \frac{\rho(\Delta x)}{\Delta x^2}\right] / (1 - rK_z a)$$

for every $(x,y) \in T$. Due to the arbitrary choice of $b \in \mathbb{R}$ and therefore of the trapezium T we reach the thesis.

The following is immediately deduced:

COROLLARY 2. If $\rho(h) = \mathcal{O}(h^3)$, the sequence $(\pi_n)_n$ converges uniformly together with its derivatives $\frac{\Delta \pi_n}{\Delta x}$, $\frac{\Delta \pi_n}{\Delta y}$ to the surfaces z, z_x, z_y respectively, where z is the exact solution of the problem (j).

REMARK 1. If the relation (1) of [3] does not determine a reduction of the strip I, Baiada's method establishes the global existence of the solution and gives a numerical approximation. For example, in the case of semilinear equations and systems, the relations (1) are reduced to inequalities that do not imply restrictions of I and therefore the results obtained are globally valid, as proved in [5].

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GLOBAL EXISTENCE UNIQUENESS AND STABILITY OF THE NONLINEAR BOLTZMANN EQUATION WITH ALMOST GENERAL GAS-PARTICLES INTERACTION POTENTIAL

G. Toscani and N. Bellomo

1. Introduction

The Boltzmann equation describes the time-space evolution of the one-particle distribution function of a diluite monoatomic gas. The relevant properties of this equation can be found, for instance, in the book of Cercignani /4/ or in the recently edited lecture by Zweifel /14/.

The analysis of the initial value problem for the full nonlinear Boltzmann equation, as carefully reviewed in refs./5, 8/, has been developed in the last 40 years with the final objective of supplying a proof of global existence under reasonable assumptions on the initial data. Such a proof does not involve a purely mathematical interest. In fact, it can also be regarded as an undirect validation of the mathematical model defined by the Boltzmann equation and hopefully give some light towards the rigorous derivation of this equation.

In spite of the relevance of the subject and of several previous attempts, as documented in the large bibliography cited in refs./5, 8/ as well as in /14/, a proof of global existence for the nonlinear unmodified Boltzmann equation has been just very recently supplied by Illner and Shinbrot /9/ for initial data which decay exponentially to zero at infinity and by the Authors /1/ for initial data which go to zero at infinity in terms of $1/|\underset{\sim}{x}|^p$, $p > 1$, where $\underset{\sim}{x}$ is the space coordinate.

Both these papers /9, 1/ deal with the nonlinear Boltzmann equation in absence of an external field; the method which is applied in each paper towards the final objective is, however, quite different. The first paper applies the so called "iterative scheme" by Kaniel and Shinbrot /10/ and proves global existence, uniqueness and asymptotic behaviour of the solutions of the initial value problem for the "hard spheres" Boltzmann equation. The second paper applies the fixed point Cacciopoli Banach theorem /11/ to obtain the same proof for the nonlinear Boltzmann equation characterized by a large class of "hard" gas-particles interaction potential when the initial data obey the afore mentioned decay at infinity.

As far as the asymptotic behaviour is concerned, the first paper shows that the number density tends to zero as the inverse cubic power of time, i.e. as $1/t^3$. On the other hand, the second paper shows a decay in terms of $1/t^p$ with the weaker constraint: $p > 1$. This slowest decay is due to the smoother assumption on the initial data.

This paper is in final form and will not be submitted elsewhere for publication

It is worth pointing out that the physical idea of attempting a succesfull proof of the global existence of the solutions of the initial value problem for the nonlinear Boltzmann equation under the assumption that the gas is initially confined in a central region has been independently produced by Illner and Shinbrot in the already cited paper /9/ and by the first author /12/ who supplied an analogous result, qualitatively very closed to the one of ref./9/, for the nonlinear discrete velocity Boltzmann equation.

Both method have shown wide generality. In fact, the iterative scheme has been applied /2/ to the analysis of the initial value problem for the semidiscrete Boltzmann equation, a model proposed by Cabannes /3/, whereas Toscani /13/ has applied the fixed point theorems towards the analysis of the Discrete Velocity Boltzmann equation without any restriction on the number of admissible velocities and on the space distribution of the initial data as far as the decay at infinity is assured in terms analogous to the ones of /1/.

The objective of this paper consists in supplying a proof for global existence and asymptotic behaviour, by means of a suitable application of fixed point theorems, i.e. with the same method of /1/, for initial data which decay at infinity, but with very general potential, with cut-off, for the intermolecular forces. The second section of the paper is then a brief summary of the physical and mathematical structure of the Boltzmann equation. The third section defines some relevant properties of the non-linear collisional operator related to the mathematical models defining the intermolecular forces interaction potential and supplies a local existence theorem. The fourth section contains the main result: global existence, uniqueness and asymptotic stability are, in fact, proven, showing that the local solution, which exists unique and positive, can be globally extended. The last section contains a detailed analysis, including a discussion on its consistency, about the assumptions on the initial data as well as on the asymptotic behaviour of the solutions.

2. The Boltzmann Equation

The integral form of the full nonlinear Boltzmann equation, which defines the time-space evolution of the one particle distribution function $f = f(t, \underset{\sim}{x}, \vec{V})$ of a diluite monoatomic gas can be written, in absence of external fields and for potentials with cut-off, in the following integral form:

$$(1)\qquad f(t, \underset{\sim}{x}, \vec{V}) = f_o(\underset{\sim}{x} - \vec{V}t, \vec{V}) + \int_0^t J_1(s, \underset{\sim}{x} - \vec{V}(t-s), \vec{V})\, ds - \int_0^t J_2(s, \underset{\sim}{x} - \vec{V}(t-s), \vec{V})\, f(s, \underset{\sim}{x} - \vec{V}(t-s), \vec{V})\, ds$$

where f is a non-negative real valued function of the time $t \in [0, T]$, of the space $\underset{\sim}{x} \in \mathbb{R}^3$ and of the vector velocity $\vec{V} \in \mathbb{R}^3$ and where the "gain" and "loss" terms have, respectively, the following expressions:

$$(2)\qquad J_1(t, \underset{\sim}{x}, \vec{V}) = \int_D B(\theta, \vec{q})\, f(t, \underset{\sim}{x}, \vec{V}')\, f(t, \underset{\sim}{x}, \vec{V}_1')\, d\varepsilon\, d\theta\, d\vec{V}_1$$

$$(3)\qquad f(t, \underset{\sim}{x}, \vec{V})\, J_2(t, \underset{\sim}{x}, \vec{V})$$

where, in eq.(3) the term J_2 is given by

$$(4)\qquad J_2 = \int_D B(\theta, \vec{q})\, f(t, \underset{\sim}{x}, \vec{V}_1)\, d\varepsilon\, d\theta\, d\vec{V}_1$$

$(\vec{V}, \vec{V}_1)$ are the precollisional velocities, $(\vec{V}', \vec{V}_1')$ the postcollisional velocities, $\vec{q}$ is the relative velocity in the collision:

$$\vec{q} = (\vec{V}_1 - \vec{V})$$

and the angles "ε" and "θ" are, respectively the polar and azimuthal angles in a spherical coordinate system attached to $\vec{V}$ with the z-axis oriented in the direction of $\vec{q}$. Consequently, the domain D is

$$D = [0, 2\pi] \oplus [0, \pi/2] \oplus \mathbb{R}^3$$

The actual structure of $B(\theta, \vec{q})$ depends upon the interaction force potential for pairs of gas-particles. Usually this term, for inverse power interaction potential, is written in the form:

$$(5)\qquad B = B(\theta, q; \alpha) = \beta_\alpha(\theta)\, q^{(\alpha - 4)/\alpha}$$

(Soft interactions: $2 \leq \alpha < 4$; Maxwell molecules $\alpha = 4$; Hard interactions: $\alpha > 4$).

Equation (1) is the integral formulation of the differential form of the Boltzmann equation:

$$(6)\qquad f = f(t, \underset{\sim}{x}, \vec{V}), \qquad \partial f/\partial t + \vec{V} \cdot \underset{\sim}{\nabla} f = J(f, f) = J_1(f, f) - f\, J_2(f)$$

In order to deal with the Cauchy problem (1) some function space has to be defined with a definition of solution. The analysis of this paper is found on the concept of mild solution, which was already defined in ref./1/ for the Boltzmann equation. However, for completeness, some further details are also given here.

Let $\underset{\sim}{C}_b(\mathbb{R}^3 \oplus \mathbb{R}^3)$ be the space of all continuous functions on $\mathbb{R}^3 \oplus \mathbb{R}^3$ which tend to zero at infinity and let $\mathbb{B}_T$ be the space

$$(7)\qquad \mathbb{B}_T = \underset{\sim}{C}_b([0, T] \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$$

of all functions mapping $(t, \underset{\sim}{x}, \vec{V})$ into $\mathbb{R}$ equipped, respectively, with the norms:

$$(8)\qquad \| f \|_\infty = \sup_{(\underset{\sim}{x}, \vec{V}) \in \mathbb{R}^3 \oplus \mathbb{R}^3} | f |$$

$$(9)\qquad \| f \| = \sup_{t \in T} \| f \|_\infty$$

Moreover, the following complete Banach space can be considered:

(10) $$\mathfrak{B}_T^r = \{f(t, \underset{\sim}{x}, \vec{V}) = f^*(t, \underset{\sim}{x}, \vec{V}) \exp(-r^2 V^2) \; ; \quad f^* \in \mathbb{B}_T\}$$

equipped with the norm

(11) $$\| f \|^r = \sup_{t \in T} \| f^* \|_\infty$$

obviously: $f \in \mathfrak{B}_T^r \Rightarrow f \in \mathbb{B}_T$.

The following definition of "mild solution" can now be supplied:

- *A function $f = f(t, \underset{\sim}{x}, \vec{V})$ is defined "mild solution" of the initial value problem for the Boltzmann equation (6) with given initial conditions $f_o = f(0, \underset{\sim}{x}, \vec{V})$ if $0 \leq f \in \mathbb{B}_T$ and eq.(1) is satisfied* - .

3. Some preliminary results and local existence

The analysis here in after realized is founded on the following hypothesis which defines the angular cutoff for the internal forces potential:

Hypothesis: The function $B(\theta, \vec{q})$ is regular in its arguments and the following bound, see Grad/6, 7/, holds:

(12) $$|B(\theta, \vec{q})/(\sin\theta \cos\theta)| < F(1 + q)/q^\delta , \quad 0 \leq \delta < 1$$

Note that the hypothesis (12) is certainly more general than the corresponding one proposed in ref./1/. In particular both "soft" and "hard" interactions are described by the function B with the cut-off defined in (12). The various modifications to the Grad's assumption, discussed in /8/, also fullfil the hypothesis.

It is worth mentioning that in the analysis developed in this section, some of the results given in paper /1/ will be directly utilized. In this case the proof is not repeated, but full references are supplied.

A first implication of the afore-mentioned hypothesis is the following:

Proposition: If the Hypothesis holds, then the following integral:

(13) $$I(r, \delta) = \int_{\mathbb{R}^3 \cdot [0, \pi/2]} B(\theta, \vec{q})(\exp(-r^2 V_1^2)/q) \, d\vec{V}_1 d\theta$$

is bounded as follows:

(14) $$I(r, \delta) \leq c_{r,\delta} = F(\pi^3/r^2)(1/\sqrt{2}\, r)^{(1-\delta)}(1 + \sqrt{2}\, r/(1 - \delta))$$

The proof is supplied in the Appendix.

