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SUPPLEMENTO

AI

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO

DIRETTORE: **P. VETRO**

PROCEEDINGS OF THE SIXTH INTERNATIONAL CONFERENCE ON FUNCTIONAL ANALYSIS AND APPROXIMATION THEORY

Acquafredda di Maratea (Potenza- Italy), September 24-30, 2009

SERIE II - NUMERO 82 - ANNO 2010

PALERMO
SEDE DELLA SOCIETÀ
VIA ARCHIRAFI, 34

Center for Studies on
Functional Analysis and Approximation Theory
University of Basilicata
Potenza - Italy

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Preface

Following a well-established tradition initiated on 1989, during the week of September 24-30, 2009, the Sixth International Conference on Functional Analysis and Approximation Theory was held at Acquafredda di Maratea (Potenza). Earlier meetings were held in 1989, 1992, 1996, 2000 and 2004 at the same place.

The Conference was organized by the Center for Studies on Functional Analysis and Approximation Theory of the University of Basilicata (Potenza), directed by Prof. Giuseppe Mastroianni, with the collaboration of the Research Group "Real Analysis and Methods of Functional Analysis for Differential Problems and Approximation Problems" of the University of Bari, directed by Prof. Francesco Altomare.

Besides Francesco Altomare and Giuseppe Mastroianni, the organizing committee consisted also of Antonio Attalienti, Michele Campiti, Mirella Cappelletti Montano, Lorenzo D'Ambrosio, Maria Carmela De Bonis, Sabrina Diomede, Vita Leonessa, Donatella Occorsio and Maria Grazia Russo.

The meeting was devoted to some significant aspects of contemporary mathematical research on Functional Analysis, Operator Theory and Approximation Theory including the applications of these fields in other areas such as partial differential equations, integral equations and numerical analysis.

A special session was dedicated to Professor Giuseppe Mastroianni on the occasion of his 70th birthday, as a fair token of appreciation for his contribution to mathematics and to the organization of all the Maratea conferences (1989 – 2009).

The Conference was attended by over 130 mathematicians coming from more than 30 countries. The scientific program consisted of 14 plenary lectures and about 120 communications. Several thematic sessions were arranged.

Among the main topics treated therein we mention:

- Banach spaces, approximation in normed spaces.
- Fixed points theorems and best approximation.
- Banach lattices,
- Banach algebras, topological algebras and operator algebras.
- Function spaces.
- Linear and nonlinear operators on Banach spaces, positive operators, spectral theory.
- Semigroups of operators, Feller semigroups and evolution equations.
- Ordinary and partial differential equations.
- Real functions and inequalities, complex approximation.
- Approximation by positive operators, approximation by nonlinear expressions, rate of convergence.
- Polynomial approximation, rational approximation.

- Interpolation and quadrature formulas.
- Orthogonal polynomials, special functions.
- Integral equations.
- Integral operators, integral transforms and Fourier analysis.
- Integral inequalities.
- Numerical analysis.

In this volume we collect some invited lectures together with a selection of the papers corresponding to the research talks, which have been carefully refereed. Our thanks go to the referees for their collaboration as well as for their accurate work.

We gratefully acknowledge the financial support of the Institutions listed below:

- the University of Basilicata,
- the University of Bari,
- the Faculty of Sciences of the University of Basilicata,
- the Department of Mathematics and Computer Science of the University of Basilicata (Potenza),
- the Department of Mathematics of the University of Bari,
- the Department of Economic Sciences and Mathematical Methods of the University of Bari,
- the Department of Mathematics of the University of Salento,
- the National Group for Mathematical Analysis, Probability and their Applications (IndAM - G.N.A.M.P.A.),
- The Research Group "Partial Differential Equations and Mathematical Finance", University of Bari,
- the Basilicata Tourism Board.

We are particularly indebted to Sabina Milella and Graziana Musceo as well as to several other Ph.D and Post-Doc students who assisted and helped us in organizing the Conference secretariat services. To all of them we express our special thanks.

We are also grateful to Biancamaria Della Vecchia for her friendly collaboration.

Finally we would like to thank Professor P. Vetro for accepting to publish the Proceedings of the Conference in a supplement of "Rendiconti del Circolo Matematico di Palermo".

May 2010

Francesco Altomare
On behalf of the Organizing Committee

PROGRAM OF THE CONFERENCE

Friday, September 25

☑ 9:00-9:30	OPENING CEREMONY

Plenary Lectures

	Chairman: **Dany Leviatan**
☑ 9:30-10:15	**Lars-Erik Persson,** *Luleå, Sweden* Some new developments concerning characterizations of weighted multidimensional Hardy type integral inequalities
☑ 10:20-11:05	**Ben de Pagter,** *Delft, The Netherlands* Derivations on algebras of operator valued functions
☑ 11:10-11:40	*Break*
☑ 11:40-12:25	**Ioan Raşa,** *Cluj-Napoca, Romania* C_0-Semigroups and iterates of positive linear operators: asymptotic behaviour

Scientific Communications

	SECTION **A** *Chairman:* **József Szabados**	SECTION **B** *Chairman:* **Jürgen Appell**
☑ 15:00-15:15	**Miguel Antonio Jimenez Pozo,** *Puebla, Mexico* Hölder weighted integrable periodic functions and Approximation	**Dragan Djordjevic** *Nis, Serbia* Recent results on generalized inverses
☑ 15:20-15:35	**Marie-Madeleine Derriennic,** *Rennes, France* Identities and inequalities for quasi-interpolants of Bernstein-type	**Łukasz Piasecki** *Lublin, Poland* Full characterization of the best Lipschitz constants of all iterations of mean lipschitzian mappings
☑ 15:40-15:55	**Mircea Ivan,** *Cluj-Napoca, Romania* Some remarks on generalized interpolation	**Giulio Trombetta,** *Cosenza, Italy* On a measure of noncompactness in topological linear spaces
☑ 16:00-16:15	**Ulrich Abel,** *Friedberg, Germany* Asymptotic properties of some positive operators of Altomare	**Henrik Laurberg Pedersen,** *Copenhagen, Denmark* Pre Stieltjes functions
☑ 16:20-16:35	**Ioan Gavrea,** *Cluj-Napoca, Romania* Converse inequalities for Meyer-König and Zeller operators of finite type	**Fatma Diaba**, *Annaba, Algeria* On rank one perturbation of continuous spectrum which generates prescribed finite point spectrum
☑ 16:40-16:55	**Octavian Agratini,** *Cluj – Napoca, Romania* q- Approximation Processes on Unbounded Intervals	**Alessandro Trombetta,** *Cosenza, Italy* Boundary conditions for k-ψ-contractive wedge maps
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **Michele Campiti**	SECTION **B** *Chairman:* **Lech Maligranda**
☑ 17:15-17:30	**Daniel Cárdenas-Morales,** *Jaén, Spain* On general asymptotic formulae: a converse result.	**Elgiz Bayram,** *Ankara, Turkey* Dissipative Sturm-Liouville operators
☑ 17:35-17:50	**Radu Păltaneă,** *Brasov, Romania* Quantitative Results in Weighted Approximation	**Wolfgang Erb,** *Munich, Germany* Uncertainty principles on compact Riemannian manifolds
☑ 17:55-18:10	**Cihan Orhan,** *Ankara, Turkey* Some Variations of the Bohman-Korovkin Theorem	**Alexey Uglanov,** *Saint-Petersburg, Russia* Vector Integrals and Absolutely Summing Operators in Locally Convex Spaces
☑ 18:15-18:30	**Mirella Cappelletti Montano,** *Bari, Italy* On a generalization of Kantorovich operators on simplices and hypercubes	**Sidi Mohammed Bahri,** *Mostaganem, Algeria* On a Certain Class of Carleman Integral Operators
☑ 18:35-18:50	**Elena Berdysheva,** *Stuttgart , Germany* Multivariate Bernstein-Durrmeyer Operators with Arbitrary Weight Functions	
☑ 18:55-19:10	**Vita Leonessa,** *Potenza, Italy* Kantorovich-type operators associated with positive projection and their limit semigroup	
☑ 19:10-19:25	**Sergey Tikhonov,** *Barcelona, Spain* Sharp inequalities for moduli of smoothness	

Saturday, September 26

Plenary Lectures

	Chairman: **Peter Junghanns**
☑ 9:00-9:45	**Bernd Silbermann,** *Chemnitz, Germany* Banach algebras of structured matrix sequences and topics in asymptotic spectral theory.
☑ 9:50-10:35	**Giovanni Monegato,** *Torino, Italy* Integral equation formulations of some 2D contact problems
☑ 10:40-11:00	*Break*

Saturday, September 26

Scientific Communications

	SECTION **A** *Chairman:* **Mihai Voicu**	SECTION **B** *Chairman:* **Daniel Potts**
☑ 11:00-11:15	**Sankatha Singh,** *St. John's, Canada* Ky Fan's best approximation theorem and applications	**Teresa E. Perez,** *Granada, Spain* Difference Equation for a Stieltjes function associated with Discrete Classical Orthogonal Polynomials
☑ 11:20-11:35	**Fowzi Sejeeni,** *Makkah, Saudi Arabia* On the Strict Best Approximation	**Valeriy Galkin,** *Obninsk, Russia* The metric theory of functional solutions of conservation laws systems
☑ 11:40-11:55	**Maryam Beyg Mohammadi,** *Kermanshah, Iran* Some New Fixed Point Theorems in Modular Spaces	**Tadeusz Jankowski,** *Gdansk, Poland* Nonlinear differential equations with boundary conditions
☑ 12:00-12:15		**Sara Remogna,** *Torino, Italy* Quasi-interpolant operators based on trivariate C^2 quartic box splines

	SECTION **A** *Chairman:* **Giuseppe Mastroianni**	SECTION **B** *Chairman:* **Ben de Pagter**
☑ 15:00-15:15	**Peter Junghanns,** *Chemnitz, Germany* On Mapping Properties of Integral Operators with Logarithmic Kernels	**Jacobus Grobler,** *Potchefstroom, South Africa* Stochastic processes in Riesz spaces
☑ 15:20-15:35	**Markus Seidel,** *Chemnitz , Germany* On the stability of collocation methods for Cauchy singular integral equations in weighted L^p spaces	**Richard Becker,** *Paris, France* The role of Conical Measures in Functional Analysis
☑ 15:40-15:55	**Helena Mascarenhas,** *Lisboa, Portugal* Spectral Approximation and index for bidimensional convolution type operators	**Mirosława Zima,** *Rzeszów, Poland* On commutator of positive linear operators
☑ 16:00-16:15	**Concetta Laurita,** *Potenza, Italy* A Nyström method for Cauchy singular integral equations with negative index	**Anna Betiuk-Pilarska,** *Lublin, Poland* Banach lattices which are order uniformly noncreasy
☑ 16:20-16:35	**Carmelina Frammartino,** *Potenza, Italy* A Nyström method for 2s-th order BVP	**Mihai Voicu,** *Bucharest, Romania* Remarks on Pseudoresolvents
☑ 16:40-16:55	**Maria Grazia Russo** *Potenza, Italy* Nyström methods for weakly singular Fredholm integral equations with singular right-hand sides	**Rachida Amiar,** *Annaba, Algeria* Approximation by positive operators of the C_0-semigroups associated with one-dimensional diffusion equations
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **Péter Vértesi**	SECTION **B** *Chairman:* **Ioan Raşa**
☑ 17:15-17:30	**Christian Berg,** *Copenhagen, Denmark* Eigenvalues of large Hankel matrices	**Sabrina Diomede,** *Bari, Italy* On Feller semigroups associated with modified Bernstein-Schnabl operators
☑ 17:35-17:50	**Kathy Driver,** *Cape Town, South Africa* The interlacing of zeros and orthogonality	**Kirsti Mattila,** *Stockholm, Sweden* Ergodicity for operators generating a bounded resolvent family on Banach spaces
☑ 17:55-18:10	**Donatella Occorsio,** *Potenza, Italy* Approximation of the Hilbert transform on the real positive semi-axis	**Sabina Milella,** *Bari, Italy* C_0-semigroups and modified Szász-Mirakjan operators
☑ 18:15-18:30	**Pietro Pastore,** *Potenza, Italy* Numerical treatment of integrals with highly oscillatory density function	**Cristian Tacelli,** *Lecce, Italy* On the order of convergence of the Voronovskaja-type formulas and applications
☑ 18:35-18:50	**Maria Carmela De Bonis,** *Potenza, Italy* Quadrature rules for high-oscillatory and periodic functions and some applications to integral equations.	**Graziana Musceo,** *Bari, Italy* Degenerate Second Order Differential Operators on Weighted Continuous Function Spaces
☑ 18:55-19:10	**Ferenc Toókos,** *Munich, Germany* Mixed recurrence relations and interlacing theorems for orthogonal and q-orthogonal polynomials	**Ali Farajzadeh,** *Kermanshah, Iran* On the approximate convexity and submonotonicity in locally convex spaces
☑ 19:10-19:25	**Letizia Scuderi,** *Torino, Italy* Lubich convolutions quadratures and their application to space-time BIEs	**Helmi Temimi,** *Kuwai, Kuwait* Performance of A New Numerical Solver for Differential equations based on Discontinuous Galerkin Finite Element Method

Sunday, September 27

Special Session dedicated to
Professor Giuseppe Mastroianni
on the occasion of his 70th birthday

☑ 9:00-9:30	OPENING CEREMONY

Plenary Lectures

	Chairman: **Gradimir Milovanović**
☑ 9:30-10:20	**József Szabados,** *Budapest, Hungary* Eighteen years of cooperation with Giuseppe Mastroianni – A survey
☑ 10:25-11:15	**Vilmos Totik,** *Szeged, Hungary and Tampa,Florida* The polynomial inverse image method or how to prove results for general compact sets ?
☑ 11:20-11:50	*Break*
☑ 11:50-12:40	**Péter Vértesi,** *Budapest, Hungary* My Italian Connections

☑ 19:00	***Social Dinner***

Monday, September 28

Plenary Lectures

	Chairman: **Jean-Pierre Antoine**
☑ 9:00-9:50	**Francisco Marcellan,** *Madrid, Spain* Orthogonal polynomials on the unit circle: Spectral transformations and integrable systems
☑ 9:55-10:45	**Jürgen Appell,** *Würzburg, Germany* Some approaches to the p-Laplace operator
☑ 10:45-11:15	*Break*
☑ 11:15-12:05	**Lech Maligranda,** *Luleå, Sweden* Structure of Cesàro function spaces

Scientific Communications

	SECTION **A** *Chairman:* **Vilmos Totik**	SECTION **B** *Chairman:* **Camillo Trapani**
☑ 15:00-15:15	**Dany Leviatan,** *Tel Aviv, Israel* Approximation of Sobolev and other classes by ridge functions	**Jan Fourie,** *Potchefstroom, South Africa* Classes of operator valued functions, with relation to integral multipliers
☑ 15:20-15:35	**Aldric Brown,** *London, United Kingdom* An extension of Haar's Theorem	**Natasha Samko,** *Faro, Portugal* Weighted Hardy inequalities in Morrey spaces
☑ 15:40-15:55	**Herbert Stahl,** *Berlin, Germany* AAK Approximants to Functions with Branch Points	**Jean-Marie Aubry,** *Paris, France* Multifractality, local p-convexity and diametral dimensions
☑ 16:00-16:15	**András Kroó,** *Budapest, Hungary* On approximation of convex bodies by convex algebraic level surfaces	**Stanislaw Prus,** *Lublin, Poland* Constructing separated sequences in Banach spaces
☑ 16:20-16:35	**Tamas Erdelyi,** *College Station, Texas, USA* Zeros of polynomials with restricted coefficients	**Angela Albanese,** *Lecce, Italy* Schauder bases in Fréchet spaces and mean ergodic operators
☑ 16:40-16:55	**Luisa Fermo,** *Potenza, Italy* The Shepard operator on the real semiaxis	**Olav Nygaard,** *Kristiansand, Norway* Observations on Ideals in Banach spaces
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **András Kroó**	SECTION **B** *Chairman:* **Jacobus Grobler**
☑ 17:15-17:30	**Viktor Kolyada,** *Karlstad, Sweden* Fubini type theorems for L^p-moduli of continuity	**Antonio Fernández,** *Sevilla, Spain* Interpolation of spaces of integrable functions with respect to vector measures
☑ 17:35-17:50	**Frank Filbir,** *Munich, Germany* Localized kernels and approximation processes on Riemannian manifolds	**Francisco Naranjo** *Sevilla, Spain* Bartle-Dunford-Schwartz integrability versus Bochner, Pettis, Dunford integrability
☑ 17:55-18:10	**Igor Shevchuk,** *Kyiv, Ukraine* Pointwise estimates in coconvex approximation	**Paola Cavaliere,** *Salerno, Italy* Approximation of finitely additive functions valued into topological groups
☑ 18:15-18:30	**Incoronata Notarangelo,** *Potenza, Italy* Polynomial approximation with Pollaczek-type weights	**Andrzej Kryczka,** *Lublin, Poland* Alternate signs Banach-Saks property and real interpolation of operators
☑ 18:35-18:50	**Anna Napoli** *Cosenza, Italy* New kernels of Sard type and applications	**A. Sri Ranga,** *Sao Jose do Rio Preto, SP, Brazil* Continued Fractions and Birth and Death Processes
☑ 18:55-19:10		**Mohamed Jalel Atia,** *Gabes, Tunisia* Analytic construction of an example of orthogonal polynomials of class one

Tuesday, September 29

Plenary Lectures

	Chairman: **Christian Berg**
☑ 9:00-9:50	**Gradimir Milovanovic,** *Novi Beograd, Serbia* Some classes of generalized quadratures: construction and analysis
☑ 9:55-10:45	**Daniel Potts,** *Chemnitz, Germany* Efficient reconstruction of functions on the sphere from scattered data
☑ 10:45-11:15	*Break*
☑ 11:15-12:05	**Mikio Kato,** *Kitakyushu, Japan* Some recent results on Banach space geometry

Scientific Communications

	SECTION **A** *Chairman:* **Mikio Kato**	SECTION **B** *Chairman:* **Natasha Samko**
☑ 15:00-15:15	**Jean-Pierre Antoine,** *Louvain-la-Neuve, Belgium* Partial inner product spaces and operators on them	**Katsuo Matsuoka,** *Tokyo, Japan* On some Herz-type spaces and the boundedness for classical operators
☑ 15:20-15:35	**Camillo Trapani,** *Palermo, Italy* The completion of a C*-algebra under a weaker locally convex topology	**Ushangi Goginava,** *Tbilisi, Georgia* Maximal operators of logarithmic means of two-dimensional Walsh-Fourier series
☑ 15:40-15:55	**Lourdes Palacios,** *Mexico City, Mexico* On some generalized properties in Q-topological algebras	**Marija Stanic,** *Kragujevac, Serbia* Trigonometric Polynomials of Semi-Integer Degree Orthogonal with Respect to a Moment Functional
☑ 16:00-16:15	**Francesco Tschinke,** *Palermo, Italy* Spectral properties of partial *-algebras	**Vladimir Vasilyev,** *Bryansk, Russia* Spectral resolution for Calderon-Zygmund operators
☑ 16:20-16:35	**Giorgia Bellomonte,** *Palermo, Italy* Absolutely Convergent Extensions of Nonclosable Positive Linear Functionals	**István Blahota,** *Nyiregyhaza, Hungary* On boundedness of maximal operators with respect to Vilenkin systems
☑ 16:40-16:55		**Rodolfo Toledo,** *Nyíregyháza, Hungary* Representative product systems
☑ 16:55-17:15	*Break*	

	SECTION **A** *Chairman:* **Tamas Erdelyi**	SECTION **B** *Chairman:* **Cihan Orhan**
☑ 17:15-17:30	**Oxana Motorna,** *Kyiv,Ukraine* Generalized Lebesgue constants for Fourier-Jacobi partial sums	**György Gát,** *Nyiregyhaza, Hungary* Approximation properties of one and two dimensional Fejér and Marczinkiewicz means of Fourier series with respect to Walsh and Vilenkin systems
☑ 17:35-17:50	**Jacob Christiansen,** *Copenhagen, Denmark ,* Szegő Asymptotics	**Károly Nagy,** *Nyiregyhaza, Hungary* Approximation by Nörlund means of Walsh-Kaczmarz-Fourier series
☑ 17:55-18:10	**Karla Rost,** *Chemnitz , Germany* Split Bezoutians and inverses of symmetric Toeplitz or centrosymmetric Toeplitz-plus-Hankel matrices	**Givi Nadibaidze,** *Tbilisi, Georgia* On the coefficients Fourier and absolute summability almost everywhere of series with respect to block orthonormal system
☑ 18:15-18:30	**Tengiz Kopaliani,** *Tbilisi, Georgia* Gagliardo-Nirenberg inequality in some Banach function spaces	**Teimuraz Akhobadze,** *Tbilisi, Georgia* Uniform convergence and summability of Fourier series
☑ 18:35-18:50	**Mehrdad Simkani,** *Flint, USA* A General Theorem for Rational Approximation on the Extended Complex Plane	

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PART I
INVITED LECTURES

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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Linear spectral transformations, Hessenberg matrices, and orthogonal polynomials

Kenier Castillo, Luis Garza and Francisco Marcellán

Abstract. In this manuscript we analyze some linear spectral transformations of a positive definite Hermitian linear functional. In particular we focus our attention in the behavior of the Verblunsky parameters of the perturbed linear functional when we add the linear functional defined by the Lebesgue measure on the unit circle. Some illustrative examples are pointed out.

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1 Introduction

Orthogonal polynomials with respect to nontrivial probability measures supported on the real line have attracted the interest of researchers long time ago. Their relations with continued fractions and rational approximation, their role in the spectral theory of linear differential operators and Sturm-Liouville problems, as well as their applications in the theory of integrable systems (Toda lattices and Lax pairs, among others) constitute an illustrative sample of their impact in many domains of Mathematics and Physics.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Surprisingly, the theory of orthogonal polynomials with respect to non-trivial probability measures supported on the unit circle had not been so popular until the mid 80's. The monographs by G. Szegő (see [22] and [13]) and Ya. L. Geronimus (see [10] and [9]) were the main (and the few) major contributions to the subject despite the fact that people working in linear prediction theory and digital signal processing used as a basic background orthogonal polynomials on the unit circle (see [7] and references therein). The very recent monograph by B. Simon ([20]) constitutes an updated overview of the most remarkable directions of research in the theory, both from a theoretical approach (extensions of the Szegő theory from an analytic point of view), as well as from their applications in the spectral analysis of unitary operators and the (GGT and CMV) matrix representations of the multiplication operator, quadrature formulas, and integrable systems (Schur flows and Ablowitz-Ladik equations), among others.

In the last years, some attention has been paid to the connections between some canonical perturbations of measures supported on the unit circle, the corresponding sequences of orthogonal polynomials, the GGT matrix representations, and Carathéodory functions (see [6], [8], [11], [12], [15], [17], [18], [19] among others) in the framework of the theory of the so called rational spectral transformations. It is the natural counterpart of the theory developed in [21] and [24] for measures supported on the real line.

The aim of our review is twofold. First, we present the state of the art in the theory of linear spectral transformations (Christoffel, Uvarov, and Geronimus as particular examples) and the connection with the LU and QR factorization of Jacobi and Hessenberg matrices, respectively. Second, we study an interesting example of a linear spectral transformation on the unit circle far away of the above canonical cases. The key idea is the analysis of the perturbation of a measure supported on the unit circle by a new measure such that the corresponding Carathéodory functions differ in a polynomial. In particular, when we add a constant, i.e. the corresponding Toeplitz matrix is the result of the addition of a constant diagonal matrix to the initial Toeplitz matrix, we study some new analytic properties of the corresponding sequences of orthogonal polynomials in a different way to the work done in [3] and [4].

The structure of the paper is as follows. In Section 2 we present a basic background about orthogonal polynomials with respect to measures

supported on the real line. We emphasize the role of the Jacobi matrices associated with such polynomials. The connection between canonical spectral linear transforms of the measures and LU and UL factorizations of the corresponding Jacobi matrices is stated. In Section 3 we deal with some analog problems for polynomials orthogonal with respect to measures supported on the unit circle. In this case, the connection between canonical linear spectral transforms of the measures and QR factorizations of the corresponding Hessenberg matrices is given. In Section 4 we focus our attention in a perturbation of the Toeplitz matrices associated with such measures. In particular, we analyze the perturbation of a Toeplitz matrix by the addition of a scalar matrix. The expression of the resulting orthogonal polynomials in terms of the original sequence is deduced. Finally, Section 5 is devoted to the study of such a kind of perturbation when the initial Toeplitz matrix is associated with the Bernstein-Szegő measure and the Tchebychev measure, respectively.

2 Orthogonal polynomials on the real line and Jacobi matrices

Given a nontrivial probability measure μ supported on some infinite subset E of the real line, a (unique) sequence of polynomials $\{p_n\}_{n\geqslant 0}$ can be defined as

$$\int_E p_m(x)p_n(x)d\mu(x) = \delta_{m,n}, \quad n, m \geqslant 0, \tag{2.1}$$

where $\delta_{m,n}$ is the Kronecker delta. $\{p_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to μ and

$$p_n(x) = \gamma_n x^n + \zeta_n x^{n-1} + \text{lower degree terms}, \tag{2.2}$$

with $\gamma_n > 0$ for every $n \geqslant 0$. Starting from the initial conditions $p_0(x) = 1$ and $p_{-1}(x) = 0$, one can obtain this family of polynomials by means of the so-called three term recurrence relation [5]

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{2.3}$$

with the assumptions $P_0(x) = 1$ and $P_{-1}(x) = 0$, where the recurrence coefficients are

$$a_n = \int_E xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0, \quad n \geqslant 1,$$

and

$$b_n = \int_E x p_n^2(x) d\mu(x) = \frac{\zeta_n}{\gamma_n} - \frac{\zeta_{n+1}}{\gamma_{n+1}}, \quad n \geqslant 0.$$

(2.3) has a matrix representation of the form

$$xp(x) = \hat{\mathbf{J}} p(x),$$

where $p(x) = [p_0(x), p_1(x), \ldots]^t$ and $\hat{\mathbf{J}}$ is the symmetric tridiagonal matrix

$$\hat{\mathbf{J}} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \ddots \\ 0 & 0 & a_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

that is known in the literature as Jacobi matrix [5]. It represents the multiplication operator with respect to the basis of orthonormal polynomials defined as above.

On the other hand, the monic orthogonal polynomials with respect to μ are given by $P_n(x) = p_n(x)/\gamma_n$, $n \geqslant 0$. In such a case, (2.3) becomes

$$P_{n+1}(x) = (x - b_n)P_n(x) - d_n P_{n-1}(x), \quad n \geqslant 0, \tag{2.4}$$

with $d_n = a_n^2$, and its corresponding matrix representation is

$$\mathbf{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ d_1 & b_1 & 1 & 0 & \cdots \\ 0 & d_2 & b_2 & 1 & \ddots \\ 0 & 0 & d_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

that is known as monic Jacobi matrix.

2.1 Spectral transformations

Spectral transformations of nontrivial probability measures supported on the real line have been extensively studied during the last years and within a wide variety of frameworks including bispectral problems, integrable systems, matrix theory, group theory, and Stieltjes functions (see [1], [14], [16], [21] among others). We are interested here in the effect of such transformations on the corresponding Jacobi matrices (see [2], [23], among others).

We consider three canonical transformations as follows

(*i*) Christoffel transformation

$$d\tilde{\mu} = (x-\beta)d\mu, \quad \beta \notin supp(\mu).$$

(*ii*) Uvarov transformation

$$d\tilde{\mu} = d\mu + M_r\delta(x-\beta), \quad M_r \in \mathbb{R}.$$

(*iii*) Geronimus transformation

$$d\tilde{\mu} = \frac{d\mu}{x-\beta} + M_r\delta(x-\beta), \quad \beta \notin supp(\mu), M_r \in \mathbb{R}.$$

We will denote the previous transformations by $\mathcal{R}_C(\beta)$, $\mathcal{R}_U(\beta, M_r)$, and $\mathcal{R}_G(\beta, M_r)$, respectively. They are related as follows

$$\begin{aligned} \mathcal{R}_C(\beta) \circ \mathcal{R}_G(\beta, M_r) &= \mathcal{I} \quad \text{(Identity transformation)}, \\ \mathcal{R}_G(\beta, M_r) \circ \mathcal{R}_C(\beta) &= \mathcal{R}_U(\beta, M_r). \end{aligned}$$

If we denote by $S(x)$ the Stieltjes function associated with μ, i.e. $S(x) = \frac{\mu_0}{x} + \sum_{k=1}^{\infty} \frac{\mu_k}{x^{k+1}}$, where $\mu_k = \int_E x^k d\mu(x)$ are the moments of μ, then these transformations can be expressed in terms of the corresponding Stieltjes functions as

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{D(x)}, \tag{2.5}$$

where $\widetilde{S}(x)$ is the Stieltjes function associated with $\tilde{\mu}$, and $A(x), B(x), D(x)$ are polynomials in the variable x, which are explicitly given (see [24]) for each of the above transformations. Furthermore, [24] shows that the linear spectral transformations (2.5) constitute a non-commutative group whose generators are transformations (*i*) and (*iii*).

On the other hand, it has been shown that the Jacobi matrices associated with the perturbed measures $\tilde{\mu}$ for the above transformations can be obtained from the original Jacobi matrix via LU and UL factorizations using the Darboux transformation, defined as follows.

Consider the monic Jacobi matrix associated with some nontrivial probability measure μ. Notice that, if all of its principal leading submatrices are nonsingular, then $\mathbf{J}$ has a unique LU factorization where $\mathbf{L}$ and $\mathbf{U}$ are bidiagonal matrices

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & 0 & \cdots \\ 0 & l_2 & 1 & 0 & \ddots \\ 0 & 0 & l_3 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_2 & 1 & 0 & \cdots \\ 0 & 0 & u_3 & 1 & \ddots \\ 0 & 0 & 0 & u_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.6}$$

Next, we construct, by reversing the order of the factors from the above factorization, a new matrix $\mathbf{J}_D := \mathbf{UL}$ that is said to be the *Darboux transformation of* $\mathbf{J}$ *without parameter* . Notice that $\mathbf{J}_D$ is again a tridiagonal matrix with ones as entries on the upper diagonal and, according to Favard's theorem, it is a monic Jacobi matrix associated with some nontrivial measure $\tilde{\mu}$.

In a similar way, we consider the UL factorization of the matrix $\mathbf{J}$, where $\mathbf{U}$ and $\mathbf{L}$ are of the form given in (2.6), $\mathbf{J} = \mathbf{UL}$. Now let $\mathbf{J}_d := \mathbf{LU}$, and thus we obtain again a monic Jacobi matrix. In this situation, $\mathbf{J}_d$ is said to be the *Darboux transformation* of $\mathbf{J}$. Contrary to the LU factorization, the UL factorization is not unique and depends on a free parameter.

Remark 2.1 *A necessary and sufficient condition for the existence of the LU factorization of* $\mathbf{J}$ *is* $P_n(0) \neq 0$, $n \geqslant 1$. *Furthermore, assuming the LU factorization exists, the elements of* $\mathbf{L}$ *and* $\mathbf{U}$ *are given by*

$$l_1 = \frac{d_1}{b_0}, \quad l_n = \frac{d_n}{b_{n-1} - l_{n-1}}, \quad n \geqslant 2, \tag{2.7}$$

$$u_1 = b_0, \quad u_n = b_{n-1} - l_{n-1}, \quad n \geqslant 2. \tag{2.8}$$

Next, we describe transformations $(i) - (iii)$ in terms of the corresponding monic Jacobi matrices.

Proposition 2.2 *[2] Let* μ *be a nontrivial probability measure and denote by* $\{P_n\}_{n\geqslant 0}$ *and* $\mathbf{J}$ *its corresponding sequence of monic orthogonal polynomials and monic Jacobi matrix, respectively. Let* $\beta \in \mathbb{R}$ *such that* $P_n(\beta) \neq 0$, $n \geqslant 1$. *Then, if we apply the transformation*

$$\mathbf{J} - \beta\mathbf{I} = \mathbf{LU}, \quad \mathbf{J}_1 := \mathbf{UL} + \beta\mathbf{I},$$

then $\mathbf{J}_1$ *is the monic Jacobi matrix associated with* $d\tilde{\mu} = (x-\beta)d\mu$, *i.e. the Christoffel transformation.*

Notice that if β does not belong to the convex hull of $supp(\mu)$ then $P_n(\beta) \neq 0$ for every $n \geq 1$. It is not difficult to show that we can iterate the previous result and extend it to describe the monic Jacobi matrix associated with m consecutive Christoffel transformations. Indeed,

Proposition 2.3 *[2] Let* μ, $\{P_n\}_{n\geqslant 0}$, *and* $\mathbf{J}$ *be as in the previous Propo-*

sition. Consider the following transformations of $\mathbf{J}$

$$\begin{aligned}
\mathbf{C}_1 &:= \mathbf{J} - \beta_1\mathbf{I} = \mathbf{L_1U_1}, \qquad \tilde{\mathbf{C}}_1 := \mathbf{U_1L_1} + \beta_1\mathbf{I},\\
\mathbf{C}_2 &:= \tilde{\mathbf{C}}_1 - \beta_2\mathbf{I} = \mathbf{L_2U_2}, \qquad \tilde{\mathbf{C}}_2 := \mathbf{U_2L_2} + \beta_2\mathbf{I},\\
&\vdots\\
\mathbf{C}_m &:= \tilde{\mathbf{C}}_{m-1} - \beta_m\mathbf{I} = \mathbf{L_mU_m}, \quad \tilde{\mathbf{C}}_m := \mathbf{U_mL_m} + \beta_m\mathbf{I},
\end{aligned}$$

with $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$. *If we denote by* $\{P_{n,i}\}$ *the sequence of monic orthogonal polynomials associated with* $\tilde{\mathbf{C}}_i$, $1 \leqslant i \leqslant m-1$, *and assuming that* $P_n(\beta) \neq 0$, $P_{n,i}(\beta_{i+1}) \neq 0$, $n \geqslant 1$, $1 \leqslant i \leqslant m-1$, *then* $\tilde{\mathbf{C}}_m$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = (x-\beta_1)(x-\beta_2)\ldots(x-\beta_m)d\mu.$$

Notice that in the previous results for the Christoffel transformation, only the Darboux transformation without parameter was used. On the other hand, for transformations (*ii*) and (*iii*) there are analogous results that use both the Darboux transformation and the Darboux transformation without parameter, as follows.

Proposition 2.4 *[2] Let* $\mathbf{J}$ *be the monic Jacobi matrix associated with the nontrivial probability measure* μ. *Consider the following transformations of* $\mathbf{J}$

$$\begin{aligned}
\mathbf{J} - \beta\mathbf{I} &= \mathbf{L_1U_1}, \qquad \mathbf{J}_1 := \mathbf{U_1L_1},\\
\mathbf{J}_1 &= \mathbf{U_2L_2}, \qquad \mathbf{J}_2 := \mathbf{L_2U_2} + \beta\mathbf{I}.
\end{aligned}$$

Then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = d\mu + M_r\delta(x-\beta),$$

i.e. the Uvarov transformation of μ, *where*

$$M_r = \frac{\mu_0(b_0 - \beta - s)}{s},$$

with $\mu_0 = \int_E d\mu(x)$ *and* s *is the free parameter associated with the* UL *factorization of* $\mathbf{J}_1$.