After these preliminaries, we can now show in the same fashion as in ref./1/, that the "gain" term of the Boltzmann equation is globally bounded for some special

classes of functions. Accordingly, the following Lemma can be proposed:

LEMMA 1 : Let $G_{ar}(t, \underset{\sim}{x}, \vec{V}) = \exp(-r^2 V^2) \exp(-a^2(\underset{\sim}{x} - \vec{V} t)^2)$, if the Hypothesis holds, then the integral:

(15) $$K_{ar} = K_{ar}(t, \underset{\sim}{x}, \vec{V}) = \int_0^t J_1(G_{ar}, G_{ar})(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V}) \, ds$$

is bounded as follows:

(16) $$K_{ar} \leq (2\pi\sqrt{\pi}/|a|) \, c_{r,\delta} \exp(-r^2 V^2) \exp(-a^2(\underset{\sim}{x} - \vec{V} t)^2)$$

LEMMA 2: Let $B_{pr}(t, \underset{\sim}{x}, \vec{V}) = \exp(-r^2 V^2)/(1 + a^2|\underset{\sim}{x} - \vec{V} t|^2)^{p/2}$, if the Hypothesis holds then the integral:

(17) $$H_{pr} = H_{pr}(t, \underset{\sim}{x}, \vec{V}) = \int_0^t J_1(B_{pr}, B_{pr})(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V}) \, ds$$

is bounded as follows:

(18) $$H_{pr} \leq \frac{16}{|a|(p - 1)} \, c_{r,\delta} \exp(-r^2 V^2)/(1 + a^2|\underset{\sim}{x} - \vec{V} t|^2)^{p/2}$$

with $p > 1$.

Proof of Lemma 1: Replacing the actual expression of G_{ar} into eq.(15) yields:

(19) $$K_{ar} = \int_0^t \int_D B(\theta, \vec{q}) \exp(-r^2(V'^2 + V_1'^2)) \exp(-a^2(\underset{\sim}{x} - \vec{V}(t - s) - \vec{V}'s)^2)$$

$$\exp(-a^2(\underset{\sim}{x} - \vec{V}(t - s) - \vec{V}_1's)) \, d\varepsilon \, d\theta \, d\vec{V}_1 \, ds$$

setting $\underset{\sim}{y} = \underset{\sim}{x} - \vec{V} t$ and recalling the energy conservation in the collision:

$$V'^2 + V_1'^2 = V^2 + V_1^2$$

the following expression of K_{ar} is obtained:

(20) $$K_{ar} = 2\pi \exp(-r^2 V^2) \int_0^t \int_{[0, \frac{1}{2}\pi] \cdot \mathbb{R}^3} B(\theta, \vec{q}) \exp(-a^2(\underset{\sim}{y} + (\vec{V} - \vec{V}')s)^2 +$$

$$(\underset{\sim}{y} + (\vec{V} - \vec{V}_1')s)^2)) \exp(-r^2 V_1^2) \, d\varepsilon \, d\vec{V}_1 \, ds$$

Moreover, the following equality (the proof is supplied in the Appendix) holds:

(21) $$|\underset{\sim}{y} + (\vec{V} - \vec{V}')s|^2 + |\underset{\sim}{y} + (\vec{V} - \vec{V}_1')s|^2 = y^2 + |\underset{\sim}{y} + \vec{q}\, s|^2 .$$

Consequently, casting the above equality into the expression of K_{ar} yields:

$$(22)\quad K_{ar} = 2\pi \exp(-r^2V^2)\int_0^t\int_{[0,\pi/2]\cdot\mathbb{R}^3} B(\theta, \vec{q})\exp(-r^2 V_1^2)\exp(-a^2(y^2 + |\underset{\sim}{y} + \vec{qs}|^2))$$

$$d\theta\, d\vec{V}_1\, ds \leq 2\pi \exp(-r^2 V^2)\int_{[0,\pi/2]\cdot\mathbb{R}^3} B(\theta, \vec{q})\exp(-r^2 V_1^2)$$

$$\exp(-a^2 y^2)\, d\theta\, d\vec{V}_1 \int_0^t \exp(-a^2(\underset{\sim}{y} + \vec{qs})^2)\, d\theta\, dV_1\, ds$$

where the integration over time can be easily recovered with the final result:

$$(23)\quad K_{ar} \leq (2\pi\sqrt{\pi}/|a|)\exp(-r^2 V^2)\exp(-a^2 y^2)\int_{[0,\pi/2]\cdot\mathbb{R}^3} (B(\theta, \vec{q})/q)\exp(-r^2V_1^2)\, d\theta\, d\vec{V}_1$$

Lemma 1 is then proven if the inequality of the Proposition is applied and the expression of $\underset{\sim}{y}$ is cast into inequality (23).

Proof of Lemma 2: The proof follows exactly the same steps of Lemma 1 and 2 of ref./1/ with the obvious implications due to the more general Hypothesis concerning the term "B" and the consequences implied by the Proposition. Such a proof is not then repeated here.

Let us now consider the problem of the local existence, uniqueness and positivity of the solutions of the Cauchy problem in unbounded domains for initial conditions specialized in the same form of the functions G_{ar} and B_{ar} as in the above Lemma. Accordingly, the following Theorem can be proposed:

THEOREM 1: Let the initial conditions be supplied in the form:

$$(24)\quad 0 < f_o(\underset{\sim}{x}, \vec{V};\, t=0)\, \exp(r^2 V^2) \in \tilde{C}_b(\mathbb{R}^3 \oplus \mathbb{R}^3)\ ,\quad r > 0$$

if the hypothesis holds and for soft interaction $\alpha \leq 4$, the "mild" solution of eq.(6) exists unique in $\mathbb{D}_T$ for $t \in [0,T]$, where

$$(25)\quad T \leq \frac{1}{16\, c_{r\delta}\| f_o\|^r}$$

$$(26)\quad \mathbb{D}_T = \{f \in \mathfrak{B}_T^r:\ f \geq 0\ ,\quad \| f\|^r \leq 2\| f_o\|^r\}$$

Proof: For every real valued positive constant λ, let f^λ denote the function:

$$f^\lambda(t, \underset{\sim}{x}, \vec{V}) = \exp(\lambda t)\, f(t, \underset{\sim}{x}, \vec{V})$$

Obviously: $f \in \mathfrak{B}_T^r \Rightarrow f^\lambda \in \mathfrak{B}_T^r$.

Casting f^λ into eq.(1) supplies the following "mild formulation" of the evolution of f^λ:

$$(27)\quad f^\lambda(t, \underset{\sim}{x}, \vec{V}) = f_o(\underset{\sim}{x} - \vec{V}t, \vec{V};t=0) + \int_0^t J^\lambda(s, \underset{\sim}{x} - \vec{V}(t-s), \vec{V})\, ds$$

where

$$(28)\quad J^\lambda(t, \underset{\sim}{x}, \vec{V}) = \lambda f^\lambda(t, \underset{\sim}{x}, \vec{V}) + \exp(-\lambda t)\, J_1(f^\lambda, f^\lambda)(t, \underset{\sim}{x}, \vec{V}) -$$

$$\exp(-\lambda t) f^\lambda(t, \underset{\sim}{x}, \vec{V})\, J_2(f^\lambda)(t, \underset{\sim}{x}, \vec{V})$$

where the expressions of J_1 and J_2 are supplied by eqs.(2-4) with f replaced by f^λ.

Equation (27) can be rewritten, in operator form, as follows:

$$(29)\quad f^\lambda = G_\lambda f^\lambda$$

where the meaning of G_λ is clearly defined by the above equations.

If the Hypothesis holds and if T is bounded as stated by eq.(25) then for $\lambda = 4\, c_{r,\delta}\| f_o\|^r$, the following implications and inequalities can be proven:

$$(30)\quad f^\lambda \in \mathfrak{B}_T^r \Rightarrow G_\lambda f^\lambda \in \mathfrak{B}_T^r$$

$$(31)\quad f^\lambda \in \mathbb{D}_T \Rightarrow G_\lambda f^\lambda \in \mathbb{D}_T$$

$$(32)\quad \| G_\lambda f_1^\lambda - G_\lambda f_2^\lambda\|^r \leq (5/8)\| f_1^\lambda - f_2^\lambda\|^r$$

The proof of (30-32) is given in the Appendix.

Consequently, according to the Cacciopoli-Banach fixed point theorem /11/, eq.(29) has a unique nonnegative mild solution for $t \in [0, T]$. In the same fashion, existence and uniqueness can be proven in the same time interval and for initial data as specified in the Theorem for the Boltzmann equation. Therefore, since f mantains the same sign of f^λ, then the unique solution, for non negative initial conditions, is also non negative. Moreover,

$$(33)\quad \| f(t = T)\|^r \leq (2/\exp(1/4))\| f_o\|^r$$

Remark: Both functions specialized in Lemma 1 and 2 satisfy the conditions of the Theorem. The condition (33) indicates, analogously to ref./2/, that if the condition at t = T is assumed as initial conditions for the following interval and so on, then the solution itself cannot be globally extended without some further smallness assumptions.

Remark: An analogous proof can be realized for "hard" interaction with the further hypothesis on the initial data that f_o tends to zero at infinity in space as in Lemma 1 and 2.

2. Global Existence and Asymptotic Stability

After the local existence Theorem supplied in the preceeding section, one has to prove that the solution, which exists unique and positive locally, is globally bounded and stays in the considered function space. The positiveness of the local solution implies that the above objective can also be reached considering the truncated equation:

$$f = U f \quad : \qquad f(t, \underset{\sim}{x}, \vec{V}) = f_o(t=0, \underset{\sim}{x} - \vec{V}t, \vec{V}) + \int_0^t J_1(s, \underset{\sim}{x} - \vec{V}(t-s), \vec{V})ds \tag{34}$$

which is written in a fixed point form.

The line which is here followed is the same of the local existence Theorem. Namely a suitable closed convex subset of $\mathfrak{B}_T^r$ is chosen in a fashion that the operator U can be proved to define a mapping from the said subset into itself. I. e. the operator equation : f = Uf admittes a unique fixed point in the considered subset of the Banach space defined in (10).