Proposition 2.5 *[2] Let* $\mathbf{J}_1$ *be the monic Jacobi matrix associated with the measure* $\hat{\mu}$. *Suppose there exists a measure* μ *such that* $d\hat{\mu} = (x-\beta)d\mu$. *If we apply the following transformation to* $\mathbf{J}_1$

$$\mathbf{J}_1 - \beta\mathbf{I} = \mathbf{U_1L_1}, \quad \mathbf{J}_2 := \mathbf{L_1U_1} + \beta\mathbf{I},$$

then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = \frac{d\hat{\mu}}{x-\beta} + M_r\delta(x-\beta),$$

i.e. the Geronimus transformation of $\hat{\mu}$*, where* $M_r = \frac{\int_E d\hat{\mu}}{s}$ *and* s *is the free parameter associated with the* UL *factorization of* $\mathbf{J}_1$*.*

3 Orthogonal polynomials on the unit circle and Hessenberg matrices

3.1 Background

Let $\mathcal{L}$ be a linear functional in the linear space of Laurent polynomials satisfying $c_n = \langle \mathcal{L}, z^n\rangle = \overline{\langle \mathcal{L}, z^{-n}\rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$, i.e. $\mathcal{L}$ is Hermitian. The complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ are said to be the *moments* associated with $\mathcal{L}$ and the matrix

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \tag{3.1}$$

is known in the literature as a Toeplitz matrix [13]. $\mathbb{P}$ will denote the linear space of polynomials with complex coefficients. If $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$, is non-singular for every $n \geqslant 0$, then $\mathcal{L}$ is said to be quasi-definite and the existence of a sequence of monic polynomials, orthogonal with respect to $\mathcal{L}$, is guaranteed. On the other hand, if $\det \mathbf{T}_n > 0$, for every $n \geqslant 0$, then $\mathcal{L}$ is said to be positive definite and it has the following integral representation

$$\langle \mathcal{L}, p(z)\rangle = \int_{\mathbb{T}} p(z) d\sigma(z), \quad p \in \mathbb{P}, \tag{3.2}$$

where σ is a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z : |z| = 1\}$.

The concepts presented in the previous Section concerning measures supported on the real line have a counterpart for measures supported on the unit circle. If σ is a nontrivial probability measure supported on $\mathbb{T}$, then

there exists a (unique) family of polynomials $\{\varphi_n\}_{n\geqslant 0}$, with $deg\,\varphi_n = n$ and positive leading coefficient, such that

$$\int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)}d\sigma(z) = \delta_{m,n}. \tag{3.3}$$

$\{\varphi_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to σ. Denoting by κ_n the leading coefficient of $\varphi_n(z)$, $\Phi_n(z) = \varphi_n(z)/\kappa_n$ is the corresponding sequence of monic orthogonal polynomials. These polynomials satisfy the following forward and backward recurrence relations (see [10], [13], [20], [22])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \tag{3.4}$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \tag{3.5}$$

where $\Phi_n^*(z) = z^n\bar{\Phi}_n(z^{-1})$ is the so-called reversed polynomial and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are known as Verblunsky or reflection parameters. It is important to notice that in the positive definite case we get $|\Phi_n(0)| < 1$, $n \geqslant 1$.

The multiplication operator with respect to $\{\varphi_n\}_{n\geqslant 0}$ is represented in a matrix form by

$$z\varphi(z) = \mathbf{H}_\varphi\varphi(z), \tag{3.6}$$

where $\varphi(z) = [\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z), \ldots]^t$ and $\mathbf{H}_\varphi$ is a lower Hessenberg matrix whose entries are

$$h_{n,j} = \begin{cases} \frac{\kappa_n}{\kappa_{n+1}} & \text{if} \quad j = n+1, \\ -\frac{\kappa_j}{\kappa_n}\Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if} \quad j \leqslant n, \\ 0 & \text{if} \quad j > n+1. \end{cases} \tag{3.7}$$

Proposition 3.1 $\mathbf{H}_\varphi$ *satisfies*

(i) $\mathbf{H}_\varphi\mathbf{H}_\varphi^* = \mathbf{I}$,

(ii) $\mathbf{H}_\varphi^*\mathbf{H}_\varphi = \mathbf{I} - \lambda_\infty(0)\varphi(0)\varphi(0)^*$,

where $\mathbf{I}$ *is the semi-infinite identity matrix and* $\lambda_\infty(0) = \prod_{n=0}^{\infty}(1-|\Phi_{n+1}(0)|^2)$.

Remark 3.2 *Proposition 3.1 states that the infinite matrix* $\mathbf{H}_\varphi$ *is unitary if and only if* $\sum_{n=0}^{\infty}|\Phi_n(0)|^2 = +\infty$. *In terms of the measure* σ *this fact is equivalent to* $\log\sigma' \notin L^1\left(\frac{d\theta}{2\pi}\right)$. *In other words,* σ *does not belong to the Szegő class (see [20]).*

Remark 3.3 *From* (3.6)*, is not difficult to show that* $\mathbf{H}_\Phi$*, the Hessenberg matrix associated with* $\{\Phi_n\}_{n\geqslant 0}$*, has as entries*

$$h_{n,j}=\begin{cases}1 & \text{if} \quad j=n+1,\\ -\frac{\mathbf{k}_n}{\mathbf{k}_j}\Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if} \quad j\leqslant n,\\ 0 & \text{if} \quad j>n+1,\end{cases} \tag{3.8}$$

where

$$\mathbf{k}_n=\frac{1}{\kappa_n^2}.$$

3.2 Spectral transformations

Transformations $(i)-(iii)$ defined in the previous Section for measures supported on the real line have also a counterpart for measures supported on the unit circle. They are defined as follows

(i) Christoffel transformation

$$d\sigma_C=|z-\alpha|^2 d\sigma,\quad \alpha\in\mathbb{C},|z|=1.$$

(ii) Uvarov transformation with one mass

$$d\sigma_U=d\sigma+M_c\delta(z-\alpha),\quad |\alpha|=1, M_c\in\mathbb{R}_+.$$

(iii) Uvarov transformation with two masses

$$d\sigma_U=d\sigma+M_c\delta(z-\alpha)+\bar{M}_c\delta(z-\bar{\alpha}^{-1}),\quad |\alpha|\in\mathbb{R}_+\setminus\{0,1\}, M_c\in\mathbb{C}.$$

(iv) Geronimus transformation

$$d\sigma_G=\frac{1}{|z-\alpha|^2}d\sigma+M_c\delta(z-\alpha)+\bar{M}_c\delta(z-\bar{\alpha}^{-1}),\quad |\alpha|>1, M_c\in\mathbb{C}.$$

We will denote these transformations by $\mathcal{F}_C(\alpha)$, $\mathcal{F}_U(\alpha,M_c)$, and $\mathcal{F}_G(\alpha,M_c)$, respectively. As in the case of the real line, these transformations are related by

$$\begin{aligned}\mathcal{F}_C(\alpha)\circ\mathcal{F}_G(\alpha,M_c) &= \mathcal{I}\quad \text{(Identity transformation)},\\ \mathcal{F}_G(\alpha,M_c)\circ\mathcal{F}_C(\alpha) &= \mathcal{F}_U(\alpha,M_c).\end{aligned}$$

Furthermore, introducing the Carathéodory function associated with σ

$$F(z)=c_0+2\sum_{k=1}^{\infty}c_{-k}z^k,$$

these transformations can be expressed in terms of the Carathéodory functions as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}, \tag{3.9}$$

where $\widetilde{F}(z)$ is the Carathéodory function associated with the perturbed measure and $A(z), B(z),\ D(z)$ are polynomials that are explicitly given (see [18]) for the above transformations.

Next, we focus our attention in the study of these transformations from the point of view of the corresponding Hessenberg matrices. We state a relation between the Hessenberg matrices associated with the original and perturbed measures by means of a QR factorization. Next, we describe some results in this direction.

We begin with the Christoffel transformation. Let denote by $\{\psi_n\}_{n\geqslant 0}$ the sequence of orthonormal polynomials with respect to σ_C. The relation between both families of polynomials is (see [11])

$$(z-\alpha)\psi_n(z) = \sqrt{\frac{K_n(\alpha,\alpha)}{K_{n+1}(\alpha,\alpha)}}\varphi_{n+1}(z) - \sum_{j=0}^{n} \frac{\varphi_{n+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{n+1}(\alpha,\alpha)K_n(\alpha,\alpha)}}\varphi_j(z), \tag{3.10}$$

where $K_n(z,y) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(y)}$. In the sequel,

$$K_n^{(i,j)}(z,y) = \sum_{k=0}^{n} \varphi_k^{(i)}(z)\overline{\varphi_k^{(j)}(y)},$$

i.e., the $i-th$ $(resp.\ j-th)$ partial derivative of $K_n(z,y)$ with respect to the variable z $(resp.\ y)$ is taken.

On the other hand, If

$$\varphi(z) = [\varphi_0(z),\ \ldots, \varphi_n(z), \ldots]^t \quad \text{and} \quad \psi(z) = [\psi_0(z),\ \ldots, \psi_n(z), \ldots]^t,$$

then the matrix representation of (3.10) is

$$(z-\alpha)\psi(z) = \mathbf{M}_C\varphi(z), \tag{3.11}$$

where $\mathbf{M}_C$ is a lower Hessenberg matrix with entries

$$m_{i,j} = \begin{cases} -\frac{\varphi_{i+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{i+1}(\alpha,\alpha)K_i(\alpha,\alpha)}}, & \text{if} \quad j \leqslant i, \\ \sqrt{\frac{K_i(\alpha,\alpha)}{K_{i+1}(\alpha,\alpha)}}, & \text{if} \quad j = i+1, \\ 0, & \text{if} \quad j > i+1. \end{cases} \tag{3.12}$$

Notice that $(\mathbf{M}_C)_n$, the $n\times n$ principal leading submatrix of $\mathbf{M}_C$, is a quasi-unitary matrix, i.e. its first $n-1$ rows constitute an orthonormal set, and the last row is orthogonal with respect to this set, but is not normalized.

Furthermore, if we denote by $\mathbf{L}_{\varphi\psi}$ the lower triangular matrix such that $\varphi(z) = \mathbf{L}_{\varphi\psi}\psi(z)$, then $\mathbf{L}_{\varphi\psi}$ can be computed from $\mathbf{H}_\varphi$ and $\mathbf{M}_C$ as follows

Proposition 3.4 *[6],[19]*

$$\mathbf{L}_{\varphi\psi} = (\mathbf{H}_\varphi - \alpha\mathbf{I})\mathbf{M}_C^{*}. \tag{3.13}$$

Now it is possible to determine the relation between $\mathbf{H}_\psi$, the Hessenberg matrix associated with σ_C, and $\mathbf{H}_\varphi$.

Proposition 3.5 *[6],[19]*

$$\mathbf{H}_\varphi - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_C, \tag{3.14}$$

$$\mathbf{H}_\psi - \alpha\mathbf{I} = \mathbf{M}_C\mathbf{L}_{\varphi\psi}. \tag{3.15}$$

Notice that an "almost" QR factorization appears, since $\mathbf{M}_C$ is not a unitary matrix (but it is very close). The iteration of the canonical Christoffel transformation has been analyzed in [11], [15], and [17].

The results for the Christoffel transformation can be used to obtain a similar result for the Uvarov transformation. If we assume that there exists a sequence of polynomials $\{\upsilon_n\}_{n\geqslant 0}$ orthonormal with respect to σ_U as defined in (ii), then the relation between the Hessenberg matrices $\mathbf{H}_\varphi$ and $\mathbf{H}_\upsilon$ associated with σ and σ_U, respectively, is (see [6], [18])

Proposition 3.6

$$\mathbf{H}_\varphi - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_C, \tag{3.16}$$

$$\mathbf{H}_\upsilon - \alpha\mathbf{I} = \mathbf{L}_U\mathbf{M}_U, \tag{3.17}$$

where $\mathbf{L}_U = \mathbf{L}_{\upsilon\varphi}\mathbf{L}_{\varphi\psi}$, $\mathbf{M}_U = \mathbf{M}_C\mathbf{L}_{\upsilon\varphi}^{-1}$, *and* $\mathbf{L}$ *are the matrices of change of bases for the orthonormal polynomial families denoted by their subindices.*

The iteration of the canonical Uvarov transformation, with $|\alpha| = 1$, has been studied in [10] and [17].

Finally, let us consider the Geronimus transformation defined in (iii). Necessary and sufficient conditions for the existence of a sequence of monic polynomials orthogonal with respect to σ_G were studied in [8], as well as the relation between the corresponding families of monic orthogonal polynomials and their associated Hessenberg matrices. If we denote by $\{G_n\}_{n\geqslant 0}$ the sequence of monic polynomials orthogonal with respect to σ_G and by $\mathbf{M}_G$ the corresponding Hessenberg matrix, from the relation between $\{G_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$, then we get

Proposition 3.7 *[8] Let* $\mathbf{L}_G$ *be the lower triangular matrix with* 1 *on the diagonal elements such that* $G(z) = \mathbf{L}_G\Phi(z)$ *and denote by* $\mathbf{H}_G$ *the Hessenberg matrix associated with* $\{G_n\}_{n\geqslant 0}$. *Then,*

$$\mathbf{H}_\Phi - \alpha\mathbf{I} = \mathbf{M}_G\mathbf{L}_G \tag{3.18}$$

and

$$\mathbf{H}_G - \alpha\mathbf{I} = \mathbf{L}_G\mathbf{M}_G. \tag{3.19}$$

4 A diagonal perturbation of a Toeplitz matrix

Let $\mathcal{L}$ be a linear functional in the linear space Λ of Laurent polynomials. Define a linear functional $\widetilde{\mathcal{L}}$ such that its associated bilinear functional satisfies

$$\langle p(z), q(z)\rangle_{\widetilde{\mathcal{L}}} := \langle p(z), q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}\int_{\mathbb{T}} p(z)\overline{q(z)}\frac{dz}{2\pi iz}, \tag{4.1}$$

with $\boldsymbol{m} \in \mathbb{R}$ and $p, q \in \mathbb{P}$. Assume that $\mathcal{L}$ is a positive definite linear functional and let σ be its corresponding measure as defined in (3.2). Then, this transformation, that will be denoted by $\mathcal{F}_D(\boldsymbol{m})$, can be expressed in terms of the corresponding measure σ as

$$d\tilde{\sigma} = d\sigma + \boldsymbol{m}\frac{dz}{2\pi iz}, \quad z = e^{i\theta}, \tag{4.2}$$

i.e. the addition of a Lebesgue measure to σ. We will assume $\boldsymbol{m} \in \mathbb{R}_+$ in order for $\tilde{\sigma}$ to be a positive Borel measure supported in $\mathbb{T}$. On the other hand, the moments $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ associated with $\widetilde{\mathcal{L}}$ are given by

$$\tilde{c}_0 = c_0 + \boldsymbol{m}, \quad \tilde{c}_k = c_k, \quad k = \pm 1, 2, \ldots. \tag{4.3}$$

As a consequence, the Carathéodory function of the linear functional $\widetilde{\mathcal{L}}$ is

$$\widetilde{F}(z) = F(z) + \boldsymbol{m}. \tag{4.4}$$

Notice that $\widetilde{\mathbf{T}}$, the Toeplitz matrix associated with $\widetilde{\mathcal{L}}$, is

$$\widetilde{\mathbf{T}} = \mathbf{T} + \boldsymbol{m}\mathbf{I}, \tag{4.5}$$

i.e. a constant is added to the main diagonal of $\mathbf{T}$. Now we proceed to obtain the sequence of monic orthogonal polynomials with respect to $\widetilde{\mathcal{L}}$.

Proposition 4.1 *Let $\mathcal{L}$ be a positive definite linear functional and denote by $\{\Phi_n\}_{n\geqslant 0}$ its associated sequence of monic orthogonal polynomials. Then, $\{\Psi_n\}_{n\geqslant 0}$, the sequence of monic polynomials orthogonal with respect to $\widetilde{\mathcal{L}}$ defined by* (4.1)*, is given by*

$$\Psi_n(z) = \Phi_n(z) - \mathbf{K}_{n-1}^t(z,0)(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\mathbf{\Phi}_n(0), \tag{4.6}$$

with $\mathbf{K}_{n-1}(z,0) = [K_{n-1}(z,0), K_{n-1}^{(0,1)}(z,0), \ldots, K_{n-1}^{(0,n-1)}(z,0)]^t$, $\mathbf{D}_n = diag\{\frac{1}{0!}, \ldots, \frac{1}{(n-1)!}\}$, $\mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(n-1)}(0)]^t$, *and*

$$\mathbf{P}_{n-1} = \begin{pmatrix} \varphi_0(0) & \varphi_1(0) & \cdots & \varphi_{n-1}(0) \\ 0 & \varphi_1'(0) & \cdots & \varphi_{n-1}'(0) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \varphi_{n-1}^{(n-1)}(0) \end{pmatrix}.$$

Proof. Set

$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z), \tag{4.7}$$

where, for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned} \lambda_{n,k} &= \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_k} = \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\widetilde{\mathcal{L}}} - \boldsymbol{m}\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}}{\mathbf{k}_k}, \\ &= -\frac{\boldsymbol{m}}{\mathbf{k}_k}\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}. \end{aligned}$$

Thus,

$$\Psi_n(z) = \Phi_n(z) - \boldsymbol{m}\int_{\mathbb{T}} \Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}, \tag{4.8}$$

$$= \Phi_n(z) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi_n^{(j)}(0)}{j!}\frac{K_{n-1}^{(0,j)}(z,0)}{j!}. \tag{4.9}$$

In particular, for $0 \leqslant i \leqslant n-1$, we get

$$\Psi_n^{(i)}(0) = \Phi_n^{(i)}(0) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi_n^{(j)}(0)}{j!}\frac{K_{n-1}^{(i,j)}(0,0)}{j!}. \tag{4.10}$$

So, we have the following linear system of n equations and n unknowns

$$\begin{aligned}
\Psi_n(0) &= \Phi_n(0) - \boldsymbol{m}\sum_{j=0}^{n-1}\frac{\Psi_n(0)}{j!}\frac{K_{n-1}^{(0,j)}(0,0)}{j!},\\
\Psi_n^{'}(0) &= \Phi_n^{'}(0) - \boldsymbol{m}\sum_{j=0}^{n-1}\frac{\Psi_n^{(j)}(0)}{j!}\frac{K_{n-1}^{(1,j)}(0,0)}{j!},\\
\Psi_n^{''}(0) &= \Phi_n^{''}(0) - \boldsymbol{m}\sum_{j=0}^{n-1}\frac{\Psi_n^{(j)}(0)}{j!}\frac{K_{n-1}^{(2,j)}(0,0)}{j!},\\
&\vdots\\
\Psi_n^{(n-1)}(0) &= \Phi_n^{(n-1)}(0) - \boldsymbol{m}\sum_{j=0}^{n-1}\frac{\Psi_n^{(j)}(0)}{j!}\frac{K_{n-1}^{(n-1,j)}(0,0)}{j!},
\end{aligned}$$

which reads as

$$(\mathbf{I}_n + \boldsymbol{m}\mathbf{R}_{n-1}\mathbf{D}_n^2)\boldsymbol{\Psi}_n(0) = \boldsymbol{\Phi}_n(0), \tag{4.11}$$

where

$$\boldsymbol{\Psi}_n(0) = [\Psi_n(0),\ldots,\Psi_n^{(n-1)}(0)]^t, \quad \boldsymbol{\Phi}_n(0) = [\Phi_n(0),\ldots,\Phi_n^{(n-1)}(0)]^t,$$

$$\mathbf{R}_{n-1} = \begin{pmatrix}
K_{n-1}^{(0,0)}(0,0) & K_{n-1}^{(0,1)}(0,0) & \cdots & K_{n-1}^{(0,n-1)}(0,0)\\
K_{n-1}^{(1,0)}(0,0) & K_{n-1}^{(1,1)}(0,0) & \cdots & K_{n-1}^{(1,n-1)}(0,0)\\
\vdots & \vdots & \ddots & \vdots\\
K_{n-1}^{(n-1,0)}(0,0) & K_{n-1}^{(n-1,1)}(0,0) & \cdots & K_{n-1}^{(n-1,n-1)}(0,0)
\end{pmatrix}, \tag{4.12}$$

and $\mathbf{D}_n = diag\{\frac{1}{0!},\frac{1}{1!},\ldots,\frac{1}{(n-1)!}\}$. As a consequence, (4.11) becomes

$$\boldsymbol{\Psi}_n(0) = \boldsymbol{m}^{-1}(\boldsymbol{m}^{-1}\mathbf{I}_n + \mathbf{R}_{n-1}\mathbf{D}_n^2)^{-1}\boldsymbol{\Phi}_n(0). \tag{4.13}$$

Thus, if $\mathbf{K}_{n-1}(z,0) = [K_{n-1}(z,0), K_{n-1}^{(0,1)}(z,0),\ldots,K_{n-1}^{(0,n-1)}(z,0)]^t$, then (4.9) can be written

$$\begin{aligned}
\Psi_n(z) &= \Phi_n(z) - \boldsymbol{m}\mathbf{K}_{n-1}^t(z,0)\mathbf{D}_n^2\boldsymbol{\Psi}_n(0),\\
&= \Phi_n(z) - \mathbf{K}_{n-1}^t(z,0)(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_{n-1})^{-1}\boldsymbol{\Phi}_n(0),
\end{aligned}$$

which is (4.6), since $\mathbf{R}_{n-1} = \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t$. □

In particular, for the corresponding Verblunsky parameters we get

$$\Psi_n(0) = \Phi_n(0) - \mathbf{K}_{n-1}^t(0,0)(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\mathbf{\Phi}_n(0).$$

On the other hand, considering the derivatives of order j with respect to the variable z in the Christoffel-Darboux formula, (see [10], [20], and [22])

$$K_{n-1}(z,y) = \frac{\overline{\Phi_n^*(y)}\Phi_n^*(z) - \overline{\Phi_n(y)}\Phi_n(z)}{\mathbf{k}_n(1-\bar{y}z)},$$

we get

$$K_{n-1}^{(j,0)}(z,y) = \frac{\overline{\Phi_n^*(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n^*(z)}{1-\bar{y}z}\right)^{(j)} - \frac{\overline{\Phi_n(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n(z)}{1-\bar{y}z}\right)^{(j)}.$$

Thus

$$K_{n-1}^{(j,0)}(0,y) = \frac{\overline{\Phi_n^*(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n^*(z)}{1-\bar{y}z}\right)^{(j)}(0) - \frac{\overline{\Phi_n(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n(z)}{1-\bar{y}z}\right)^{(j)}(0).$$

Now, taking into account the Leibniz rule

$$\left(\frac{\Phi_n(z)}{1-\bar{y}z}\right)^{(j)} = \sum_{k=0}^{j}\binom{j}{k}\Phi_n^{(j-k)}(z)\frac{k!\bar{y}^k}{(1-\bar{y}z)^{k+1}},$$

the evaluation at $z=0$ yields

$$\begin{aligned} j!\sum_{k=0}^{j}\frac{\Phi_n^{(j-k)}(0)}{(j-k)!}\bar{y}^k &= \overline{j!\sum_{k=0}^{j}\frac{\overline{\Phi_n^{(j-k)}(0)}}{(j-k)!}y^k} \\ &= j!\overline{T_j^*(\Phi_n(y);0)}, \end{aligned}$$

where $T_j(p(y);0)$ denotes the $j-th$ Taylor polynomial of $p(y)$ around $y=0$. In an analog way

$$j!\sum_{k=0}^{j}\frac{\Phi_n^{*(j-k)}(0)}{(j-k)!}\bar{y}^k = j!\overline{T_j^*(\Phi_n^*(y);0)}.$$

Therefore, we have proved

Proposition 4.2

$$K_{n-1}^{(0,j)}(z,0) = j!\left[\frac{\Phi_n^*(z)}{\mathbf{k}_n}T_j^*(\Phi_n^*(z);0) - \frac{\Phi_n(z)}{\mathbf{k}_n}T_j^*(\Phi_n(z);0)\right].$$

From the previous Proposition, if we denote

$$\mathbf{T}(\Phi_n(z);0) = [T_0^*(\Phi_n(z);0), T_1^*(\Phi_n(z);0), \ldots, T_{n-1}^*(\Phi_n(z);0)]^t,$$

then (4.6) becomes

$$\begin{aligned}\Psi_n(z) &= \Phi_n(z) - \frac{\Phi_n(z)}{\mathbf{k}_n}\mathbf{T}^t(\Phi_n(z);0)\mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\mathbf{\Phi}_n(0)\\ &- \frac{\Phi_n^*(z)}{\mathbf{k}_n}\mathbf{T}^t(\Phi_n^*(z);0)\mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\mathbf{\Phi}_n(0)\\ &= a_n(z)\Phi_n(z) + b_n(z)\Phi_n^*(z), \qquad (4.14)\end{aligned}$$

with

$$\begin{aligned}a_n(z) &= 1 - \frac{1}{\mathbf{k}_n}\mathbf{T}^t(\Phi_n(z);0)\mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\mathbf{\Phi}_n(0),\\ b_n(z) &= -\frac{1}{\mathbf{k}_n}\mathbf{T}^t(\Phi_n^*(z);0)\mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\mathbf{\Phi}_n(0).\end{aligned}$$

5 Examples

5.1 A Bernstein-Szegő case

Let us consider the measure σ such that

$$d\sigma = \frac{1-|\beta|^2}{|z-\beta|^2}\frac{d\theta}{2\pi}, \quad |\beta| < 1, \quad z = e^{i\theta}, \qquad (5.1)$$

and apply to it the transformation considered in the previous Section, i.e. let us define a new measure $\tilde{\sigma}$

$$d\tilde{\sigma} = \frac{1-|\beta|^2}{|z-\beta|^2}\frac{d\theta}{2\pi} + \boldsymbol{m}\frac{d\theta}{2\pi}, \quad \boldsymbol{m} \in \mathbb{R}_+. \qquad (5.2)$$

Our goal is to find the sequence of monic polynomials orthogonal with respect to (5.2), that will be denoted by $\{\Psi_n\}_{n\geqslant 0}$. (5.1) is known in the literature as Bernstein-Szegő measure, and its corresponding sequence of monic orthogonal polynomials is (see [20])

$$\Phi_n(z) = z^{n-1}(z-\beta), \quad n \geqslant 1.$$

Furthermore, we have $\Phi_n^*(z) = 1 - \bar{\beta}z$, $n \geqslant 1$. Notice that, for $n = 1$, from (4.14) we get

$$\begin{aligned}\Psi_1(z) &= a_1(z)\Phi_1(z) + b_1(z)\Phi_1^*(z),\\ &= [1 - \beta^2(1-|\beta|^2)(\boldsymbol{m} + (1-|\beta|^2)^{-1}](z-\beta)\\ &- [\beta(1-|\beta|^2)(\boldsymbol{m} + (1-|\beta|^2)^{-1}](1-\bar{\beta}z). \qquad (5.3)\end{aligned}$$

For $n \geqslant 2$, $T_j(\Phi_n(0);0) = 0$, $0 \leqslant j \leqslant n-2$, and

$$\begin{aligned} T_{n-1}(\Phi_n(z);0) &= \Phi_n(0) + \Phi_n^{'}(0)z + \Phi_n^{''}(0)\frac{z^2}{2!} + \ldots + \Phi_n^{(n-1)}(0)\frac{z^{n-1}}{(n-1)!}, \\ &= -\beta z^{n-1}. \end{aligned}$$

Thus, $\mathbf{T}(\Phi_n(z);0) = [0,0,\ldots,-\bar{\beta}]^t$. On the other hand, $T_0(\Phi_n^*(0);0) = 1$, and $T_j(\Phi_n^*(z);0) = 1 - \overline{\beta}z$, $1 \leqslant j \leqslant n-1$.

Therefore, $\mathbf{T}(\Phi_n^*(z);0) = [1, z-\beta, z(z-\beta), \ldots, z^{n-2}(z-\beta)]^t$. Furthermore, in this case

$$\mathbf{P}_{n-1} = \frac{1}{\sqrt{1-|\beta|^2}}\mathbf{D}_n^{-1}\mathbf{B}_n,$$

where

$$\mathbf{B}_n = \begin{pmatrix} 1 & \beta & 0 & \cdots & 0 \\ 0 & 1 & \beta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & \beta \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{5.4}$$

On the other hand, denoting by

$$\begin{aligned} \mathbf{N}_n &= \mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}, & (5.5) \\ &= [\boldsymbol{m}^{-1}\mathbf{I}_n + \frac{1}{1-|\beta|^2}\mathbf{B}_n\mathbf{B}_n^t]^{-1}\mathbf{D}_n, & (5.6) \end{aligned}$$

and taking into account that $\boldsymbol{\Phi}_n(0) = [0,0,\cdots,-(n-1)!\beta]^t$, we get

$$\begin{aligned} a_n(z) &= 1 - \frac{|\beta|^2}{1-|\beta|^2}(n-1)!n_{n-1,n-1}, \\ b_n(z) &= -\frac{\beta}{1-|\beta|^2}(n-1)!\left[n_{0,n-1} + (z-\beta)\sum_{i=0}^{n-2} n_{i+1,n-1}z^i\right], \end{aligned} \tag{5.7}$$

where $n_{i,j}$, $0 \leqslant i,j \leqslant n-1$ are the entries of $\mathbf{N}_n$. From this, all elements on (4.14) are known and we can compute $\Psi_n(z)$. The corresponding Verblunsky parameters, for $n \geqslant 2$, are

$$\begin{aligned} \Psi_n(0) &= b_n(0), \\ &= \frac{\beta}{1-|\beta|^2}(n-1)!(\beta n_{1,n-1} - n_{0,n-1}). \end{aligned}$$

The first 20 Verblunsky coefficients for different values of $\boldsymbol{m}$ and β are shown in Figure 1.

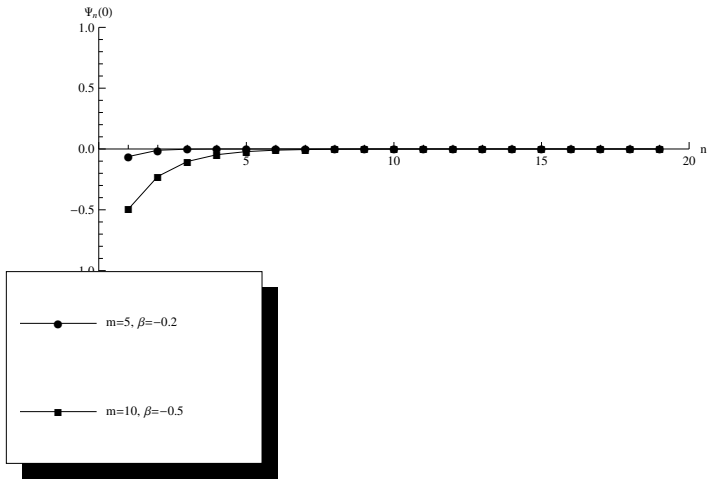


Figure 1: Verblunsky coefficients for Example 5.1

5.2 Tchebychev case

Let us consider the measure

$$d\sigma = |z-1|^2 \frac{d\theta}{2\pi}, \quad z = e^{i\theta}, \tag{5.8}$$

and its corresponding sequence of monic orthogonal polynomials, given by (see [20])

$$\Phi_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} (k+1) z^k, \qquad n \geqslant 0. \tag{5.9}$$

We introduce the perturbation defined in the previous Section and obtain

$$d\tilde{\sigma} = d\sigma + \boldsymbol{m} \frac{d\theta}{2\pi}. \tag{5.10}$$

We now proceed to get an explicit expression for the sequence of monic polynomials orthogonal with respect to $\tilde{\sigma}$. Notice that

$$T_j(\Phi_n(z); 0) = \frac{1}{n+1} \sum_{k=0}^{j} (k+1) z^k, \quad 0 \leqslant j \leqslant n-1,$$

and, as a consequence,

$$T_j^*(\Phi_n(z); 0) = \frac{1}{n+1} \sum_{k=0}^{j} (k+1) z^{j-k}, \quad 0 \leqslant j \leqslant n-1.$$

On the other hand, since

$$\Phi_n^*(z) = \frac{1}{n+1}\sum_{k=0}^{n}(k+1)z^{n-k}, \quad n \geqslant 0,$$

we get

$$T_j(\Phi_n^*(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(n+1-k)z^k, \quad 1 \leqslant j \leqslant n-1,$$

and, thus,

$$T_j^*(\Phi_n^*(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(n+1-k)z^{j-k}, \quad 1 \leqslant j \leqslant n-1.$$

Therefore, $\mathbf{T}(\Phi_n(0);0) = \frac{1}{n+1}[1,2,\ldots,n]^t$ and $\mathbf{T}(\Phi_n^*(0);0) = \frac{1}{n+1}[n+1,n,n-1,\ldots,2]^t$. Furthermore, since

$$\varphi_n(z) = \sqrt{\frac{2}{(n+1)(n+2)}}\sum_{k=0}^{n}(k+1)z^k, \qquad n \geqslant 0, \tag{5.11}$$

are the orthonormal polynomials with respect to σ, we obtain

$$\mathbf{P}_{n-1} = \mathbf{D}_n^{-1}\mathbf{A}_n\mathbf{\Lambda}_n \tag{5.12}$$

where

$$\mathbf{A}_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \mathbf{\Lambda}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_{n-1} \end{pmatrix},$$

and $p_j = \sqrt{\frac{2}{(j+1)(j+2)}}$.

Finally, we have $\mathbf{\Phi}_n(0) = \frac{1}{n+1}[1,2!,\cdots,n!]^t$ and we can obtain $\Psi_n(z)$ explicitly from (4.14). Figure 2 shows the behavior of the corresponding Verblunsky coefficients for different values of $\boldsymbol{m}$.

Remark 5.1 *Notice that* (5.8) *is a Christoffel transformation of the Lebesgue measure with* $\alpha = 1$ *(see Section 2). This example can be generalized as*

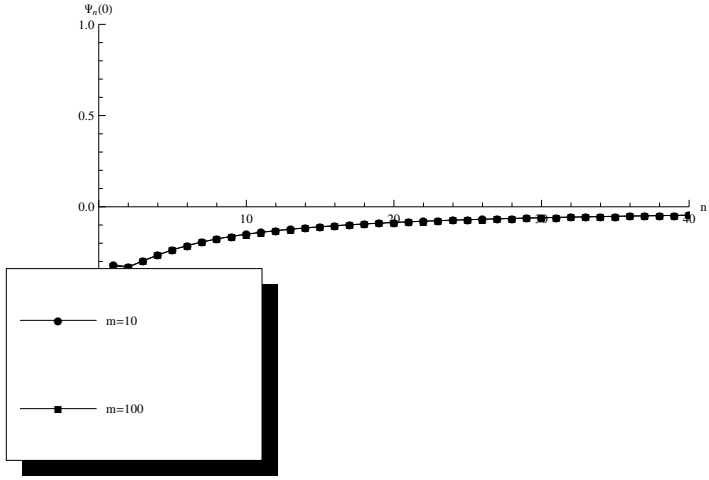


Figure 2: Verblunsky coefficients for Example 5.2

follows. Let σ *be an absolutely continuous measure whose Radon-Nikodyn derivative with respect to the Lebesgue measure is* $\sigma' = |z-\alpha|^2$, $z = e^{i\theta}$, *i.e. a positive trigonometric polynomial of degree* 1. *Applying the transformation in Section* 3, *with* $\boldsymbol{m} \in \mathbb{R}_+$, *and assuming* $\alpha \in \mathbb{C}$, *we obtain*

$$\boldsymbol{m} + |z-\alpha|^2 = \boldsymbol{m} + 1 - \alpha z^{-1} - \bar{\alpha} z + |\alpha|^2, \quad z = e^{i\theta}, \tag{5.13}$$

i.e. another positive trigonometric polynomial that can be represented by $|\delta z - \gamma|^2$, *where* $\delta \in \mathbb{R}, \gamma \in \mathbb{C}$. *Indeed, as*

$$|\delta z - \gamma|^2 = \delta^2 - \delta\gamma z^{-1} - \delta\bar{\gamma} z + |\gamma|^2, \quad z = e^{i\theta},$$

the comparison of the coefficients with (5.13) *yields* $1 + |\alpha|^2 + \boldsymbol{m} = \delta^2 + |\gamma|^2$ *and* $\alpha = \delta\gamma$. *Thus,*

$$1 + |\alpha|^2 + \boldsymbol{m} = \frac{|\alpha|^2}{\delta^2} + \delta^2,$$

so we can get δ *and* γ *in terms of* $\boldsymbol{m}$ *and* α. *In other words, in this case we can express the addition to a Tchebychev measure of a Lebesgue measure (multiplied by a constant* $\boldsymbol{m}$*) as a Christoffel transformation of the Lebesgue measure.*

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A weighted generalization of the classical Kantorovich operator

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Abstract. An operator of Kantorovich-type is introduced for the weighted L^p approximation with Jacobi weights, dropping the restrictions on the weight parameters appearing in case of the classical Kantorovich operator. Direct and converse theorems and a Voronovskaya-type relation are proved.