Accordingly, the following two theorems are here proposed:

THEOREM 2: If the Hypothesis holds and the initial conditions are such that:

$$0 \leq f_o(t=0, \underset{\sim}{x}, \vec{V}) \in \mathcal{C}_b^{\alpha}(\mathbb{R}^3 \oplus \mathbb{R}^3) \tag{35}$$

with

$$f_o(\underset{\sim}{x}, \vec{V}) \leq \alpha_{pr} exp(-r^2 V^2) exp(-a^2 x^2), \qquad \alpha_{pr} = \frac{|a|}{8\pi\sqrt{\pi}\, c_{r,\delta}} \tag{36}$$

then the Cauchy problem for the Boltzmann equation (6) has a unique "mild" solution:

$$\forall t \geq 0 : \quad 0 \leq f(t, \underset{\sim}{x}, \vec{V}) \in \mathbb{B}_T \tag{37}$$

moreover:

$$f(t, \underset{\sim}{x}, \vec{V}) \leq 2\alpha_{pr} exp(-r^2 V^2) exp(-a^2(\underset{\sim}{x} - \vec{V}t)^2) \tag{38}$$

THEOREM 3: If the Hypothesis holds and the initial conditions are such that:

$$0 \leq f_o(t=0, \underset{\sim}{x}, \vec{V}) \in \mathcal{C}_b^{\alpha}(\mathbb{R}^3 \oplus \mathbb{R}^3) \tag{39}$$

with

$$f_o(\underset{\sim}{x}, \vec{V}) \leq \beta_{pr} \frac{exp(-r^2 V^2)}{(1 + a^2 x^2)^{\frac{1}{2}p}} \quad , \quad p > 1, \quad \beta_{pr} = (p-1)|a|/(64\, c_{r,\delta}) \tag{40}$$

then the Cauchy problem for the Boltzmann equation (6) has a unique "mild" solution:

$$\forall t \geq 0 : \quad 0 \leq f(t, \underset{\sim}{x}, \vec{V}) \in \mathbb{B}_T \tag{41}$$

moreover

$$f(t, \underset{\sim}{x}, \vec{V}) \leq 2\beta_{pr} \frac{exp(-r^2 V^2)}{(1 + a^2|\underset{\sim}{x} - \vec{V}t|^2)^{\frac{1}{2}p}} \tag{42}$$

Proof of Theorem 2: Consider the complete Banach space defined in (10) and the subset $\mathfrak{D}_T^r$ of such a space defined by:

$$\mathfrak{D}_T^r = \{f \in \mathfrak{B}_T^r : \; 0 \leq f(t, \underset{\sim}{x}, \vec{V}) \leq 2\alpha_{ar} \exp(-r^2V^2)\exp(-a^2(\underset{\sim}{x} - \vec{V}t)^2)\} \tag{43}$$

If the initial conditions are specialized as in the Theorem, then the operator "U", defined in (34), is a contractive mapping from $\mathfrak{D}_T^r$, convex closed subset of $\mathfrak{B}_T^r$, into itself. Therefore U has a unique fixed point, for every $t \geq 0$, in $\mathfrak{D}_T^r$.

The proof is in two steps:

a) $f \in \mathfrak{B}_T^r \implies U f \in \mathfrak{B}_T^r, \quad U : \mathfrak{B}_T^r \to \mathfrak{B}_T^r$

b) U is a contractive operator on $\mathfrak{D}_T^r$.

The proof of step a) is already contained in the proof of the local existence Theorem. The proof of step b) is founded on the result of Lemma 1. In fact, after Lemma 1,

$$\int_0^t J_1(G_{ar}, G_{ar})(s, \underset{\sim}{x} - \vec{V}(t-s), \vec{V})\, ds \leq (1/4\,\alpha_{ar})\, G_{ar}(t, \underset{\sim}{x}, \vec{V}) \tag{44}$$

consequently:

$$f \in \mathfrak{D}_T^r \implies U f \in \mathfrak{D}_T^r \tag{45}$$

Let now $f, g \in \mathfrak{D}_T^r$, then the following inequality holds:

$$|U f - U g| \leq \int_0^t \int_D B(\theta, \vec{q}) |f(s, \underset{\sim}{x} + \vec{V}s, \vec{V}')f(s, \underset{\sim}{x} + \vec{V}s, \vec{V}'_1) - g(s, \underset{\sim}{x} + \vec{V}s, \vec{V}')g(s, \underset{\sim}{x} + \vec{V}s, \vec{V}'_1)|\, d\varepsilon\, d\theta\, d\vec{V}_1\, ds \leq \tag{46}$$

$$\int_0^t \int_D B(\theta, \vec{q})\{|f^*(s, \underset{\sim}{x} + \vec{V}s, \vec{V}') - g^*(s, \underset{\sim}{x} + \vec{V}s, \vec{V}')|g^*(s, \underset{\sim}{x} + \vec{V}s, \vec{V}'_1)$$

$$+ |f^*(s, \underset{\sim}{x} + \vec{V}s, \vec{V}'_1) - g^*(s, \underset{\sim}{x} + \vec{V}s, \vec{V}'_1)|f^*(s, \underset{\sim}{x} + \vec{V}s, \vec{V}'_1)\}.$$

$$\exp(-r^2V_1^2)\exp(-r^2V^2)\, d\varepsilon\, d\theta\, d\vec{V}_1\, ds$$

consequently, according also to Lemma 2,

$$|Uf - Ug| \leq \tfrac{1}{2}\| f - g \|^r \exp(-r^2V^2) \tag{47}$$

and considering the proposed definition of norm:

$$\| Uf - Ug\|^r \leq \tfrac{1}{2}\| f - g\|^r \tag{48}$$

which proves the second step of the Theorem, which is then finally proven.

Proof of Theorem 3: The proof, which is not repeated here, is the same of the proof of the main Theorem of paper /1/ besides the only difference of the new constant β_{rp}, which takes into account the more general Hypothesis and replaces the constant of the corresponding Theorem in ref./1/.

Both Theorems 2 and 3 imply the following stability result:

THEOREM 4: Let the initial conditions be supplied as in Theorem 2 or 3, then in both cases there exists a non empty set of initial conditions f_o, with $f_o \in \Phi$, such that:

$$f_o \in \Phi \Rightarrow \forall f_o \in \Phi : \ \| f_o\| > 0\,, \quad \lim_{t\to\infty} \| f \| \to 0$$

where Φ is the set of functions which obey the constraint indicated in Theorems 2 and 3.

Proof of Theorem 4: The proof is simply a direct consequence of the inequalities (38) and (42) defined in the statements of Theorems 2 and 3.

5. Discussion

Two results supplying global existence, uniqueness and asymptotic behaviour have been supplied in this paper for a gas which is confined in a central region in a fashion that the gas density tends to zero at infinity in space and velocity. The proofs have been supplied for very general gas-particles forces interaction potentials with the only requirement of satisfying the angular "cut-off" condition detailled in the hypothesis of sec. 2.

As stated in the introduction of this paper, the idea of confining the gas into a central region was produced independently in papers /9, 12/. The first one dealing with the "hard spheres" nonlinear Boltzmann equation and the second one on the Discrete Boltzmann equation.

This present research work supplies a new proof and generalization to a larger class of interaction potentials of the results ofpaper /9/ (which considered the "exponential decay" at infinity of the initial conditions) as well as of the results of paper /1/. It is also worth pointing out that a local existence and positivity proof has been supplied in this present work, so that global existence

follows, after some further smallness assumptions, from the local existence proof. In other words, the method supplies, without any smallness assumption, a local existence and uniqueness proof for positive defined solutions of the initial value problem.

At this end, a discussion on the consistency about the physics of the initial conditions is necessary with reference both to the initial data and to the asymptotic behaviour of the solutions. In particular, the discussion is realized with reference to the following quantities:

local number density at t=0 : $n_o(x, t = 0) = \int_{\mathbb{R}^3} f_o(x, \vec{V}; t=0)\, d\vec{V}$

local mean free path at t=0 : $\ell_o(x, t = 0) \propto 1/n_o(x, t = 0)$

total number of particles : $N = \int_{\mathbb{R}^3} n_o(x, t=0)\, dx$

asymptotic behaviour of $n(0, t)$: $n(x = 0;\ t \to \infty)$

The above quantities can be recovered after standard calculation well known in the kinetic theory of gases. The result is visualized in the table below, which reports the said quantities for both cases:
exponential decay of the initial data at infinity in space (case A),
inverse power decay of the initial data at infinity in space (case B).

	A : Exponential decay Theorem 2	B: Inverse power decay Theorem 3
n_o	$(\lvert a\rvert/8c_{r,\delta}r^3)\exp(-a^2x^2)$	$\frac{(p-1)\lvert a\rvert\pi\sqrt{\pi}}{64\, c_{r,\delta} r^3}(1+a^2x^2)^{-\frac{1}{2}p}$
ℓ_o	$\propto (8c_{r,\delta}r^3/\lvert a\rvert)\exp(a^2x^2)$	$\frac{64\, c_{r,\delta} r^3}{(p-1)\lvert a\rvert\pi\sqrt{\pi}}(1 + a^2x^2)^{\frac{1}{2}p}$
N	$\pi\sqrt{\pi}/(8\, a^2 r^3 c_{r,\delta})$	$\forall\, a,\ p < 5 : N \to \infty$
$n(0, t\to\infty)$	$\propto 1/t^3$	$\propto 1/t^p$

The result summarized in the table clearly indicates the following features:

a) For both cases the mean free path in the origin can be very small if the parameters "a" in case A and "p" B are large:

A: $a\uparrow \Rightarrow \ell\downarrow$, B: $p\uparrow \Rightarrow \ell\downarrow$

i.e. the higherst is the decay in space of the initial density, the highest can be the gas density in the origin.

b) The total number of particles in the case A is finite (a gas in a vacuum), on the other hand, in case B the number of particles is finite for $p < 5$.

c) The rate of decay for the number density is of the order of $1/t^3$ in case A and of the order of $1/t^p$ in case B. So that for $p < 3$ a smoother decay can be obtained even for small mean free path in the origin.

Both points b,c) show, in the comparison, that the conditions ot Theorem 3 are less restrictive on the basis of the physics of the system, than the ones of Theorem 2. This comparison is even more evident if one realizes the exercise of specializing the constant p in order to have the same values of the factor for the densities in both cases; or specializing the constant $c_{r,\delta}$ for a particular interaction potential.

The main difference remains that global existence is obtained for a finite number of particles in case A and for an infinite number of particles in case B.

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APPENDIX

Proof of Inequality (14):

Consider the Integral $I(r,\delta)$ defined in (13). Then if the Hypothesis holds:

$$I \leq F(\pi/2) \int_{\mathbb{R}^3} ((1 + q)/q^{(\delta+1)}) \exp(-r^2 V_1^2)\, d\vec{V}_1$$

choosing a frame (x, y, z) with the x-axis in the direction of $\vec{V}$ we get, as a consequence:

$$|\vec{q}| = |\vec{V}_1 - \vec{V}| = ((V_1 - x)^2 + y^2 + z^2)^{\frac{1}{2}} \geq (x^2 + y^2)^{\frac{1}{2}}$$

and, if $\rho = (x^2 + y^2)^{\frac{1}{2}}$:

$$(1 + q)/q^{(1+\delta)} \leq (1 + \rho)/\rho^{(1+\delta)}$$

consequently:

$$I \leq F(\pi/2) \int_{-\infty}^{\infty} \exp(-r^2 x^2)\, dx \int_0^{2\pi} \int_0^{\infty} \exp(-r^2 \rho^2)((1 + \rho)/\rho^{\delta})\, d\rho\, d\theta =$$

$$= (\pi^2\sqrt{\pi}/r)F \int_0^{\infty} ((1 + \rho)/\rho^{\delta}) \exp(-r^2\rho^2)\, d\rho$$

which yields:

$$I \leq (\pi^2\sqrt{\pi}F/r) \int_0^{\infty} (1/\rho^{\delta}) \exp(-r^2\rho^2)\, d\rho + \int_0^{\infty} \rho^{(1-\delta)} \exp(-r^2\rho^2)\, d\rho$$

the execution of both integrals in the same fashion of the analogous integrals calculated in the Appendix of ref./1/ supplies the result defined in eq.(14).