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1 Introduction

Since the Bernstein polynomials are not defined for $f \in L^p([0,1])$, $1 \le p < \infty$, the Kantorovich polynomials K_n, given by

$$K_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{I_k} f(u)\,du, \tag{1.1}$$

$$I_k := \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right], \qquad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1],$$

were introduced and studied deeply (see, e.g., [2, p. 115 ff.]). The case of weighted approximation by K_n operator with the Jacobi weight

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$$w(x) = w^{\alpha,\beta}(x) = x^{\alpha}(1-x)^{\beta}, \qquad \alpha, \beta > -1,$$

was also investigated (see [2, p. 159]). Some equivalence results were given under the restrictions

$$-\frac{1}{p} < \alpha,\ \beta < 1 - \frac{1}{p} \tag{1.2}$$

on the weight parameters. Here the left side inequalities for the weight parameters are necessary. In order to eliminate the right hand side inequalities, we construct a weighted generalization of the classical Kantorovich operator and give convergence, direct and converse approximation results involving the weighted modulus of smoothness of second order, as well as a Voronovskaya-type asymptotic relation.

2 Main results

In the following C denotes a positive constant which can take different values even in subsequent formulae. We point out that throughout this paper the weight is always the Jacobi weight quoted in the Introduction.

In all the theorems and lemmas in this paper, we always assume the conditions

$$-\frac{1}{p} < \alpha,\ \beta \qquad \text{and} \qquad 1 \le p \le \infty.$$

Let

$$L_w^p := \begin{cases} \{f \,|\, wf \in L^p(0,1)\} & \text{if} \quad 1 \le p < \infty, \\ \{f \,|\, f \in C(0,1),\ \lim_{x(1-x)\to 0}(wf)(x) = 0\} & \text{if} \quad p = \infty. \end{cases}$$

The norm in L_w^p is defined as

$$\|wf\|_p = \begin{cases} \left(\int_0^1 |(wf)(x)|^p\, dx\right)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \sup_{0\le x\le 1} |(wf)(x)| & \text{if} \quad p = \infty. \end{cases}$$

We also put $\|wf\|_{L^p(I)}$ as the weighted L^p norm on an interval I. For smoother functions we introduce the Sobolev type space $W_w^{2,p}$ defined as

$$W_w^{2,p} := \{f \in L_w^p |\ f' \in \mathrm{AC}(0,1),\ \|w\varphi^2(x)f''\|_p < \infty\},$$

where

$$\varphi(x) = \sqrt{x(1-x)},$$

and $\mathrm{AC}(I)$ is the set of all absolutely continuous functions on I. For $f \in L_w^p$ we consider the weighted modulus of smoothness defined by

$$\begin{aligned}\omega_\varphi^2(f;t)_{w,p} = \sup_{0<h\le t} &\|w\Delta_{h\varphi}^2\|_{L^p([Ch^2,1-Ch^2])} \\ &+ \|w\overleftarrow{\Delta}_h^2 f\|_{L^p([1-Ch^2,1])} + \|w\overrightarrow{\Delta}_h^2 f\|_{L^p([0,Ch^2])},\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\overrightarrow{\Delta}_h^2 f(x) &= f(x) - 2f(x+h) + f(x+2h),\\ \overleftarrow{\Delta}_h^2 f(x) &= f(x) - 2f(x-h) + f(x-2h)\end{aligned}$$

and

$$\Delta_{h\varphi}^2 f(x) = f\left(x + h\frac{\varphi(x)}{2}\right) - 2f(x) + f\left(x - h\frac{\varphi(x)}{2}\right)$$

(cf. [2, p. 218]).

Now for every $f \in L_w^1$ introduce the operator

$$B_n^*(f;x) = \sum_{k=0}^{n} \frac{\int_{I_k}(wf)(t)\,dt}{\int_{I_k} w(t)\,dt} p_{n,k}(x), \qquad x \in [0,1]. \tag{2.2}$$

Obviously, if $\alpha = \beta = 0$, then B_n^* reduces to the classical Kantorovich operator (1.1).

Note that the definition (2.2) allows a wider class of functions than the original Kantorovich operator (cf. [2, p. 158]). From (2.2) it follows that B_n^* is a positive linear operator preserving constants.

By the mean value theorem

$$\begin{aligned}a_{n,k}(f) := \frac{\int_{I_k}(wf)(t)\,dt}{\int_{I_k} w(t)\,dt} = &\frac{w(\xi_k)\int_{I_k} f(t)\,dt}{w(\eta_k)/(n+1)} \sim (n+1)\int_{I_k} f(t)\,dt,\\ &\xi_k, \eta_k \in I_k,\ 1 \le k \le n-1,\end{aligned} \tag{2.3}$$

hence, in some sense, B_n^* is not far from the operator K_n.

For $f \in L_w^p$ define

$$E_0(f)_{w,p} := \inf_{c\in\mathbf{R}} \|w(f-c)\|_p$$

to be the best approximation of f in weighted L_w^p spaces by constants.

We have the following approximation results.

Theorem 2.1 *If $f \in L^p_w$ then*

$$\|w[f - B^*_n(f)]\|_p \le C \left[\omega^2_\varphi \left(f; \frac{1}{\sqrt{n}} \right)_{w,p} + \frac{E_0(f)_{w,p}}{n} \right]. \tag{2.4}$$

Hence we deduce the weighted convergence of $B^*_n(f)$ to f.

Compare this with [2, p. 159] for the analogous result for the classical Kantorovich operator. In our result we do not need the strong upper restriction in (1.2), and we have (2.4) which is not known for the classical weighted Kantorovich operator.

Theorem 2.2 *If $f \in L^p_w$ then*

$$\|w[f - B^*_n(f)]\|_p = O(n^{-\gamma/2}) \;\Leftrightarrow\; \omega^2_\varphi(f;h)_{w,p} = O(h^\gamma), \quad 0 < \gamma < 2.$$

This equivalence provides a converse result for $B^\star_n$.

We also prove an asymptotic relation of Voronovskaya-type.

Theorem 2.3 *For $f'' \in C[0,1]$ we have*

$$\lim_{n\to\infty} n[B^*_n(f;x) - f(x)] = \begin{cases} \frac{\alpha+1}{\alpha+2} f'(0) & \text{if } x = 0, \\ \frac{1-2x}{2} f'(x) + \frac{x(1-x)}{2} f''(x) & \text{if } 0 < x < 1, \\ -\frac{\beta+1}{\beta+2} f'(1), & \text{if } x = 1. \end{cases} \tag{2.5}$$

Note that if $\alpha = \beta = 0$, then this is the limit relation

$$\lim_{n\to\infty} n[K_n(f;x) - f(x)] = \frac{1-2x}{2} f'(x) + \frac{x(1-x)}{2} f''(x), \; 0 \le x \le 1$$

for the classical Kantorovich operator. (We could not find an explicit reference for the latter relation, but it follows from the arguments used in [2, p. 176].) If $\alpha, \beta \ne 0$, then the limit functions on the right hand side in (2.5) present a discontinuity at the endpoints, which is affected by the weight parameters.

To prove our main approximation results we need some preliminary lemmas interesting in themselves, since they prove the weighted boundedness and weighted Markov–Bernstein type inequalities for B^*_n.

Lemma 2.4 *For $f \in L^p_w$, $1 \le p \le \infty$,*

$$\|wB^*_n(f)\|_p \le C\|wf\|_p.$$

(Cf. [2, p. 159, (10.1.4)].)

Lemma 2.5 *For $f \in L^p_w$, $1 \le p \le \infty$,*

$$\|w\varphi^2 B_n^{*''}(f)\|_p \le Cn\|wf\|_p. \tag{2.6}$$

(Cf. [2, p. 159, (10.1.5)].)

Lemma 2.6 *For $f \in W^{2,p}_w$, $1 \le p \le \infty$,*

$$\|w\varphi^2 B_n^{*''}(f)\|_p \le C\left(\|w\varphi^2 f''\|_p + \|wf'\|_p\right). \tag{2.7}$$

Remark. Comparing this lemma with [2, p. 159, (10.1.4)], we can see that we have $\|wf'\|_p$ instead of $\|wf\|_p$ on the right hand side of (2.7). Our statement is more appropriate, since the right hand side of (2.7) becomes zero when f=const. In any case, we can get (10.4.1) of [2] by using the inequality

$$\|wf'\|_p \le C(\|w\varphi^2 f''\|_p + \|wf\|_p), \qquad 1 \le p \le \infty \tag{2.8}$$

(cf. [2, Lemma 10.5.1]).

3 Proofs

3.1 Proof of Lemma 2.4

By symmetry, it is sufficient to estimate the sum figuring in (2.2) for $0 \le k \le n/2$ (and this remark applies to all subsequent proofs). For brevity, such sums will always be denoted by $\sum'$. We obtain

$$d_k := \int_{I_k} w(t)\,dt \sim \frac{(k+1)^\alpha}{n^{\alpha+1}}, \qquad 0 \le k \le n/2, \tag{3.1}$$

and thus by (2.3)

$$|a_{n,k}(f)| \le C\frac{n^{\alpha+1/p}}{(k+1)^\alpha}\|wf\|_{L^p(I_k)}, \qquad 0 \le k \le n/2. \tag{3.2}$$

Let first $1 \le p < \infty$. By Hölder inequality and $\sum_{k=0}^n p_{n,k}(x) = 1$ we get

$$\begin{aligned}\left\|w\sum{}' a_{n,k}(f)p_{n,k}\right\|_p^p &\le \left\|w\left(\sum{}' |a_{n,k}(f)|^p p_{n,k}\right)^{1/p}\right\|_p^p \\ &\le Cn^{\alpha p+1}\sum{}' \frac{\|wf\|_{L^p(I_k)}}{(k+1)^{\alpha p}}\int_0^1 w(x)^p p_{n,k}(x)\,dx.\end{aligned}$$

Here, calculating the Euler integral,

$$\int_0^1 w(x)^p p_{n,k}(x)\,dx = \binom{n}{k}\frac{\Gamma(k+\alpha p+1)\Gamma(n-k+\beta p+1)}{\Gamma(n+(\alpha+\beta)p+2)}$$
$$\leq \frac{(k+1)^{\alpha p}(n-k)^{\beta p}}{n^{(\alpha+\beta)p+1}} \leq \frac{(k+1)^{\alpha p}}{n^{\alpha p+1}}, \qquad 0 \leq k \leq n/2.$$

Hence

$$\left\| w \sum{}' a_{n,k}(f) p_{n,k} \right\|_p^p \leq C \sum{}' \|wf\|_{L^p(I_k)} \leq C\|wf\|_p^p, 1 \leq p < \infty.$$

Now let $p = \infty$. Then α, $\beta \geq 0$, and from (3.2) one gets

$$w(x) \sum{}' a_{n,k} p_{n,k}(x)$$
$$\leq C\|wf\|_\infty \sum{}' \left(\frac{n}{k+1}\right)^\alpha \binom{n}{k} x^{k+\alpha}(1-x)^{n-k+\beta}.$$

Thus

$$w(x) \sum{}' a_{n,k} p_{n,k}(x)$$
$$\leq C\|wf\|_\infty \left\{ \sum_{k+1\leq nx} + \sum_{k+1>nx} \right\} \frac{n^\alpha \binom{n}{k} x^{k+\alpha}(1-x)^{n-k+\beta}}{(k+1)^\alpha} \tag{3.3}$$
$$:= C\|wf\|_\infty \{W_1 + W_2\}.$$

Obviously

$$W_2 \leq C \sum_{k+1>nx} \binom{n}{k} x^k (1-x)^{n-k} \leq C.$$

If $k+1 \leq nx$, then with $a = [\alpha] + 1$,

$$\left(\frac{nx}{k+1}\right)^\alpha \leq \left(\frac{nx}{k+1}\right)^a$$
$$\leq \frac{x^a (n+a)(n+a-1)\cdots(n+1)}{(k+a)(k+a-1)\cdots(k+1)} \frac{(k+a)(k+a-1)\cdots(k+1)}{(k+1)^a}$$
$$\leq C x^a \frac{(n+a)(n+a-1)\cdots(n+1)}{(k+a)(k+a-1)\cdots(k+1)}.$$

Hence

$$\begin{aligned} W_1 &\le \sum_{k+1\le nx} \frac{(n+a)(n+a-1)\cdots(n+1)}{(k+a)(k+a-1)\cdots(k+1)} \binom{n}{k} x^{k+a}(1-x)^{n-k} \\ &\le C \sum_{k=0}^{n} \binom{n+a}{k+a} x^{k+a}(1-x)^{(n+a)-(k+a)} \\ &= C \sum_{k=0}^{n} p_{n+a,k+a}(x) \le C. \end{aligned} \tag{3.4}$$

By (3.3)–(3.4), we get the statement for $p = \infty$. □

3.2 Proof of Lemma 2.5

First we prove (2.6) when $1 \le p < \infty$. Since (see [1, formula (32), p. 32])

$$\begin{aligned} B_n^{*''}(g;x) = &- \frac{n}{\varphi^2(x)} B_n(g;x) + \left(\frac{n}{\varphi(x)^2}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) g\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2 \\ &- \frac{n(1-2x)}{x^2(1-x)^2} \sum_{k=0}^{n} p_{n,k}(x) g\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right), \end{aligned}$$

with B_n being the classical Bernstein operator for $g \in C([0,1])$, analogously we obtain

$$\begin{aligned} w(x)\varphi^2(x)|B_n^{*''}(f;x)| \le C \Big[& nw(x)|B_n^*(f;x)| \\ &+ \sum_{k=0}^{n} |a_{n,k}(f)| \frac{w(x)}{\varphi^2(x)} p_{n,k}(x)(k-nx)^2 \\ &+ \sum_{k=0}^{n} |a_{n,k}(f)| \frac{w(x)}{\varphi^2(x)} p_{n,k}(x)|k-nx| \Big]. \end{aligned} \tag{3.5}$$

Hence, using

$$\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^\gamma \le C n^{\gamma/2} \varphi^\gamma(x), \qquad \gamma = 0,1,2, \tag{3.6}$$

(cf. [3], (6) on p. 6) with $\gamma = 1$ and $\gamma = 2$ as well as (3.2) we get

$$\begin{aligned}
\|w\varphi^2 B_n^{*''}(f)\|_p^p \leq & C \left[n^p \|w B_n^{*''}(f)\|_p^p + \left\| \frac{w(x)}{\varphi^2(x)} \sum_{k=0}^{n} a_{n,k}(f) p_{n,k}(x)(k-nx)^2 \right\|_p^p \right. \\
& \left. + \left\| \frac{w(x)}{\varphi^2(x)} \sum_{k=0}^{n} a_{n,k}(f) p_{n,k}(x)|k-nx| \right\|_p^p \right. \\
\leq & C \left[n^p \|wf\|_p^p + \left\| \frac{w(x)}{\varphi^2(x)} \left(\sum_{k=0}^{n} |a_{n,k}(f)|^p p_{n,k}(x)(k-nx)^2 \right)^{1/p} \right. \right. \\
& \left. \left. \left(\sum_{k=0}^{n} p_{n,k}(x)(k-nx)^2 \right)^{1-1/p} \right\|_p^p \right. \\
& \left. + \left\| \frac{w(x)}{\varphi^2(x)} \left(\sum_{k=0}^{n} |a_{n,k}(f)|^p p_{n,k}(x)|k-nx| \right)^{1/p} \right. \right. \\
& \left. \left. \left(\sum_{k=0}^{n} p_{n,k}(x)|k-nx| \right)^{1-1/p} \right\|_p^p \right],
\end{aligned}$$

whence

$$\begin{aligned}
& \|w\varphi^2 B_n^{*''}(f)\|_p^p \\
& \leq C \left[n^p \|wf\|_p^p + n^{p-1} \left\| \frac{w(x)}{\varphi^{2/p}(x)} \left(\sum_{k=0}^{n} |a_{n,k}(f)|^p p_{n,k}(x)(k-nx)^2 \right)^{1/p} \right\|_p^p \right. \\
& \quad \left. + n^{\frac{p-1}{2}} \left\| \frac{w(x)}{\varphi^{1+1/p}(x)} \left(\sum_{k=0}^{n} |a_{n,k}(f)|^p p_{n,k}(x)|k-nx| \right)^{1/p} \right\|_p^p \right] \\
& \leq C \left[n^p \|wf\|_p^p + n^{(\alpha+1)p} \sum{}' \frac{\|wf\|_{L^p(I_k)}}{(k+1)^{\alpha p}} \int_0^1 \frac{w(x)^p}{\varphi^2(x)} p_{n,k}(x)(k-nx)^2 \, dx \right. \\
& \quad \left. + n^{\alpha p + \frac{p+1}{2}} \sum{}' \frac{\|wf\|_{L^p(I_k)}}{(k+1)^{\alpha p}} \int_0^1 \frac{w(x)^p}{\varphi^{p+1}(x)} p_{n,k}(x)|k-nx| \, dx \right].
\end{aligned}$$

Here, an easy calculation yields

$$\begin{aligned}\int_0^1 \frac{w(x)^p}{\varphi^2(x)} p_{n,k}(x)(k-nx)^2\,dx \binom{n}{k} \Bigg[& k^2 \frac{\Gamma(k+\alpha p)\Gamma(n-k+\beta p)}{\Gamma(n+\alpha p+\beta p)} \\ & - 2kn \frac{\Gamma(k+\alpha p+1)\Gamma(n-k+\beta p)}{\Gamma(n+\alpha p+\beta p+1)} + n^2 \frac{\Gamma(k+\alpha p+2)\Gamma(n-k+\beta p)}{\Gamma(n+\alpha p+\beta p+2)} \Bigg] \\ \leq & C\left(\frac{k+1}{n}\right)^{\alpha p}, \qquad 0 \leq k \leq n/2.\end{aligned}$$

Similarly, by Hölder inequality

$$\begin{aligned}\int_0^1 \frac{w(x)^p}{\varphi^{p+1}(x)} p_{n,k}(x)(k-nx)^2\,dx \leq & C \left\{ \int_0^1 \frac{w(x)^p}{\varphi^2(x)} p_{n,k}(x)(k-nx)^2\,dx \right. \\ & \left. \int_0^1 \frac{w(x)^p}{\varphi^{2p}(x)} p_{n,k}(x)(k-nx)^2\,dx \right\}^{1/2} \\ \leq & C \left\{ \left(\frac{k+1}{n}\right)^{\alpha p} \left(\frac{k+1}{n}\right)^{(\alpha-1)p+1} \right\}^{1/2} \\ \leq & C \left(\frac{k+1}{n}\right)^{\alpha p - \frac{p-1}{2}}.\end{aligned}$$

Thus

$$\begin{aligned}\|w\varphi^2 B_n^{*''}(f)\|_p^p \leq & C n^p \left[\|wf\|_p^p + n^p \sum{}' \|wf\|_{L^p(I_k)} + n^p \sum{}' \frac{\|wf\|_{L^p(I_k)}}{(k+1)^{\frac{p-1}{2}}} \right] \\ \leq & C n^p \|wf\|_p^p,\end{aligned}$$

which proves the statement for $1 \leq p < \infty$.

Now let $p = \infty$. By Lemma 2.4

$$nw(x)|B_n^*(f;x)| \leq Cn\|wf\|_\infty.$$

Again, it is sufficient to estimate the sums in (3.5) from 0 to $n/2$. Using (3.2), the corresponding part of the first sum in (3.5) is

$$\leq C\|wf\|_\infty n^\alpha \frac{w(x)}{\varphi^2(x)} \sum{}' p_{n,k}(x) \frac{(k-nx)^2}{(k+1)^\alpha}.$$

Since for $k+1 \leq nx$ we have

$$nx - k = (n+a)x - (k+a) + a(1-x), \qquad \text{where } a = [\alpha]+1,$$

it follows that

$$(nx-k)^2 \le 2[(n+a)x-(k+a)]^2 + 2a^2(1-x)^2, \qquad k+1 \le nx.$$

Thus by the same argument as in (3.4) we obtain

$$\begin{aligned}\frac{1}{\varphi^2(x)}\sum{}'\left(\frac{nx}{k+1}\right)^{\alpha} p_{n,k}(x)(k-nx)^2 \le C\frac{1}{\varphi^2(x)}\Big[&(nx)^{\alpha+2}(1-x)^n \\ &+\sum_{k+1\le nx}\binom{n+a}{k+a}p_{n+a,k+a}(x)(nx-k)^2 \\ &+(1-x)^2\sum_{k=1}^{n}p_{n,k}(x)+\sum_{nx\le k+1\le n+1}p_{n,k}(x)(k-nx)^2\Big] \\ \le C\left[n+\frac{1-(1-x)^n}{x}+n\right] &\le Cn,\end{aligned}$$

where we have made use of (3.6) with $\gamma=0$ and $\gamma=2$.

Similarly, the corresponding part of the second sum in (3.5) is

$$\le C\|wf\|_\infty n^\alpha \frac{w(x)}{\varphi^2(x)}\sum{}' p_{n,k}(x)\frac{|k-nx|}{(k+1)^\alpha}.$$

Since $|k-nx| \le |k+a-(n+a)x| + a(1-x)$,

$$\begin{aligned}\frac{1}{\varphi^2(x)}\sum{}'\left(\frac{nx}{k+1}\right)^{\alpha} p_{n,k}(x)|k-nx| \le C\frac{1}{\varphi^2(x)}\Big[&(nx)^{\alpha+1}(1-x)^n \\ &+\sum_{k+1\le nx}\binom{n+a}{k+a}p_{n+a,k+a}(x)|nx-k| \\ &+(1-x)\sum_{k=1}^{n}p_{n,k}(x)+\sum_{k+1>nx}p_{n,k}(x)|k-nx|\Big] \\ \le C\left[n+\frac{1}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)|k-nx|\right].&\end{aligned}$$

Now if $\varphi^2(x) \ge 1/n$, then using

$$\frac{1}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)|k-nx| \le \frac{\sqrt{n}}{\varphi(x)} \le n$$

(this is a consequence of (3.6)) we are done. If $0 \le x \le C/n$, then

$$\frac{1}{\varphi^2(x)}\sum_{k=0}^{n} p_{n,k}(x)|k-nx| \le n\sum_{k=0}^{n} p_{n,k}(x)(k+C) = n(nx+C) \le 2Cn$$

because of the reproducing property of the Bernstein polynomials. Similar consideration settles the case $1 - C/n \le x \le 1$.

Collecting these estimates, we get (2.6) for $p = \infty$. □

3.3 Proof of Lemma 2.6

First assume $p = \infty$. From the analogue of [2, p. 125, formula (9.4.4)]

$$w(x)B_n^{*''}(f;x) = n(n-1)w(x)\varphi^2(x)\sum_{k=0}^{n-2}\overrightarrow{\Delta}^2 a_{n,k}(f)p_{n-2,k}(x), \tag{3.7}$$

where $\overrightarrow{\Delta}^2 a_{n,k}(f) = a_{n,k+2}(f) - 2a_{n,k+1}(f) + a_{n,k}(f)$.

Again, it will be sufficient to estimate here the sum from 0 to $[n/2]$. Using first order Taylor expansion at $\frac{k+3}{n+1}$ we get by (2.3)

$$\begin{aligned} a_{n,k}(f) =& \frac{1}{d_k}\left[\int_{I_k}\left[f\left(\frac{k+3}{n+1}\right) - \left(t - \frac{k+3}{n+1}\right)f'\left(\frac{k+3}{n+1}\right)\right.\right. \\ & \left.\left. - \int_t^{\frac{k+3}{n+1}} (u-t)f''(u)\,du\right] w(t)\,dt\right]. \end{aligned} \tag{3.8}$$

Here, for $1 \le k \le n/2$,

$$\begin{aligned} &\frac{1}{d_k}\left|\int_{I_k} w(t)\int_t^{\frac{k+3}{n+1}} (u-t)f''(u)\,du\,dt\right| \\ &\le \frac{C}{d_k}\|w\varphi^2 f''\|_\infty \int_{I_k} w(t)\int_t^{\frac{k+3}{n+1}} \frac{u-t}{\varphi^2(u)w(u)}\,du\,dt \\ &\le \frac{C}{n}\|w\varphi^2 f''\|_\infty \int_{\frac{k}{n+1}}^{\frac{k+3}{n+1}} \frac{du}{u^{\alpha+1}} \\ &\le C\|w\varphi^2 f''\|_\infty \frac{n^{\alpha-1}}{(k+1)^{\alpha+1}}, \end{aligned} \tag{3.9}$$

and for $k=0$

$$\begin{aligned}
&\frac{1}{d_0}\left|\int_{I_0} w(t)\int_t^{\frac{3}{n+1}}(u-t)f''(u)\,du\,dt\right| \\
&\leq Cn^{\alpha}\|w\varphi^2 f''\|_\infty \int_{I_0} t^{\alpha}\int_t^{\frac{3}{n+1}}\frac{u-t}{w(u)\varphi^2(u)}\,du\,dt \\
&\leq Cn^{\alpha-1}\|w\varphi^2 f''\|_\infty \int_{I_0} t^{\alpha}\int_t^{\frac{3}{n+1}}\frac{du}{u^{\alpha+1}}\,dt \\
&\leq Cn^{\alpha-1}\|w\varphi^2 f''\|_\infty,
\end{aligned}$$

i.e. (3.9) holds for all $0\leq k\leq n/2$. Hence

$$\begin{aligned}
a_{n,k}(f) =& f\left(\frac{k+3}{n+1}\right) - f'\left(\frac{k+3}{n+1}\right)\frac{1}{d_k}\int_{I_k}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt \\
&+ O\left(\frac{n^{\alpha-1}\|w\varphi^2 f''\|_\infty}{(k+1)^{\alpha+1}}\right), \qquad 0\leq k\leq n/2.
\end{aligned}$$

Analogously

$$\begin{aligned}
a_{n,k+1}(f) =& f\left(\frac{k+3}{n+1}\right) - f'\left(\frac{k+3}{n+1}\right)\frac{1}{d_{k+1}}\int_{I_{k+1}}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt \\
&+ O\left(\frac{n^{\alpha-1}\|w\varphi^2 f''\|_\infty}{(k+1)^{\alpha+1}}\right), \qquad 0\leq k\leq n/2
\end{aligned}$$

and

$$\begin{aligned}
a_{n,k+2}(f) =& f\left(\frac{k+3}{n+1}\right) - f'\left(\frac{k+3}{n+1}\right)\frac{1}{d_{k+2}}\int_{I_{k+2}}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt \\
&+ O\left(\frac{n^{\alpha-1}\|w\varphi^2 f''\|_\infty}{(k+1)^{\alpha+1}}\right), \qquad 0\leq k\leq n/2.
\end{aligned}$$

Thus for $0\leq k\leq n/2$

$$\overrightarrow{\Delta}^2 a_{n,k}(f) = f'\left(\frac{k+3}{n+1}\right)\Delta + O\left(\frac{n^{\alpha-1}\|w\varphi^2 f''\|_\infty}{(k+1)^{\alpha+1}}\right), \tag{3.10}$$

where

$$\begin{aligned}\Delta :=&\frac{1}{d_k}\int_{I_k}\left(t-\frac{k+3}{n}\right)w(t)\,dt\\&-\frac{2}{d_{k+1}}\int_{I_{k+1}}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt\\&+\frac{1}{d_{k+2}}\int_{I_{k+2}}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt\\=&\frac{1}{d_{k+1}}\left\{\int_{I_k}-2\int_{I_{k+1}}+\int_{I_{k+2}}\right\}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt\\&+D_k\int_{I_k}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt\\&-D_{k+1}\int_{I_{k+2}}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt,\end{aligned}\tag{3.11}$$

and $D_k := 1/d_k - 1/d_{k+1}$. Here by (3.1),

$$\begin{aligned}|D_k| \le& C\frac{n^{2\alpha+2}}{(k+1)^{2\alpha}}\left|\left[\int_{I_{k+1}}w(t)\,dt-\int_{I_k}w(t)dt\right]\right|\\\le& C\frac{n^{2\alpha+2}}{(k+1)^{2\alpha}}\int_{I_k}\left|w(t)-w\left(t+\frac{1}{n+1}\right)\right|dt\\\le& C\frac{n^{2\alpha+2}}{(k+1)^{2\alpha}}\int_{I_k}\frac{t^{\alpha-1}}{n+1}\,dt\le C\left(\frac{n}{k+1}\right)^{\alpha+1},\ 0\le k\le n/2.\end{aligned}\tag{3.12}$$

Moreover

$$\int_{J_k}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt\le\frac{C}{n^2}\frac{(k+1)^\alpha}{n^\alpha},\qquad 0\le k\le n/2,\tag{3.13}$$

where $J_k := I_k \cup I_{k+1} \cup I_{k+2}$. Thus (3.11) yields

$$\begin{aligned}\Delta =&\frac{1}{d_{k+1}}\left\{\int_{I_k}-2\int_{I_{k+1}}+\int_{I_{k+2}}\right\}\left(t-\frac{k+3}{n+1}\right)w(t)\,dt\\&+O\left(\frac{1}{n(k+1)}\right),\qquad 0\le k\le n/2.\end{aligned}\tag{3.14}$$

Denoting

$$W(t)=\left(t-\frac{k+3}{n+1}\right)w(t),$$

we have by (3.1)

$$\begin{aligned}\Delta =&\frac{1}{d_{k+1}}\int_{I_{k+1}}\left\{\left(u-\frac{k+4}{n+1}\right)w\left(u-\frac{1}{n+1}\right)-2\left(u-\frac{k+3}{n+1}\right)w(u)\right.\\&\left.-\left(u-\frac{k+2}{n+1}\right)w\left(u+\frac{1}{n+1}\right)\right\}du+O\left(\frac{1}{n(k+1)}\right)\\=&\frac{1}{d_{k+1}}\int_{I_{k+1}}\left[W\left(u-\frac{1}{n+1}\right)-2W(u)+W\left(u+\frac{1}{n+1}\right)\right]du\\&+O\left(\frac{1}{n(k+1)}\right)\\\le&C\frac{n^{\alpha+1}}{(k+1)^{\alpha}}\frac{1}{n^3}|W''(\xi)|+O\left(\frac{1}{n(k+1)}\right)\\\le&C\frac{n^{\alpha-2}}{(k+1)^{\alpha}}\frac{(k+1)^{\alpha-2}}{n^{\alpha-1}}+O\left(\frac{1}{n(k+1)}\right)\\=&O\left(\frac{1}{n(k+1)}\right),\qquad\frac{k}{n+1}<\xi<\frac{k+3}{n+1},\quad 0\le k\le n/2.\end{aligned}$$

Thus (3.10) yields for $0\le k\le n/2$

$$\begin{aligned}\overrightarrow{\Delta}^2a_{n,k}(f)=&f'\left(\frac{k+3}{n+1}\right)O\left(\frac{1}{n(k+1)}\right)+O\left(\frac{n^{\alpha-1}\|w\varphi^2f''\|_\infty}{(k+1)^{\alpha+1}}\right)\\=&(\|wf'\|_\infty+\|w\varphi^2f''\|_\infty)O\left(\frac{n^{\alpha-1}}{(k+1)^{\alpha+1}}\right).\end{aligned}$$

Substituting into (3.7) we get for the first part of the sum

$$\begin{aligned}&(\|wf'\|_\infty+\|w\varphi^2f''\|_\infty)n(n-1)w(x)\varphi^2(x)\sum{}'p_{n-2,k}(x)\frac{n^{\alpha-1}}{(k+1)^{\alpha+1}}\\&\qquad\le C(\|wf'\|_\infty+\|w\varphi^2f''\|_\infty)w(x)\varphi^2(x)\sum{}'p_{n-2,k}(x)\left(\frac{n}{k+1}\right)^{\alpha+1}.\end{aligned}$$

Here

$$w(x)\varphi^2(x)\sum{}'p_{n-2,k}(x)\left(\frac{n}{k+1}\right)^{\alpha+1}$$

$$\le C\left\{\sum_{k+1\le nx}+\sum_{k+1>nx}\right\}\left(\frac{n}{k+1}\right)^{\alpha+1}\binom{n-2}{k}x^{k+\alpha+1}(1-x)^{n-k+\beta+1}.$$

Denoting $a = [\alpha] + 1$ we get

$$\sum_{k+1\le nx} \left(\frac{n}{k+1}\right)^{\alpha+1} \binom{n-2}{k} x^{k+\alpha+1}(1-x)^{n-k+\beta+1}$$
$$\le C \sum_{k=0}^{n-2} \binom{n+a-1}{k+a+1} x^{k+a+1}(1-x)^{n-k-2} \le C,$$

while

$$\sum_{k+1>nx} \left(\frac{n}{k+1}\right)^{\alpha+1} \binom{n-2}{k} x^{k+\alpha+1}(1-x)^{n-k+\beta+1}$$
$$\le \sum_{k=0}^{n-2} \binom{n-2}{k} x^{k}(1-x)^{n-k-2} \le 1.$$

This proves the assertion for $p = \infty$.