Proof of Equality (21)

$$|\underset{\sim}{y} + (\vec{V} - \vec{V}')s|^2 + |\underset{\sim}{y} + (\vec{V} - \vec{V}_1)s|^2 =$$

$$y^2 + s^2(\vec{V} - \vec{V}')^2 + 2s\underset{\sim}{y}\cdot(\vec{V} - \vec{V}') + y^2 + s^2(\vec{V} - \vec{V}_1)^2 + 2s\underset{\sim}{y}\cdot(\vec{V} - \vec{V}_1) =$$

$$y^2 + (y^2 + 2s\underset{\sim}{y}\cdot((\vec{V} - \vec{V}') + (\vec{V} - \vec{V}_1)) + s^2((\vec{V} - \vec{V}')^2 + (\vec{V} - \vec{V}_1)^2))$$

recalling now, see again the Appendix of ref./1/, that:

$$\vec{q} = (\vec{V} - \vec{V}') + (\vec{V} - \vec{V}_1)$$

and casting the above expression of $\vec{q}$ into the considered equality proves eq.(21).

Proof of Implications (30, 31):

Let $f \in \mathfrak{G}_T^r$, then the following inequalities hold:

$$\exp(r^2V^2) \int_0^t f(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V})\, ds \leq$$

$$\lambda\, T \|f\|^r \exp(r^2V^2) \left| \int_0^t \exp(-\lambda s)\, J_1(f,f)(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V}) ds \right| <$$

$$\exp(r^2V^2) \int_0^t \int_{\mathbb{R}^3} \cdot \int_0^{\frac{1}{2}\pi} B(\theta, \vec{q}) |f(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V}')f(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V}_1')| \cdot$$

$$d\vec{V}_1\, d\theta\, ds \leq \int_0^t \int_{\mathbb{R}^3} \int_0^{\frac{1}{2}\pi} B(\theta, q) \|f\|^{r2} \exp(-r^2V_1^2)\, d\vec{V}_1\, d\theta\, ds \leq c_{r,\delta} T \|f\|^r$$

analogously for the last term in J^{λ}, the implication (30) is then proven.

Let now $f \in \mathbb{D}_T$, obviously:

$$\int_0^{\frac{1}{2}\pi} \int_{\mathbb{R}^3} B(\theta, q)\, f(s, \underset{\sim}{x} - \vec{V}(t - s), \vec{V}_1)\, d\theta\, d\vec{V}_1 \leq c_{r,\delta} \|f_o\|^r$$

Consequently if $\lambda = 4\, c_{r,\delta} \|f_o\|^r$, then $J^{\lambda}(f,f) \geq 0$. Moreover, if $f > 0$, the

following inequality holds for $f \in \mathbb{D}_T$ and $T \leq 1/(16\, c_{r,\delta} \| f_o \|^r)$

$$\| G_\lambda f \|^r \leq \| f_o \|^r (1 + 2\lambda T + 4\, c_{r,\delta} \| f_o \|^r T) \leq 2 \| f_o \|^r$$

and implication (31) is also proven.

Proof of Inequality (32):

Consider the expression: $\exp(r^2 V^2)\ |G_\lambda f - G_\lambda g|$ with $f, g \in \mathbb{D}_T$, then, according to the various inequalities indicated in the proof of implications (30,31), the following further inequality holds:

$$\exp(r^2 V^2)\ |G_\lambda f - G_\lambda g| \leq$$

$$\lambda T \| f - g \|^r + (2\, c_{r,\delta} T \| f \|^r + 2\, c_{r,\delta} T \| g \|^r) \| f - g \|^r \leq$$

$$(\lambda T + 8\, c_{r,\delta} T \| f_o \|^r) \| f - g \|^r$$

Consequently, setting the actual value of T as indicated in the Theorem and considering the norm of the right-side term in the above inequality, proves the inequality (32).

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HOMEOMORPHISMS OF THE CIRCLE AND THE MAGNITUDE OF FOURIER COEFFICIENTS

Daniel Waterman

A very useful estimate of the Fourier coefficient is that made in terms of the integral modulus of continuity. If $f \in L^1$ on the circle group T, then the integral modulus of continuity in L^1 is

$$\omega_1(\delta) = \omega_1(\delta; f) = \sup_{0 \le h \le \delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)| \, dx \right\}$$

and we have the wellknown result

$$|\hat{f}(n)| \le \frac{1}{2} \omega_1(\pi/n) .$$

If one examines the proof of this result, it will be seen that what is crucial is that we are integrating the values of an interval function, and a simple modification yields the result that for a function of bounded variation we have

$$(*) \qquad \hat{f}(n) = O(1/n) .$$

Function classes defined by conditions on the suprema of sums of intervals functions are invariant under change of variable, i.e., if g is a homeomorphism of T onto itself, then for any interval I, $g(I) = J$ is an interval and $f \circ g(I) = f(J)$. Here if $J = [a,b]$, by $f(\ )$ we mean $f(b) - f(a)$. Thus the class of functions of bounded variation is mapped into itself by such compositions as are other classes of functions of generalized bounded variation. An example of such a class for which an estimate may be derived for the Fourier coefficient is the class of functions of Λ-bounded variation (ΛBV). Let $\Lambda = \{\lambda_n\}$, be an increasing sequence of positive numbers such that $\Sigma 1/\lambda_n = \infty$. A function f is said to be in ΛBV if, for every sequence $\{I_n\}$ of nonoverlapping intervals, we have

$$\sum_1^\infty |f(I_n)|/\lambda_n < \infty .$$

We have shown [2,3] that for $f \in \Lambda$BV we have

(**) $$|\hat{f}(n)| = O\Big(1/\sum_{1}^{n} 1/\lambda_k\Big).$$

It is natural to consider the following question: What characterizes the class of functions f such that $f\circ g$ satisfies condition (*) or condition (**) for every homeomorphism g? In the case of (*) this has been studied. We have shown [4,5] that for a regulated function, i.e., a function with only simple discontinuities, (*) holds for every g if and only if f is of bounded variation. If we do not assume that f is regulated, we find that f must be equal to a function of bounded variation except on a set of universal measure zero, i.e., a set all of whose homeomorphic images are of measure zero.

It should be noted that the question we have posed above is somewhat ambiguous. It has two possible interpretations. Considering condition (*), we may seek to characterize the functions f such that

(A) $$|\widehat{f\circ g}(n)| \le C_f/n \quad \text{for every } g$$

or

(B) $$|\widehat{f\circ g}(n)| \le C_{f,g}/n \quad \text{for every } g,$$

where we indicate dependence of the constant on functions by subscripts. We have shown that these conditions are, in fact, equivalent [5].

When we considered this question in relation to the condition (**), we found it more natural to express the condition in the following way.

Let $h(n)$ be a nondecreasing positive function which is concave downward. Čanturia [1] has considered the functions of class $V[h]$, the regulated functions which have the property

$$\sup \sum_{1}^{n} |f(I_k)| \le Ch(n),$$

the supremum being extended over all sequences $\{I_k\}_1^n$ of nonoverlapping intervals, and has shown that the Fourier coefficient of an $f \in V[h]$ satisfies

(***) $$|\hat{f}(n)| \le Ch(n)/n.$$

We have recently shown [6] that $V[h]$ is the class of regulated functions for which

$$|\widehat{f\circ g}(n)| \le Ch(n)/n$$

for every homeomorphism g. As above, the requirements $C = C_f$ and $C = C_{f,g}$ are equivalent. If we do not assume that f is regulated, then f must equal a function of $V[h]$ except on a set of universal measure zero.

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SUI TEOREMI DI VITALI - HAHN - SAKS, DIEUDONNE', E NIKODYM

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INTRODUZIONE.

In Teoria della Misura, sono ben noti i teoremi che vanno sotto il nome di Vitali-Hahn-Saks, (v. [9]), di Nikodym (v. [8]) e di Dieudonné (v. [4]). Lo scopo di questa nota è di fornire una panoramica di alcuni recenti risultati che generalizzano tali teoremi in situazioni astratte. Prenderemo in considerazione misure finitamente additive, definite su algebre booleane, e a valori in gruppi topologici. Quasi tutte le dimostrazioni sono omesse (per esse si rimanda alla letteratura), eccetto quella del teorema 2.6., che è nuovo.

1. NOTAZIONI E DEFINIZIONI.

Sia Γ un gruppo abeliano topologico separato, il cui elemento neutro verrà denotato con 0. Su Γ faremo anche l'ipotesi di completezza, benché non sempre sia necessaria.

Denotiamo con $\mathcal{I}(0)$ una base d'intorni, chiusi e simmetrici, del lo 0 in Γ.

Se (B_n) è una successione di parti di Γ, con la scrittura

$$\lim_{n\to\infty} B_n = 0 ,$$

intenderemo che, per ogni intorno U di 0, si ha definitivamente $B_n \subset U$.

Se U è un intorno di 0, e k è un intero positivo, denoteremo con k U l'insieme

$$k\,U = \{u_1+\ldots+u_k : (u_1,\ldots,u_k)\in U^k\} .$$

1.1. DEFINIZIONE.

Se B è un sottoinsieme di Γ, diremo che B è *limitato* se, per ogni elemento $U\in\mathcal{I}(0)$, esiste un intero $k>0$ verificante la relazio-

ne $B \subset k\ U$.

Sia ora $\mathcal{A}$ una fissata algebra di insiemi, avente Ω come elemento massimo, e $\emptyset$ come minimo.

Su $\mathcal{A}$ supporremo fissati due reticoli, $\mathcal{F}$ e $\mathcal{G}$, tali che ogni elemento di $\mathcal{F}$ sia il complementare di qualche elemento di $\mathcal{G}$.

1.2. DEFINIZIONI.

Diremo *massa* (a valori in Γ), ogni applicazione finitamente additiva $m: \mathcal{A} \to \Gamma$. Per ogni elemento $A \in \mathcal{A}$, denoteremo con $m^+(A)$ l'insieme $\{m(A \cap E) : E \in \mathcal{A}\}$.

Data una massa $m: \mathcal{A} \to \Gamma$, e una parte $\mathcal{H}$ di $\mathcal{A}$, diremo che m è *$\mathcal{H}$-esaustiva* se, per ogni successione (H_n) di elementi di $\mathcal{H}$, a due a due disgiunti, si ha $\lim_{n \to \infty} m(H_n) = 0$.

Data una famiglia $\{m_i\}_{i \in I}$ di masse a valori in Γ, diremo che le m_i sono *uniformemente $\mathcal{H}$-esaustive* se, per ogni successione (H_n) di elementi di $\mathcal{H}$, a due a due disgiunti, risulta

$$\lim_{n \to \infty} \{m_i(H_n) : i \in I\} = 0$$

Qualora si abbia $\mathcal{H} = \mathcal{A}$, si parlerà semplicemente di *esaustività* e di *uniforme esaustività*.

1.3. DEFINIZIONE.

Data una massa $m: \mathcal{A} \to \Gamma$, diremo che essa è *regolare* (rispetto a $\mathcal{F}$ e $\mathcal{G}$) se sono soddisfatte le seguenti condizioni:

(a) per ogni elemento $A \in \mathcal{A}$, esistono due successioni, (F_n) e (G_n), per le quali si ha:

$F_n \in \mathcal{F}$, $G_n \in \mathcal{G}$, $F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n$ per ogni $n \in \mathbb{N}$, e

$$\lim_{n \to +\infty} m^+(G_n \setminus F_n) = 0 ;$$

(b) per ogni elemento $B \in \mathcal{F}$, esistono due successioni, (F_n) e (G_n), per le quali si ha:

$F_n \in \mathcal{F}$, $G_n \in \mathcal{G}$, $B \subset F_{n+1} \subset G_n \subset F_n$ per ogni $n \in \mathbb{N}$, e

$$\lim_{n \to +\infty} m^+(F_n \setminus B) = 0$$

Nel caso particolare in cui $\mathcal{A}$ sia la tribù dei boreliani in uno spazio T_4, $\mathcal{G}$ il reticolo degli insiemi aperti, e $\mathcal{F}$ quello dei chiusi, la definizione 1.3. si riduce a quella usuale di regolarità per una misura (v. [4], [1], [2]).