Now let $1 \le p < \infty$. First note that in (3.8) the absolute value of the last term is

$$\begin{aligned}
&\le \frac{1}{d_k} \int_{I_k} \int_t^{\frac{k+3}{n+1}} \frac{u-t}{u^{\alpha+1}} w(u)\varphi^2(u)|f''(u)|\,du\, w(t)\,dt \\
&\le C \frac{n^{\alpha+1}}{(k+1)^{\alpha}} \int_{I_k} \frac{1}{t^{\alpha}} \frac{\frac{k+3}{n+1}-t}{\frac{k+3}{n+1}} t^{\alpha}\,dt \int_{J_k} w(u)\varphi^2(u)|f''(u)|\,du \\
&\le C \frac{n^{\alpha}}{(k+1)^{\alpha+1}} \|w\varphi^2 f''\|_{L_1(J_k)} \\
&\le C \frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}} \|w\varphi^2 f''\|_{L_p(J_k)}.
\end{aligned}$$

Hence

$$\begin{aligned}
a_{n,k}(f) =& f\left(\frac{k+3}{n+1}\right) - \frac{1}{d_k} \int_{I_k} \left(t - \frac{k+3}{n+1}\right) f'(t) w(t)\,dt \\
&+ O\left(\frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}} \|w\varphi^2 f''\|_{L_p(J_k)}\right), \qquad 0 \le k \le n/2.
\end{aligned}$$

Working as in case of $p=\infty$ we have

$$\begin{aligned}\overrightarrow{\Delta}^2 a_{n,k}(f) =&\frac{1}{d_k}\int_{I_k}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ &-\frac{2}{d_{k+1}}\int_{I_{k+1}}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ &+\frac{1}{d_{k+2}}\int_{I_{k+2}}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ &+O\left(\frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}}\|w\varphi^2 f''\|_{L_p(J_k)}\right),\qquad 0\le k\le n/2.\end{aligned}\tag{3.15}$$

Now, similarly to (3.11),

$$\begin{aligned}\tilde{\Delta} :=&\frac{1}{d_k}\int_{I_k}\left(t-\frac{k+3}{n}\right)f'(t)w(t)\,dt\\ &-\frac{2}{d_{k+1}}\int_{I_{k+1}}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ &+\frac{1}{d_{k+2}}\int_{I_{k+2}}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ =&\frac{1}{d_{k+1}}\left\{\int_{I_k}-2\int_{I_{k+1}}+\int_{I_{k+2}}\right\}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ &+D_k\int_{I_k}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\\ &-D_{k+1}\int_{I_{k+2}}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt.\end{aligned}\tag{3.16}$$

Here, using (3.12), the last two terms can be estimated (similarly to (3.13)) by

$$C\left(\frac{n}{k+1}\right)^{\alpha+1}\int_{J_k}\left(t-\frac{k+3}{n+1}\right)f'(t)w(t)\,dt\le C\frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}}\|wf'\|_{L_p(J_k)}.$$

Thus (3.16) yields for $0 \le k \le n/2$

$$\begin{aligned}
\tilde{\Delta} =& \frac{1}{d_{k+1}} \left\{ \int_{I_k} -2 \int_{I_{k+1}} + \int_{I_{k+2}} \right\} f'(t) W(t)\, dt + O\left(\frac{n^\alpha}{(k+1)^{\alpha+1}} \right) \|wf'\|_{L_1(J_k)} \\
=& \frac{1}{d_{k+1}} \int_{I_{k+1}} \left[f'\left(t - \frac{1}{n+1} \right) W\left(t - \frac{1}{n+1} \right) - 2f'(t)W(t) \right] \\
& + f'\left(t + \frac{1}{n+1} \right) W\left(t + \frac{1}{n+1} \right) \Bigg] \, dt + O\left(\frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}} \right) \|wf'\|_{L_p(J_k)}.
\end{aligned}$$

Now

$$\begin{aligned}
& \left| \int_{I_{k+1}} \left[f'\left(t - \frac{1}{n+1} \right) W\left(t - \frac{1}{n+1} \right) - 2f'(t)W(t) \right. \right. \\
& \left. \left. + f'\left(t + \frac{1}{n+1} \right) W\left(t + \frac{1}{n+1} \right) \right] \, dt \right| \\
\le & \int_{I_{k+1}} \left| f'\left(t - \frac{1}{n+1} \right) \right| \left| W\left(t - \frac{1}{n+1} \right) - 2W(t) + W\left(t + \frac{1}{n+1} \right) \right| \, dt \\
& + 2 \left| \int_{I_{k+1}} W(t) \left[f'\left(t - \frac{1}{n+1} \right) - f'(t) \right] \, dt \right| \\
& + \left| \int_{I_{k+1}} W\left(t + \frac{1}{n+1} \right) \left(t - \frac{1}{n+1} \right) \left[f'\left(t + \frac{1}{n+1} \right) - f'\left(t - \frac{1}{n+1} \right) \right] \, dt \right| \\
\le & \frac{C}{n^2} \int_{I_{k+1}} \left| f'\left(t - \frac{1}{n+1} \right) W''(\xi) \right| \, dt + \frac{C}{n} \int_{I_{k+1}} w\left(t + \frac{1}{n+1} \right) \int_{t - \frac{1}{n+1}}^{t + \frac{1}{n+1}} f''(u)\, du\, dt \\
\le & \frac{C}{n^2} \frac{1}{n} \left(\frac{n}{k+1} \right)^2 \|wf'\|_{L_1(J_k)} + \frac{C}{n} \frac{n}{k+1} \|w\varphi^2 f''\|_{L_1(J_k)} \\
= & \frac{C}{n^{2-1/p}(k+1)} (\|wf'\|_{L_p(J_k)} + \|w\varphi^2 f''\|_{L_p(J_k)}), \qquad 0 \le k \le n/2.
\end{aligned}$$

Thus

$$\tilde{\Delta} = O\left(\frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}} \right) (\|wf'\|_{L_p(J_k)} + \|w\varphi^2 f''\|_{L_p(J_k)}), \qquad 0 \le k \le n/2,$$

which yields from (3.15) for $0 \le k \le n/2$,

$$\overrightarrow{\Delta}^2 a_{n,k}(f) = O\left(\frac{n^{\alpha-1+1/p}}{(k+1)^{\alpha+1}} \right) \left(\|wf'\|_{L_p(J_k)} + \|w\varphi^2 f''\|_{L_p(J_k)} \right).$$

Finally, by Hölder inequality and $\sum' p_{n-2,k}(x) \le 1$ we get

$$\begin{aligned}
&\int_0^1 \left[n(n-1)\ w(x)\varphi^2(x) \sum{}' |\overrightarrow{\Delta}^2 a_{n,k}(f)| p_{n-2,k}(x)\right]^p dx \\
&\le C n^{(\alpha+1)p+1} \\
&\quad \times \sum{}' \int_0^1 \frac{1}{(k+1)^{(\alpha+1)p}} \binom{n-2}{k} x^{k+(\alpha+1)p} (1-x)^{n-k-2+(\beta+1)p}\, dx \\
&\quad \times (\|wf'\|_{L_p(J_k)} + \|w\varphi^2 f''\|_{L_p(J_k)})^p \\
&= C \sum{}' \frac{n^{(\alpha+1)p+1}\Gamma(n-1)\Gamma(k+(\alpha+1)p+1)\Gamma(n-k-1+(\beta+1)p)}{(k+1)^{(\alpha+1)p}\Gamma(n+(\alpha+\beta+2)p)\Gamma(k+1)\Gamma(n-k-1)} \\
&\quad \times (\|wf'\|_{L_p(J_k)} + \|w\varphi^2 f''\|_{L_p(J_k)})^p \\
&\le C \sum{}' \frac{k^{(\alpha+1)p}(n-k)^{(\beta+1)p}}{k^{(\alpha+1)p} n^{(\beta+1)p}} (\|wf'\|_{L_p(J_k)} + \|w\varphi^2 f''\|_{L_p(J_k)})^p \\
&\le C(\|wf'\|_p + \|w\varphi^2 f''\|_p)^p.
\end{aligned}$$

The other part of the sum can be estimated similarly, and this proves the lemma for $1 \le p < \infty$. □

3.4 Proof of Theorem 2.1

It is sufficient to prove

$$\|w[B_n^*(f) - f]\|_p \le \frac{C_{p,w}}{n}[\|w\varphi^2 f''\|_p + \|wf'\|_p] \tag{3.17}$$

for $w\varphi^2 f'' \in L^p([0,1])$. Namely, (2.8) and (3.17) imply

$$\|w[B_n^*(f) - f]\|_p \le \frac{C_{p,w}}{n}[\|w\varphi^2 f''\|_p + \|wf\|_p].$$

Here, using the equivalence of the K-functional

$$K_{2,\varphi}(f, t^2)_{w,p} = \inf_{g' \in \mathrm{AC}[0,1]} \{\|w(f-g)\|_p + t^2 \|w\varphi^2 g]|_p\} \tag{3.18}$$

and modulus of smoothness (2.1) (cf. [2, Theorem 6.1.1,]), hence Theorem 2.1 follows by usual arguments.

Following [2, Section 10.5, p. 168] by (2.5) applied to the function $f(x) = x$, we get

$$\|w[B_n^*(f) - f]\|_p \le \frac{\|wf'\|_p}{n} + \|wB_n^*(R)\|_p,$$

where

$$R(f;t,x) := \int_x^t (t-v)f''(v)\,dv.$$

Case 1: $p = 1$. With the notation $g(v) := w(v)\varphi^2(v)f''(v)$ we get for $t \in I_k$, $0 \le k \le n/2$

$$|R(f;t,x)| \le \left| \int_x^{\frac{k^\star}{n+1}} g(v)\,dv \right| |t-x| \left| \frac{1}{w(x)\varphi^2(x)} + \frac{1}{t^{\alpha+1}} \right|,$$

where $k^\star$ is either k or $k+1$, depending on which value makes the interval of integration larger. Hence for $0 \le k \le n/2$

$$\begin{aligned} &\frac{w(x)}{d_k} \left| \int_{I_k} w(t)R(f;t,x)\,dt \right| \\ &\quad \le C \left| \int_x^{\frac{k^\star}{n+1}} g(v)\,dv \right| \left| \frac{k^\star}{n+1} - x \right| \varphi^{-2}(x) \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right]. \end{aligned} \tag{3.19}$$

Thus with the notation $E_n := [1/n, 1-1/n]$ we have

$$\begin{aligned} \|wB_n^\star(R)\|_1 &\le \int_0^1 w(x) \sum_{k=0}^n \frac{p_{n,k}(x)}{d_k} \left| \int_{I_k} w(t)R(f;t,x)\,dt \right| dx \\ &\le C \int_0^1 {\sum_{}}' p_{n,k}(x) \left| \frac{k^\star}{n+1} - x \right| \varphi^{-2}(x) \left| \int_x^{\frac{k^\star}{n+1}} g(v)\,dv \right| \\ &\quad \times \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right] dx + S \\ &= \int_{E_n} {\sum_{}}' p_{n,k}(x) \left| \frac{k^\star}{n+1} - x \right| \varphi^{-2}(x) \left| \int_x^{\frac{k^\star}{n+1}} g(v)\,dv \right| \\ &\quad \times \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right] dx + C\frac{\|g\|_1}{n} \int_0^{1/n} \sum_{k=1}^n \frac{p_{n,k}(x)}{x}\,dx \\ &\quad + n^{\alpha+1} \int_0^{1/n} x^\alpha \int_0^{1/n} t^\alpha \left| \int_t^x (t-v)f''(v)\,dv \right| dt\,dx \\ &\quad + C\|g\|_1 \int_{1-1/n}^1 \sum_{k=0} np_{n,k}(x)\,dx + S \\ &= \sum_{i=1}^5 I_i + S, \end{aligned} \tag{3.20}$$

where S denotes the integral of the other part of the sum which can be similarly estimated.

For the first integral, we further split the first sum into two parts:

$$I_1 = \sum_{0\le k\le nx} + \sum_{nx\le k\le n/2} =: A_1 + A_2. \tag{3.21}$$

Similarly to the proof of Lemma 1 we obtain, with $b := [\alpha] + 2$,

$$\begin{aligned} A_1 \le & C \int_{E_n} \sum_{k=0}^{n} p_{n,k}(x) \left| \frac{k^\star}{n+1} - x \right| \varphi^{-2}(x) \left| \int_x^{\frac{k^\star}{n+1}} g(v)\, dv \right| \left(\frac{nx}{k+1} \right)^b dx \\ \le & C \sum_{k=0}^{n} \int_{E_n} p_{n+b,k+b}(x) \left(\left| \frac{(k+b)^\star}{n+b+1} - x \right| + \frac{1}{n+b} \right) \varphi^{-2}(x) \\ & \times \left(\left| \int_x^{\frac{(k+b)^\star}{n+b+1}} g(v)\, dv \right| + \left| \int_{\frac{k^\star}{n+1}}^{\frac{(k+b)^\star}{n+b+1}} g(v)\, dv \right| \right) dx. \end{aligned}$$

Here we split again:

$$\int_{E_n} = \int_{E_n \setminus \left\{ x:\, \left| x - \frac{(k+b)^\star}{n+b+1} \right| \le \frac{(k+b)^\star}{n+b+1} \right\}} + \int_{\left\{ x:\, \left| x - \frac{(k+b)^\star}{n+b+1} \right| \le \frac{(k+b)^\star}{n+b+1} \right\}} =: B_1 + B_2$$

and get

$$\begin{aligned} B_1 \le & C \sum_{k=0}^{n} \int_{E_n \setminus \left\{ x:\, \left| x - \frac{(k+b)^\star}{n+b+1} \right| \le \frac{(k+b)^\star}{n+b+1} \right\}} p_{n+b,k+b}(x) \left(\left| \frac{(k+b)^\star}{n+b+1} - x \right| + \frac{1}{n+b} \right) \\ & \times \varphi^{-2}(x) \left| \int_x^{\frac{(k+b)^\star}{n+b+1}} g(v)\, dv \right| dx \\ \le & C \sum_{k=0}^{n} \int_{E_n} p_{n+b,k+b}(x) \left(\left| \frac{(k+b)^\star}{n+b+1} - x \right| + \frac{1}{n+b} \right) \varphi^{-2}(x) \\ & \times \left| \int_x^{\frac{(k+b)^\star}{n+b+1}} g(v)\, dv \right| dx. \end{aligned}$$

The last integral is of the same form as (10.5.6) in [2] (n and k replaced by $n+b$ and $k+b$, resp.), and can be estimated by $\frac{C}{n}\|g\|_1$.

As for B_2, we get

$$\begin{aligned}
B_2 \leq & C \sum_{k=0}^{n} \int_{\left\{x:\left|x-\frac{(k+b)^\star}{n+b+1}\right|\leq\frac{(k+b)^\star}{n+b+1}\right\}} p_{n+b,k+b}(x) \left(\left|\frac{(k+b)^\star}{n+b+1}-x\right|+\frac{1}{n+b}\right) \\
& \times \varphi^{-2}(x) \left| \int_{\frac{k^\star}{n+1}}^{\frac{(k+b)^\star}{n+b+1}} g(v)\,dv \right| dx \\
\leq & C \sum_{k=0}^{n} \int_{0}^{1} p_{n+b,k+b}(x) \left(\left|\frac{(k+b)^\star}{n+b+1}-x\right|+\frac{1}{n+b}\right) \\
& \times \varphi^{-2}(x) \left| \int_{\frac{k^\star}{n+1}}^{\frac{(k+b)^\star}{n+b+1}} g(v)\,dv \right| dx \\
\leq & C \sum_{k=0}^{n} (x) \left(\left|\frac{(k+b)^\star}{n+b+1}-x\right|+\frac{1}{n+b}\right) \varphi^{-2}(x) \left| \int_{\frac{k^\star}{n+1}}^{\frac{(k+b)^\star}{n+b+1}} g(v)\,dv \right| dx \\
\leq & \frac{C}{n} \sum_{k=0}^{n} \left(\left|\frac{(k+b)^\star}{n+b+1}-x\right|+\frac{1}{n+b}\right) \varphi^{-2}(x) \left| \int_{\frac{k^\star}{n+1}}^{\frac{(k+b)^\star}{n+b+1}} g(v)\,dv \right| \\
\leq & \frac{C}{n} \|g\|_1,
\end{aligned}$$

where we have used the estimate

$$\int_0^1 p_{n,k}(x) \left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right) dx \leq \frac{C}{n},$$

which can be easily verified by Hölder's inequality.

The estimate of A_2 in (3.21) is simpler:

$$A_2 \leq C \int_{E_n} \sum_{k=0}^{n} p_{n,k}(x) \left|\frac{k^\star}{n+1}-x\right| \varphi^{-2}(x) \left| \int_x^{\frac{k^\star}{n+1}} g(v)\,dv \right| dx \leq \frac{C}{n}\|g\|_1,$$

again with reference to (10.5.6) of [2].

Now we estimate the second integral, I_2 in (3.20). Since for $1 \leq k \leq n$,

$$\int_0^{1/n} \frac{p_{n,k}(x)}{x}\,dx \leq \binom{n}{k} \int_0^{1/n} x^{k-1}\,dx \leq C\binom{n}{k}\frac{1}{n^k} \leq \frac{C}{k!}$$

it follows that

$$I_2 \leq \frac{C}{n} \sum_{k=1}^{n} \frac{1}{(k-1)!} \left| \int_0^1 g(v)\,dv \right| dx \leq \frac{C}{n}\|g\|_1.$$

Next we estimate I_3, by integration by parts:

$$\begin{aligned}
I_3 \le & Cn^{\alpha+1} \int_0^{1/n} x^\alpha \int_0^{1/n} t^\alpha \left[|t-x||f'(x)| + \left| \int_x^t f'(v)\,dv \right| \right] dt\,dx \\
\le & Cn^{\alpha+1} \left[\frac{1}{n} \int_0^{1/n} x^\alpha |f'(x)|\,dx \int_0^{1/n} t^\alpha\,dt \right. \\
& \left. + \int_0^{1/n} \int_0^{1/n} (x^\alpha + t^\alpha) \left| \int_x^t v^\alpha |f'(v)|\,dv \right| dt\,dx \right] \\
\le & \frac{C}{n} \|wf'\|_1.
\end{aligned}$$

Evidently, $I_4 \le \frac{C}{n}\|g\|_1$. Collecting our estimates, we get the statement for $p = 1$.

Case 2: $1 < p \le \infty$. With $M(g,x)$ denoting the maximal function of g, (3.19) implies

$$\begin{aligned}
\|wB_n^\star(R(f;t,\cdot))\|_p \le & C \left\| \sum{}' \left| \int_x^{\frac{k^\star}{n+1}} g(v)\,dv \right| \left| \frac{k^\star}{n+1} - x \right| \varphi^{-2}(x) \right. \\
& \left. \times \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right] p_{n,k}(x) \right\|_p + S \\
\le & C \left\| M(g) \sum{}' \left(\frac{k^\star}{n+1} - x \right)^2 \varphi^{-2}(x) \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right] p_{n,k}(x) \right\|_p + S \\
\le & \left\| \sum{}' \left[\left(\frac{k}{n+1} - x \right)^2 + \frac{1}{n^2} \right] \varphi^{-2}(x) \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right] p_{n,k}(x) \right\|_\infty \\
& \times \|M(g)\|_p + S,
\end{aligned}$$

where S is the other part of the sum which can be estimated similarly to $\sum'$. Using the maximal function inequality we get

$$\|wB_n^\star(R(f;\cdot))\|_p \le CA_n\|g\|_p + S,$$

where

$$\begin{aligned}
A_n := & \left\| \sum{}' \left[\left(\frac{k}{n+1} - x \right)^2 + \frac{1}{n^2} \right] \varphi^{-2}(x) \left[1 + \left(\frac{nx}{k+1} \right)^{\alpha+1} \right] p_{n,k}(x) \right\|_\infty \\
:= & A_{1n} + A_{2n}.
\end{aligned}$$

Here using (3.6) with $\gamma = 2$, we get $A_{1n} \le C/n$. As for A_{2n}, it can be settled by using the technics before: denoting $b := [\alpha] + 2$ we have

$$\begin{aligned} A_{2n} =& \left\| \sum{}' \left[\left(\frac{k}{n+1} - x \right)^2 + \frac{1}{n^2} \right] \varphi^{-2}(x) \left(\frac{nx}{k+1} \right)^{\alpha+1} p_{n,k}(x) \right\|_\infty \\ \le& C \left\| \sum_{k=0}^{n} \left[\left(\frac{k+b}{n+b+1} - x \right)^2 + \frac{1}{n^2} \right] \varphi^{-2}(x) \left(\frac{nx}{k+1} \right)^{b} p_{n,k}(x) \right\|_\infty \\ \le& C \left\| \sum_{k=0}^{n} \left[\left(\frac{k+b}{n+b+1} - x \right)^2 + \frac{1}{n^2} \right] \varphi^{-2}(x) p_{n+b,k+b}(x) \right\|_\infty \\ \le& C/n. \end{aligned}$$

This proves the statement for $1 < p \le \infty$, and completes the proof of Theorem 2.1. □

3.5 Proof of Theorem 2.2

We use the argument of proof of Theorem 9.3.2 in [2]. By the definition (3.18) of the K-functional, as well as Lemmas 2.5 and 2.6, we get

$$\begin{aligned} K_{2,\varphi}(f, 1/n)_{w,p} \le& \|w(f - B_n^*(f))\|_p + n^{-1} \|w\varphi^2 B_n^{*"}(f)\|_p \\ \le& \|w(f - B_n^*(f))\|_p + n^{-1} [\|w\varphi^2 B_n^{*"}(f - g)\|_p + \|w\varphi^2 B_n^{*"}(g)\|_p] \\ \le& \|w(f - B_n^*(f))\|_p + C \frac{k}{n} [\|w(f - g)\|_p + k^{-1} \|w(g)\|_p] \\ \le& \|w(f - B_n^*(f))\|_p + C \frac{k}{n} K_{2,\varphi}(f, 1/k)_{w,p} \end{aligned}$$

provided g is the function minimizing the last K-functional. Now applying the Berens–Lorentz lemma (cf. Lemma 9.3.4 in [2]), we obtain the "⇒" part of Theorem 2.2. The "⇐" part follows from Theorem 1. □

3.6 Proof of Theorem 2.3

Letting

$$f_n(x) = \frac{\int_x^{x+1/n} (wf)(t)\, dt}{\int_x^{x+1/n} w(t) dt}, \qquad 0 < x < 1,$$

it is easy to see that for sufficiently large n,

$$\begin{aligned} f_n(x) =& \frac{\int_x^{x+1/n} w(t)[f(x)+(t-x)f'(x)+(t-x)^2 f''(\xi_n)/2]\,dt}{\int_x^{x+1/n} w(t)\,dt} \\ =& f(x)+f'(x)\frac{\int_x^{x+1/n} w(t)(t-x)\,dt}{\int_x^{x+1/n} w(t)\,dt}+\frac{\|f''\|_\infty}{2n^2} \\ =& f(x)+f'(x)w(\xi_n)\frac{\int_x^{x+1/n}(t-x)\,dt}{w(\eta_n)/n}+O\left(\frac{1}{n^2}\right) \\ =& f(x)+f'(x)\frac{w(\xi_n)}{w(\eta_n)}\frac{1}{2n}+O\left(\frac{1}{n^2}\right),\ x<\xi_n,\ \eta_n<x+\frac{1}{n}. \end{aligned}$$

Hence, if $0<x<1$ is fixed, then

$$\left|f_n(x)-f(x)-\frac{f'(x)}{2n}\right| \le \left|1-\frac{w(\xi_n)}{w(\eta_n)}\right|\frac{|f'(x)|}{n}+O\left(\frac{1}{n^2}\right)=o\left(\frac{1}{n}\right),$$

that is

$$f_n(x)=f(x)+\frac{f'(x)}{2n}+o\left(\frac{1}{n}\right). \tag{3.22}$$

Differentiating the identity

$$f_n(x)\int_x^{x+1/n} w(t)\,dt=\int_x^{x+1/n}(wf)(t)\,dt,$$

it follows that

$$\begin{aligned} f_n(x)\left[w\left(x+\frac{1}{n}\right)-w(x)\right]+f_n'(x)\int_x^{x+1/n} w(t)\,dt& \\ =(wf)\left(x+\frac{1}{n}\right)-(wf)(x).& \end{aligned} \tag{3.23}$$

Dividing (3.23) by $1/n$ and letting n tend to infinity we get

$$f(x)w'(x)+w(x)\lim_{n\to\infty} f_n'(x)=(wf)'(x),\quad 0<x<1.$$

and consequently

$$\lim_{n\to\infty} f_n'(x)=f'(x),\qquad 0<x<1.$$

By differentiating (3.23),

$$\begin{aligned} f_n''(x)\int_x^{x+1/n} w(t)\,dt + 2f_n'(x)\left[w\left(x+\frac{1}{n}\right)-w(x)\right] \\ + f_n(x)\left[w'\left(x+\frac{1}{n}\right)-w'(x)\right] \\ =(wf)'\left(x+\frac{1}{n}\right)-(wf)'(x) \end{aligned} \tag{3.24}$$

and again dividing by $1/n$ and letting n tend to ∞ we get

$$\lim_{n\to\infty} f_n''(x)w(x)+2f'(x)w'(x)+f(x)w''(x)=(wf)''(x).$$

Hence

$$\lim_{n\to\infty} f_n''(x)=f''(x),\ 0<x<1,$$

Now following [3, p. 22], with $\eta(\delta)\to 0$, when $\delta\to 0$,

$$\begin{aligned} f_n\left(\frac{k}{n}\right) =& f_n(x)+\left(\frac{k}{n}-x\right)f_n'(x)+\frac{1}{2}\left(\frac{k}{n}-x\right)^2\left[f_n''(x)+\eta\left(\left|x-\frac{k}{n}\right|\right)\right] \\ =& f_n(x)+\left(\frac{k}{n-1}-x\right)f_n'(x)+\frac{1}{2}\left(\frac{k}{n-1}-x\right)^2\left[f_n''(x)\right. \\ &\left.+\eta\left(\left|x-\frac{k}{n}\right|\right)\right]-\frac{k}{n(n-1)}f_n'(x) \\ &+\left[\frac{x}{n}\frac{k}{n-1}-\frac{2n-1}{2n^2}\frac{k^2}{(n-1)^2}\right]f_n''(x)+O\left(\frac{1}{n}\right)\eta\left(\left|x-\frac{k}{n}\right|\right), \end{aligned}$$

where we have used the fact that

$$\sum_{k=0}^{n}\eta\left(\left|\frac{k}{n}-x\right|\right)p_{n,k}(x)=o(1).$$

Hence by the preservation property of linear functions for Bernstein polynomials

$$\begin{aligned} B_{n-1}^*(f;x) =& B_n(f_n;x)=f_n(x)+x\frac{1-x}{2(n-1)}f_n''(x)-\frac{x}{n}f_n'(x) \\ &+\left[\frac{x^2}{n}-\frac{2n-1}{2n^2}\left(x^2+O\left(\frac{1}{n}\right)\right)\right]f''(x)+o\left(\frac{1}{n}\right) \\ =& f_n(x)-\frac{x}{n}f_n'(x)+\frac{x(1-x)}{2(n-1)}f_n''(x)+o\left(\frac{1}{n}\right) \end{aligned}$$

from which by (3.22)

$$\lim_{n\to\infty} n[B^*_{n-1}(f;x) - f(x)] = \frac{1-2x}{2}f'(x) + \frac{x(1-x)}{2}f''(x),\ 0 < x < 1.$$

Finally, let $x = 0$. Applying again Taylor expansion we get for $0 < \xi_n < 1/n$

$$\begin{aligned}
n[B^*_{n-1}(f;0) - f(0)] =& n\left[\frac{\int_0^{1/n}(wf)(t)\,dt}{\int_0^{1/n} w(t)\,dt} - f(0)\right] \\
=& n\frac{\int_0^{1/n}[f(t)-f(0)]w(t)\,dt}{\int_0^{1/n} w(t)\,dt} \\
=& n\frac{\int_0^{1/n}[tf'(0) + \frac{1}{2}t^2 f''(\xi_n)]\,dt}{\int_0^{1/n} w(t)\,dt} \\
=& nf'(0)\frac{\int_0^{1/n} tw(t)\,dt}{\int_0^{1/n} w(t)\,dt} + O\left(\frac{\|f''\|_\infty}{n}\right) \\
=& nf'(0)\left(\frac{1-a_n}{1-b_n}\right)^\beta \frac{\int_0^{1/n} t^{\alpha+1}\,dt}{\int_0^{1/n} t^\alpha\,dt} + O\left(\frac{\|f''\|_\infty}{n}\right) \\
=& \frac{\alpha+1}{\alpha+2}f'(0)\left(\frac{1-a_n}{1-b_n}\right)^\beta + O\left(\frac{\|f''\|_\infty}{n}\right),
\end{aligned}$$

where $0 < a_n,\ b_n < 1/n$, and thus here the right hand side tends to $\frac{\alpha+1}{\alpha+2}f'(0)$ as $n \to \infty$. □

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On the numerical solution of integral equations with logarithmic kernels

Peter Junghanns and Giovanni Monegato

Dedicated to the 70th birthday of our friend Giuseppe Mastroianni

Abstract. The convergence of a collocation and a collocation-quadrature method for the numerical solution of integral equations of the form

$$-\frac{1}{\pi}\int_{-1}^{1}\left[(y-x)^{\kappa}\ln|y-x|-h(x,y)\right]\frac{u(y)\,dy}{\sqrt{1-y^2}}=f(x)\,,\ -1<x<1\,,$$

where $\kappa=0,1,2$, is considered. The main ideas are presented in cases $\kappa=0$ and $\kappa=1$ and can be easily translated to case $\kappa=2$.

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1 Introduction

In [4] the mechanical response of a 2D plate, supported along a line given by a function $\delta(x)$ defined on the segment $[-1,1]$ and subject to a transverse concentrated force, is examined by reducing the problem to an integral equation. In particular when the (unknown) distributed contact reaction is assumed to be described by an integrable distributed shear force $g(y)$, the

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following governing integral equation has been derived

$$\int_{-1}^{1} g(y)(x-y)^2 \ln|x-y|\, dy = h(x) - 8\pi D\delta(x) + a_2 + b_2 x \qquad (1.1)$$

where $h(x)$ is a known function depending on the kind of plate loading, D is the (known) plate flexural rigidity and a_2, b_2 are unknown constants, having a precise physical meaning, to be determined together with the function $g(y)$.

In that paper, an alternative situation is examined in which the plate is again supported along a line defined on a segment of length 2, but the distributed contact reaction is assumed to be described by a distributed (integrable) twisting moment m. In this case the governing integral equation takes the form

$$\int_{-1}^{1} m(y)(x-y)[1+2\ln|x-y|]\, dy = h(x) - 8\pi D\delta(x) + a_1 \qquad (1.2)$$

where a_1 is an unknown constant, to be determined together with $m(y)$.

As shown in [3], for arbitrarily given values of the constants a_2, b_2, a_1 equations (1.1), (1.2) do not have integrable solutions $g(y), m(y)$, respectively. These equations have unique integrable solutions if and only if their right hand side constants take uniquely defined values, that ought to be determined together with the corresponding solutions.

To examine the existence and uniqueness of solutions of these equations, in [3] integral operators of the form

$$(\mathcal{A}_\kappa u)(x) = -\frac{1}{\pi}\int_{-1}^{1} (y-x)^\kappa \ln|y-x| \,\frac{u(y)\,dy}{\sqrt{1-y^2}}, \quad -1 < x < 1,$$

are considered in weighted Sobolev spaces $\mathbf{L}_\sigma^{2,s}$ for $\kappa = 0, 1, 2$. Here, for $s \geq 0$, $\mathbf{L}_\sigma^{2,s}$ denotes the Hilbert space of all $f : (-1,1) \longrightarrow \mathbb{C}$ satisfying

$$\|f\|_{\sigma,s} := \sqrt{\sum_{n=0}^{\infty} (n+1)^{2s} \left|\langle f, p_n^\sigma\rangle_\sigma\right|^2} < \infty$$

equipped with the inner product

$$\langle f, g\rangle_{\sigma,s} = \sum_{n=0}^{\infty} (n+1)^{2s} \langle f, p_n^\sigma\rangle_\sigma \overline{\langle g, p_n^\sigma\rangle_\sigma},$$

where $\sigma(x) = (1-x^2)^{-\frac{1}{2}}$, $\langle f, g\rangle_\sigma = \int_{-1}^{1} f(x)\overline{g(x)}\sigma(x)\,dx$, and $p_n^\sigma(x)$ denotes the n th normalized Chebyshev polynomial of first kind,

$$p_0^\sigma(x) = \frac{1}{\sqrt{\pi}}, \quad p_n^\sigma(\cos s) = \sqrt{\frac{2}{\pi}}\cos(ns).$$

Note that $\mathbf{L}_\sigma^{2,0} = \mathbf{L}_\sigma^2$. For $s < 0$, $\mathbf{L}_\sigma^{2,s}$ is defined as the dual space $\left(\mathbf{L}_\sigma^{2,-s}\right)^*$. In addition to these spaces we use the sequence spaces

$$\ell_s^2 = \left\{\xi = (\xi_n)_{n=0}^\infty : \sum_{n=0}^\infty |\xi_n|^2(n+1)^{2s} < \infty\right\}, \quad s \in \mathbb{R},$$

equipped with the inner product $\langle \xi, \eta\rangle_s = \sum_{n=0}^\infty (n+1)^{2s}\xi_n\overline{\eta_n}$ and the norm $\|\xi\|_s = \sqrt{\langle \xi, \xi\rangle_s}$. Clearly, for $s \geq 0$, the operator $\mathcal{J} : \mathbf{L}_\sigma^{2,s} \longrightarrow \ell_s^2$, $f \mapsto (\langle f, p_n^\sigma\rangle_\sigma)_{n=0}^\infty$ is an isometrical isomorphism. Moreover, the dual space $\left(\mathbf{L}_\sigma^{2,s}\right)^*$ is isometric isomorphic to ℓ_{-s}^2 via the map $\mathcal{J} : \left(\mathbf{L}_\sigma^{2,s}\right)^* \longrightarrow \ell_{-s}^2$, $v \mapsto (v(p_n^\sigma))_{n=0}^\infty$ (cf. [3, Section 5]). This allows us to consider the operators $\mathcal{A}_\kappa$ in all spaces $\mathbf{L}_\sigma^{2,s}$, $s \in \mathbb{R}$ by identifying with the operator $\mathcal{J}^{-1}\mathbf{A}_\kappa\mathcal{J}$, where $\mathbf{A}_\kappa\xi$ is defined for any complex number sequence $\xi = (\xi_n)_{n=0}^\infty$ by the condition $\mathbf{A}_\kappa\xi = \mathcal{J}\mathcal{A}_\kappa\mathcal{J}^{-1}\xi$ for all $\xi \in \ell^2$ (see also Prop. 1.1).

Let us summarize some results from [3] which are important for our investigations here.

Proposition 1.1 *The operators $\mathcal{A}_\kappa : \mathbf{L}_\sigma^{2,s} \longrightarrow \mathbf{L}_\sigma^{2,s+\kappa+1}$ are linear and bounded operators. Moreover:*

(a) *The operator $\mathcal{A}_0 : \mathbf{L}_\sigma^{2,s} \longrightarrow \mathbf{L}_\sigma^{2,s+1}$ is invertible and $\mathbf{A}_0\xi = (\alpha_n\xi_n)_{n=0}^\infty$ with $\alpha_0 = \ln 2$ as well as $\alpha_n = \frac{1}{n}$, $n > 0$,*

(b) *For $\kappa = 1, 2$, the image of the operator $\mathcal{A}_\kappa : \mathbf{L}_\sigma^{2,s} \longrightarrow \mathbf{L}_\sigma^{2,s+\kappa+1}$ is not closed. Furthermore,*

$$\mathbf{A}_1\xi = (\beta_n\xi_{n+1} - \beta_{n-1}\xi_{n-1})_{n=0}^\infty$$

with $\xi_{-1} := 0$, $\beta_{-1} := 0$, $\beta_0 = \dfrac{\ln 2 - 1}{\sqrt{2}}$, $\beta_n = \dfrac{1}{2n(n+1)}$, and

$$\mathbf{A}_2\xi = (\eta_n)_{n=0}^\infty$$

with $\eta_0 = -\gamma_0\xi_0 + \gamma_1\xi_2$, $\eta_1 = -\gamma\xi_1 + \gamma_2\xi_3$,

$$\eta_n = \gamma_{n-1}\xi_{n-2} - 2\gamma_n\xi_n + \gamma_{n+1}\xi_{n+2},\ n > 1\,,$$

and

$$\gamma = \ln 2 - \frac{5}{4}\,,\ \gamma_0 = 1 - \ln 2\,,\ \gamma_1 = \frac{2\ln 2 - 3}{4\sqrt{2}}\,,$$

$$\gamma_n = \frac{1}{2(n-1)n(n+1)},\ n > 1\,.$$

(c) *Let* $\kappa = 1$ *or* $\kappa = 2$. *If* $f \in \mathbf{L}^{2,s}_\sigma$ *for some* $s > \tau + 1 + 2\kappa$ *and* $\tau \geq -1$, *then the equation*

$$\mathcal{A}_\kappa u = f + \sum_{j=0}^{\kappa-1} \zeta_j p_j^\sigma \tag{1.3}$$

has a unique solution $(u, \zeta) \in \mathbf{L}^{2,\tau}_\sigma \times \mathbb{C}^\kappa$. *Moreover, for* $f \equiv 0$, *equation* (1.3) *has only the trivial solution in* $\mathbf{L}^{2,-\frac{3}{2}}_\sigma \times \mathbb{C}^\kappa$.

Remark 1.2 *For* $\eta \in \ell^2_s$, $s > \tau + 3$, $\tau \geq -1$, *the solution of* $\mathbf{A}_1\xi = \eta + (\zeta_0, 0, \ldots)$ *is given by*

$$\zeta_0 = -4\beta_0 \sum_{m=1}^{\infty} m\,\eta_{2m} - \eta_0\,, \quad \xi_0 = -\frac{1}{\beta_0}\sum_{m=0}^{\infty}(2m+1)\eta_{2m+1}\,,$$

and

$$\xi_{2j+1} = -4(2j+1)\sum_{m=j+1}^{\infty} m\,\eta_{2m}\,, \quad \xi_{2(j+1)} = -4(j+1)\sum_{m=j+1}^{\infty}(2m+1)\eta_{2m+1}\,,$$

$j = 0, 1, \ldots$ *(cf. [3, eq.s in (5.6) and what follows]).*

Denote $\mathcal{A} = \mathcal{A}_0$, $\mathbf{A} = \mathbf{A}_0$, and let us first consider the simple equation $\mathcal{A}u = f$ for $f \in \mathbf{L}^{2,s+1}_\sigma$, which is equivalent to the equation $\mathbf{A}\xi = \eta$ with $\eta = \mathcal{J}f \in \ell^2_{s+1}$ and $\xi = \mathcal{J}u \in \ell^2_s$. Define $\mathbf{P}_n : \ell^2_s \longrightarrow \ell^2_s$, $\xi \mapsto (\xi_0, \ldots, \xi_{n-1}, 0, 0, \ldots)$ and look for an approximate solution $\xi^n \in \operatorname{im}\mathbf{P}_n$ by solving $\mathbf{A}_n\xi^n = \eta^n \in \operatorname{im}\mathbf{P}_n$ with $\widetilde{\mathbf{A}}_n = \mathbf{P}_n\mathbf{A}|_{\operatorname{im}\mathbf{P}_n} = \mathbf{A}|_{\operatorname{im}\mathbf{P}_n}$. Of course, $\xi^n = \mathbf{A}^{-1}\eta^n$, which implies

$$\|\xi^* - \xi^n\|_{\ell^2_t} \leq \left\|\mathbf{A}^{-1}\right\|_{\ell^2_{t+1}\to\ell^2_t} \|\eta - \eta^n\|_{\ell^2_{t+1}}\,,$$

where $\left\|\mathbf{A}^{-1}\right\|_{\ell^2_{t+1}\to\ell^2_t} = (\ln 2)^{-1}$ for all $t \in \mathbb{R}$. Consequently, we can approximate the solution $u^* \in \mathbf{L}^{2,s}_\sigma$ of $\mathcal{A}u = f \in \mathbf{L}^{2,s+1}_\sigma$ by $u^*_n = \sum_{k=0}^{n-1} \xi^n_k p^\sigma_k$, if $f_n = \sum_{k=0}^{n-1} \eta^n_k p^\sigma_k$ is an appropriate approximation of f. For example, $f_n(x)$ can be chosen as the interpolation polynomial $(\mathcal{L}^\sigma_n f)(x)$ of $f(x)$ w.r.t. the Chebyshev nodes $x^\sigma_{nk} = \cos\frac{2k-1}{2n}\pi$, $k = 1,\dots,n$, of the first kind. If $f \in \mathbf{L}^{2,s}_\sigma$ for some $s > \frac{1}{2}$, then (see, for example [1, Theorem 3.4])

$$\lim_{n\to\infty} \|\mathcal{L}^\sigma_n f - f\|_{\sigma,s} = 0 \tag{1.4}$$

and

$$\|\mathcal{L}^\sigma_n f - f\|_{\sigma,t} \le \operatorname{const} n^{t-s} \|f\|_{\sigma,s} \quad \text{for} \quad 0 \le t \le s\,. \tag{1.5}$$

It follows

$$\|u^* - u^*_n\|_{\sigma,t} \le \operatorname{const} n^{t-s} \|f\|_{\sigma,s+1}\,, \quad -1 \le t \le s\,,$$

if $s > -\frac{1}{2}$ and $f \in \mathbf{L}^{2,s+1}_\sigma$. The numbers $\eta^n_k = \langle \mathcal{L}^\sigma_n f, p^\sigma_k\rangle_\sigma$ can be computed efficiently by

$$\begin{aligned} \eta^n_k &= \frac{\pi}{n}\sum_{j=1}^{n} f(x^\sigma_{nj}) p^\sigma_k(x^\sigma_{nj}) \\ &= \begin{cases} \dfrac{\sqrt{\pi}}{n}\displaystyle\sum_{j=1}^{n} f(x^\sigma_{nj}) & : \quad k = 0\,, \\ \dfrac{\sqrt{2\pi}}{n}\displaystyle\sum_{j=1}^{n} \cos\frac{k(2j-1)\pi}{2n}\, f(x^\sigma_{nj}) & : \quad k = 1,\dots,n-1\,, \end{cases} \end{aligned} \tag{1.6}$$

using the respective fast discrete cosine transform

$$\mathbf{C}^2_n = \left[\ \cos\frac{k(2j-1)\pi}{2n}\ \right]^{n-1,\ n}_{k=0,j=1}\,.$$

Since the case $\kappa = 0$ is a classical one that has been studied by several authors, in the next sections we examine the numerical solution of integral equations of the form

$$-\frac{1}{\pi}\int_{-1}^{1} [(y-x)^\kappa \ln|y-x| - h(x,y)]\,\frac{u(y)\,dy}{\sqrt{1-y^2}} = f(x) + a_\kappa + (\kappa-1)b_\kappa x\,,$$

where $-1 < x < 1$, $\kappa = 1, 2$ and a_κ, b_κ are unknown constants to be determined together with $u(y)$, by means of a collocation and a collocation-quadrature method. In particular we derive convergence estimates. The main ideas are explicitly presented in case $\kappa = 1$, since this is then easily translated to case $\kappa = 2$.

2 Collocation for the dominant equation

Set $\mathcal{B} = \mathcal{A}_1$ and $\mathbf{B} = \mathbf{A}_1$. Due to Prop. 1.1,(c) we are interested in the numerical approximation of the solution $(u, \zeta_0) \in \mathbf{L}^2_\sigma \times \mathbb{C}$ of the equation

$$\mathcal{B}u = f + \zeta_0 p_0^\sigma , \tag{2.1}$$

which is equivalent to the equation

$$\mathbf{B}\xi = \eta + (\zeta_0, 0, 0, \ldots) , \ \eta = \mathcal{J}f, \ \xi = \mathcal{J}u . \tag{2.2}$$

We look for $u_n \in \operatorname{im} \mathcal{P}_{n-1}$, where $\mathcal{P}_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ denotes the orthoprojection

$$\mathcal{P}_n f = \sum_{k=0}^{n-1} \langle f, p_k^\sigma \rangle_\sigma \, p_k^\sigma ,$$

by solving the collocation equations

$$(\mathcal{B}u_n)(x_{nj}^\sigma) = f(x_{nj}^\sigma) + \frac{\zeta_0^n}{\sqrt{\pi}} , \quad j = 1, \ldots, n . \tag{2.3}$$

The system (2.3) can be written equivalently as

$$\mathcal{B}u_n = \mathcal{L}_n^\sigma f + \zeta_0^n p_0^\sigma , \tag{2.4}$$

taking into account $\mathcal{B}u_n \in \operatorname{im} \mathcal{P}_n$ for all $u_n \in \operatorname{im} \mathcal{P}_{n-1}$ (cf. Prop. 1.1,(b)), or as

$$\mathbf{B}\xi^n = (\eta_0^n + \zeta_0^n, \eta_1^n, \ldots, \eta_{n-1}^n, 0, 0, \ldots) , \quad \xi^n = \mathcal{J}u_n \in \operatorname{im} \mathbf{P}_{n-1} , \tag{2.5}$$

with η_k^n defined in (1.6). Due to Remark 1.2 and the fact, that $\eta_j^n = 0$ for $j \geq n$, for the solution of (2.5) we have that $\zeta_0^n = -4\beta_0 \sum_{m=1}^{\left[\frac{n-1}{2}\right]} m \, \eta_{2m}^n - \eta_0^n$,

$$\xi_{2j+1}^n = -4(2j+1) \sum_{m=j+1}^{\left[\frac{n-1}{2}\right]} m \, \eta_{2m}^n , \ j = 0, \ldots, \left[\frac{n-1}{2}\right] - 1 , \tag{2.6}$$

and that $\xi^n_{2(j+1)}$ is equal to

$$\begin{cases} -\dfrac{1}{\beta_0}\displaystyle\sum_{m=0}^{\left[\frac{n}{2}\right]-1}(2m+1)\eta^n_{2m+1} & : \quad j=-1\,, \\ -4(j+1)\displaystyle\sum_{m=j+1}^{\left[\frac{n}{2}\right]-1}(2m+1)\eta^n_{2m+1}\,, & : \quad j=0,\dots,\left[\frac{n}{2}\right]-2\,. \end{cases} \tag{2.7}$$

Consequently, provided that the assumptions of Remark 1.2 are satisfied, $\lim\limits_{n\to\infty}\zeta^n_0=\zeta_0$, where, taking into account (1.5),

$$\begin{aligned} |\zeta^n_0-\zeta_0| &\le \text{const}\left(\sum_{m=0}^{\left[\frac{n-1}{2}\right]}2(m+1)\,|\eta^n_{2m}-\eta_{2m}|+\sum_{m=\left[\frac{n-1}{2}\right]+1}^{\infty}2m\,|\eta_{2m}|\right)\\ &\le \text{const}\left(n^{\frac12}\,\|\mathcal{L}^\sigma_n f-\mathcal{P}_n f\|_{\sigma,1}+n^{\frac32-s}\,\|f\|_{\sigma,s}\right)\\ &\le \text{const}\,n^{\frac32-s}\,\|f\|_{\sigma,s}\,. \end{aligned}$$

Moreover, for $j=0,\dots,\left[\dfrac{n-1}{2}\right]-1$,

$$\begin{aligned} \frac{1}{2j+1}\left|\xi^n_{2j+1}-\xi_{2j+1}\right| &= 2\sum_{m=j+1}^{\left[\frac{n-1}{2}\right]}2m\,|\eta^n_{2m}-\eta_{2m}|+2\sum_{m=\left[\frac{n-1}{2}\right]+1}^{\infty}2m\,|\eta_{2m}|\\ &\le \text{const}\left(n^{\frac12}\,\|\mathcal{L}^\sigma_n f-\mathcal{P}_n f\|_{\sigma,1}+n^{\frac32-s}\,\|f\|_{\sigma,s}\right)\\ &\le \text{const}\,n^{\frac32-s}\,\|f\|_{\sigma,s} \end{aligned}$$

and, analogously,

$$\frac{1}{2(j+1)}\left|\xi^n_{2j}-\xi_{2j}\right|\le\text{const}\,n^{\frac32-s}\,\|f\|_{\sigma,s}\,,\quad j=0,\dots,\left[\frac{n}{2}\right]-1$$

such that

$$\|u_n-\mathcal{P}_n u\|_{\sigma,t}=\sqrt{\sum_{k=0}^{n-1}(k+1)^{2t}|\xi^n_k-\xi_k|^2}\le\text{const}\,n^{3+t-s}\,\|f\|_{\sigma,s}\,,\quad 0\le t\,.$$

Taking into account $\|u - \mathcal{P}_n u\|_{\sigma,t} \le \text{const}\, n^{t-\tau} \|u\|_{\sigma,\tau}$, $t \le \tau$, we get

$$\|u_n - u\|_{\sigma,t} \le \text{const} \left(n^{3+t-s} \|f\|_{\sigma,s} + n^{t-\tau} \|u\|_{\sigma,\tau} \right) , \quad 0 \le t \le \tau .$$

Now, one can use $s > \tau + 3$ and $\|u\|_{\sigma,\tau} \le \text{const}\, \|f\|_{\sigma,s}$ (cf. Lemma 3.2,(a)). Hence, the following proposition has been proved.

Proposition 2.1 *Assume that $f \in \mathbf{L}^{2,s}_\sigma$ for some $s > \tau + 3$, $\tau \ge 0$. Then the unique solution (u_n, ζ_0^n) of equation (2.4) converges in $\mathbf{L}^2_\sigma \times \mathbb{C}$ to the unique solution $(u, \zeta_0) \in \mathbf{L}^2_\sigma \times \mathbb{C}$ of equation (2.1), where, for $0 \le t \le \tau$,*

$$|\zeta_0^n - \zeta_0| \le \text{const}\, n^{\frac{3}{2}-s} \|f\|_{\sigma,s} \quad \text{and} \quad \|u_n - u\|_{\sigma,t} \le \text{const}\, n^{t-\tau} \|f\|_{\sigma,s} ,$$

where the constants do not depend on f and t .

3 Collocation for the complete equation

Assume $s_0 > 3$ and that the continuous function $h : [-1,1]^2 \longrightarrow \mathbb{C}$ satisfies

(A) $h(\cdot, y) \in \mathbf{L}^{2,s_0}_\sigma$ uniformly w.r.t. $y \in [-1,1]$.

Define $\mathcal{H} : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ by

$$(\mathcal{H}u)(x) := \frac{1}{\pi} \int_{-1}^{1} h(x,y) \frac{u(y)\, dy}{\sqrt{1-y^2}} , \quad -1 < x < 1 .$$

Then ([1, Lemma 4.2]) $\mathcal{H} \in \mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{L}^{2,s_0}_\sigma)$. At first we look for an approximate solution $(u_n, \zeta_0^n) \in \operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C}$ of the equation

$$(\mathcal{B} + \mathcal{H})u = f + \zeta_0 p_0^\sigma \tag{3.1}$$

by solving the collocation equations

$$-\frac{1}{\pi} \int_{-1}^{1} \left[(y - x_{nj}^\sigma) \ln |y - x_{nj}^\sigma| - h(x_{nj}^\sigma, y) \right] \frac{u_n(y)\, dy}{\sqrt{1-y^2}} = f(x_{nj}^\sigma) + \frac{\zeta_0^n}{\sqrt{\pi}} ,$$

$j = 1, \dots, n$. We write this system of equations in the form

$$(\widehat{\mathcal{B}} + \widehat{\mathcal{H}}_n)(u_n, \zeta_0) = \mathcal{L}_n^\sigma f , \quad (u_n, \zeta_0^n) \in \operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C} , \tag{3.2}$$

where $\widehat{\mathcal{B}}(u, \zeta_0) = \mathcal{B}u - \zeta_0 p_0^\sigma$, $\widehat{\mathcal{H}}(u, \zeta_0) = \mathcal{H}u$, and $\widehat{\mathcal{H}}_n(u, \zeta_0) = \mathcal{L}_n^\sigma \widehat{\mathcal{H}}(u, \zeta_0)$. Equip $\widehat{\mathbf{L}}^{2,s}_\sigma = \mathbf{L}^{2,s}_\sigma \times \mathbb{C}$ with the norm $\|(u, \zeta_0)\|_{\sigma,s} = \sqrt{\|u\|^2_{\sigma,s} + |\zeta_0|^2}$. In what follows we will assume that

(B) Equation (3.1) with $f \equiv 0$ has only the trivial solution in $\widehat{\mathbf{L}}^2_\sigma$.

Lemma 3.1 *Let $2 < t+3 < s$. Then the operator $\widehat{\mathcal{B}}_0 : \mathbf{L}^{2,s}_\sigma \longrightarrow \widehat{\mathbf{L}}^{2,t}_\sigma$ defined by $\widehat{\mathcal{J}}^{-1}\widehat{\mathbf{B}}_0\mathcal{J}$, where $\widehat{\mathcal{J}}(u,\zeta_0) = (\mathcal{J}u,\zeta_0)$ and $(\xi,\zeta_0) = \widehat{\mathbf{B}}_0\eta$ is given by Remark 1.2, is linear and bounded and satiesfies*

$$\widehat{\mathcal{B}}\,\widehat{\mathcal{B}}_0 f = f \quad \forall\, f \in \mathbf{L}^{2,s}_\sigma$$

and

$$\widehat{\mathcal{B}}_0\widehat{\mathcal{B}}(u,\zeta_0) = (u,\zeta_0) \quad \forall\,(u,\zeta_0) \in \widehat{\mathbf{L}}^{2,-1}_\sigma : \mathcal{B}u \in \mathbf{L}^{2,s}_\sigma\,.$$

Proof. With $\widehat{\mathbf{B}}(\xi,\zeta_0) := \mathbf{B}\xi - (\zeta_0,0,\ldots)$, $\mathbf{B} = \mathbf{A}_1$, from Remark 1.2 we have $\widehat{\mathbf{B}}\,\widehat{\mathbf{B}}_0\eta = \eta$ for all $\eta \in \ell^2_s$. Moreover, for $\eta \in \ell^2_s$,

$$\begin{aligned}\left\|\widehat{\mathbf{B}}_0\eta\right\|^2_{\ell^2_t} &\le \text{const}\sum_{n=0}^{\infty}(n+1)^{2t+2}\left|\sum_{m=n}^{\infty}(m+1)\eta_m\right|^2\\ &\le \text{const}\sum_{n=0}^{\infty}(n+1)^{2t+2}(n+1)^{1+2(1-s)}\,\|\eta\|^2_{\ell^2_s} \le \text{const}\,\|\eta\|^2_{\ell^2_s}\,,\end{aligned}$$

since $2t+5-2s < -1$. If $\xi \in \ell^2_{-1}$ such that $\mathbf{B}\xi \in \ell^2_s$, then $\widehat{\mathbf{B}}\widehat{\mathbf{B}}_0\widehat{\mathbf{B}}(\xi,\zeta_0) = \widehat{\mathbf{B}}(\xi,\zeta_0)$ and consequently, due to Prop. 1.1,(c), $\widehat{\mathbf{B}}_0\widehat{\mathbf{B}}(\xi,\zeta_0) = (\xi,\zeta_0)$. □

Lemma 3.2 *Assume that $3 \le t+3 < s < s_0$ and that conditions (A) and (B) are fulfilled.*

(a) *There exists a constant $c_{ts} > 0$ such that*

$$\left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})(u,\zeta_0)\right\|_{\sigma,s} \ge c_{ts}\,\|(u,\zeta_0)\|_{\sigma,t} \quad \forall\,(u,\zeta_0) \in \widehat{\mathbf{L}}^2_\sigma : \mathcal{B}u \in \mathbf{L}^{2,s}_\sigma\,.$$

(b) *For all sufficiently large n,*

$$\left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}}_n)(u,\zeta_0)\right\|_{\sigma,s} \ge \frac{c_{ts}}{2}\,\|(u,\zeta_0)\|_{\sigma,t} \quad \forall\,(u,\zeta_0) \in \widehat{\mathbf{L}}^2_\sigma : \mathcal{B}u \in \mathbf{L}^{2,s}_\sigma\,.$$

In particular, equation (3.2) is uniquely solvable for such n.

Proof. (a) By Lemma 3.1 we have that $u \in \mathbf{L}^2_\sigma$ and $\mathcal{B}u \in \mathbf{L}^{2,s}_\sigma$ imply $u \in \mathbf{L}^{2,t}_\sigma$. Thus, if we assume that such a constant $c_{ts} > 0$ does not exist, then there are $(u_n,\zeta^n_0) \in \widehat{\mathbf{L}}^{2,t}_\sigma$ with $\|(u_n,\zeta^n_0)\|_{\sigma,t} = 1$, $\mathcal{B}u_n \in \mathbf{L}^{2,s}_\sigma$, and

$\lim_{n\to\infty} \left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})(u_n,\zeta_0^n)\right\|_{\sigma,s} = 0$. Since $\mathcal{H}: \mathbf{L}_\sigma^{2,t} \longrightarrow \mathbf{L}_\sigma^{2,s}$ is a compact operator, we can assume that $\widehat{\mathcal{H}}(u_n,\zeta_0^n)$ converges to some $f \in \mathbf{L}_\sigma^{2,s}$ w.r.t. $\|.\|_{\sigma,s}$. Consequently, $\lim_{n\to\infty} \left\|\widehat{\mathcal{B}}(u_n,\zeta_0^n)+f\right\|_{\sigma,s} = 0$. Using Lemma 3.1 we conclude that $\lim_{n\to\infty} \left\|(u_n,\zeta_0^n)+\widehat{\mathcal{B}}_0 f\right\|_{\sigma,t} = 0$. Together with

$$
\begin{aligned}
&\left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})\widehat{\mathcal{B}}_0 f\right\|_{\sigma,s} \\
&= \left\|f+\widehat{\mathcal{H}}\widehat{\mathcal{B}}_0 f\right\|_{\sigma,s} \\
&\le \left\|f+\widehat{\mathcal{B}}(u_n,\zeta_0^n)\right\|_{\sigma,s} + \left\|\widehat{\mathcal{H}}\widehat{\mathcal{B}}_0 f-\widehat{\mathcal{B}}(u_n,\zeta_0^n)\right\|_{\sigma,s} \\
&\le \left\|f+\widehat{\mathcal{B}}(u_n,\zeta_0^n)\right\|_{\sigma,s} + \left\|\widehat{\mathcal{H}}\widehat{\mathcal{B}}_0 f+\widehat{\mathcal{H}}(u_n,\zeta_0^n)\right\|_{\sigma,s} + \left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})(u_n,\zeta_0^n)\right\|_{\sigma,s} \\
&\le \left\|f+\widehat{\mathcal{B}}(u_n,\zeta_0^n)\right\|_{\sigma,s} + \text{const}\left\|\widehat{\mathcal{B}}_0 f+(u_n,\zeta_0^n)\right\|_{\sigma,t} + \left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})(u_n,\zeta_0^n)\right\|_{\sigma,s}
\end{aligned}
$$

we get $\left\|\widehat{\mathcal{B}}_0 f\right\|_{\sigma,t} = 1$ and $(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})\widehat{\mathcal{B}}_0 f = \Theta$ in contradiction to condition (B).

(b) Due to assumption (A) we have $\widehat{\mathcal{H}} \in \mathcal{L}(\widehat{\mathbf{L}}_\sigma^2, \mathbf{L}_\sigma^{2,s_0})$ and consequently, taking into account (1.5),

$$
\begin{aligned}
\left\|(\widehat{\mathcal{H}}-\widehat{\mathcal{H}}_n)(u,\zeta_0)\right\|_{\sigma,r} &\le \text{const}\, n^{r-s_0}\left\|\widehat{\mathcal{H}}(u,\zeta_0)\right\|_{\sigma,s} \\
&\le \text{const}\, n^{r-s_0}\|(u,\zeta_0)\|_\sigma\,, \quad 0\le r\le s_0\,,
\end{aligned}
\tag{3.3}
$$

which implies together with (b)

$$
\begin{aligned}
c_{ts}\|(u,\zeta_0)\|_{\sigma,t} &\le \left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}}_n)(u,\zeta_0)\right\|_{\sigma,s} + \left\|(\widehat{\mathcal{H}}-\widehat{\mathcal{H}}_n)(u,\zeta_0)\right\|_{\sigma,s} \\
&\le \left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}}_n)(u,\zeta_0)\right\|_{\sigma,s} + \frac{c_{ts}}{2}\|(u,\zeta_0)\|_{\sigma,t}
\end{aligned}
$$

for $(u,\zeta_0) \in \widehat{\mathbf{L}}_\sigma^2$ with $\mathcal{B}u \in \mathbf{L}_\sigma^{2,s}$ and for all sufficiently large n. □

Proposition 3.3 *Let $3 \le 3+\tau < s < s_0$, $0 \le t \le \tau$, $f \in \mathbf{L}_\sigma^{2,s}$, and assume that conditions (A) and (B) are fulfilled. Then, for all sufficiently*

large n, equation (3.2) has a unique solution (u_n, ζ_0^n). Moreover, there is a constant $c \neq c(f,t)$ such that

$$|\zeta_0^n - \zeta_0^*| \le c\, n^{-\tau}\, \|f\|_{\sigma,s} \quad \text{and} \quad \|u_n - u^*\|_{\sigma,t} \le c\, n^{t-\tau}\, \|f\|_{\sigma,s}$$

where $(u^*, \zeta_0^*) \in \widehat{\mathbf{L}}_\sigma^{2,\tau}$ is the unique solution of (3.1).

Proof. Choose some $\varepsilon \in (0, s-\tau-3)$ and let $(u_n, \zeta_0^n) \in \widehat{\mathbf{L}}_\sigma^{2,\tau}$ be the solution of (3.2), which exists uniquely for all sufficiently large n (cf. Lemma 3.2,(b)) and which belongs automatically to $\operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C}$ since $\mathcal{L}_n^\sigma f - \widehat{\mathcal{H}}_n u \in \operatorname{im} \mathcal{P}_n$. With the help of Lemma 3.2,(b) we get, for all sufficiently large n and m,

$$\begin{aligned} &\frac{c_{t,3+t+\varepsilon}}{2}\, \|(u_n, \zeta_0^n) - (u_m, \zeta_0^m)\|_{\sigma,t} \\ &\le \|\mathcal{L}_n^\sigma f - \mathcal{L}_m^\sigma f\|_{\sigma,3+t+\varepsilon} + \left\|(\widehat{\mathcal{H}}_m - \widehat{\mathcal{H}}_n)(u_m, \zeta_0^m)\right\|_{\sigma,3+t+\varepsilon}. \end{aligned} \tag{3.4}$$

Since the norms $\|(u_m, \zeta_0^m)\|_\sigma$ are bounded (cf. Lemma 3.2,(b)), the estimate (3.4) together with (3.3) implies that (u_n, ζ_0^n) converges in the norm of $\widehat{\mathbf{L}}_\sigma^{2,\tau}$ to some $(u^*, \zeta_0^*) \in \widehat{\mathbf{L}}_\sigma^{2,\tau}$. Set $f^* = (\widehat{\mathcal{B}} + \widehat{\mathcal{H}})(u^*, \zeta_0^*)$. The estimation

$$\begin{aligned} &\|f^* - f\|_{\sigma,2} \\ &\le \left\|f^* - (\widehat{\mathcal{B}} + \widehat{\mathcal{H}})(u_n, \zeta_0^n)\right\|_{\sigma,2} + \left\|(\widehat{\mathcal{H}} - \widehat{\mathcal{H}}_n)(u_n, \zeta_0^n)\right\|_{\sigma,2} + \|\mathcal{L}_n^\sigma f - f\|_{\sigma,2} \end{aligned}$$

shows that $f^* = f$. Relations (3.4), (1.5), and (3.3) lead to

$$\begin{aligned} &\|(u_n, \zeta_0^n) - (u^*, \zeta_0^*)\|_\sigma \\ &\le \frac{2}{c_{0,3+\varepsilon}} \left(\|\mathcal{L}_n^\sigma f - f\|_{\sigma,3+\varepsilon} + \left\|\widehat{\mathcal{H}} - \widehat{\mathcal{H}}_n\right\|_{\widehat{\mathbf{L}}_\sigma^2 \to \mathbf{L}_\sigma^{2,3+\varepsilon}} \|(u^*, \zeta_0^*)\|_\sigma \right) \\ &\le \text{const}\, n^{3+\varepsilon-s}\, \|f\|_{\sigma,s}. \end{aligned}$$

Since $\mathcal{B}u^* \in \mathbf{L}_\sigma^{2,s}$ we can apply Lemma 3.2,(a) and obtain

$$\begin{aligned} \|u_n - u^*\|_{\sigma,t} &\le \|u_n - \mathcal{P}_{n-1}u^*\|_{\sigma,t} + \|\mathcal{P}_{n-1}u^* - u^*\|_{\sigma,t} \\ &\le n^t\, \|u_n - \mathcal{P}_{n-1}u^*\|_\sigma + n^{t-\tau}\, \|u^*\|_{\sigma,\tau} \\ &\le n^t\, \|u_n - u^*\|_\sigma + \frac{n^{t-\tau}}{c_{\tau,s}}\, \|f\|_{\sigma,s} \\ &\le \text{const}\, \left(n^{3+t+\varepsilon-s} + n^{t-\tau}\right) \|f\|_{\sigma,s}, \end{aligned}$$

which completes the proof. □

In what follows we improve the convergence rate of ζ_0^n given by Prop. 3.3.

Lemma 3.4 *Let the conditions (A) and (B) be fulfilled and let* $3 < r < s_0$. *Then*

$$\dim \ker_{\mathbf{L}_\sigma^{2,-r}} (\mathcal{B}+\mathcal{H})^* = 1\,,$$

where $(\mathcal{B}+\mathcal{H})^* : \mathbf{L}_\sigma^{2,-r} \longrightarrow \mathbf{L}_\sigma^{2,2-r}$ *denotes the* $\langle\cdot,\cdot\rangle$*-adjoint operator to* $\mathcal{B}+\mathcal{H} : \mathbf{L}_\sigma^{2,r-2} \longrightarrow \mathbf{L}_\sigma^{2,r}$.

Proof. We show that $p_0^\sigma \notin \overline{(\mathcal{B}+\mathcal{H})(\mathbf{L}_\sigma^{2,r-2})}^{\mathbf{L}_\sigma^{2,r}}$. Given that there are elements $u_n \in \mathbf{L}_\sigma^{2,r-2}$ with $\lim\limits_{n\to\infty} \|(\mathcal{B}+\mathcal{H})u_n - p_0^\sigma\|_{\sigma,r} = 0$. Since, in view of Lemma 3.2,(a), the sequence $(u_n)_{n=1}^\infty$ is bounded in $\mathbf{L}_\sigma^2$, we can assume, due to the compactness of $\mathcal{H} : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^{2,r}$, that $\lim\limits_{n\to\infty} \|\mathcal{H}u_n - f\|_{\sigma,r} = 0$ for some $f \in \mathbf{L}_\sigma^{2,r}$. Hence, $\lim\limits_{n\to\infty} \|\mathcal{B}u_n - p_0^\sigma + f\|_{\sigma,r} = 0$, which implies together with Lemma 3.1 $\lim\limits_{n\to\infty} \left\|(u_n,1) + \widehat{\mathcal{B}}_0 f\right\|_\sigma = 0$. Consequently, $\widehat{\mathcal{B}}_0 f \neq (\Theta, 0)$ and (cf. the proof of Lemma 3.2,(a))

$$\left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})\widehat{\mathcal{B}}_0 f\right\|_{\sigma,r}$$

$$\leq \left\|f + \widehat{\mathcal{B}}(u_n,1)\right\|_{\sigma,r} + \text{const}\left\|\widehat{\mathcal{B}}_0 f + (u_n,1)\right\|_\sigma + \left\|(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})(u_n,1)\right\|_{\sigma,r} \longrightarrow 0\,,$$

such that $(\widehat{\mathcal{B}}+\widehat{\mathcal{H}})\widehat{\mathcal{B}}_0 f = \Theta$ in contradiction to condition (B). So we can conclude that the kernel of $(\mathcal{B}+\mathcal{H})^* : \mathbf{L}_\sigma^{2,-r} \longrightarrow \mathbf{L}_\sigma^{2,2-r}$ is non trivial. Let $v_1, v_2 \in \ker_{\mathbf{L}_\sigma^{2,-r}} (\mathcal{B}+\mathcal{H})^* \setminus \{\Theta\}$ and let $(u_n^*, \mu_n^*) \in \widehat{\mathbf{L}}_\sigma^2$, $n = 0,1,2,\dots$, be the solutions of $(\mathcal{B}+\mathcal{H})u_n = p_n^\sigma + \mu_n p_0^\sigma$, which exist uniquely in accordance with Prop. 3.3. It follows $\mu_n^* v_j(p_0^\sigma) = -v_j(p_n^\sigma)$, $n = 0,1,2,\dots$, $j = 1,2$. This implies $v_j(p_0^\sigma) \neq 0$, $j = 1,2$, and $v_1(p_n^\sigma) = \alpha\, v_2(p_n^\sigma)$, $n = 0,1,2,\dots$, where $\alpha = \frac{v_1(p_0^\sigma)}{v_2(p_0^\sigma)}$. Consequently, since the set of polynomials is dense in $\mathbf{L}_\sigma^{2,r}$, $v_1 = \alpha\, v_2$, i.e. $\dim \ker_{\mathbf{L}_\sigma^{2,-r}} (\mathcal{B}+\mathcal{H})^* = 1$. □

Corollary 3.5 *Let the conditions of Prop. 3.3 and Lemma 3.4 be fulfilled and assume that*

$$\ker_{\mathbf{L}_\sigma^{2,-r}} (\mathcal{B}+\mathcal{H})^* \subset \mathbf{L}_\sigma^{2,-r_0} \tag{3.5}$$

for some $r_0 \in [0,3]$. *Then*

$$|\zeta_0^n - \zeta_0^*| \leq \text{const}\, n^{r_0 - s}\, \|f\|_{\sigma,s}\,, \tag{3.6}$$

which improves the estimate in Prop. 3.3.

Proof. Let $v \in \mathbf{L}_\sigma^{2,-r_0}$ with $\|v\|_{\sigma,-r_0} = 1$ span the kernel of $(\mathcal{B}+\mathcal{H})^*$ in $\mathbf{L}_\sigma^{2,-r}$. Then (cf. the proof of Lemma 3.4) $v(p_0^\sigma) \neq 0$ and, since $(\mathcal{B}+\mathcal{H})u^* = f+\zeta_0^* p_0^\sigma$ and $(\mathcal{B}+\mathcal{H}_n)u_n = \mathcal{L}_n^\sigma f + \zeta_0^n p_0^\sigma$,

$$v(f+\zeta_0^* p_0^\sigma) = v(\mathcal{L}_n^\sigma f + (\mathcal{H}-\mathcal{H}_n)u_n + \zeta_0^n p_0^\sigma) = 0\,.$$

This implies

$$\zeta_0^n - \zeta_0^* = \frac{v(f-\mathcal{L}_n^\sigma f + (\mathcal{H}_n-\mathcal{H})u_n)}{v(p_0^\sigma)}$$

and, taking into account (3.3) together with Lemma 3.2,(b),

$$\begin{aligned} |\zeta_0^n - \zeta_0^*| &\leq \frac{\|f-\mathcal{L}_n^\sigma f\|_{\sigma,r_0} + \|(\mathcal{H}_n-\mathcal{H})u_n\|_{\sigma,r_0}}{|v(p_0^\sigma)|} \\ &\leq \text{const}\left(n^{r_0-s}\,\|f\|_{\sigma,s} + n^{r_0-s_0}\,\|u_n\|_\sigma\right) \leq \text{const}\, n^{r_0-s}\,\|f\|_{\sigma,s}\,, \end{aligned}$$

which completes the proof. □

In the following proposition we formulate conditions which ensure that condition (3.5) is fulfilled. For this we use the notations $\varphi(y) = \sqrt{1-y^2}$ as well as $h_j(x,y) = [\varphi(y)]^j \dfrac{\partial^j h(x,y)}{\partial^j y}$, $-1 < y < 1$, and in what follows we assume that $h_j : [-1,1]^2 \longrightarrow \mathbb{C}$ is continuous for $j = 1,\dots,j_0$ and for some $j_0 \in \mathbb{N}$. Recall the following lemma.