2. RECENTI RISULTATI.

Cominceremo riportando un teorema di [2], il quale illustra alcune conseguenze della regolarità, combinata con l'esaustività.

2.1. TEOREMA ([2]).

Siano $m_n: \mathcal{A} \to \Gamma$ masse regolari, per $n \in \mathbb{N}$. Se le m_n sono uniformemente $\mathcal{G}$-esaustive, allora esse sono anche uniformemente esaustive.

(Da questo enunciato si può dedurre facilmente che lo stesso teorema vale anche per famiglie generiche di masse, e non solo per successioni, ma noi qui lo useremo in questa forma).

Come vedremo, il teorema 2.1., abbinato ad un'opportuna formulazione del teorema di Vitali-Hahn-Saks, conduce ad una versione astratta del teorema di Dieudonné, circa la convergenza di misure regolari. Un approccio di questo tipo è già stato operato in [2], ma sotto condizioni meno generali di quelle che daremo qui.

2.2. DEFINIZIONE (v. [3]).

Data una parte $\mathcal{H}$ di $\mathcal{A}$, diremo che $\mathcal{H}$ gode della *proprietà* (f_1) se, per ogni coppia (A_n), (B_n) di successioni in $\mathcal{A}$, verificanti le relazioni:

$B_n \in \mathcal{H}$ $\forall n$; $A_n \cap A_m = B_n \cap B_m = \emptyset$ per $n \neq m$; $A_n \cap B_k = \emptyset$ $\forall n$ e k;

esistono una parte infinita $P \subset \mathbb{N}$, e un elemento $B \in \mathcal{H}$, tali da aversi:

$A_n \cap B = \emptyset$ per ogni n; $B_n \cap B = \emptyset$ per $n \in P$; $B_n \subset B$ per ogni $n \in P$.

In lavori precedenti, ([5], [6], [7], [10]) si sono introdotte varie proprietà del genere della (f_1), tutte relative al caso $\mathcal{H} = \mathcal{A}$: la proprietà (E), la (SCP), la (f). Non è difficile provare che, sempre nel caso $\mathcal{H} = \mathcal{A}$, ciascuna di queste implica la (f_1).

La seguente proposizione è di facile verifica.

2.3. PROPOSIZIONE.

Supponiamo che $\mathcal{H} \subset \mathcal{A}$ goda della proprietà (f_1), e sia (H_n) una successione di elementi di $\mathcal{H}$, a due a due disgiunti. Esistono allora una successione (P_k) di parti infinite di $\mathbb{N}$, a due a due disgiunte, e una successione (F_k) di elementi di $\mathcal{H}$, a due a due disgiunti, per i quali si ha:

$$H_n \subset F_k, \text{ per ogni } n \in P_k, \text{ e per ogni } k \in \mathbb{N}.$$

Il lemma che segue si può provare procedendo esattamente come

nella dimostrazione del Lemma 3.4. di [3].

2.4. LEMMA.

Sia $\mathscr{H}$ un reticolo, che goda della proprietà (f_1), e sia (H_p) una successione di elementi di $\mathscr{H}$, a due a due disgiunti. Sia poi (P_k) una successione decrescente di parti infinite di $\mathbb{N}$, i cui elementi minimi siano denotati (p_k), e soddisfino alla condizione: $p_k \notin P_{k+1}$ per ogni $k \in \mathbb{N}$. Sia infine (F_k) una successione decrescente di elementi di $\mathscr{H}$, per la quale risulti, $\forall k \in \mathbb{N}$,

$$H_p \subset F_k \text{ per } p \in P_k; \text{ e } H_p \cap F_{k+1} = \emptyset, \text{ per } p \leq p_k.$$

Allora esistono una successione crescente (k_j) d'interi positivi, e un elemento $F \in \mathscr{H}$, per i quali si ha, per ogni $j \in \mathbb{N}$:

$$H_{p_{k_j}} \subset F; \text{ e } F \setminus (H_{p_{k_1}} \cup \ldots \cup H_{p_{k_j}}) \subset F_{k_j+1}.$$

In [3], utilizzando 2.3. e 2.4. per il caso $\mathscr{H} = \mathscr{A}$, si prova la seguente estensione del teorema di Nikodym.

2.5. TEOREMA.

Sia $\{m_i\}_{i \in I}$ una famiglia di masse esaustive, definite su $\mathscr{A}$ e a valori in Γ. Supponiamo che la stessa algebra $\mathscr{A}$ goda della proprietà (f_1), e che sia limitato in Γ l'insieme

$$\{m_i(A) : i \in I\}$$

per ogni elemento $A \in \mathscr{A}$. Allora è limitato anche l'insieme

$$\bigcup_{A \in \mathscr{A}} \{m_i(A) : i \in I\}.$$

Proveremo ora un'estensione del teorema di Vitali-Hahn-Saks.

2.6. TEOREMA.

Sia $\mathscr{H} \subset \mathscr{A}$ un reticolo che goda della proprietà (f_1). Sia data una successione (m_n) di masse $\mathscr{H}$-esaustive, definite su $\mathscr{A}$ e a valori in Γ. Se esiste il limite

$$\lim_{n \to +\infty} m_n(H)$$

per ogni elemento $H \in \mathscr{H}$, allora le m_n sono uniformemente $\mathscr{H}$-esaustive.

DIMOSTRAZIONE. Sia (H_n) una successione di elementi di $\mathscr{H}$, a due a due disgiunti. Procedendo per assurdo, supporremo, senza perdita di generalità, che esista un intorno $U \in \mathscr{I}(0)$ tale da aversi

$$m_n(H_n) \notin U + U, \text{ per ogni } n.$$

Scegliamo ora una successione decrescente (U_h) in $\mathscr{I}(0)$, con $U_0 = U$, e verificante: $U_h + U_h \subset U_{h-1}$ per ogni h.

Ora, utilizzando la proposizione 2.3., e l'esaustività di m_1, è possibile determinare una parte infinita $P_1 \subset \mathbb{N}$, e un elemento $F_1 \in \mathscr{H}$ in modo che risulti: $H_p \subset F_1$ per $p \in P_1$, e $m_1^+(F_1) \subset U_1$.

Poniamo $p_1 = \min P_1$: in relazione a H_{p_1}, esiste un intero $N_1 > p_1$ per il quale si abbia: $m_i(H_{p_1}) - m_j(H_{p_1}) \in U_1$, per $i,j \geq N_1$.

Per l'esaustività di m_{p_1}, e usando ancora 2.3., si può trovare una parte infinita $P_2 \subset P_1$, il cui minimo verrà denotato p_2, e un elemento $F_2 \in \mathscr{H}$, in modo da aversi:

$H_p \subset F_2 \quad \forall p \in P_2;\ p_2 > N_1 + p_1;\ m_{p_1}^+(F_2) \subset U_2$.

Inoltre, eventualmente usando ancora la (f_1), si può fare in modo che sia $F_2 \cap H_p = \emptyset$ per ogni $p \leq p_1$.

In relazione a H_{p_2}, esiste poi un intero $N_2 > p_2$ per cui risulti:

$m_i(H_{p_2}) - m_j(H_{p_2}) \in U_2$, per ogni $i, j \geq N_2$.

Così procedendo, si verrà a determinare una successione decrescente (P_k) di parti infinite di $\mathbb{N}$, i cui minimi saranno denotati (p_k), una successione decrescente (F_k) di elementi di $\mathscr{H}$, e una successione crescente (N_k) di numeri interi positivi, soddisfacenti alle quattro condizioni seguenti:

(1) $N_k > p_k$; $p_{k+1} > N_k + p_k$ per ogni k;

(2) $H_p \subset F_k$ per $p \in P_k$, e $H_p \cap F_{k+1} = \emptyset$ per $p \leq p_k$, per ogni k;

(3) $m_{p_k}^+(F_{k+1}) \subset U_{k+1}$, per ogni k;

(4) $m_i(H_{p_k}) - m_j(H_{p_k}) \notin U_k$, per $i, j \geq N_k$, e per ogni k.

Per semplicità, scriveremo m_k' in luogo di m_{p_k}, e H_k' anziché H_{p_k}. Avremo allora, per $k > 2$:

$m_k'(H_k') \notin U + U$, ma $m_{k-1}'(H_k') \in U_k$ per la (3), da cui

$$(\circ) \quad m_k'(H_k') - m_{k-1}'(H_k') \notin U + U_1.$$

Ora, utilizzando il Lemma 2.4., è possibile determinare una successione crescente (k_r) di interi pari, e un elemento $F \in \mathcal{H}$, tali da aversi, per ogni r:

$$H'_{k_r} \subset F \quad , \text{ e } \quad F \setminus (H'_{k_1} \cup \ldots \cup H'_{k_r}) \subset F_{k_r+1} \ .$$

Allora avremo, per ogni .i e ogni r in $\mathbb{N}$:

$$m_i(H'_{k_r}) = m_i(F) - \sum_{\rho=1}^{r-1} m_i(H'_{k_\rho}) - m_i\left[F \setminus (H'_{k_1} \cup \ldots \cup H'_{k_r})\right] \ .$$

Essendo $F_{k_r+1} \subset F_{k_r}$, avremo $m'_{k_r}\left[F \setminus (H'_{k_1} \cup \ldots \cup H'_{k_r})\right] \in U_{r+1}$, e $m'_{k_r-1}\left[F\ (H'_{k_1} \cup \ldots \cup H'_{k_r})\right] \in U_r$.

Inoltre, poiché i k_r sono pari, avremo $p_{k_r-1} > N_{k_r-2} \geq N_{k_{r-1}}$, per per cui è $m'_{k_r}(H'_{k_\rho}) - m'_{k_r-1}(H'_{k_\rho}) \in U_\rho$ per $\rho = 1,\ldots,r-1$.

Infine, per r sufficientemente grande, è anche $m'_{k_r}(F) - m'_{k_r-1}(F) \in U_2$, poiché è $F \in \mathcal{H}$, e quindi

$$m'_{k_r}(H'_{k_r}) - m'_{k_r-1}(H'_{k_r}) \in U_2 + U + U_r + U_{r+1} \subset U + U_1 ,$$

e ciò è in contrasto con la (°).

L'assurdo trovato prova la validità dell'asserto.

Combinando il teorema 2.6. con 2.1., e tenendo presente la completezza di Γ, non è difficile ora dedurre la seguente estensione del teorema di Dieudonné.

2.7. TEOREMA.

Supponiamo che il reticolo $\mathcal{G}$ goda della proprietà (f_1), e siano $m_n : \mathcal{A} \to \Gamma$ masse $\mathcal{G}$-esaustive e regolari, per ogni $n \in \mathbb{N}$. Se esiste il limite

$$\lim_{n\to+\infty} m_n(G) \ ,$$

per ogni elemento $G \in \mathcal{G}$, allora le m_n sono uniformemente esaustive, e il limite

$$\lim_{n\to+\infty} m_n(A)$$

esiste per ogni elemento $A \in \mathcal{A}$.