Lemma 3.6 ([1],pp. 196,197) *For an integer $r \geq 0$, $u \in \mathbf{L}_\sigma^{2,r}$ if and only if $\varphi^j D^j u \in \mathbf{L}_\sigma^2$ for all $j = 0,1,\dots,r$, where D denotes the operator of generalized differentiation.*

Proposition 3.7 *Assume (A), (B), and $h_j(\cdot,y) \in \mathbf{L}_\sigma^{2,r_1}$ uniformly w.r.t. $y \in [-1,1]$ for $j = 1,\dots,j_0$ and for some $r_1 \in (3,s_0)$ and $j_0 \geq 3$. Then condition* (3.5) *is fulfilled for $r_0 > 3-j_0$, i.e., the estimate* (3.6) *holds for $r_0 = 0$ if $j_0 > 3$ and $r_0 = \varepsilon > 0$ arbitrarily small if $j_0 = 3$.*

Proof. Since $\mathcal{H} \in \mathcal{L}(\mathbf{L}_\sigma^2, \mathbf{L}_\sigma^{2,s_0})$ we have $\mathcal{H}^* \in \mathcal{L}(\mathbf{L}_\sigma^{2,-r}, \mathbf{L}_\sigma^2)$ for all $r \in [0,s_0]$, where, for $v \in \mathbf{L}_\sigma^{2,-r}$, i.e. $(v(p_n^\sigma))_{n=0}^\infty \in \ell_{-r}^2$, and $u \in \mathbf{L}_\sigma^2$,

$$(\mathcal{H}^* v)(u) = v(\mathcal{H}u) = \sum_{k=0}^\infty \langle \mathcal{H}u, p_k^\sigma\rangle_\sigma\, v(p_k^\sigma)\,.$$

Consequently, for each $v \in \mathbf{L}_\sigma^{2,-r}$ there exists an element $u_v \in \mathbf{L}_\sigma^2$ such that

$$\langle u, u_v\rangle_\sigma = (\mathcal{H}^* v)(u) = \lim_{n\to\infty} \left\langle u, \sum_{k=0}^n \overline{v(p_k^\sigma)}\mathcal{H}^* p_k^\sigma \right\rangle_\sigma\,,$$

where

$$(\mathcal{H}^* p_k^\sigma)(y) = \frac{1}{\pi} \int_{-1}^{1} \overline{h(x,y)} p_k^\sigma(x) \, \frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi} \left\langle p_k^\sigma, h(\cdot, y) \right\rangle_\sigma .$$

Since, due to condition (A), $\sum_{k=0}^{\infty} \left| \langle h(\cdot, y), p_k^\sigma \rangle_\sigma \right|^2 (k+1)^{2r} \le M_r^2 < \infty$, the series $\sum_{k=0}^{\infty} \overline{v(p_k^\sigma)} (\mathcal{H}^* p_k^\sigma)(y)$ converges uniformly w.r.t. $y \in [-1,1]$, where

$$\sum_{k=0}^{\infty} |v(p_k^\sigma)(\mathcal{H}^* p_k^\sigma)(y)| \le \frac{M_r \, \|v\|_{\sigma,-r}}{\pi} .$$

Hence, $u_v = \sum_{k=0}^{\infty} v(p_k^\sigma) \mathcal{H}^* p_k^\sigma$ in the $\|.\|_\infty$ sense, and $u_v : [-1,1] \longrightarrow \mathbb{C}$ is continuous, since the functions $y \mapsto \left\langle p_k^\sigma, h(\cdot, y) \right\rangle_\sigma$, $k = 0, 1, 2, \ldots$, are continuous for $y \in [-1,1]$. Moreover, due to our assumptions we have

$$\varphi^j D^j u_v = \sum_{k=0}^{\infty} \overline{v(p_k^\sigma)} \varphi^j D^j \mathcal{H}^* p_k^\sigma \quad \text{for} \quad v \in \mathbf{L}_\sigma^{2,-r_1}, \; j = 1, \ldots, j_0 ,$$

in the $\|.\|_\infty$ sense, since

$$\sum_{k=0}^{\infty} \left| v(p_k^\sigma) \left(\varphi^j D^j \mathcal{H}^* p_k^\sigma \right)(y) \right| \le \frac{\|h_j(\cdot, y)\|_{\sigma,r_1} \, \|v\|_{\sigma,-r_1}}{\pi} .$$

Hence, by Lemma 3.6 $u_v \in \mathbf{L}_\sigma^{2,j_0}$. Now, let $v \in \mathbf{L}_\sigma^{2,-r_1}$, $\|v\|_{\sigma,r_1} = 1$ span $\ker_{\mathbf{L}_\sigma^{2,-r_1}} (\mathcal{B} + \mathcal{H})^*$ (cf. Lemma 3.4). Since $\mathcal{B}^* = -\mathcal{B}$ we have $\mathcal{B} v = \mathcal{H}^* v = u_v \in \mathbf{L}_\sigma^{2,j_0}$ and, in view of Lemma 3.1, $(v, 0) = \widehat{\mathcal{B}}_0 \mathcal{H}^* v \in \widehat{\mathbf{L}}_\sigma^{2,t}$ for $t < j_0 - 3$, which completes the proof. □

4 A quadrature method

For $(u, \zeta_0) \in \mathbf{L}_\sigma^2$ we define $\widehat{\mathcal{H}}_n^0 (u, \zeta_0) = \mathcal{H}_n^0 u$ by $(\mathcal{H}_n^0 u)(x) = (\mathcal{L}_n^\sigma g_n)(x)$ with

$$g_n(x) = \frac{1}{\pi} \int_{-1}^{1} \left[\mathcal{L}_{n-1}^\sigma h(x, .) \right](y) \, \frac{u(y) \, dy}{\sqrt{1-y^2}} , \quad -1 < x < 1 .$$

We look for an approximate solution $(u_n, \zeta_0^n) \in \operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C}$ of equation (3.1) by solving

$$(\widehat{\mathcal{B}} + \widehat{\mathcal{H}}_n^0)(u_n, \zeta_0^n) = \mathcal{L}_n^\sigma f\,. \tag{4.1}$$

Taking into account the algebraic accuracy of the Gauss-Chebyshev quadrature we can write equation (4.1) as the system

$$-\frac{1}{\pi}\int_{-1}^{1}(y - x_{nj}^\sigma)\ln|y - x_{nj}^\sigma|\,\frac{u_n(y)\,dy}{\sqrt{1-y^2}} + \frac{1}{n-1}\sum_{k=1}^{n-1} h(x_{nj}^\sigma, x_{n-1,k}^\sigma)\chi_k^n$$

$$= f(x_{nj}^\sigma) + \frac{\zeta_0^n}{\sqrt{\pi}}\,, \quad j = 1, \dots, n\,,\ \chi_k^n = u_n(x_{n-1,k}^\sigma)\,.$$

Assume that

(C) $h(x,.) \in \mathbf{L}_\sigma^{2,s_0}$ uniformly w.r.t. $x \in [-1,1]$.

Proposition 4.1 *Let, together with (C), all the conditions of Prop. 3.3 be satisfied. Then, for all sufficiently large n, equation (4.1) has a unique solution (u_n, ζ_0^n) and all assertions of Prop. 3.3 remain true. Of course, also Cor. 3.5 is in force.*

Proof. From conditions (A) and (C) we get

$$\lim_{n\to\infty} \left\| (\widehat{\mathcal{H}} - \widehat{\mathcal{H}}_n^0)(u, \zeta_0) \right\|_{\widehat{\mathbf{L}}_\sigma^{2,t} \to \mathbf{L}_\sigma^{2,s}} \le \text{const}\, n^{t-s}\, \|(u, \zeta_0)\|_\sigma$$

(cf. [2, Lemma 3.4]). Hence, Lemma 3.2,(b) remains true if $\widehat{\mathcal{H}}_n$ is replaced by $\widehat{\mathcal{H}}_n^0$ (cf. the proof of Lemma 3.2,(b)), and the proof of Prop. 4.1 goes on the same lines as the proof of Prop. 3.3. □

Remark 4.2 *Using Prop. 1.1,(b),(c) in case $\kappa = 2$ as well as formulae analogous to Remark 1.2 for the solution of $\mathbf{A}_2\xi = (\zeta_0, \zeta_1, 0, \dots)$ (see [3, Section 5, Equ. (5.17) and Cor. 5.14]), one can formulate and prove propositions like Prop. 2.1, Prop. 3.3, and Prop. 4.1 for the equation*

$$-\frac{1}{\pi}\int_{-1}^{1}[(y-x)^2\ln|y-x| - h(x,y)]\frac{u(y)\,dy}{\sqrt{1-y^2}} = f(x) + \zeta_0 p_0^\sigma(x) + \zeta_1 p_1^\sigma(x)\,.$$

Remark 4.3 *Since $\chi_i^n = \sum_{k=0}^{n-2} p_k^\sigma(x_{n-1,i}^\sigma)\xi_k^n$, a matrix version of equation* (4.1) *is given by*

$$(\mathbf{B} + \mathbf{D}_n\mathbf{C}_n^2\mathbf{H}_n\mathbf{C}_{n-1}^3\mathbf{D}_{n-1})\xi^n = \eta^n + (\zeta_0^n, 0, \dots, 0)\,,$$

where

$$\mathbf{H}_n = \left[\ h(x_{nj}^{\sigma}, x_{n-1,k}^{\sigma})\ \right]_{j=1,k=1}^{n,\ n-1}, \quad \mathbf{C}_n^2 = \left[\ \cos\frac{j(2k-1)\pi}{2n}\ \right]_{j=0,k=1}^{n-1,\ n},$$

$\mathbf{C}_{n-1}^3 = \left(\mathbf{C}_{n-1}^2\right)^T$, $\mathbf{D}_n = \frac{1}{n}\ \mathrm{diag}\left[\ 1\ \ \sqrt{2}\ \ \cdots\ \ \sqrt{2}\ \right]$, *and the vector* $\eta^n = \left[\ \eta_k^n\ \right]_{k=1}^{n}$ *is defined by* (1.6).

5 Numerical examples

As an example let us consider the equation

$$-\frac{1}{\pi}\int_{-1}^{1}\left[(y-x)\ln|y-x| - \cos(xy+2y)\right]\frac{u(y)\,dy}{\sqrt{1-y^2}} = f(x) + \frac{\zeta_0}{\sqrt{\pi}} \qquad (5.1)$$

for different right hand sides $f(x)$ and apply the collocation quadrature method (4.1) for its numerical solution.

n	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
4	-5.40112961	-0.48449341	-5.21815407	0.00000000	0.00000000
8	-8.43254040	-1.11412102	-6.97308165	0.14453160	-1.30716032
16	-8.47596429	-1.12423470	-7.02314233	0.14623119	-1.38230256
32	-8.47857251	-1.12485039	-7.02604962	0.14632202	-1.38671366
64	-8.47873350	-1.12488837	-7.02622831	0.14632760	-1.38698578
128	-8.47874353	-1.12489074	-7.02623943	0.14632795	-1.38700273
256	-8.47874416	-1.12489088	-7.02624013	0.14632797	-1.38700379
512	-8.47874420	-1.12489089	-7.02624017	0.14632797	-1.38700386

Equation (5.1), Fourier coefficients, $f(x) = x^3|x|$

n	ζ_0^n	$u_n(-0.9)$	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
4	1.56360580	-5.28070867	1.11622350	-1.15880369	-6.76610747
8	1.35009089	-7.46665743	-0.53942159	-1.71255964	-10.42210437
16	1.36028283	-7.33128912	-0.62465602	-1.80938307	-10.46165518
32	1.36084850	-7.34419883	-0.64187474	-1.79710269	-10.43954148
64	1.36088304	-7.34654510	-0.64594776	-1.79917382	-10.43672800
128	1.36088519	-7.34633790	-0.64695190	-1.79892639	-10.43710217
256	1.36088532	-7.34636881	-0.64720206	-1.79895946	-10.43711238
512	1.36088533	-7.34637260	-0.64726455	-1.79895542	-10.43711063

Equation (5.1), ζ_0^n and function values, $f(x) = x^3|x|$

n	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
4	0.00000000	-2.35696767	0.00000000	0.00000000	0.00000000
8	0.00000000	-4.27093405	0.00000000	-5.51829717	0.00000000
16	0.00000000	-4.25553787	0.00000000	-5.47167380	0.00000000
32	0.00000000	-4.25538652	0.00000000	-5.47121507	0.00000000
64	0.00000000	-4.25538436	0.00000000	-5.47120852	0.00000000
128	0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000
256	0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000
512	0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000

Equation (5.1), Fourier coefficients, $f(x) = x^4|x|$

n	ζ_0^n	$u_n(-0.9)$	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
4	-0.09237946	1.69252930	0.00000000	-0.94029406	-1.78655871
8	0.32487689	3.46909238	0.00000000	2.26497025	-5.77407940
16	0.32155438	3.57407640	0.00000000	2.34797959	-5.77976528
32	0.32152179	3.57224821	0.00000000	2.34560345	-5.78331281
64	0.32152133	3.57217606	0.00000000	2.34571196	-5.78339496
128	0.32152132	3.57217755	0.00000000	2.34570873	-5.78339207
256	0.32152132	3.57217748	0.00000000	2.34570884	-5.78339205
512	0.32152132	3.57217748	0.00000000	2.34570883	-5.78339205

Equation (5.1), ζ_0^n and function values, $f(x) = x^4|x|$

We observe that the convergence (of the Fourier coefficients and of the function values) becomes better if the right hand side becomes smoother. In case of

$$f(x) = \frac{1}{\pi}\left[\frac{2\sin(x+2)}{x+2} - \frac{1}{2}(1-x)^2\left(\ln(1-x) - \frac{1}{2}\right)\right. \tag{5.2}$$
$$\left. + \frac{1}{2}(1+x)^2\left(\ln(1+x) - \frac{1}{2}\right)\right]$$

the function $u^*(x) = \sqrt{1-x^2}$ is the exact solution, where $\zeta_0^* = 0$.

n	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
4	1.44851367	0.07966143	-0.37489673	0.00000000	0.00000000
8	1.13963910	0.00265930	-0.51943238	-0.00038250	-0.08772781
16	1.12978800	0.00033218	-0.53036099	-0.00004873	-0.10401085
32	1.12855539	0.00004156	-0.53172768	-0.00000610	-0.10608699
64	1.12840120	0.00000520	-0.53189862	-0.00000076	-0.10634738
128	1.12838192	0.00000065	-0.53191999	-0.00000010	-0.10637995
256	1.12837951	0.00000008	-0.53192266	-0.00000001	-0.10638403
512	1.12837921	0.00000001	-0.53192299	-0.00000000	-0.10638454

Equation (5.1), Fourier coefficients, $f(x)$ from (5.2)

n	ζ_0^n	$u_n(-0.9)$	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
4	0.03101251	0.57457469	1.11636063	0.99857879	0.63682384
8	-0.00242665	0.41433610	1.00283824	0.87114862	0.29552061
16	-0.00030175	0.43740970	1.00028530	0.86633439	0.30846637
32	-0.00003770	0.43596566	1.00003272	0.86609941	0.31170267
64	-0.00000471	0.43587373	1.00000399	0.86603259	0.31226353
128	-0.00000059	0.43588962	1.00000050	0.86602645	0.31224771
256	-0.00000007	0.43588980	1.00000006	0.86602553	0.31224956
512	-0.00000001	0.43588988	1.00000001	0.86602542	0.31224986
$u^*(x)$		0.43588989	1.00000000	0.86602540	0.31224990

Equation (5.1), ζ_0^n and function values, $f(x)$ from (5.2)

As a second example we deal with the equation

$$-\frac{1}{\pi}\int_{-1}^{1}\left[(y-x)^2\ln|y-x|-\cos(xy+2y)\right]\frac{u(y)\,dy}{\sqrt{1-y^2}} = f(x)+\frac{\zeta_0}{\sqrt{\pi}}+\frac{\sqrt{2}\zeta_1 x}{\sqrt{\pi}} \tag{5.3}$$

again using the collocation quadrature method analogous to (4.1). The numerical results in case $f(x)=x^3|x|$ and $f(x)=x^4|x|$ are presented in the following four tables. Moreover, the last two tables are concerned with the case

$$f(x)=\frac{1}{\pi}\left[\frac{2\sin(x+2)}{x+2}-\frac{1}{3}(1-x)^3\left(\ln(1-x)-\frac{1}{3}\right)-\frac{1}{3}(1+x)^3\left(\ln(1+x)-\frac{1}{3}\right)\right] \tag{5.4}$$

in which $u^*(x)=\sqrt{1-x^2}$ is the exact solution, where $\zeta_0^*=\zeta_1^*=0$.

n	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
4	0.00000000	3.89516842	0.00000000	0.00000000	0.00000000
8	0.00000000	6.06352352	0.00000000	2.99539862	0.00000000
16	0.00000000	6.30796573	0.00000000	3.61165572	0.00000000
32	0.00000000	6.36442444	0.00000000	3.77432865	0.00000000
64	0.00000000	6.37841811	0.00000000	3.81589808	0.00000000
128	0.00000000	6.38191213	0.00000000	3.82635452	0.00000000
256	0.00000000	6.38278541	0.00000000	3.82897278	0.00000000
512	0.00000001	6.38300375	-0.00000002	3.82962770	-0.00000004

Equation (5.3), Fourier coefficients, $f(x)=x^3|x|$

n	ζ_0^n	ζ_1^n	$u_n(-0.9)$	$u_n(0.5)$	$u_n(0.95)$
4	-0.00000000	3.53746377	-2.79710527	1.55394737	2.95250001
8	-0.00000000	2.77500388	-5.82901534	-0.72916989	6.00687279
16	-0.00000000	2.96250258	-5.74292291	-0.58450110	7.41386459
32	0.00000000	3.00749821	-5.47426811	-0.54753047	6.51404218
64	-0.00000000	3.01875475	-5.46727406	-0.54075715	6.54254851
128	0.00000000	3.02157177	-5.47955038	-0.53760774	6.52001095
256	0.00000000	3.02227625	-5.47504222	-0.53698198	6.52890323
512	0.00000001	3.02245241	-5.47562556	-0.53680014	6.52615747

Equation (5.3), ζ_0^n and function values, $f(x) = x^3|x|$

n	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
4	-0.58280657	0.72882972	-2.14508166	0.00000000	0.00000000
8	-1.40188257	0.85249743	0.32708937	0.06894267	0.01413619
16	-1.40191993	0.85249555	0.32721284	0.06896456	0.01432215
32	-1.40191993	0.85249555	0.32721284	0.06896456	0.01432215
64	-1.40191993	0.85249555	0.32721284	0.06896456	0.01432215
128	-1.40191993	0.85249555	0.32721284	0.06896456	0.01432215
256	-1.40191993	0.85249556	0.32721284	0.06896457	0.01432214
512	-1.40191994	0.85249559	0.32721287	0.06896467	0.01432221

Equation (5.3), Fourier coefficients, $f(x) = e^x$

n	ζ_0^n	ζ_1^n	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
4	-2.14508166	-0.32474949	-0.32881339	-0.03805240	0.22363249
8	-2.39164719	-0.47532359	-1.04062806	-0.64119672	0.10053606
16	-2.39172319	-0.47532077	-1.04072747	-0.64121781	0.10059761
32	-2.39172319	-0.47532077	-1.04072747	-0.64121781	0.10059761
64	-2.39172319	-0.47532077	-1.04072747	-0.64121781	0.10059761
128	-2.39172319	-0.47532077	-1.04072747	-0.64121781	0.10059761
256	-2.39172319	-0.47532076	-1.04072747	-0.64121781	0.10059755
512	-2.39172321	-0.47532074	-1.04072749	-0.64121787	0.10059716

Equation (5.3), ζ_0^n and function values, $f(x) = e^x$

n	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
4	0.40998695	0.04201924	-0.38240806	0.00000000	0.00000000
8	1.11994191	-0.00200748	-0.50425705	0.00042534	-0.06204493
16	1.12757437	-0.00020515	-0.52906441	0.00004660	-0.10115735
32	1.12828511	-0.00002445	-0.53158241	0.00000562	-0.10574444
64	1.12836761	-0.00000302	-0.53188097	0.00000070	-0.10630502
128	1.12837773	-0.00000038	-0.53191780	0.00000009	-0.10637467
256	1.12837899	-0.00000005	-0.53192239	0.00000001	-0.10638337
512	1.12837914	-0.00000001	-0.53192294	-0.00000001	-0.10638442

Equation (5.3), Fourier coefficients, $f(x)$ from (5.4)

n	ζ_0^n	ζ_1^n	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
4	-0.38240806	0.18170011	0.23131037	0.24807362	0.26316054
8	-0.01709214	-0.00070921	0.98469378	0.85663685	0.29199603
16	-0.00173656	-0.00006887	0.99875332	0.86470886	0.29949307
32	-0.00020612	-0.00000812	0.99986381	0.86586928	0.31155025
64	-0.00002543	-0.00000100	0.99998359	0.86600576	0.31216344
128	-0.00000317	-0.00000012	0.99999797	0.86602295	0.31223920
256	-0.00000040	-0.00000002	0.99999975	0.86602510	0.31224851
512	-0.00000006	-0.00000000	0.99999996	0.86602537	0.31224977
$u^*(x)$			1.00000000	0.86602540	0.31224990

Equation (5.3), ζ_0^n and function values, $f(x)$ from (5.4)

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On sharp estimates concerning von Neumann-Jordan and James constants for a Banach space

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Dedicated to our wives Keiko and Midori

Abstract. Recently the inequality $C_{NJ}(X) \leq J(X)$ was proved in Takahashi and Kato [15], where $C_{NJ}(X)$ and $J(X)$ are the von Neumann-Jordan and James constants of a Banach space X respectively. We shall present its another proof, which is much simpler, and also a sharper estimate than this inequality. To do this we shall estimate $C_{NJ}(X)$ by both of the modified von Neumann-Jordan constant $C'_{NJ}(X)$ and the constant $\rho_X(1)$, where $\rho_X(\tau)$ is the modulus of smoothness of X. By this we shall have a unifying sight on a sequence of previous results as well.

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1 Introduction

Recently many geometric constants for a Banach space X have been investigated, which enable us to make precise descriptions on various geometric properties of X. Among them the von Neumann-Jordan constant $C_{NJ}(X)$

This paper is in final form and no version of it will be submitted for publication elsewhere.

([5]) and the James constant $J(X)$ ([7, 8]) are most widely treated. Concerning the relation between these two constants Kato, Maligranda and Takahashi [10] showed in 2001 the following couple of inequalities:

$$\frac{J(X)^2}{2} \le C_{NJ}(X) \le \frac{J(X)^2}{1+(J(X)-1)^2}, \tag{1.1}$$

where the first inequality attains equality for many uniformly non-square spaces, while the second is strict for all uniformly non-square spaces. Since then several authors improved the right side inequality. First Nikolova, Persson and Zachariades [13, 2004] showed

$$C_{NJ}(X) \le \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}\left\{\sqrt{J(X)^2 - 4J(X) + 8} - 2\right\} \tag{1.2}$$

(see also Maligranda et al. [12, 2008]). In 2008 Alonso, Martín and Papini [2] proved the stronger inequality

$$C_{NJ}(X) \le 2\left\{1 + J(X) - \sqrt{2J(X)}\right\}, \tag{1.3}$$

which in particular answered affirmatively "Maligranda's conjecture":

$$C_{NJ}(X) \le \frac{J(X)^2}{4} + 1 \tag{1.4}$$

(cf. Yang [18]). In 2009 Wang and Pang [17] improved (1.3) to obtain that

$$C_{NJ}(X) \le J(X) + \sqrt{J(X)-1}\left[\sqrt{1+\left(1-\sqrt{J(X)-1}\right)^2} - 1\right]. \tag{1.5}$$

Recently Takahashi and Kato [15] proved that the following quite simple inequality

$$C_{NJ}(X) \le J(X) \tag{1.6}$$

holds true for all Banach spaces X, which answered affirmatively the question posed in Alonso et al. [2, Question 1]. The inequality (1.6) was also proved simultaneously by Yang and Li [19] and by Wang [16]. In the above Alonso et al. [2] used the modified von Neumann-Jordan constant $C'_{NJ}(X)$, in particular they showed the weaker estimate than (1.6),

$$C'_{NJ}(X) \le J(X); \tag{1.7}$$

Wang-Pang [17] and Takahashi-Kato [15] used the constants $A_2(X) = \rho_X(1) + 1$ and $\rho_X(1)$ respectively, where $\rho_X(\tau)$ is the modulus of smoothness.

The aim of this paper is to present another proof of the inequality (1.6) by using $C'_{NJ}(X)$ on the line of Alonso et al. [2], which is much simpler than that of [15], and also the following sharper inequality than (1.6):

$$C_{NJ}(X) \le \left(\frac{\sqrt{2}(J(X)-1) + \sqrt{(J(X)-1)^2+1}}{J(X)} \right)^2 \le J(X). \tag{1.8}$$

Section 2 is devoted to preliminary definitions and results. In Section 3 we shall first show the following inequality which estimates $C_{NJ}(X)$ by both of $C'_{NJ}(X)$ and $\rho_X(1)$:

$$C_{NJ}(X) \le C'_{NJ}(X) - \rho_X(1) + \sqrt{\left(C'_{NJ}(X) - \rho_X(1) - 1\right)^2 + \rho_X(1)^2} \tag{1.9}$$

(Theorem 3.4). As is expected, this inequality yields both of the previous estimates of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and by $\rho_X(1)$:

$$C_{NJ}(X) \le 1 + \left(\sqrt{2C'_{NJ}(X)} - 1 \right)^2 \tag{1.10}$$

(Alonso et al. [2, Theorem 1]) and

$$C_{NJ}(X) \le 1 + \rho_X(1) \left[\sqrt{(1-\rho_X(1))^2 + 1} - (1 - \rho_X(1)) \right] \tag{1.11}$$

(Takahashi and Kato [15, Theorem 2]). In Theorem 3.10 the following estimate for $C'_{NJ}(X)$ by $J(X)$ will be shown:

$$C'_{NJ}(X) \le 1 + 4 \left\{ 1 - \frac{1}{J(X)} \right\}^2, \tag{1.12}$$

from which it follows that

$$C'_{NJ}(X) \le \frac{1}{2} \left(1 + \sqrt{J(X) - 1} \right)^2 \tag{1.13}$$

(see also Yang and Li [19]). Combining the inequalities (1.10) and (1.13), we shall obtain the inequality (1.6) in Theorem 3.12. The inequality (1.7) by Alonso et al. [2] is also a direct consequence of (1.13). In Theorem 3.14 the inequality (1.8) will be shown.

In Section 4 owing to Theorem 3.4 we shall have a unifying sight on the above-mentioned progress concerning the relation between $C_{NJ}(X)$ and $J(X)$, which will be helpful to understand it since it is not obvious at all to see that each result improves its former one. In the final Section 5 we shall answer the question of Alonso et al. [2, cf. Corollary 2] negatively which states whether the following estimate is sharp: $C_{NJ}(X) - C'_{NJ}(X) \le 3 - 2\sqrt{2}$.

2 Preliminaries

Throughout the paper let X be a real Banach space with $\dim X \geq 2$. The closed unit ball and unit sphere are denoted by B_X and S_X respectively. The *von Neumann-Jordan constant* of X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : \ x, y \text{ are not both } 0\right\} \tag{2.1}$$

([5]), where the supremum can be taken over all $x \in S_X$ and $y \in B_X$ ([10]). The first systematic work on this constant was done in Kato and Takahashi [11]. The constant defined by taking supremum over all $x, y \in S_X$ in (2.1) is denoted by $C'_{NJ}(X)$, that is,

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : \ x, y \ \in S_X\right\}, \tag{2.2}$$

which was discussed intensively in Takahashi [14] and Alonso et al. [2]. It is clear that $C'_{NJ}(X) \leq C_{NJ}(X)$ and they do not coincide in general (see Remark 3.5 below). A Banach space X is called *uniformly non-square* ([9]) if there exists $\delta > 0$ such that

$$\min(\|x+y\|, \|x-y\|) \leq 2(1-\delta) \ \text{ if } \ \|x\| = \|y\| = 1.$$

The *James constant* of X ([7, 8]) is defined by

$$J(X) = \sup\big\{\min(\|x+y\|, \|x-y\|) : \ x, y \in S_X\big\}. \tag{2.3}$$

It is well known (Casini [4]) that $J(X) = \sup\{\epsilon \in (0,2) : \delta_X(\epsilon) \leq 1 - \epsilon/2\}$, where $\delta_X(\varepsilon)$ is the *modulus of convexity* of X:

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in S_X, \ \|x-y\| = \epsilon\right\} \ (0 \leq \epsilon \leq 2);$$

S_X may be replaced by B_X. In particular if $J(X) < 2$, we have

$$\delta_X(J(X)) = 1 - \frac{J(X)}{2}. \tag{2.4}$$

The *modulus of smoothness* of X is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \ x, y \in S_X\right\} \quad (\tau > 0).$$

We shall collect some known facts about these constants (cf. [10, 2], etc.).

(a) $1 \le C'_{NJ}(X) \le C_{NJ}(X) \le 2$ and $\sqrt{2} \le J(X) \le 2$.

$C_{NJ}(X) = 1$ (resp. $C'_{NJ}(X) = 1$) if and only if X is a Hilbert space.

$J(X) = \sqrt{2}$ if X is a Hilbert space; the converse is not true.

(b) $\sqrt{2} - 1 \le \rho_X(1) \le 1$ ([3, Proposition 2.3]). $\rho_X(1) = \sqrt{2} - 1$ if X is a Hilbert space (cf. [3, esp., p.126]); the converse is not true ([3, esp., p.135]).

(c) The following are equivalent.

(i) X is uniformly non-square.

(ii) $C_{NJ}(X) < 2$.

(iii) $C'_{NJ}(X) < 2$.

(iv) $J(X) < 2$.

(v) $\delta_X(\varepsilon) > 0$ with some $0 < \varepsilon < 2$.

(vi) $\rho_X(1) < 1$.

3 Sharp estimates concerning $C_{NJ}(X)$ and $J(X)$

We start with recalling the following lemma.

Lemma 3.1 ([15, Lemma 3]). (i) *Let $0 \le a \le 2$ and $0 \le b \le 2$. Then*

$$\frac{a^2 + b^2}{4} \le \left(\frac{a+b}{2} - 1\right)^2 + 1. \tag{3.1}$$

(ii) *Let a be a real number and let $b > 0$. Then*

$$\frac{at^2 + bt}{1 + t^2} \le \frac{a + \sqrt{a^2 + b^2}}{2} \quad \text{for all } t \ge 0. \tag{3.2}$$

For convenience of the reader we mention the proof. For (i) we have

$$\left(\frac{a+b}{2} - 1\right)^2 + 1 - \frac{a^2 + b^2}{4} = \frac{(2-a)(2-b)}{2} \ge 0.$$

For (ii) let $f(t) = (at^2 + bt)/(1 + t^2)$. Then $f(t)$ attains its maximum at $t_0 := (a + \sqrt{a^2 + b^2})/b$. Noting that $bt_0^2 - 2at_0 - b = 0$, we have

$$f(t_0) = \frac{t_0(2at_0 + 2b)}{2(1 + t_0^2)} = \frac{t_0(bt_0^2 + b)}{2(1 + t_0^2)} = \frac{b}{2}t_0 = \frac{a + \sqrt{a^2 + b^2}}{2},$$

which implies (3.2).

Proposition 3.2 *Let X be a Banach space. Then*

$$\frac{(1+\rho_X(1))^2}{2} \le C'_{NJ}(X) \le 1+\rho_X(1)^2. \tag{3.3}$$

Proof. The first inequality, which is found in [17], is immediate to see. For the second let $x, y \in S_X$. Then by Lemma 3.1 (i)

$$\frac{\|x+y\|^2+\|x-y\|^2}{4} \le 1+\left(\frac{\|x+y\|+\|x-y\|}{2}-1\right)^2 \le 1+\rho_X(1)^2,$$

from which we have the conclusion. □

Remark 3.3 (i) *The second inequality in (3.3) is sharper than $C'_{NJ}(X) \le 1+\rho_X(1)(= A_2(X))$ which was shown in Wang and Pang [17, esp., p. 308].*

(ii) *Each inequality in (3.3) is sharp. In fact if X is not uniformly non-square, equality holds in the both inequalities in (3.3) as $C'_{NJ}(X) = 2$ and $\rho_X(1) = 1$. The first inequality attains equality with L_p-spaces, $1 \le p \le \infty$ since $\rho_{L_p}(1) = \max\{2^{1/p}, 2^{1-1/p}\} - 1$ ([3, Proposition 5.1]) and $C'_{NJ}(X) = \max\{2^{2/p-1}, 2^{1-2/p}\}$ ([2, Example 5]). Another example for the first inequality is the Day-James ℓ_2-ℓ_∞ space: Let $X = \mathbb{R}^2$ equipped with the norm*

$$\|x\| = \begin{cases} \|x\|_2 & \text{if } x_1x_2 \ge 0, \\ \|x\|_\infty & \text{if } x_1x_2 \le 0 \end{cases}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. Then $\rho_X(1) = 1/\sqrt{2}$ ([14, Example 3]; see also [1, Example 25]). In [2] it is shown that $C'_{NJ}(X) = (3+2\sqrt{2})/4$. Therefore the first inequality attains equality with the space ℓ_2-ℓ_∞. The second inequality of (3.3) attains equality with the ℓ_∞-ℓ_1 space: Let $X = \mathbb{R}^2$ equipped with the norm

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \ge 0, \\ \|x\|_1 & \text{if } x_1x_2 \le 0 \end{cases}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. Then $\rho_X(1) = 1/2$ ([3, Example 3.2], [10, Example 4]) and $C'_{NJ}(X) = 5/4$ ([14, Example 4], [2, Example 1]). Hence equality holds in the second inequality.

Theorem 3.4 *Let X be a Banach space. Then*

$$C_{NJ}(X) \le C'_{NJ}(X) - \rho_X(1) + \sqrt{(C'_{NJ}(X)-\rho_X(1)-1)^2+\rho_X(1)^2}. \tag{1.9}$$

Proof. First we note that the von Neumann-Jordan constant $C_{NJ}(X)$ is reformulated as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2(1+t^2)} : \ x,y\in S_X, 0\le t\le 1\right\}. \quad (3.4)$$

Let $x, y \in S_X$ and let $0 \le t \le 1$. Since

$$\|x+y\|^2+\|x-y\|^2 \le 4C'_{NJ}(X)$$

and

$$\|x+y\|+\|x-y\| \le 2(1+\rho_X(1)),$$

we have

$$\begin{aligned}
& \|x+ty\|^2+\|x-ty\|^2 \\
\le\ & (t\|x+y\|+(1-t))^2+(t\|x-y\|+(1-t))^2 \\
=\ & (\|x+y\|^2+\|x-y\|^2)t^2+2(\|x+y\|+\|x-y\|)t(1-t)+2(1-t)^2 \\
\le\ & 4C'_{NJ}(X)t^2+4(1+\rho_X(1))t(1-t)+2(1-t)^2 \\
=\ & 2\left[(2C'_{NJ}(X)-2\rho_X(1)-2)t^2+2\rho_X(1)t+(1+t^2)\right].
\end{aligned}$$

Therefore by Lemma 3.1 (ii)

$$\begin{aligned}
& \frac{\|x+ty\|^2+\|x-ty\|^2}{2(1+t^2)} \\
\le\ & \frac{2\left[(C'_{NJ}(X)-\rho_X(1)-1)t^2+\rho_X(1)t\right]}{1+t^2}+1 \\
\le\ & (C'_{NJ}(X)-\rho_X(1)-1)+\sqrt{\left(C'_{NJ}(X)-\rho_X(1)-1\right)^2+\rho_X(1)^2}+1 \\
=\ & C'_{NJ}(X)-\rho_X(1)+\sqrt{\left(C'_{NJ}(X)-\rho_X(1)-1\right)^2+\rho_X(1)^2},
\end{aligned}$$

by which we obtain the inequality (1.9). □

Remark 3.5 (i) *If X is not uniformly non-square, we have equality in (1.9) as $C_{NJ}(X) = C'_{NJ}(X) = 2$ and $\rho_X(1) = 1$.*

(ii) *The inequality (1.9) attains equality with both of the Day-James ℓ_∞-ℓ_1 and ℓ_2-ℓ_∞ spaces, which are uniformly non-square. In fact let X be the ℓ_∞-ℓ_1 space. Then $\rho_X(1) = 1/2$ and $C'_{NJ}(X) = 5/4$ as mentioned in Remark 3.3. On the other hand it is known that $C_{NJ}(X) = (3+\sqrt{5})/4$ ([20, Example 3.2], [14, Example 4]). Thus we have equality in (1.9). We note*

that the above value of $C_{NJ}(X)$ obtained in [20] and [14] is immediately derived from Theorem 3.4. Indeed since $\rho_X(1) = 1/2$ and $C'_{NJ}(X) = 5/4$, we have $C_{NJ}(X) \le (3+\sqrt{5})/4$ by (1.9). Taking $x = (1,0)$ and $y = (0,(\sqrt{5}-1)/2)$ in the definition of $C_{NJ}(X)$, we have $C_{NJ}(X) \ge (3+\sqrt{5})/4$.

Next let X be the ℓ_2-ℓ_∞ space. Then $\rho_X(1) = 1/\sqrt{2}$ and $C'_{NJ}(X) = (3+2\sqrt{2})/4$ (Remark 3.3). On the other hand $C_{NJ}(X) = 3/2$ ([20, Example 3.3]). Therefore we have equality in (1.9). We note again that the above result $C_{NJ}(X) = 3/2$ is immediately obtained by Theorem 3.4. Indeed since $\rho_X(1) = 1/\sqrt{2}$ and $C'_{NJ}(X) = (3+2\sqrt{2})/4$, we have $C_{NJ}(X) \le 3/2$ by (1.9). Taking $x = (-1,1)$ and $y = (1/2,1/2)$ in the definition of $C_{NJ}(X)$, we have $C_{NJ}(X) \ge 3/2$.

The following elementary facts will be useful in our later discussion.

Lemma 3.6 *Let $f(u,v) = u - v + \sqrt{(u-v-1)^2+v^2}$. Then for all $u_2 \ge u_1 \ge 1$ and $v_2 \ge v_1 \ge u_1 - 1$,*

$$f(u_1,v_1) \le f(u_2,v_2),$$

where equality holds only when $u_1 = u_2$ and $v_1 = v_2$.

In fact, for each $v \ne 0$ the function $f(u,v)$ is strictly increasing in $u \in (-\infty,+\infty)$, and for each $u \ge 1$ the function $f(u,v)$ is strictly increasing in $v \in [u-1,+\infty)$.

As is expected, Theorem 3.4 yields both of the next previous results concerning the estimates of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and by $\rho_X(1)$.

Corollary 3.7 (Alonso et al. [2, Theorem 1]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \le 1 + \left[\sqrt{2C'_{NJ}(X)} - 1\right]^2, \tag{1.10}$$

where the inequality is strict if $\rho_X(1) < \sqrt{2C'_{NJ}(X)} - 1$.

Proof. Let $f(u,v)$ be as in Lemma 3.6. Take $u_1 = u_2 = C'_{NJ}(X)$ and $v_1 = \rho_X(1), v_2 = \sqrt{2C'_{NJ}(X)} - 1$. By Proposition 3.2 we have $v_2 \ge v_1 \ge u_1 - 1$. Then by Theorem 3.4 and Lemma 3.6 we have

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(C'_{NJ}(X), \sqrt{2C'_{NJ}(X)} - 1) = 1 + \left[\sqrt{2C'_{NJ}(X)} - 1\right]^2, \end{aligned}$$

where the second inequality is strict if $\rho_X(1) < \sqrt{2C'_{NJ}(X)} - 1$. This completes the proof. □

Corollary 3.8 (Takahashi and Kato [15, Theorem 2]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \le 1 + \rho_X(1)\left[\sqrt{(1-\rho_X(1))^2+1} - (1-\rho_X(1))\right], \tag{1.11}$$

where the inequality is strict if $C'_{NJ}(X) < 1 + \rho_X(1)^2$.

Proof. Let $f(u,v)$ be as in Lemma 3.6. Then since $C'_{NJ}(X) \le 1 + \rho_X(1)^2$ by Proposition 3.2, we have

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(1+\rho_X(1)^2, \rho_X(1)) \\ &= 1 + \rho_X(1)\left[\sqrt{(1-\rho_X(1))^2+1} - (1-\rho_X(1))\right] \end{aligned}$$

by Theorem 3.4 and Lemma 3.6, where the second inequality is strict if $C'_{NJ}(X) < 1 + \rho_X(1)^2$. This completes the proof. □

Remark 3.9 *As stated in [2] and [15], the estimates given in Corollaries 3.7 and 3.8 are sharp. In fact, if X is not uniformly non-square, equality holds in (1.10) and (1.11) as $C_{NJ}(X) = C'_{NJ}(X) = 2$ and $\rho_X(1) = 1$. The inequality (1.10) attains equality with the ℓ_2-ℓ_∞ space X as $C_{NJ}(X) = 3/2$ and $C'_{NJ}(X) = (3+2\sqrt{2})/4$. The inequality (1.11) attains equality with the ℓ_∞-ℓ_1 space X as $C_{NJ}(X) = (3+\sqrt{5})/4$ and $\rho_X(1) = 1/2$.*

Theorem 3.10 *Let X be a Banach space. Then*

$$C'_{NJ}(X) \le 1 + 4\left\{1 - \frac{1}{J(X)}\right\}^2. \tag{1.12}$$

Proof. We shall show that for all $x, y \in S_X$

$$\frac{\|x+y\|^2 + \|x-y\|^2}{4} \le 1 + 4\left\{1 - \frac{1}{J(X)}\right\}^2. \tag{3.5}$$

Let $x, y \in S_X$ and assume $\|x+y\| \le \|x-y\|$ without loss of generality. Let $\varepsilon = \|x-y\|$. Then $\|x+y\| \le 2(1-\delta_X(\varepsilon))$. We write J and $\delta(\varepsilon)$ for $J(X)$ and $\delta_X(\varepsilon)$ respectively. If $\varepsilon \le J$, we have

$$\frac{\|x+y\|^2 + \|x-y\|^2}{4} \le \frac{J^2}{2} \le 1 + 4\left(1 - \frac{1}{J}\right)^2.$$

Let $J \leq \varepsilon \leq 2$. Then, as the function $\delta(\varepsilon)/\varepsilon$ is non-decreasing ([6]), we have $\delta(J)/J \leq \delta(\varepsilon)/\varepsilon$, from which it follows that

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &\leq 4(1-\delta(\varepsilon))^2 + \varepsilon^2 \\ &\leq 4\left\{1 - \frac{\delta(J)}{J}\varepsilon\right\}^2 + \varepsilon^2 \\ &\leq 4\left\{1 - \frac{2\delta(J)}{J}\right\}^2 + 4 \\ &= 4 + 16\left\{1 - \frac{1}{J}\right\}^2, \end{aligned}$$

where the above equality follows from the fact (2.4), $2\delta(J) = 2 - J$. Hence we have (3.5). This completes the proof. □

The following corollary is useful, which was also independently shown in [19].

Corollary 3.11 *Let X be a Banach space. Then*

$$C'_{NJ}(X) \leq \frac{1}{2}\left(1 + \sqrt{J(X) - 1}\right)^2, \tag{1.13}$$

where equality holds only when X is not uniformly non-square.

Proof. By Theorem 3.10 we have

$$2C'_{NJ}(X) \leq 2 + 8\left\{1 - \frac{1}{J(X)}\right\}^2. \tag{3.6}$$

So we show that

$$2 + 8\left\{1 - \frac{1}{J(X)}\right\}^2 \leq \left(1 + \sqrt{J(X) - 1}\right)^2. \tag{3.7}$$

Put $u = \sqrt{J(X) - 1}$. Then $\sqrt{\sqrt{2} - 1} \leq u \leq 1$ and the inequality (3.7) is reformulated as

$$2 + 8\left(\frac{u^2}{u^2 + 1}\right)^2 \leq (1 + u)^2,$$

or

$$2(u^2 + 1)^2 + 8u^4 \leq (u + 1)^2(u^2 + 1)^2. \tag{3.8}$$

Noting that $4u^3 \geq 4(\sqrt{2} - 1)\sqrt{\sqrt{2} - 1} > 1$, we have

$$(u + 1)^2(u^2 + 1)^2 - 2(u^2 + 1)^2 - 8u^4 = (u - 1)^2(u^4 + 4u^3 - 1) \geq 0.$$

Hence the inequality (3.8) holds and equality attains in (3.8) only when $u = 1$, or $J(X) = 2$. This completes the proof. □

Theorem 3.12 (Takahashi and Kato [15, Theorem 3]; see also [16, 19]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \le J(X), \tag{1.6}$$

where equality holds only when X is not uniformly non-square.

Proof. By Corollaries 3.7 and 3.11 we have

$$C_{NJ}(X) \le 1 + \left[\sqrt{2C'_{NJ}(X)} - 1\right]^2 \le J(X). \tag{3.9}$$

Though the latter assertion is shown in [15], it can be also easily seen from our discussion. Indeed if X is uniformly non-square, the second inequality of (3.9) is strict by Corollary 3.11. □

We shall close this section with presenting a sharper inequality than (1.6) which is a consequence of Theorem 3.4 and the following recent result of the present authors [15].

Lemma 3.13 ([15, Theorem 1]). *Let X be a Banach space. Then*

$$\rho_X(1) \le 2\left\{1 - \frac{1}{J(X)}\right\}. \tag{3.10}$$

Theorem 3.14 *Let X be a Banach space. Then*

$$C_{NJ}(X) \le \left(\frac{\sqrt{2}(J(X)-1) + \sqrt{(J(X)-1)^2+1}}{J(X)}\right)^2 \le J(X), \tag{1.8}$$

where the second inequality is strict if and only if X is uniformly non-square.

Proof. Let $f(u, v)$ be as in Lemma 3.6. Then Theorem 3.4 says that $C_{NJ}(X) \le f(C'_{NJ}(X), \rho_X(1))$. We write J for $J(X)$. By Theorem 3.10 and Lemma 3.13

$$C'_{NJ}(X) \le 1 + 4\left(1 - \frac{1}{J}\right)^2 \quad \text{and} \quad \rho_X(1) \le 2\left(1 - \frac{1}{J}\right)$$

respectively. Hence we have

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(1+4(1-1/J)^2, 2(1-1/J)) \\ &= \frac{1}{J^2}\left[3J^2-6J+4+2\sqrt{2}(J-1)\sqrt{(J-1)^2+1}\right] \\ &= \frac{1}{J^2}\left[\sqrt{2}(J-1)+\sqrt{(J-1)^2+1}\right]^2, \end{aligned}$$

which proves the first inequality. The second inequality of (1.8) attains equality if $J=2$. Assume that $J<2$. Then

$$1+4\left(1-\frac{1}{J}\right)^2 \le \frac{1}{2}(1+\sqrt{J-1})^2$$

by (3.7) and

$$2\left(1-\frac{1}{J}\right) < \sqrt{J-1}.$$

Therefore by Lemma 3.6 we have

$$f(1+4(1-1/J)^2, 2(1-1/J)) < f((1+\sqrt{J-1})^2/2, \sqrt{J-1}) = J.$$

This completes the proof. □

4 Unifying sight on previous results

First we note the following.

Proposition 4.1 *Let X be a Banach space. Then*

$$C'_{NJ}(X) \le \frac{1}{2}\left(1+\sqrt{J(X)-1}\right)^2 \le J(X) \le 1+\frac{J(X)^2}{4} \tag{4.1}$$

and

$$\rho_X(1) \le \sqrt{J(X)-1} \le \sqrt{2J(X)}-1 \le \frac{J(X)}{2}. \tag{4.2}$$

Indeed the first inequality of (4.1) is (1.13) of Corollary 3.11 and that of (4.2) is the result of Wang and Pang [17, Theorem 1], which is also a direct consequence of Lemma 3.13 ([15, Theorem 1]). The other inequalities are easy to see.

Remark 4.2 (i) $C'_{NJ}(X) = J(X)$ *if and only if* X *is not uniformly non-square. Indeed if* X *is uniformly non-square, the second inequality of (4.1) is strict by Corollary 3.11.*

(ii) $\rho_X(1) = \sqrt{J(X) - 1}$ *if and only if* X *is not uniformly non-square (Takahashi and Kato [15, Corollary 1]).*

Owing to Theorem 3.4 we can see all the previous results (1.2), (1.3), (1.5), (1.6) on $C_{NJ}(X)$ and $J(X)$ in a unifying sight.

Corollary 4.3 (i) (Nikolova-Persson-Zachariades [13, Remark]; see also [12])

$$C_{NJ}(X) \le \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}\left\{\sqrt{J(X)^2 - 4J(X) + 8} - 2\right\}. \tag{1.2}$$

(ii) (Alonso et al. [2, Theorem 2])

$$C_{NJ}(X) \le 2\left\{1 + J(X) - \sqrt{2J(X))}\right\}, \tag{1.3}$$

(iii) (Wang and Pang [17, Theorem 4])

$$C_{NJ}(X) \le J(X) + \sqrt{J(X) - 1}\left[\sqrt{1 + \left(1 - \sqrt{J(X) - 1}\right)^2} - 1\right]. \tag{1.5}$$

(iv) (Takahashi and Kato [15, Theorem 3])

$$C_{NJ}(X) \le J(X). \tag{1.6}$$

Proof. Let $f(u, v)$ be as in Lemma 3.6. Then by Theorem 3.4

$$C_{NJ}(X) \le f(C'_{NJ}(X), \rho_X(1)), \tag{4.3}$$

from which, by noting Proposition 4.1 and Lemma 3.6, all the results stated above will be derived as follows.

(i) Since $C'_{NJ}(X) \le 1 + J(X)^2/4$ and $\rho_X(1) \le J(X)/2$, we have

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(1 + J(X)^2/4, J(X)/2) \\ &= \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}\left\{\sqrt{J(X)^2 - 4J(X) + 8} - 2\right\}. \end{aligned}$$

(ii) Since $C'_{NJ}(X) \le J(X)$ and $\rho_X(1) \le \sqrt{2J(X)} - 1$,

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(J(X), \sqrt{2J(X)} - 1), \\ &= 2\left\{1 + J(X) - \sqrt{2J(X)}\right\}. \end{aligned}$$

(iii) Since $C'_{NJ}(X) \le J(X)$ and $\rho_X(1) \le \sqrt{J(X) - 1}$,

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(J(X), \sqrt{J(X) - 1}) \\ &= J(X) + \sqrt{J(X) - 1}\left[\sqrt{1 + \left(1 - \sqrt{J(X) - 1}\right)^2} - 1\right]. \end{aligned}$$

(iv) Since $C'_{NJ}(X) \le \left(1 + \sqrt{J(X) - 1}\right)^2/2$ and $\rho_X(1) \le \sqrt{J(X) - 1}$,

$$\begin{aligned} C_{NJ}(X) &\le f(C'_{NJ}(X), \rho_X(1)) \\ &\le f(\left(1 + \sqrt{J(X) - 1}\right)^2/2, \sqrt{J(X) - 1}) \\ &= J(X). \end{aligned}$$

This completes the proof. □

5 Some other consequences

Alonso et al. [2, Corollary 2] showed that

$$C_{NJ}(X) - C'_{NJ}(X) \le 3 - 2\sqrt{2} \tag{5.1}$$

and they asked whether this estimate is sharp. We have the following.

Theorem 5.1 *Let X be a Banach space. Then*

$$C_{NJ}(X) - C'_{NJ}(X) \le \frac{1}{2}\left(1 - \rho_X(1)\right)^2, \tag{5.2}$$

where equality holds if and only if $C_{NJ}(X) = 1 + \rho_X(1)^2$ and $C'_{NJ}(X) = (1 + \rho_X(1))^2/2$.

Proof. By Theorem 3.4 and Proposition 3.2 we have

$$\begin{aligned} C_{NJ}(X) - C'_{NJ}(X) &\leq \sqrt{(1+\rho_X(1)-C'_{NJ}(X))^2+\rho_X(1)^2} - \rho_X(1) \\ &\leq \sqrt{[1+\rho_X(1)-(1+\rho_X(1))^2/2]^2+\rho_X(1)^2} - \rho_X(1) \\ &= \frac{1}{2}(1-\rho_X(1))^2. \end{aligned}$$

Next suppose that equality holds in (5.2). Then by the above observation we have $C'_{NJ}(X) = (1+\rho_X(1))^2/2$ and $C_{NJ}(X) = C'_{NJ}(X) + (1-\rho_X(1))^2/2 = 1+\rho_X(1)^2$. The converse is clearly true, which completes the proof. □

Remark 5.2 *The inequality (5.2) attains equality with the spaces which are not uniformly non-square and also with the ℓ_2-ℓ_∞ space X as $C_{NJ}(X) = 3/2$, $C'_N(X) = (3+2\sqrt{2})/4$ and $\rho_X(1) = 1/\sqrt{2}$ (see Remark 3.5).*

Finally we shall present a negative answer to the above question by Alonso et al. [2].

Corollary 5.3 *For any Banach space X*

$$C_{NJ}(X) - C'_{NJ}(X) < 3 - 2\sqrt{2}. \tag{5.3}$$

Proof. Since $\rho_X(1) \geq \sqrt{2}-1$, by Theorem 5.1 we have

$$C_{NJ}(X) - C'_{NJ}(X) \leq \frac{1}{2}(1-\rho_X(1))^2 \leq 3-2\sqrt{2}. \tag{5.4}$$

Suppose that $C_{NJ}(X) - C'_{NJ}(X) = 3-2\sqrt{2}$. Then by (5.4) we have $\rho_X(1) = \sqrt{2}-1$ and equality holds in (5.2). By Theorem 5.1 we have $C_{NJ}(X) = 1+\rho_X(1)^2 = 1+(\sqrt{2}-1)^2$ and $C'_{NJ}(X) = (1+\rho_X(1))^2/2 = 1$. From the latter identity X is a Hilbert space, which is impossible as $C_{NJ}(X) > 1$. This completes the proof. □

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Some remarks and new developments concerning Hardy-type inequalities

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Abstract. In this paper we present some complementery information to that given in the recent historical paper [19] and book [20]. A simple convexity argument is used to prove Hardy's inequality. This unified approach also implies a number of other classical inequalities (e.g. Hardy's discrete inequality, Pólya-Knopp's inequality, Carleman's inequality, some weighted variants of Hardy's inequality etc.). Some new results e.g. from recent PhD theses concerning characterizations of multidimensional forms of Hardy's inequality are presented and discussed in this light. Moreover, some new results concerning Hardy type inequalities in Morrey spaces are presented and a number of examples, remarks and ideas for further research are included.

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1 Introduction

The famous Hardy inequality in continuous form reads

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \qquad (1.1)$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

where $p > 1$ and f is a non-negative p-integrable function. The dramatic more than 10 years period until G.H. Hardy finally formulated and proved (1.1) in his 1925 paper [11] (see also [10]) has recently been described in detail in [19]. After that this inequality has been generalized and applied in an almost unbelievable way, see e.g. the monographs [16], [20], [21], [31] and the references given therein. And the interest seems to even have increased during the very last years. In this article we complement the historical article [19] and book [20] in various ways and hope that this complementary information can both give some interesting new information and serve as a source of inspiration for further research in this fascinating area.

In **Section 2** we present a very simple proof of Hardy's inequality, which was first discovered by E.K. Godunova already 1965 in [8] but this paper seems not to have influenced the development so much but after this result was rediscovered and complemented in a paper by S.Kaijser et al [14] in 2002 it has been presented several results in this direction, see e.g. the review article [28] and Theorems 3.17 and 3.20 in this paper. If Hardy himself had discovered this simple proof (i.e. that (1.1) follows almost immediately from Jensen's inequality (from 1906) and a variable transformation) we are sure that the prehistory described in [19] had been very different. This fact could also have influenced in an essential way the continued development of Hardy type inequalities as described e.g. in the books [16], [20], [21] and [31]. Moreover, we show that this simple convexity argument also directly gives many other classical inequalities like Hardy's discrete inequality, Carleman's inequality and Pólya-Knopp's inequality and even the first weighted form of (1.1), which obviously Hardy believed was a genuine generalization of (1.1) but our argument shows that in fact this weighted form is equivalent to (1.1) (see Remark 2.11).

In **Section 3** we discuss some recent results concerning characterizations of multidimensional weighted forms of (1.1) for different parameters p and q . We start with the famous two-dimensional result of E. Sawyer in [37]. After that we review and comment upon a number of fairly new results e.g. those obtained in PhD theses by A. Wedestig [40], E. Ushakova [39] and M. Johansson [13]. Hence this paper complement the book [20] in an essential way. See also the book [16], where some complementary information can be found.

In the book [20] a number of Hardy type inequalities for various function spaces (other than weighted L^p-spaces) are presented and discussed. Note that weighted estimates of the Hardy operators with respect to the norm of the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent were obtained

e.g. in [6], [17] and [24].

In **Section 4** in this paper we give a new complement to this, namely we present and discuss some recent results concerning Hardy type inequalities in weighted Morrey spaces. Hardy inequalities with respect to the norm of Morrey spaces are quite recent results even in both the one-dimensional and non-weighted cases. Such inequalities in the one-dimensional case and for a class of weights were given in the paper [35] from 2009 for $p \to p$-inequalities, and in [34] for $p \to q$-inequalities. We survey these recent results and also present some corresponding multi-dimensional results.

2 A convexity approach to classical Hardy type inequalities

2.1 Remarks on four classical inequalities

A). Hardy's inequality (continuous form:)

If f is non-negative and $p-$ integrable over $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx), \quad p>1. \tag{2.1}$$

B). Hardy's inequality (discrete form:)

If $\{a_n\}_1^\infty$ is a sequence of non-negative numbers, then

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{n=1}^n a_i\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p, \quad p>1. \tag{2.2}$$

Remark 2.1 *The dramatic more than 10 years period of research until Hardy stated in [10] and proved in [11] his inequality* (2.1) *was recently described in [19]. It is historically clear that Hardy's original motivation when he discovered his inequalities was to find a simple proof of Hilbert's double series inequality.*

Remark 2.2 *It is clear that* (2.2) $\Rightarrow$ (2.1). *More exactly, by applying* (2.1) *with step functions we obtain* (2.2). *This was pointed out to Hardy in a private letter from F. Landau already in 1921 and here Landau even included a proof of* (2.2) *so it should not be wrong to even call* (2.2) *the Hardy-Landau inequality.*

C). Carleman's inequality:

If $\{a_n\}_1^\infty$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdots a_n} \quad \leq \quad e \sum_{n=1}^{\infty} a_n. \tag{2.3}$$

Remark 2.3 *This inequality was proved by T. Carleman in 1922 ([4]) in connection to this important work on quasianalytical functions. Carleman's idea of proof was to find maximum of* $\sum_{i=1}^{n}(a_1 \cdots a_i)^{1/i}$ *under the constraint* $\sum_{i=1}^{n} a_i = 1, n \in Z_+$. *However,* (2.3) *is in fact a limit inequality (as* $p \to \infty$*) of the inequalities* (2.2) *for* $p > 1$ *according to the following:*

Replace a_i *with* $a_i^{1/p}$ *in the Hardy discrete inequality and we obtain that*

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{n=1}^{n} a_i^{1/p} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n, \ p > 1.$$

Moreover, when $p \to \infty$ *we have that*

$$\left(\frac{1}{n} \sum_{i=1}^{n} a_i^{1/p} \right)^p \to \left(\prod_{i=1}^{n} a_i \right)^{1/n} \quad \text{and} \quad \left(\frac{p}{p-1} \right)^p \to e.$$

In view of the fact that Carleman and Hardy had a direct cooperation at that time (see e.g. [5]) it is maybe a surprise that Carleman did not mention this fact in his paper.

D). Pólya-Knopp's inequality:

If f is a positive and integrable function on $(0, \infty)$, then

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x ln f(y) dy \right) dx \leq e \int_0^\infty f(x) dx. \tag{2.4}$$

Remark 2.4 *Sometimes* (2.4) *is referred to as the Knopp inequality with reference to his 1928 paper [15]. But it is clear that it was known before and in his 1925 paper [11] Hardy informed that G. Pólya had pointed out the fact that* (2.4) *is in fact a limit inequality (as* $p \to \infty$*) of the inequality* (1.1) *for* $p > 1$ *and the proof is literally the same as that above that* (2.2) *implies* (2.3). *Accordingly, nowadays* (2.4) *is many times referred to as Pólya-Knopp's inequality and we have adopted this term.*

2.2 An equivalent form of Hardy's inequality (1.1) and its limit inequality (2.4)

In the sequel of this Section we assume that f is a non-negative and measurable function over $\mathbb{R}_+$.

Before we perform the discussion in this Section we present two crucial observations:

Observation 2.1: We note that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1 \tag{2.5}$$

$$\Leftrightarrow$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1\cdot\int_0^\infty g^p(x)\frac{dx}{x}, \tag{2.6}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$.

In fact, consider (2.5) and we find that

$$RHS = \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx = \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x^{1-1/p})\frac{dx}{x} =$$

$$= \left(\frac{p}{p-1}\right)^{p+1} \int_0^\infty g^p(y)\frac{dy}{y},$$

$$LHS = \int_0^\infty \left(\frac{1}{x}\int_0^x g(t^{1-1/p})t^{-1/p}dt\right)^p dx =$$

$$= \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(\frac{1}{x}\int_0^{x^{1-1/p}} g(s)ds\right)^p dx =$$

$$= \left(\frac{p}{p-1}\right)^{p+1} \int_0^\infty \left(\frac{1}{y}\int_0^y g(s)ds\right)^p \frac{dy}{y}.$$

Remark 2.5 *We see that Hardy's inequality* (1.1) *is equivalent to* (2.6) *for* $p>1$ *and, thus, that Hardy's inequality can be proved in the following simple way (see form* (2.6)*): By Jensen's inequality and Fubini's theorem we have that*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x} =$$

$$= \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.$$

In a similar and even simpler way we can verify the following:

Observation 2.2: It yields that

$$\int_0^\infty exp\left(\frac{1}{x}\int_0^x lnf(y)dy\right)dx \le e\int_0^\infty f(x)dx \tag{2.7}$$

$$\Leftrightarrow$$

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x lng(y)dy\right)\frac{dx}{x} \le 1\cdot\int_0^\infty g(x)\frac{dx}{x}, \tag{2.8}$$

where $f(x)=\frac{g(x)}{x}$.

Remark 2.6 *According to this Observation, Jensen's inequality and Fubini's theorem we see that also the limit Pólya-Knopp's inequality* (2.4) *follows directly from Jensen's inequality.*

In the same way as above, we also have the following more general Statement (see also [14]):

Proposition 2.7 *Let f be a non-negative and measurable function on $I\subset\mathbb{R}$ and let $\Phi: I\to\mathbb{R}_+$ be a convex function. Then*

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} \le \int_0^\infty \Phi\big(f(x)\big)\frac{dx}{x}. \tag{2.9}$$

If Φ instead is positive and concave, then the reversed inequality holds.

Proof. By Jensen's inequality and Fubini's theorems we have that

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} \le \int_0^\infty\int_0^x \Phi(f(y))dy\frac{dx}{x^2} =$$

$$=\int_0^\infty \Phi(f(y))\int_y^\infty \frac{1}{x^2}dx\ dy = \int_0^\infty \Phi(f(y))\frac{dy}{y}.$$

If Φ is concave the only inequality holds in the reversed direction.

2.3 Some examples and further remarks

Example 2.8 *Consider the convex function* $\Phi(u) = u^p, p \geq 1 \quad or \quad p < 0$. *Then* (2.9) *reads*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p \frac{dx}{x} \leq \int_0^\infty f^p(x)\frac{dx}{x}, \tag{2.10}$$

i.e. according to Observation 2.1, we obtain Hardy's original inequality (1.1) *for* $p > 1$. *For* $0 < p < 1$ (2.10) *holds in the reversed direction.*

Example 2.9 *Consider the convex function* $\Phi(u) = e^u$ *and replace* $f(y)$ *with* $ln\ f(y)$. *Then* (2.9) *reads*

$$\int_0^\infty exp\left(\frac{1}{x}\int_0^x lnf(y)dy\right)\frac{dx}{x} \leq \int_0^\infty f(x)\frac{dx}{x},$$

i.e. according to Observation 2.2, we obtain also Pólya-Knopp's inequality directly without going via some limit argument.

In 1928 G.H. Hardy himself (see [12]) proved the first generalization of his inequality (1.1) namely the following:

The inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a dx \tag{2.11}$$

holds for all measurable and non-negative functions f on $(0,\infty)$ whenever $a < p-1, p > 1$.

We remark that (2.11) in fact also is equivalent to (2.5) and, thus, to (1.1) . More generally the following statement follows directly from (2.6):

Example 2.10 *Let* f *be a measurable and non-negative function on* $(0,\infty)$. *Then*

a) the inequality (2.11) *holds whenever* $a < p-1$ *and* $p > 1$ *or* $p < 0$.

b) the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^a dx \leq \left(\frac{p}{a+1-p}\right)^p \int_0^\infty f^p(x)x^a dx \tag{2.12}$$

holds whenever $a > p-1, \quad p > 1$ *or* $p < 0$.

c) For the case $0 < p < 1$ *the inequalities* (2.11) *and* (2.12) *hold in the reversed direction.*

In fact, statement a) follows by just using the substitution $f(t) = g(t^{\frac{p-1-a}{p}})t^{-\frac{1+a}{p}}$, *making some straightforward calculations as in Observation 2.1 to see that* (2.11) *is equivalent to* (2.6) *and, thus, to* (2.5). *Moreover, b) follows from a) by making some other obvious standard substitutions.*

The proof of c) is the same as that of a) and b) and the fact that now the basic inequality (2.6) *holds in the reversed direction (see Example 2.8).*

Remark 2.11 *The above means in particular that the first generalization* (2.11) *by Hardy in fact is not more general than the original form* (1.1) *because both are equivalent to* (2.6).

By using the simple technique presented in this Subsection and the refined Jensen type inequality for superquadratic or subquadratic functions recently proved in the paper [1] the following refined Hardy type inequality can be proved (see [27] , Theorem 3.1):

Theorem 2.12 *Let the function* f *be a non-negative and measurable on* $(0,\infty)$ *and let* $k>1$.

(i) If $p \geq 2$, *then*

$$\int_0^\infty x^{-k}\left(\int_0^x f(t)dt\right)^p\,dx + \tag{2.13}$$

$$\frac{k-1}{p}\int_0^\infty\int_t^\infty \left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right|^p x^{p-k-\frac{k-1}{p}}dx\, t^{\frac{k-1}{p}-1}dt$$

$$\leq \left(\frac{p}{k-1}\right)^p \int_0^\infty f^p(x)\; x^{p-k}dx.$$

(ii) If $1<p\leq 2$, *then the inequality* (2.13) *holds in the reversed direction.*

Remark 2.13 *Note that this result means that if* $p \geq 2$, *then the classical weighted Hardy inequality* (2.11) *for* $k = p-a > 1$ *can be refined by adding a second term on the left-hand side. In fact, this factor is so big that the inequality holds in the reversed direction for* $1 < p \leq 2$ *so that, in particular, for* $p = 2$ (2.13) *reduces to an identity.*

Hence, in this case the "breaking point", where the sign of the inequality is changed, is $p = 2$, *while in the classical (non-refined) Hardy's inequality the natural breaking point is* $p = 1$.

Remark 2.14 *As expected a similar dual refined Hardy inequality holds when* $\int_0^x f(t)dt$ *is replaced by* $\int_x^\infty f(t)dt$ *and* $k < 1$ *(see [28], Theorem 3.2).*

Remark 2.15 *The simple technique presented above can be used to obtain multidimensional generalizations of several results in this Section. We give some examples of this fact in our Subsection 3.3 (see Theorems 3.17 and 3.20).*

Remark 2.16 *The constants in all inequalities presented in this Subsection are sharp. In particular, this means that this simple convexity technique also gives the best possible result in all cases derived so far.*

Summing up our investigations presented so far in this Section we strongly believe that both the prehistory of Hardy's inequalities and the continued development of Hardy type inequalities could have been very different if Hardy himself had discovered this simple convexity technique of proof.

2.4 Some modern forms of the four classical inequalities.

Here and in the sequel of this Section by a weight (weight sequence) we mean a non-negative, measurable and locally integrable function over the considered interval (a sequence $v = \{v_i\}_1^\infty$, with $v_i \geq 0, i = 1, 2, ...$).

It is natural to raise the following questions connected to the inequalities A) - D) in Subsection 2.1:

1. For which weights u and v does it hold that

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p}, \quad 0 < b \leq \infty,$$

for some finite constant C ?

2. For which weights u and v does it hold that

$$\left(\int_0^b \left[\exp\frac{1}{x}\int_0^x lnf(t)dt\right]^q u(x)dx\right)^{1/q} \leq C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p}, 0 < b \leq \infty,$$

for some finite constant C ?

3. For which weight sequences $u = \{u_k\}$ and $v = \{v_k\}$ does it hold that

$$\left(\sum_{n=1}^\infty \left(\sum_{i=1}^n a_i\right)^q u_n\right)^{1/q} \leq C\left(\sum_{n=1}^\infty a_n^p v_n\right)^{1/p}$$

for some finite constant C ?

4. For which weight sequences $u = \{u_k\}$ and $v = \{v_k\}$ does it hold that

$$\left(\sum_{n=1}^{\infty} (\sqrt[n]{a_1 \cdots a_n})^q \, u_n\right)^{1/q} \quad \leq \quad C \left(\sum_{n=1}^{\infty} a_n^p v_n\right)^{1/p}$$

for some finite constant C ?

During the last 80 years it has been a lot of activities to answer these and more general questions concerning Hardy type inequalities and a lot of interesting results have been presented and applied (see e.g. the books [16], [18], [20], [21], [23], [31], [38] and the references given there).

Just as one example and for the sake of comparison later on with the multi-dimensional case we mention the following well known result:

Theorem 2.17 *Let $1 < p \leq q < \infty$ and u and v be weights on $\mathbb{R}_+$. Then each of the following conditions is necessary and sufficient for the inequality*

$$\left(\int_0^{\infty} \left(\int_0^x f(t)\,dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_0^{\infty} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{2.14}$$

to hold for all positive and measurable functions on $\mathbb{R}_+$:

a) The Muckenhoupt-Bradley condition,

$$A_M := \sup_{x>0} \left(\int_x^{\infty} u(t)\,dt\right)^{\frac{1}{q}} \left(V(x)\right)^{\frac{1}{p'}} < \infty, \quad V(x) := \int_0^x v(t)^{1-p'}\,dt, \tag{2.15}$$

with the estimation $C \in [A_M, \lambda A_M]$ for the best constant C in (2.14)*, where*

$$\lambda = min(p^{1/q}(p')^{1/p'}, q^{1/q}(q')^{1/p'}).$$

b) The condition

$$A_{PS} := \sup_{x>0} V(x)^{-\frac{1}{p}} \left(\int_0^x u(t)V(t)^q\,dt\right)^{\frac{1}{q}} < \infty, \tag{2.16}$$

with $C \in [A_{PS}, p'A_{PS}]$ for the best constant in (2.14).

Remark 2.18 *The dramatic history until* (2.15) *was derived was recently described in the book [20]. The condition* (2.16) *was derived e.g. by L.E. Persson and V. Stepanov in [32]. It is nowadays known that also these two conditions are not unique and can in fact be replaced by infinite many conditions or more exactly by four different scales of conditions (see [9]).*

3 Multidimensional Hardy-type inequalities

In this Section by a weight we mean a non-negative, measurable and locally integrable function on $\mathbb{R}^n_+,\ n \in \mathbb{Z}$.

3.1 Some two-dimensional results

We first recall that the following two-dimensional inequality, which was proved by E.T. Sawyer, see [37], Theorem 1:

Theorem 3.1 *Let $1 < p \le q < \infty$ and u and v be weights on $\mathbb{R}^2_+$. Then the inequality*

$$\left(\int_0^\infty\int_0^\infty \left(\int_0^{x_1}\int_0^{x_2} f(t_1,t_2)dt_1dt_2\right)^q u(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{q}} \le$$

$$\le C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{p}} \tag{3.1}$$

holds for all non-negative and measurable functions on $\mathbb{R}^2_+$, if and only if the following three conditions are satisfied:

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\left(\int_{y_1}^\infty\int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p'}} < \infty, \tag{3.2}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\frac{\left(\int_0^{y_1}\int_0^{y_2}\left(\int_0^{x_1}\int_0^{x_2} v(t_1,t_2)^{1-p'}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p}}} < \infty, \tag{3.3}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \frac{\left(\int\limits_{y_1}^{\infty}\int\limits_{y_2}^{\infty}\left(\int\limits_{x_1}^{\infty}\int\limits_{x_2}^{\infty}u(t_1,t_2)dt_1dt_2\right)^{p'} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int\limits_{y_1}^{\infty}\int\limits_{y_2}^{\infty}u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q'}}} < \infty. \tag{3.4}$$

All three conditions (3.2)-(3.4) are independent and no one may be removed.