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EXISTENCE, SEMICONTINUITY AND REPRESENTATION FOR THE INTEGRALS OF THE CALCULUS OF VARIATIONS. THE BV CASE. (')

Primo Brandi - Anna Salvadori

1. *INTRODUCTION*. The Weierstrass integral of the Calculus of Variations (CV-W integral) consists on the Burkill-Cesari integral of a suitable set function. The advantages it allows, over the Lebesgue one, are well-known and they are already conspicuous even in the elementary case of the lenght of a curve.

Many Authors studied this integral, both in the parametric and in the non parametric case, and the bibliography on the subject is very rich; as regard the more recent results see Vinti ([14]).

In a rather general setting, the set function they consider is of the type

$$\Phi(I) = F(x(\omega(I)),\phi(I)) ,$$

where x is a point function, ω is a choice, i.e. $\omega(I) \in I$, and ϕ is a set function, not necessarily connected to x.

Up to now, all the results were given with the intention of achieving a complete independence of the integral on the choice function. Therefore they found a natural application in the case of the CV-W integrals over continuous and BV curves or surfaces.

While, in the present note, we sharpen the theory in order to widen the application-field up to contain the case of BV varieties (not necessarily continuous).

The idea consists on releasing the problem from the choice function, by substituting $x(\omega(I))$ with a generical set function $X(I)$. In this way, a natural adaptation of the known results gives the necessary background.

In particular we prove existence, semicontinuity and a Tonelli-

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(') This research has been carried out within the GNAFA - CNR.

type representation theorem.

Note that, coming back to the case $X(I)=x(\omega(I))$, our results take to blind the possible choices to a suitable subfamily.

In other words, it is sufficient to work on a subclass of choice functions ω, such that $X(I) = x(\omega(I))$ satisfies our hypotheses.

For instance, in the case of the CV-W integral over BV curves (see the applications - section 6), it is sufficient to avoid a suitable null set in the choice $\omega(I)\in I$.

Observe that we obtain, as a particular case, the results on the existence, semicontinuity and representation of the generalized variation, already proved by Boni in [1].

2. PRELIMINARIES.

We give here the main notations and definitions. Moreover, for convenience of the reader, we report also some well-known results that we shall use in the following.

Let $(A,\mathcal{G})$ be a topological space, $\mathcal{M}$ be the family of all the subsets of A and $\mathcal{B}$ be the σ-algebra generated by $\mathcal{G}$.

We consider a subfamily $\{I\}\subset\mathcal{M}$ and call "intervals" the sets I. A finite system $D = \left[I_1,\dots,I_N\right]$ is a finite collection of non-overlapping intervals, i.e.[(1)] $I_i \neq \emptyset$ and $I_i^\circ \cap \bar{I}_j = \emptyset$, $i\neq j$, $i,j = 1,\dots,N$.

Let $(D_t)_{t\in T}$ be a given net of finite systems.

Consider the function $s:\mathcal{M}\times\mathcal{M}\to\{0,1\}$ defined by $s(H,K) = 1$ when $H\subset K$ and $s(H,K) = 0$ otherwise.

Let $\phi : \{I\}\to R^n$ be a given interval function.

The function ϕ is Burkill-Cesari integrable (BC-integrable) ([10]) *over $M\in\mathcal{M}$ if the limit below exists*

$$\lim_T \sum_{I\in D_t} s(I,M)\phi(I)$$

In this case we shall denote its value by $BC\int_M \phi$.

The function ϕ is of bounded variations (BV) *if*

$$\overline{\lim_T} \sum_{I\in D_t} \|\phi(I)\| < +\infty$$

The function ϕ is quasi-additive (q.a.)([10]) *on $M\in\mathcal{M}$ if given $\varepsilon>0$ there exists $t_1=t_1(M,\varepsilon)$ such that, for every $t_0\gg t_1$ there is*

(1) I° and $\bar{I}$ denote respectively the $\mathcal{G}$-interior and the $\mathcal{G}$-closure of I.

$t_2=t_2(M,\varepsilon,t_0)$ *such that, if* $t\gg t_2$ *then*

$$\text{i)}\quad \sum_I s(I,M)\,\Big\|\sum_J s(J,I)\phi(J) - \phi(I)\Big\| < \varepsilon\,,$$

$$\text{ii)}\quad \sum_J s(J,M)\left[1 - \sum_I s(J,I)s(I,M)\right]\|\phi(J)\| < \varepsilon\,,$$

where $D_{t_0} = \left[I\right]$ *and* $D_t = \left[J\right]$.

The following results hold (see [10], [9], [15], [3]).

P_1. *If ϕ is q.a. on M, then it is BC-integrable over M.*

P_2. *If ϕ is q.a. and BV on M, then the function ϕ is q.a. on M.*

P_3. *If ϕ is q.a. and BV on A, then it is q.a. on M, for every $M\in\mathcal{M}$.*

REMARK 1. The present note deals with the parametric integral of the Calculus of Variations as a Weierstrass integral (see section 3). However all our results hold for the non-parametric case too. In fact it is well-known that it can be handled as a suitable parametric integral (see [7]).

Moreover for sake of semplicity and shortness, we shall state our propositions in finite dimension spaces even if they can be easily translated to the abstract setting (see analogously to [16], [3], [4], [5], [7]).

3. EXISTENCE.

3.a. - *The usual case.* Let (K,d) be a metric space and $P:\{I\}\to K$ be a given interval function.

The function P satisfies the condition (γ) ([2]) *if given $\varepsilon>0$ there exists $t_3=t_3(\varepsilon)$ such that, for every $t_0\gg t_3$ there is $t_4=t_4(\varepsilon,t_0)$ such that, if $t\gg t_4$, then*

$$\max_{J\in D_t, J\subset I, I\in D_{t_0}} d(P(I),P(J)) < \varepsilon$$

Let $F: K\times R^p\to R^q$ be given and consider the following conditions:

(F_1) *F is bounded in $K\times S_1$, where $S_1 = \{x\in R^p : \|x\| = 1\}$,*

(F_2) *F is uniformly continuous on $K\times S_1$,*

(F_3) $F(k,tx) = t\,F(k,x)$, $t\geq 0$, *on* $K\times R^p$.

We shall denote by $\Phi:\{I\}\to R^q$ the function

$$\Phi(I) = F(P(I),\phi(I)). \tag{1}$$

According to Cesari ([10], [11]) the BC-integral of Φ will be called the (parametric) *integral of the Calculus of Variations as a Weierstrass integral* (CV-W integral).

The following existence result can be proved analogously to the Theor. 6.i,ii in [10] and the Theor. 1.ii in [16].

PROPOSITION 1. *Suppose that*
F satisfies the conditions (F_1), (F_2) *and* (F_3);
P satisfies the condition (γ);
ϕ *is* q.a. *and* BV *on* A;
then the function Φ *is* q.a. *and* BV *on* A.

Note that Prop. 1 is an extension of the above mentioned existence results. In order to see that, let $p: A \to K$ be a point function and let $\omega:\{I\}\to A$ be a choice function, i.e. $\omega(I)\in I$, $I\in\{I\}$; then consider the interval function $P(I) = p\omega(I)$.

If p satisfies the Cesari's condition (P)

$$\lim_{T} \max \{ \operatorname{diam} p(I) , I\in D_t \} = 0$$

then, for every choice function ω, the interval function P satisfies the condition (γ).

While the Warner's condition (ξ), relative to a single choice function, is exactly our condition (γ).

3.b. - *The weighted case.* Suppose now that (K,d) *is compact* and consider the following functions:
$F:K\times R^p \to R^q$, $P : \{I\}\to K$, $\phi:\{I\} \to R^n$, $Q:\{I\} \to R^m$.

Moreover let $e_1\cdot e_2 : R^n\times R^m \to R^p$ denote a bilinear map with $\| e_1\cdot e_2\| \le \text{cost } \| e_1\| \, \| e_2\|$.

Thus, according to [7], the BC-integral of the interval function $\Phi_q:\{I\} \to R^q$ given by

$$\Phi_q(I) = F(P(I),\phi(I)\cdot Q(I))$$

will be called the *weighted* (parametric) *integral of the Calculus of Variations, as a Weierstrass integral.*

Note that this integral can be handled as an usual parametric CV-W integral, by introducing the integrand
$H:(K\times R^m)\times R^n \to R^q$ defined by $H(k,e_2;e_1) = F(k,e_1\cdot e_2)$.

In fact the following result can be easily proved.

LEMMA 1. *If* F *satisfies the conditions* (F_1), (F_2) *and* (F_3), *then* H *satisfies the same conditions on* $(K\times B_M)\times R^m$, *where* $B_M = \{e\in R^m : \|e\| \le M\}$, $M\in R^+$.

Thus, as a consequence of the Prop. 1, the following existence result can be proved.

PROPOSITION 2. *Suppose that*
F satisfies the conditions (F_1), (F_2) *and* (F_3);
P and Q satisfy the condition (γ) *and Q is bounded;*
ϕ *is* q.a. *and* BV *on* A;
then the function Φ_q *is* q.a. *and* BV *on* A.

4. *SEMICONTINUITY.*

In [16] Warner proposed a semicontinuity result for the CV-W integral which contains, as an application, the classic results due to Tonelli and Turner.

However the convergence he proposes in the abstract case, allows him to consider only the uniform convergence of continuous curves and surfaces, in the applications.

We give here a new version of Warner semicontinuity theorem which makes use of a weaker topology, in order to consider more general cases in the applications (see section 6).

Let's consider the following adjoint condition on the function F :

F *is real valued and*
(F_4) *for every* $(k_0,x_0)\in K\times S_1$ *and* $\varepsilon>0$ *there exists a number* $\rho_0 = \rho_0(k_0,x_0,\varepsilon)>0$ *and an element* $a_0 = a_0(k_0,x_0,\varepsilon)\in R^p$ *such that, for every* $\|k-k_0\| < \rho_0$ *it results*
$F(k,x) \ge \langle a_0,x\rangle$, $x\in R^q$,
$F(k,x) \le \langle a_0,x\rangle+\varepsilon\|x\|$, $x\in R^q$ *with* $\|(x/\|x\|)-x_0\| < \rho_0$.

Given a sequence of interval functions $(P_n)_{n\ge 0}$, we shall say that P_n M_1-*converges to* P_0 ($P_n \xrightarrow{M_1} P_0$) if

for every $\varepsilon>0$ *and* $\bar t\in T$ *there exists an integer* $n_0=n_0(\varepsilon,\bar t)$ *and an element* $t_5=t_5(\varepsilon,\bar t)\gg\bar t$ *such that, fixed any* $n\ge n_0$ *and any* $\bar{\bar t}\in T$, *then there is an element* $t_6=t_6(\varepsilon,\bar t,n,\bar{\bar t})\gg\bar{\bar t}$ *such that*

$$\max_{J\in D_{t_6},\,J\subset I,\,I\in D_{t_5}} d(P_0(I),P_n(J)) < \varepsilon$$

While we shall say that a sequence $(\phi_n)_{n\ge 0}$ of interval functions M_2-converges ($\phi_n \xrightarrow{M_2} \phi_0$) if
for every $\varepsilon>0$ *and* $I\in\{I\}$ *there exists an integer* $n_I=n_I(\varepsilon,I)$ *such that, for every* $n\ge n_I$ *there is an element* $t_I=t_I(\varepsilon,I,n)$ *such that for every* $t\gg t_I$ *it results*

$$\| \phi_0(I) - \sum_{J\in D_t} s(J,I)\phi_n(J) \| < \varepsilon .$$

PROPOSITION 3. Suppose that K is compact and F satisfies the conditions $(F_1),(F_2),(F_3)$ *and* (F_4); P_n *satisfies the condition* (γ), $n\geq 0$ *and* $P_n \xrightarrow{M_1} P_0$; ϕ_n *is q.a. and BV,* $n\geq 0$, *and* $\phi_n \xrightarrow{M_2} \phi_0$; *then* (1)

$$\underline{\lim_{n\to+\infty}}\ BC\int_A \phi_n \geq BC\int_A \phi_0 .$$

PROOF. Let $\varepsilon>0$ be fixed and denote by $M = \max\ \{\ \sup F_{/K\times S_1}\ ,\ \sup_T \sum_{I\in D_t} \|\phi_0(I)\|\ \}$.

For every $(k_0,x_0)\in K\times S_1$, let $\rho_0>0$ be given by virtue of the condition (F_4). The family $\{B(k_0,x_0;\rho_0)=\{(k,x): \|k-k_0\|<\rho_0, \|x-x_0\|<\rho_0\}, (k_0,x_0)\in K\times S_1\}$ is an open cover of the compact set $K\times S_1$; therefore it admets a finite subcover. Let (k_i,x_i), $i=1,\dots,r$, the relative centers and let $\rho_i/2$ and a_i be the corresponding elements obtained by virtue of the condition (F_4).

Note that it results

$\|a_1\| = \langle a_i, a_i/\|a_i\|\rangle \leq F(k_i, a_i/\|a_i\|) \leq M$, $i=1,\dots,r$.

Now we consider the number $\sigma=\sigma(\varepsilon) = \min\{\rho_i/2\ ,\ i=1,\dots,r\}$.

Then let $t_1=t_1(\varepsilon)$ be such that for every $t\gg t_1$ it is

$$(1)\qquad \left| BC\int_A F(P_0,\phi_0) - \sum_{I\in D_t} F(P_0(I),\phi_0(I))\right| < \varepsilon/4 .$$

Corresponding to σ and t_1, by virtue of (M_1) there exist $n_0=n_0(\varepsilon)$ and $t_5=t_5(\varepsilon)\gg t_1$ such that, for every $n\geq n_0$ and $\bar t\in T$, there is $t_6=t_6(\varepsilon,n,\bar t)\gg \bar t$ such that

$$(2)\qquad \max_{J\in D_{t_6}, J\subset I, I\in D_{t_5}} d(P_0(I),P_n(J))<\sigma .$$

Let $D_{t_5} = [I_1,\dots I_N] = [I]$. Then, by virtue of (M_2), for every $I\in D_{t_5}$ there exists an integer $n_I=n_I(\varepsilon/4MN, I)$ such that, for every $n\geq n_I$ there is $t_I=t_I(\varepsilon/4MN, I, n)$ such that

$$(3)\qquad \|\phi_0(I) - \sum_{J\in D_t} s(J,I)\phi_n(J)\| < \varepsilon/4MN \text{ , for every } t\gg t_I .$$

(1) Here $\Phi_n(I) = F(P_n(I),\phi_n(I))$, $n\geq 0$.

Observe now that we may suppose, without loss of generality, that if $t\gg t_I$ then

$$(4)\qquad \left|BC\int_I F(P_n,\phi_n) - \sum_{J\in D_t} s(J,I)\,F(P_n(J),\phi_n(J))\right| < \varepsilon/4N .$$

Put $\bar n=\bar n(\varepsilon) = \max\{n_0,\ n_I,\ I\in D_{t_5}\}$ and fix $n\geq\bar n$. Then consider any element $\bar t\in T$ such that $\bar t\gg t_I$, $I\in D_{t_5}$, and let it be $t_6=t_6(\varepsilon,n,\bar t)$.

Put $D_{t_6} = [J]$.

For every fixed $I\in D_{t_5}$ the point $(P_0(I),\phi_0(I)/\|\phi_0(I)\|)$ belongs to an open set of the finite cover, that we shall denote by the index i_I (note that w.l.o.g. we may consider only the case $\phi_0(I)\neq 0$).

Observe that it results

$$\begin{aligned}(5)\quad & BC\int_A F(P_n,\phi_n) - BC\int_A F(P_0,\phi_0) = \\ &= BC\int_A F(P_n,\phi_n) - \sum_I BC\int_I F(P_n,\phi_n) + \\ &+ \sum_I \left[BC\int_I F(P_n,\phi_n) - \sum_J s(J,I)F(P_n(J),\phi_n(J))\right] + \\ &+ \sum_I\sum_J s(J,I)\left[F(P_n(J),\phi_n(J)) - \langle a_{i_I},\phi_n(J)\rangle\right] + \\ &+ \sum_I\left[\sum_J s(J,I)\,\langle a_{i_I},\phi_n(J)\rangle - \langle a_{i_I},\phi_0(I)\rangle\right] + \\ &+ \sum_I\left[\langle a_{i_I},\phi_0(I)\rangle - F(P_0(I),\phi_0(I))\right] + \\ &+ \sum_I F(P_0(I),\phi_0(I)) - BC\int_A F(P_0,\phi_0) = \\ &= \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} + \textcircled{5} + \textcircled{6} .\end{aligned}$$

Now it is easy to see that $\textcircled{1}\geq 0$. From (4) it follows that $|\textcircled{2}|\leq N\varepsilon/4N = \varepsilon/4$. By virtue of (2) we have that $d(P_n(J),P_0(I))<\sigma$ and since we know that $d(P_0(I),k_{i_I})<\rho_{i_I}/2$, we deduce that $d(P_n(J),k_{i_I})<2\rho_{i_I}/2 = \rho_{i_I}$. Thus, from (F_4) it follows that

$$F(P_n(J),\phi_n(J))\geq\langle a_{i_I},\phi_n(J)\rangle ,$$

and this proves that $\textcircled{3}\geq 0$.

By virtue of (3) it is

$$|\,④\,| \le \sum_I |\langle a_{i_I}, \sum_J s(J,I)\phi_n(J) - \phi_0(I)\rangle| \le$$

$$\le \sum_I \| a_{i_I}\| \; \| \sum_J s(J,I)\phi_n(J) - \phi_0(I)\| \le NM\varepsilon/4MN = \varepsilon/4 \ .$$

From the condition (F_4) it follows that, since $\phi_0(I)/\|\phi_0(I)\| \in B(k_{i_I}, x_{i_I}; \rho_{i_I}/2)$, then $F(P_0(I),\phi_0(I)) \le \langle a_{i_I}, \phi_0(I)\rangle + (\varepsilon/4M)\|\phi_0(I)\|$ and therefore ⑤ $\ge -\sum_I (\varepsilon/4M)\|\phi_0(I)\| \ge -(\varepsilon/4M)\,M = -\varepsilon/4$.

From (1) it is obviously that $|\,⑥\,| \le \varepsilon/4$.
Hence we can deduce that for every $n \ge \bar{n}$

$$BC\int_A F(P_n,\phi_n) - BC\int_A F(P_0,\phi_0) \ge -4\varepsilon/4 = -\varepsilon$$

which proves the thesis.

5. *REPRESENTATION.* For sake of semplicity we shall consider the following particular case of our setting. Note that it is quite natural since it is always satisfied by the usual applications (see [8] and section 6).

However what follows holds in the general case too, provide we strengthen the conditions on the function ϕ (see [11], [4]).

Let's suppose that

- A *is a metric space;*
- *there exists a subnet* $(D_{t_n})_{n\in N}$ *such that, for every* $n\in N$,

$$I = \bigcup_{J\subset I} J,\ I\in D_{t_n} \text{ and } J\in D_{t_{n+1}}\ ;$$

- *the intervals* $\{I\}^* = \{I\in D_{t_n}\ ;\ n\in N\}$ *are compact and connected;*
- $\lim_{n\to+\infty} \max\{\mathrm{diam}(I),\ I\in D_{t_n}\} = 0$.

In the present contest the BC-integral of ϕ can be extended to a regular measure, as the following result shows (see [11], [4], [5]).

LEMMA 2. If (1) ϕ *is* (o)-q.a. *and* BV *on* A, *there exists a regular measure* $\mu: \mathcal{B} \to R^p$ *such that* $\mu(G) = BC\int_G \phi$ *and* $\|\mu\|(G) = BC\int_G \|\phi\|$ *where* $\|\mu\|$ *denotes the variation of* μ.
Moreover $\mu(I) = \mu(I^\circ)$, $I\in\{I\}^*$ *and* $\lim_{n\to+\infty} \mu(G-G_n) = 0$, *where*

$$G_n = \{\bigcup I^\circ\ ,\ I\in D_{t_n}\ ,\ I\subset G\}\ ,\ G\in\mathcal{G}.$$

(1) A function ϕ is (o)-quasi additive ((o)-q.a.) if it satisfies the condition of q.a. where $s(J,I)$ is substituted by $s(J,I^\circ)$ (see [11]).

Let's consider now the sequence $\eta_n : A \to R^p$, $n\in N$, given by

$$\eta_n(a) = \begin{cases} \mu(I)/\|\mu\|(I) & , \ a\in I^\circ\ ,\ I\in D_{t_n}\ ,\ \|\mu\|(I) \neq 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

Then, by virtue of the Prop. 1 of [4] it results that

(1) $\eta_n \to d\mu/d\|\mu\|$ μ-almost everywhere on A,

where $d\mu/d\|\mu\|$ is the Radon-Nikodym derivative.
Note that this result was proved as an application of a convergence theorem for martingales (see also [6]).

Moreover suppose that the function P is of the type $P(I) = p\circ\omega(I)$ for a given choice ω (see section 3), and define the sequence $p_n : A \to K$, $n\in N$, by

$$p_n(a) = \begin{cases} P(I) & , \ a\in I^\circ\ ,\ I\in D_{t_n} \\ p_0 & , \ \text{otherwise} \end{cases},$$

where $p_0 \in K$ is fixed.
Then it is easy to see that, if P satisfies the condition (γ), there is a function $\pi : A \to K$ such that

(2) $P_n \to \pi$ μ-almost everywhere on A.

The following representation theorem can be proved in the same way of the Prop. 2,3 and the Theor. 1 of [5] (see also the Theor. 6.iii, 6.iv, 6.v of Cesari [11]).

PROPOSITION 4. Suppose that
F satisfies the conditions (F_1), (F_2) *and* (F_3);
$P = p\circ\omega$ *satisfies the condition* (γ);
ϕ *is* (o)-q.a. *and* BV *on* A
then

$$BC\int_G \Phi = \int_G F(\pi(a), (d\mu/d\|\mu\|)(a))\, d\|\mu\|\ ,\ G\in\mathcal{G}.$$

Note that, by virtue of what we already observed in section 3, the Prop. 4 is an extension of the above mentioned representation results.

Suppose now that K is compact and $Q(I)=q\circ\omega'(I)$, where ω' is a given choice function.
If we define the sequence $(q_n)_{n\in N}$, analogously to $(p_n)_{n\in N}$, then we get a function $\xi : A \to R^m$ such that q_n converges to ξ μ-almost everywhere.

Moreover, let $m:\mathscr{B}\to R_0^+$ be any finite measure such that μ is absolutely continuous with respect to m; then, as an application of Prop. 4 (see the Coroll. 2 of [5]), we have the following result[1].

PROPOSITION 5. Suppose that K *is compact and* F *satisfies the conditions* (F_1), (F_2) *and* (F_3); $P=p\circ\omega$ *and* $Q=q\circ\omega'$ *satisfy the condition* (γ) *and* Q *is bounded;* ϕ *is* (o)-q.a. *and* BV *on* A;

then $$BC\int_G \Phi_q = \int_G F(\pi(a),(d\mu/dm)(a)\cdot\xi(a))dm \ , \quad G\in\mathscr{G}.$$

REMARK 2. Note that, if p is continuous, then $\pi=p$. But, in general, π does not coincide with p m-a.e., as the following example shows.

EXAMPLE 1. Let $E\subset[0,1]$ be a dense subset with $m(E) = 1/2$ and consider the function $p(a) = \chi_E(a)$, $a\in[0,1]$.
Let it be $\{I\}$, $\mathscr{D}$ and δ as usual and fix the choice function ω with $\omega(I)\in E\cap I^\circ$.
Then it results $p_n(a) = 1$ m-a.e., thus $\pi(a) = 1$ m-a.e. .

REMARK 3. We observe that a Tonelli-type theorem holds, which compares the CV-W integral with the usual Lebesgue one, in the general case, and it contains the above representation result as a particular case. However, for sake of shortness, we prefer to omit it. We note only that it can be proved analogously to the Tonelli-type theorem we gave in [7].

REMARK 4. Note that, if $P=p\circ\omega$ and $P'=p\circ\omega'$ are two interval

(1) If $q=1$, i.e. the weight has no effect, we can omit the compactness of K.

functions which satisfy the condition (γ) and moreover it is

$$(\gamma^*) \qquad \max_{J\in D_t, J\subset I, I\in D_{t_0}} d(p\circ\omega(J) \ , \ p\circ\omega'(J)) \ < \ \varepsilon$$

than it results $\pi(a) = \pi'(a)$ μ-a.e. .

Therefore it follows that the CV-W integral is independent by the choice function, within a class of choice functions satisfying the above condition.

6. - *SOME APPLICATIONS.*

6.1. *The* CV-W *integral over* BV *curves.*

Let $x:[a,b]\to R$ be a BV function. Then it is known ([1]) that $x\in L_1$ and it has a representative Y which is BV and right continuous on $[a,b[$ and left continuous on b.
Moreover, let $S = \{u_n \ , \ n\in N\}$ be the set of the points of discontinuity of y.
Then it is known that the function y admets the decomposition

$$(1) \qquad y(u) = y_0(u) + s(u) \quad , \quad u\in[a,b]$$

where y_0 is CBV and s is the jump-function, i.e.

$$s(u) = \begin{cases} \sum_{u_n\le u} (y(u_n) - y(u_n-0)) = \sum_{u_n\le u} y(u_n) & , \quad u\in[a,b] \\ 0 \quad , \quad u=a \end{cases}$$

Now let $\{I\}$ be the family of all the closed subintervals of $[a,b]$ and $\mathscr{D}$ be the usual family of finite systems[1]
$D = \{[u_{i-1},u_i], \ i=1,\dots,N\}$, $u_0=a < u_1 \dots < u_N = b$.

Let's consider the mesh function $\delta : \mathscr{D} \to R^+$ defined by

$$\delta(D) = \max\{\|D\| \ , \ \sum_{u_n\notin D} |\Delta y(u_n)|\}, \text{ where } \|D\| = \max\{|I| \ , \ I\in D\} .$$

As it is well-known, the mesh δ makes $\mathscr{D}$ a directed set.
We define now the two interval functions $P:\{I\}\to R$ and $\phi:\{I\}\to R$ in the following way

$$(2) \qquad \begin{aligned} P(I) &= y(u_I) \ , \text{ with } u_I\in I^\circ \\ \phi(I) &= y(b_I) - y(a_I) = \Delta y(I) \ , \ I = [a_I,b_I] . \end{aligned}$$

(1) In this case we often identify $D = [u_0,\dots,u_N]$.

Observe now that *the function* P *satisfies the condition*[1] (γ). In order to see that, we fix $\varepsilon>0$. Then let $\sigma=\sigma(\varepsilon/2)>0$ be given by virtue of the uniform continuity of y_0.
Denoted by $\delta_1=\delta_1(\varepsilon) = \min\{\sigma,\varepsilon/2\}$, we fix $D_0=\left[I\right]\in\mathcal{D}$ with $\delta(D_0)<\delta_1$. Now, for every $J\subset I$, it results

$$|y(u_I)-y(u_J)|\leq|y_0(u_I)-y_0(u_J)| + |s(u_I) - s(u_J)| <$$

$$< \varepsilon/2 + \sum_{u_n\notin D} |\Delta y(u_n)|<\varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Moreover *the function* ϕ *is* q.a. *and* BV *on* $[a,b]$.

It is obvious that ϕ is BV. In order to prove that it is q.a., we fix $\varepsilon>0$ and $D_0=\left[I\right]=\{u_i,i=0,1,\ldots,N\}$. Then let $\delta_i=\delta_i(\varepsilon/N,u_i)>0$ be given such that

$$|y(u)-y(u_i-0)|<\varepsilon/3N \text{ , with } u\in\left[u_i-\delta_i,u_i\right] ,$$

$$|y(u)-y(u_i+0)|<\varepsilon/3N \text{ , with } u\in\left[u_i,u_i+\delta_i\right] ,$$

and consider $\bar\delta = \min\{\delta_i,i=0,\ldots,N\ ,\ \varepsilon/3N\}$.

Now we fix $D=\left[J\right]$ with $\delta(D)<\bar\delta$. Note that, for every fixed $I = \left[u_i,u_{i+1}\right]\in D$, it results

$$|\phi(I) - \sum_J s(J,I)\phi(J)|\leq|y(\alpha)-y(u_i)|+|y(u_{i+1})-y(\beta)|< \varepsilon + \varepsilon +$$

$$+|\Delta y(u_{i+1})|< 3\varepsilon/3N ,$$

thus $\sum_I \left| \sum_J s(J,I)\phi(J) - \phi(I)\right|< \varepsilon$.

Now observe that

$$\sum_J\left[1 - \sum_I s(J,I)\right]|\phi(J)|\leq \sum_{i=1}^{N-1} (|\Delta y(u_i)| + 2\varepsilon/3N) < \varepsilon ,$$

and the prove is complete.

Moreover note that, *if* x *is* gAC (i.e. y *is* AC) *then the measure* μ *is absolutely continuous with respect to the Lebesgue measure* m (see [7]). Note also that, in this case, $d\mu/dm = x'$ (see [7]). Thus, as an immediate application of the Prop. 1 and 4, we can prove the following result.

(1) Actually P satisfies the condition (γ^*) with respect to any choice $u_I\in I^\circ$.

PROPOSITION 6. *Suppose that*

$F: R^{2k} \to R^q$ *satisfies the conditions* $(F_1),(F_2)$ *and* (F_3);

$x : [a,b] \to R^k$ *is* BV;

then the CV-W *integral of* F *over the curve* x *exists and it is independent by the choice* $u_I\in I^\circ$.

Moreover, if x *is* gAC, *then it results*

$$BC\int_{[a,b]} \phi = \int_a^b F(x(u),x'(u))\ dm .$$

6.2. - *The weighted variation.*

Let $x:[a,b]\to R^n$ be again a BV function and let $g:[a,b]\to R^n$ be a continuous function.

Denote by $E_x\subset[a,b]$ the set of the points of approximate continuity of x (see [12]) and let $\{I\}$ be the set of all the closed sub-intervals of $[a,b]$ whose end-points belongs to E_x. Let $\mathcal{D}$ be the family of all the finite partitions of $[a,b]$ made by intervals $\{I\}$ and let δ be the usual mesh function, i.e. $\delta(D)=\|D\|$.

We consider the functions $Q:\{I\}\to R^n$ and $\phi:\{I\}\to R^n$ defined by

$$Q(I) = \int_I g(u)\ dm\ /|I| = g(u_I) \text{ , with } u_I\in I \text{ and}$$

$$\phi(I) = \Delta x(I) .$$

Then it is easy to see that ϕ is q.a. and BV and that Q is bounded and satisfies the condition (γ) (see 6.1) .
Thus, as a consequence of the Prop. 1 and 4, we obtain the following result that extends those given by Boni ([1]) for the weighted generalized variation (see [13]).

PROPOSITION 7. *Suppose that*

$F: R^n \to R^q$ *satisfies the conditions* $(F_1),(F_2)$ *and* (F_3);

$x : [a,b] \to R^n$ *is* BV;

$g : [a,b] \to R^n$ *is continuous;*

then the weighted CV-W *integral of* F *over the curve* x *exists.*

Moreover, if x *is* gAC, *then*[1]

(1) Here $e_1\cdot e_2 = (e_1^1e_2^1,\ldots,e_1^ne_2^n)$.

$$BC\int_{[a,b]} \Phi_q = \int_a^b F(\, x'(u)\; g(u))\; dm.$$

Moreover, as an immediate application of the Prop. 3, we obtain the following semicontinuity result that contains, as a particular case, those given by Boni ([1]) for the weighted generalized variation (see also [13]).

PROPOSITION 8. *Suppose that*

$F : R^n \to R$ *satisfies the conditions* $(F_1),(F_2),(F_3)$ *and* (F_4);

$x_n : [a,b] \to R^n$, $n \geq 0$, *is a sequence of equi-BV functions such that*(1)

$x_n \to x_0$ *almost everywhere in* $[a,b]$;

$g_n : [a,b] \to R^n$, $n \geq 0$, *is a sequence of continuous functions s.t.*

$g_n \to g_0$ *uniformly in* $[a,b]$;

then

$$\underline{lim}_{n\to+\infty}\; BC\int_{[a,b]} F(\Delta x_n \cdot g_n) \geq BC\int_{[a,b]} F(\Delta x_0 \cdot g_0)$$

PROOF. Let H be the set where x_n converges to x_0 and denote by $E = \bigcup_{n\geq 0} E_{x_n} \cup H$. Then let $\{I\}^*$ be the family of all the closed subintervals whose end-points belong to E. Let $(D_m)_{m\in N}$ be a sequence of finite partitions of $[a,b]$, made by intervals of $\{I\}^*$, such that D_{m+1} is a "refinement" of D_m (i.e. $I = \bigcup_{J\subset I} J$, $I\in D_m$, $J\in D_{m+1}$) and

$$\lim_{m\to+\infty} \delta(D_m) = 0$$

In this setting, recalling that the sequence $(x_n)_{n\in N}$ is equi-BV, it is easy to see that the function $\Delta x_n(I)\cdot g_n(u_I) : \{I\}^* \to R^n$ satisfies the convergence condition (M_2).

REMARK 5. As Boni already observed ([1]), it is not possible, in general, to replace the uniform convergence of the weightes with a weaker one, even if we use the uniform convergence of the curves.

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(1) If $F(x)\geq C\|x\|$ and $g_n \geq \lambda > 0$, $n\geq 0$, then the equi-boundness of the variations of (x_n) can be omitted.

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