Remark 3.2 *Note that* (3.2) *corresponds to the Muckenhoupt-Bradley condition* (2.15), (3.3) *corresponds to the condition* (2.16) *and* (3.4) *corresponds to the dual condition of* (2.16). *According to Remark 2.18 all these conditions are equivalent in the one-dimensional case but it is not so in the two-dimensional case.*

One of the recent progresses related to Theorem 3.1 was obtained in A. Wedestig's PhD thesis [40]. It was shown there that in the case where the weight $v(x_1,x_2)$ on the right-hand side of (3.1) has the form of the product $v_1(x_1)v_2(x_2)$, then only one condition appears (but this condition is not unique and can in fact be given in infinite many forms). Namely, the following statement holds:

Theorem 3.3 *Let $1<p\le q<\infty$ and let u be a weight on $\mathbb{R}^2_+$ and v_1 and v_2 be weights on $\mathbb{R}_+$. Then the inequality*

$$\left(\int_0^{\infty}\int_0^{\infty}\left(\int_0^{x_1}\int_0^{x_2}f(t_1,t_2)dt_1dt_2\right)^{q}u(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{q}} \le$$

$$\le C\left(\int_0^{\infty}\int_0^{\infty}f^p(x_1,x_2)v_1(x_1)v_2(x_2)\,dx_1dx_2\right)^{\frac{1}{p}} \tag{3.5}$$

holds for all non-negative and measurable functions f on $\mathbb{R}^2_+$, if and only if

$$A_W(s_1,s_2) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} (V_1(t_1))^{\frac{s_1-1}{p}}\,(V_2(t_2))^{\frac{s_2-1}{p}}\times$$

$$\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} u(x_1,x_2)(V_1(x_1))^{q\frac{p-s_1}{p}}(V_2(x_2))^{q\frac{p-s_2}{p}}\,dx_1dx_2\right)^{\frac{1}{q}}<\infty \tag{3.6}$$

holds for some $s_1,s_2\in(1,p)$ *(and, hence, for all* $s_1,s_2\in(1,p)$*)), where* $V_i(t_i):=\int_0^{t_i} v_i(\xi)^{1-p'}d\xi,\ i=1,2$. *Moreover for the best constant* C *in* (3.1) *it yields that* $C\approx A_W(s_1,s_2)$.

A limit result of Theorem 3.3 is the following two-dimensional Pólya-Knopp type inequality, which was also proved in [40]:

Theorem 3.4 *Let* $0<p\le q<\infty$ *and* u *and* v *be weights on* $\mathbb{R}^2_+$. *Then the inequality*

$$\left(\int_0^{\infty}\int_0^{\infty}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log f(t_1,t_2)dt_1dt_2\right)\right]^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}$$

$$\le C\left(\int_0^{\infty}\int_0^{\infty} f^p(x_1,x_2)v(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}} \tag{3.7}$$

holds for all non-negative and measurable functions f *on* $\mathbb{R}^2_+$ *if and only if*

$$\sup_{y_1>0,y_2>0} y_1^{\frac{s_1-1}{p}}y_2^{\frac{s_2-1}{p}}\left(\int_{y_1}^{\infty}\int_{y_2}^{\infty} x_1^{-\frac{s_1q}{p}}x_2^{-\frac{s_2q}{p}}w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}<\infty, \tag{3.8}$$

holds for some $s_1>1,s_2>1$ *(and thus for all* $s_1>1,s_2>1$ *) and where*

$$w(x_1,x_2):=u(x_1,x_2)\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}dt_1dt_2\right)\right]^{\frac{q}{p}}.$$

Remark 3.5 *Observe that this limit inequality indeed holds for all weights (and not only for product weights on the right hand side) and also for* $0<p\le 1$. *The reason for this comes from the useful technical details when we perform the limit procedure, e.g. that we first do a substitution so we only need to use the case when the weight in the right hand side in* (3.5) *is equal to 1. Also here we have a good estimate of the best constant* C *in* (3.7)*, see [40].*

Corollary 3.6 *Let $0 < p \le q < \infty$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^1$. Then the inequality*

$$\left(\int_0^\infty \int_0^\infty \left[\exp\left(\frac{1}{x_1 x_2}\int_0^{x_1}\int_0^{x_2} \log f(t_1,t_2)dt_1dt_2\right)\right]^q x_1^{\alpha_1}x_2^{\alpha_2})dx_1dx_2\right)^{\frac{1}{q}} \le$$

$$\le C\left(\int_0^\infty \int_0^\infty f^p(x_1,x_2)x_1^{\beta_1}, x_2^{\beta_2}dx_1dx_2\right)^{\frac{1}{p}} \tag{3.9}$$

holds for all non-negative and measurable functions f on $\mathbb{R}^2_+$ if and only if

$$\frac{\alpha_i+1}{q} = \frac{\beta_i+1}{p}, \quad i = 1,2.$$

Moreover, the best constant C for (3.9) satisfies the inequality $C \le e^{\frac{\beta_1+\beta_2}{p}+\frac{2}{q}}$.

Remark 3.7 *Note that it can be proved that the constant $C = e^{\frac{\beta_1+\beta_2+2}{p}}$ is sharp in the case where $p = q$ and $\alpha_1 = \beta_1, \alpha_2 = \beta_2$.*

Remark 3.8 *The corresponding statements as those in Theorems 3.3 and 3.4 hold also for any dimension n (see [40]) . However, in our next Subsection we will present some results from the PhD thesis of E.Ushakova [39] (see also [33]), where also the case with product weights on the left hand side was considered. The proofs there are completely different from those in [40] and the obtained characterizations are different.*

3.2 Some multidimensional results

In the sequel we assume that f is a non-negative and measurable function.

Let $x = (x_1, ..., x_n), t = (t_1, ..., t_n) \in \mathbb{R}^n_+, n \in Z_+$ and $1 < p \le q < \infty$. We consider the $n-$ dimensional Hardy type operator

$$(H_n f)(x) = \int_0^{x_1} \cdots \int_0^{x_n} f(t)dt$$

and study the inequality

$$\left(\int_{\mathbb{R}^n_+} (H_n f)^q(x)u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n_+} f^p(x)v(x)dx\right)^{\frac{1}{p}}. \tag{3.10}$$

Sometimes we assume that one of the involved weight functions v and u is of product type, i.e. that

$$u(x) = u_1(x_1)u_2(x_2)\cdots u_n(x_n), \qquad (LP)$$

or

$$v(x) = v_1(x_1)v_2(x_2)\cdots v_n(x_n). \qquad (RP)$$

Moreover,

$$U(t) = U(t_1,\cdots,t_n) := \int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty} u(x)dx$$

and

$$V(t) = V(t_1,\cdots,t_n) := \int_0^{t_1}\cdots\int_0^{t_n} (v(x))^{1-p'}dx.$$

The next Statement gives a necessary condition for (3.10) to hold with help of some $n-$ dimensional versions of the constants A_M and A_{PS} in Theorem 2.17 (see [33] and also [39]).

Theorem 3.9 *Let $1 < p \le q < \infty$ and assume that* (3.10) *holds for all non-negative and measurable functions f on $\mathbb{R}^n_+$ with a finite constant C, which is independent on f. Then*

$$A_M^{(n)} := \sup_{t_i>0}(U(t_1,\cdots,t_n))^{1/q}(V(t_1,\cdots,t_n))^{1/p'} < \infty,$$

and

$$A_{PS}^{(n)} := \sup_{t_i>o}(V(t_1,\cdots,t_n))^{-1/p}\left(\int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty} u(x)V^q(x)dx\right)^{1/q} < \infty.$$

Proof. This statement follows by substituting the function

$$f_t(y) := \mathcal{H}i_{[0,t_1]}(y_1)\cdots\mathcal{H}i_{[0,t_n]}(v(y))^{1-p'}$$

into the inequality (3.10) . Then we get that

$$C \ge \frac{\left(\int_{\mathbb{R}^n_+}\left(\int_0^{x_1}\cdots\int_0^{x_n} f_t(y)dy\right)^q u(x)dx\right)^{1/q}}{\left(\int_{\mathbb{R}^n_+} f_t^p(y)v(y)dy\right)^{1/p}} \ge$$

$$\frac{\left(\int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty} u(x)dx\right)^{1/q}\left(\int_0^{t_1}\cdots\int_0^{t_n}(v(y))^{1-p'}dy\right)}{\left(\int_0^{t_1}\cdots\int_0^{t_n}(v(y))^{1-p'}dy\right)^{1/p}} = U^{1/p}(t)V^{1/p'}(t).$$

Thus the condition $A_M^{(n)} < \infty$ follows by just taking supremum over all $t_i > 0, i = 1, 2, ..., n$.

In a similar way the condition $A_{PS}^{(n)} < \infty$ follows by making the following estimate:

$$C \geq \frac{\int_0^{t_1} \cdots \int_0^{t_n} \left(\int_0^{x_1} \cdots \int_0^{x_n} f_t(y)dy\right)^q u(x)dx}{\left(\int_{\mathbb{R}^n_+} f_t^p(y)v(y)dy\right)^{1/p}} =$$

$$= V^{-\frac{1}{p}}(t) \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} u(x)V^q(x)dx\right)^{1/q}.$$

□

Our next result is that in the case of product weights on the right hand side we get a complete characterization of (3.10).

Theorem 3.10 *Let $1 < p \leq q < \infty$ and the weight v be of product type (RP). Then* (3.10) *holds for all non-negative and measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $A_M^{(n)} < \infty$ or $A_{PS}^{(n)} < \infty$. Moreover, $C \approx A_M^{(n)} \approx A_{PS}^{(n)}$ with constants of equivalence only depending on the parameters p and q and the dimension n.*

Note that here it yields that $V(t_1, \cdots, t_n) = V_1(x_1)V_2(x_2) \cdots V_n(x_n)$, where $V_i(t_i) := \int_0^{t_i} (v_i(x_i))^{1-p'} dx_i, i = 1, \cdots, n$. For a proof we refer to [33] (see also [39]).

In this paper also the case when u is of product type (LP) was considered and the dual of the constants $A_M^{(n)}$ and $A_{PS}^{(n)}$ were introduced:

$$A_M^{*(n)} := \sup_{t_i>0} \big(U_1(t_1) \cdots U_n(t_n)\big)^{1/q} \big(V(t_1, \cdots, t_n)\big)^{1/p'} < \infty,$$

and

$$A_{PS}^{*(n)} := \sup_{t_i>0} \big(U_1(t_1) \cdots$$

$$\cdots U_n(t_n)\big)^{-1/q'} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} v^{1-p'}(x) \big(U_1(x_1) \cdots U_n(x_n)\big)^{p'} dx\right)^{1/p'}.$$

Theorem 3.11 *Let $1 < p \leq q < \infty$ and the weight u be of product type (LP). Then* (3.10) *holds for all non-negative and measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent of f, if and only*

if $A_M^{*(n)} < \infty$ *or* $A_{PS}^{*(n)} < \infty$. *Moreover,* $C \approx A_M^{*(n)} \approx A_{PS}^{*(n)}$ *with constants of equivalence only depending on the parameters* p *and* q *and the dimension* n.

In the paper [33] also the case $1 < q < p < \infty$ was considered and the following multidimensional versions of the usual Mazya-Rosin and Persson-Stepanov constants in one dimension (see e.g. [20]) were introduced:

$$B_{MR}^{(n)} := \left(\int_{\mathbb{R}_+^n} \big(U(t)\big)^{r/q} \big(V_1(t_1)\big)^{r/q'} \cdots \big(V_n(t_n)\big)^{r/q'} dV_1(t_1) \cdots dV_n(t_n) \right)^{1/r},$$

$$B_{PS}^{(n)} := \left(\int_{\mathbb{R}_+^n} \left(\int_0^{t_1} \cdots \int_0^{t_n} u(x) (V_1(x_1) \cdots V_n(x_n))^q dx \right)^{r/q} \times \right.$$

$$\left. \times \left(V_1(t_1) \cdots V_n(t_n) \right)^{-r/q} dV_1(t_1) \cdots dV_n(t_n) \right)^{1/r}.$$

Here, as usual, $1/r = 1/q - 1/p$. For technical reasons we also need the following additional condition:

$$V_1(\infty) = \cdots = V_n(\infty) = \infty.$$

Theorem 3.12 *Let* $1 < q < p < \infty$ *and* $1/r = 1/q - 1/p$. *Assume that the weight* v *is of product type* (RP). *Then* (3.10) *holds for all non-negative and measurable functions* f *on* $\mathbb{R}_+^n$ *with some finite constant* C, *which is independent on* f, *if and only if* $B_{MR}^{(n)} < \infty$, *or* $B_{PS}^{(n)} < \infty$. *Moreover,* $C \approx B_{MR}^{(n)} \approx B_{PS}^{(n)}$ *with constants of equivalence depending only on* p *and* q *and the dimension* n.

Remark 3.13 *Also the case when the left hand side is of product type was considered in [33] and a theorem similar to Theorem 3.12 was proved by using some dual forms of the constants* $B_{MR}^{(n)}$ *and* $B_{PS}^{(n)}$ *(see [33], Theorems 3.3 and 3.4).*

We finalize this Section by discussing some limit multidimensional (Pólya-Knopp type) inequalities.

Consider the inequality

$$\left(\int_{\mathbb{R}_+^n} (G_n f)^q(x) u(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}_+^n} f^p(x) v(x) dx \right)^{1/p}, \qquad (3.11)$$

where the $n-$ dimensional geometric mean operator G_n is defined by

$$(G_n f)(x) = \exp\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} lnf(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\right).$$

We denote

$$A_G^{(n)} := \sup_{t_i>0} (t_1 \cdots t_n)^{-1/p} \left(\int_0^{t_1} \cdots \int_0^{t_n} w(x)dx\right)^{1/q}$$

with

$$w(x) := \big((G_n v)(x)\big)^{-q/p} u(x). \tag{3.12}$$

Theorem 3.14 *Let $0 < p \le q < \infty$. Then* (3.11) *holds for all non-negative and measurable functions on $\mathbb{R}^n_+$ if and only if $A_G^{(n)} < \infty$. Moreover, $C \approx A_G^{(n)}$ with constants of equivalence depending only on the parameters p and q and the dimension n.*

Remark 3.15 *The proof in [33] shows that Theorem 3.14 may be regarded as a natural limit case of Theorem 3.10 characterized by the condition $A_{PS}^{(n)} < \infty$. For $n = 2$ we get another characterization than that in Theorem 3.4. Note that also in this case the limit result holds in a wider range of parameters and for general weights.*

For the case $0 < q < p < \infty$ the following less precise result yields (see [33]):

Theorem 3.16 *Let $0 < q < p < \infty$. Then* (3.11) *holds for all non-negative and measurable functions on $\mathbb{R}^n_+$ if $B_G^{(n)} < \infty$, where*

$$B_G^{(n)} := \left(\int_{\mathbb{R}^n_+} \left(\int_0^{t_1} \cdots \int_0^{t_n} w(x)dx\right)^{r/q} (t_1 \cdots t_n)^{-r/q} dt_1 \ldots dt_n\right)^{1/r},$$

$1/r = 1/q - 1/p$ and w is defined by (3.12). *Moreover, this condition is necessary and sufficient with the additional assumption that the weight w in* (3.11) *is of product type.*

Note that this theorem is a natural limit case of Theorem 3.12 characterized with the condition $A_{PS}^{(n)} < \infty$.

3.3 Multidimensional Hardy type inequalities via the convexity approach.

First we pronounce that the convexity approach discussed in **Section 2** can also be used to derive some corresponding multidimensional Hardy-type inequalities. We present two such results and even give the results in a setting so they generalize also the corresponding results for $n = 1$.

Theorem 3.17 *Let $1 < p < \infty$ and $m = (m_1, ..., m_n) \in \mathbb{R}^n$ be such that $m_i \neq 1, i = 1, ..., n$. Let $\Phi : [0, \infty) \to R$ be a convex almost everywhere positive function such that $Ax^p \leq \Phi(x) \leq Bx^p$ holds on $[0, \infty)$ for some constants $0 < A \leq B < \infty$.*
(a) If $b = (b_1, ..., b_n), 0 < b_i \leq \infty, i = 1, ..., n$, and $m = (m_1, ..., m_n), m_i > 1, i = 1, ..., n$, then the inequality

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi\left(\int_0^{x_1} \cdots \int_0^{x_n} f(t)dt\right) \prod_{i=1}^n x_i^{-m_i} dx \leq$$

$$\frac{B^2}{A}\left(\prod_{i=1}^n \frac{p}{m_i - 1}\right)^p \int_0^{b_1} \cdots \int_0^{b_n} f^p(x) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i}\right)^{\frac{m_i - 1}{p}}\right] dx$$

holds for all non-negative and measurable functions on $(0, b_1) \times \cdots \times (0, b_n)$.
(b) If $b = (b_1, ..., b_n), 0 \leq b_i < \infty, i = 1, ..., n$, and $m = (m_1, ..., m_n), m_i < 1, i = 1, ..., n$, then the inequality

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi\left(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t)dt\right) \prod_{i=1}^n x_i^{-m_i} dx \leq$$

$$\frac{B^2}{A}\left(\prod_{i=1}^n \frac{p}{1 - m_i}\right)^p \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(x) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{b_i}{x_i}\right)^{\frac{1 - m_i}{p}}\right] dx$$

holds for all non-negative and measurable functions on $(b_1, \infty) \times \cdots \times (b_n, \infty)$.

Remark 3.18 *The proof of this theorem can be found in [30]. In the same paper also the corresponding results for the cases $p < 0$ and $0 < p < 1$ are proved and discussed. The constants are sharp in all cases.*

Example 3.19 *For the case* $n = 1, \Phi(u) = u^p,\ p > 1$ *(so that A=B=1) the inequalities in Theorem 3.17 reads:*

$$(i)\ \int_0^b x^{-m}\left(\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{m-1}\right)^p \int_0^b f^p(x)x^{p-m}\left[1-\left(\frac{x}{b}\right)^{\frac{m-1}{p}}\right]dx$$

for $0 < b \le \infty, m > 1$ *and* $1 < p < \infty$*;*

$$(ii)\ \int_b^\infty x^{-m}\left(\int_x^\infty f(t)dt\right)^p dx \le$$

$$\le \left(\frac{p}{1-m}\right)^p \int_b^\infty f^p(x)x^{p-m}\left[1-\left(\frac{x}{b}\right)^{\frac{1-m}{p}}\right]dx$$

for $0 \le b < \infty, m < 1$ *and* $1 < p < \infty$.

The inequalities (i) and (ii) are some well-known generalizations of the inequalities (2.11) *and* (2.12)*, respectively, with* $m = p - a$.

Next we present the following generalization of our Theorem 2.12 which was proved in the recent paper [29] (see also [28]):

Theorem 3.20 *Let* $1 < p < \infty$, $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ *are such that* $k_i > 1$, $b = (b_1, \dots, b_n)$, $0 < b_i \le \infty, i = 1, 2, \dots, n$, *and let the function* f *be non-negative and measurable on* $(0, b)$.

(a) If $p \ge 2$, *then*

$$\int_0^{b_1}\cdots\int_0^{b_n}\prod_{i=1}^n x_i^{-k_i}\left(\int_0^{x_1}\cdots\int_0^{x_n} f(t)dt\right)^p dx$$

$$+\prod_{i=1}^n\left(\frac{k_i-1}{p}\right)\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\left|\prod_{i=1}^n\frac{p}{k_i-1}\left(\frac{t_i}{x_i}\right)^{1-\frac{k_i-1}{p}} f(t)\right.$$

$$\left.-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t)dt\right|^p \prod_{i=1}^n x_i^{p-k_i-\frac{k_i-1}{p}}dx\prod_{i=1}^n t_i^{\frac{k_i-1}{p}-1}dt$$

$$\le \left(\prod_{i=1}^n\frac{p}{k_i-1}\right)^p\int_0^{b_1}\cdots\int_0^{b_n}\left(\prod_{i=1}^n 1-\left[\frac{x_i}{b_i}\right]^{\frac{k_i-1}{p}}\right)x_i^{p-k_i}f^p(x)dx.$$

(b) If $1 < p \le 2$, *then this inequality holds in the reversed direction.*

Remark 3.21 *Note that, in particular, this means that for the case* $p = 2$ *the inequality reduces to a new integral identity. For the case* $n = 1$ *and* $b = b_1 = \infty$ *we obtain the result in our Theorem 2.12.*

Remark 3.22 *In the paper [29] also a dual version of Theorem 3.20 was proved and also in this case we obtain a new integral identity for the case $p = 2$. The constants are sharp in all cases.*

4 Hardy type inequalities in Morrey spaces

The well known Morrey spaces $\mathcal{L}^{p,\lambda}$ were introduced in [26] in relation to the study of partial differential equations; they are presented and discussed for instance in the books [7], [18] and [38]. In the n-dimensional case, which is of interest for us, they are defined as follows:

$$\mathcal{L}^{p,\lambda} = \{f \in L^p_{loc}(\Omega) : \|f\|_{p,\lambda} < \infty\}, \quad 1 \le p < \infty, \ 0 \le \lambda < n, \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^n$ and

$$\|f\|_{p,\lambda} = \sup_{x\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} = \sup_{x\in\Omega, r>0} \frac{\|f\|_{L^p(B(x,r))}}{r^{\frac{\lambda}{p}}} \tag{4.2}$$

with $B(x,r) = \{y \in \Omega : |t - x| < r\} \cap \Omega$.

We present some Hardy type inequalities in the norm of Morrey spaces obtained in the recent paper [35] for the $p \to p$ - case and $n = 1$, and in [34] for the general $p \to q$ - case and $n \ge 1$. In the context of Morrey spaces, Hardy operators seem to have appeared only in [3] in a different aspect, namely the problem to investigate boundedness of the maximal operator in local Morrey spaces was reduced to L_p-boundedness of Hardy type operators on a cone of monotone functions.

We consider the following Hardy type operators

$$H^\alpha_\varphi f(x) := |x|^{\alpha-n}\varphi(|x|) \int\limits_{|y|<|x|} \frac{f(y)\, dy}{\varphi(|y|)},$$

$$\mathcal{H}^\alpha_\varphi f(x) := |x|^{\alpha}\varphi(|x|) \int\limits_{|y|>|x|} \frac{f(t)dt}{|y|^n\varphi(|y|)}, \quad x \in \mathbb{R}^n, \tag{4.3}$$

with radial weights and use the same notation

$$H^\alpha_\varphi f(x) = x^{\alpha-1}\varphi(x) \int\limits_0^x \frac{f(t)}{\varphi(t)}\, dt \quad \text{and} \quad \mathcal{H}^\alpha_\varphi f(x) = x^{\alpha}\varphi(x) \int\limits_x^\infty \frac{f(t)dt}{t\varphi(t)}, \quad x > 0,$$

for their one-dimensional versions on the half-axis $\mathbb{R}_+$.

In this section we give a brief survey of the results obtained in [34] and [35]. In [34] there was developed an approach which allowed both to obtain weighted $p \to q$-estimations of Hardy-type operators in Morrey spaces and also to extend them to the multidimensional Hardy inequalities (related to integration over balls).

4.1 Admitted weights

The admitted weights $\varphi(x), x > 0$ are generated by functions φ from the Bary-Stechkin-type class; they may be characterized as weights, which are continuous and positive for $x \in (0, \infty)$, with possible decay or growth at $x = 0$ and $x = \infty$ and which become almost increasing or almost decreasing after multiplication by some power function. Such weights are oscillating between two powers at the origin and infinity (with different exponents for the origin and infinity).

In the sequel, a non-negative function f on $[0, \ell], 0 < \ell \le \infty$, is called almost increasing (almost decreasing), if there exists a constant $C(\ge 1)$ such that $f(x) \le Cf(y)$ for all $x \le y$ ($x \ge y$, respectively). Equivalently, a function f is almost increasing (almost decreasing), if it is equivalent to an increasing (decreasing, resp.) function g, i.e. $c_1 f(x) \le g(x) \le c_2 f(x), c_1 > 0, c_2 > 0$.

Let $0 < \ell < \infty$. By $W_0 = W_0([0, \ell])$ we denote the class of functions φ , which are continuous and positive on $(0, \ell]$, such that there exists the finite limit $\lim\limits_{x\to 0} \varphi(x)$ and $\varphi(x)$ is almost increasing on $(0, \ell)$; by $\widetilde{W}_0 = \widetilde{W}_0([0, \ell])$ we denote the class of functions $\varphi \in W_\infty$ such $x^a\varphi(x) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}$. Similarly, by $W_\infty = W_\infty([\ell, \infty])$ we denote the class of functions φ, which are continuous and positive on $[\ell, \infty)$ and which have the finite limit $\lim_{x\to\infty} \varphi(x)$ and $\varphi(x)$ is almost increasing on $[\ell, \infty)$; by $\widetilde{W}_\infty = \widetilde{W}_\infty([\ell, \infty))$ we denote the class of functions $\varphi \in W_\infty$ such that $x^a\varphi(x) \in W_\infty$ for some $a = a(\varphi) \in \mathbb{R}$. Finally, we say that a function φ which is continuous in $(0, \infty)$ is in the class $\widetilde{W}_{0,\infty}(\mathbb{R}^1_+)$, if its restriction to $(0, 1)$ belongs to $\widetilde{W}_0([0, 1])$ and its restriction to $(1, \infty)$ belongs to $\widetilde{W}_\infty([1, \infty])$.

We say that a function $\varphi \in W_0[0, \ell], 0 < \ell < \infty$, belongs to the Zygmund classes $\mathbb{Z}^\beta$, $\mathbb{Z}_\gamma$, where $\beta, \gamma \in \mathbb{R}^1$, if

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}}dt \le c\frac{\varphi(x)}{x^\beta}, \qquad \int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}}dt \le c\frac{\varphi(x)}{x^\gamma}, \quad x \in (0, \ell),$$

respectively. Similarly, we write that a function $\varphi \in W_\infty$ belongs to the

class $\widehat{\mathbb{Z}}^{\beta}$ or $\widehat{\mathbb{Z}}_{\gamma}$, if

$$\int_x^{\infty} \Big(\frac{x}{t}\Big)^{\beta} \frac{\varphi(t)\,dt}{t} \le c\varphi(x), \qquad \int_{\ell}^{x} \Big(\frac{x}{t}\Big)^{\alpha} \frac{\varphi(t)\,dt}{t} \le c\varphi(x), \quad x \in (\ell,\infty), \tag{4.4}$$

respectively.

For functions in $\widetilde{W}_{0,\infty}(\mathbb{R}^1_+)$ the notation

$$\mathbb{Z}^{\beta_0,\beta_\infty}(\mathbb{R}^1_+) = \mathbb{Z}^{\beta_0}([0,1]) \cap \mathbb{Z}^{\beta_\infty}([1,\infty)),\ \mathbb{Z}_{\gamma_0,\gamma_\infty}(\mathbb{R}^1_+) = \mathbb{Z}_{\gamma_0}([0,1]) \cap \mathbb{Z}_{\gamma_\infty}([1,\infty))$$

has an obvious meaning. In the case where the indices coincide: $\beta_0 = \beta_\infty := \beta$, we simply write $\mathbb{Z}^{\beta}(\mathbb{R}^1_+)$ and similarly for $\mathbb{Z}_{\gamma}(\mathbb{R}^1_+)$.

4.2 Formulation of the results

We start with the following lemma on pointwise estimate of the Hardy operators proved in [34].

Lemma 4.1 *Let $1 \le p < \infty$ and $0 < \lambda < 1$.*
I. Let $\varphi \in \widetilde{W}_0$. Then the operator H^{α}_{φ} is defined on the space $\mathcal{L}^{p,\lambda}(B(0,\ell))$, $0 < \ell \le \infty$, if and only if $\int\limits_0^a \frac{t^{n-1-\frac{n-\lambda}{p}}}{\varphi(t)}\,dt < \infty$ for all $a > 0$ and in this case

$$|H^{\alpha}_{\varphi} f(x)| \le C|x|^{\alpha-n}\varphi(x) \int_0^{|x|} \frac{t^{n-1-\frac{n-\lambda}{p}}}{\varphi(t)} \|f\|_{p,\lambda}. \tag{4.5}$$

II. Let $\frac{1}{\varphi} \in \widetilde{W}_0$, or $\varphi \in \widetilde{W}_0$ and $\varphi(2t) \le C\varphi(t)$. Then the operator $\mathcal{H}^{\alpha}_{\gamma}$ is defined on the space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n \backslash B(0,\varepsilon)), \varepsilon \ge 0$, if and only if $\int\limits_{\varepsilon}^{\infty} \frac{t^{n-1-\frac{n-\lambda}{p}}}{\varphi(t)}\,dt < \infty$ and in this case

$$|\mathcal{H}^{\alpha}_{\varphi} f(x)| \le C|x|^{\alpha}\varphi(|x|) \int_{|x|}^{\infty} \frac{t^{n-1-\frac{n-\lambda}{p}}}{\varphi(t)}\,dt \|f\|_{p,\lambda}. \tag{4.6}$$

By means of the pointwise estimates of Lemma 4.1, the following statement can be proved (the case of Lebesgue space L^p, that is, $\lambda = 0$ (with $1 < p < \frac{n}{\alpha}$ in this case), is known, see for instance [21], p. 6, 54):

Theorem 4.2 *Let $0 < \lambda < n, 0 \le \alpha < n-\lambda$, $1 < p < \frac{n-\lambda}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ and $\varphi \in \widetilde{W}_0$. For the weighted Hardy type inequalities*

$$\left\| |x|^{\alpha-n}\varphi(|x|) \int\limits_{|y|<|x|} \frac{f(y)}{\varphi(|y|)}\, dy \right\|_{\mathcal{L}^{q,\lambda}(\mathbb{R}^n)} \le C \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \tag{4.7}$$

$$\left\| |x|^{\alpha}\varphi(|x|) \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n\varphi(|y|)}\, dy \right\|_{\mathcal{L}^{q,\lambda}(\mathbb{R}^n)} \le C \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \tag{4.8}$$

to hold, the conditions

$$\varphi \in \mathbb{Z}_{\frac{\lambda}{p}+\frac{n}{p'}}(\mathbb{R}^1 plus) \quad \text{and} \quad \varphi \in \mathbb{Z}^{\frac{\lambda-n}{p}}(\mathbb{R}^1 plus), \quad \text{respectively,} \tag{4.9}$$

are sufficient, and the conditions

$$\varphi \in \mathbb{Z}_{\frac{\lambda}{p}+\frac{n}{p'}+\varepsilon}(\mathbb{R}^1 plus) \quad \text{and} \quad \varphi \in \mathbb{Z}^{\frac{\lambda-n}{p}-\varepsilon}(\mathbb{R}^1 plus), \quad \text{respectively,} \tag{4.10}$$

with an arbitrary $\varepsilon > 0$, are necessary. In the case of power weight $\varphi(|x|) = |x|^{\gamma}$, the conditions

$$\gamma < \frac{\lambda}{p'} + \frac{n}{p} \quad \text{and} \quad \gamma > \frac{\lambda-n}{p}$$

are necessary and sufficient for the validity of the Hardy inequalities (4.7) *and* (4.8) *to hold, respectively.*

Theorem 4.2 was proved in [35] for the case $\alpha = 0$ and $n = 1$, and in [34] for $\alpha > 0$ and $n \ge 1$.

Remark 4.3 *The conditions* (4.9) *may be equivalently reformulated in terms of the Matuszewska-Orlicz indices of the weight function φ. Let $m(\varphi), M(\varphi)$ be the lower and upper Matuszewska-Orlicz indices of φ "responsible" for the behaviour of φ at the origin, and $m_\infty(\varphi), M_\infty(\varphi)$ be similar indices, "responsible" for the behaviour of φ at infinity. For these notions we refer to [34]-[35] and [36], Subsection 2.2, with respect to our notation and admitted class of functions φ, and to the original paper [25] and presentation*

of these notions in [22] and [23]. The conditions (4.9) *are equivalent to that*

$$\max\left(M(\varphi), M_\infty(\varphi)\right) < \frac{\lambda}{p} + \frac{n}{p'}, \qquad \min\left(m(\varphi), m_\infty(\varphi)\right) > \frac{\lambda - n}{p} \tag{4.11}$$

and the conditions (4.10) *are equivalent to that*

$$\max\left(M(\varphi), M_\infty(\varphi)\right) \le \frac{\lambda}{p} + \frac{n}{p'}, \qquad \min\left(m(\varphi), m_\infty(\varphi)\right) \ge \frac{\lambda - n}{p}, \tag{4.12}$$

respectively.

Remark 4.4 *When $\alpha = 0, n = 1$ and $\varphi(t) = t^\gamma$, the best constant C for inequalities of type* (4.7)-(4.8) *on $\mathbb{R}_+$ satisfies the estimates*

$$C \le \frac{1}{\frac{\lambda}{p'} + \frac{1}{p} - \gamma}, \quad C \le \frac{1}{\gamma + \frac{1-\lambda}{p}},$$

respectively, see [35], p. 63.

4.3 On two-dimensional "mixed-type" Hardy inequalities in Morrey spaces

Let us consider the following weighted two-dimensional Hardy operator

$$x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \varphi_1(x_1)\varphi_2(x_2) \int_0^{x_1}\int_0^{x_2} \frac{f(t_1, t_2)}{\varphi_1(t_1)\varphi_2(t_2)}\, dt_1 dt_2, \tag{4.13}$$

which is taken with separated weights: we consider this model situation just to show the necessity to modify the definition of the Morrey space in this case.

In the multidimensional case Morrey spaces over a domain $\Omega \subseteq \mathbb{R}^n$ are defined similarly to (4.1)-(4.2) with integration over balls $B(x, r), x \in \Omega$ and sup over $x \in \Omega$ and $r > 0$. The case $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ is of our special interest. It is not hard to guess that such Morrey spaces whose norm involves estimation of integral means over balls are not well adjusted for operators of the form (4.13). Simple considerations based on homogeneity with respect to each variable show that for the study of operators of the form (4.13) it is more natural to define Morrey type spaces

in a different way, namely in terms of integral means over rectangles. We introduce such *anisotropic Morrey spaces* $\mathcal{L}^{p,\lambda_1,\lambda_2}(\mathbb{R}^2_+)$ by the norm

$$\|f\|_{p,\lambda_1,\lambda_2} := \sup_{\substack{x_1>0,x_2>0\\ r_1>0,r_2>0}} \left(\frac{1}{r_1^{\lambda_1} r_2^{\lambda_2}} \int\limits_{(x_1-r_1)_+}^{x_1+r_1} \int\limits_{(x_2-r_2)_+}^{x_2+r_2} |f(y_1,y_2)|^p \, dy_1 dy_2 \right)^{\frac{1}{p}}$$

$$= \sup_{x,r\in\mathbb{R}^2_+} \frac{\|f\|_{L^p(Q(x,r))}}{r_1^{\frac{\lambda_1}{p}} r_2^{\frac{\lambda_2}{p}}}, \tag{4.14}$$

where $(x_i - r_i)_+ = \begin{cases} x_i - r_i, & \text{if } x_i - r_i \geq 0 \\ 0, & \text{if } x_i - r_i < 0, \end{cases}, i = 1,2,\ Q(x,r) = \{y = (y_1,y_2) \in \mathbb{R}^2_+ :\ (x_i - r_i)_+ < y_i < x_i + r_i,\ i = 1,2\},\ x = (x_1,x_2), r = (r_1,r_2)$.

We can go further and introduce *anisotropic mixed norm Morrey spaces* $\mathcal{L}^{\overline{p},\overline{\lambda}}(\mathbb{R}^2_+),\ \overline{p} = (p_1,p_2), \overline{\lambda} = (\lambda_1,\lambda_2)$ via the norm

$$\|f\|_{\overline{p},\overline{\lambda}} := \sup_{x,r\in\mathbb{R}^2_+} \frac{\|f\|_{L^{\overline{p}}(Q(x,r))}}{r_1^{\frac{\lambda_1}{p_1}} r_2^{\frac{\lambda_2}{p_2}}}$$

with the mixed norm $\|f\|_{L^{\overline{p}}(Q(x,r))}$ over the rectangle $Q(x,r)$ (we refer to [2] for mixed norm Lebesgue spaces).

Adjusting the proofs of Lemmas 3.5, 4.1 and Theorem 4.3 in [34] for the case of the two-dimensional Hardy operator (4.13) one can obtain the following extension of Theorem 4.2, which we formulate, for simplicity, only for the "left-hand sided" version of the operator as defined in (4.13).

Theorem 4.5 *Let* $0 \leq \lambda_i < 1, 0 \leq \alpha_i < 1 - \lambda_i,\ 1 < p_i < \frac{1-\lambda_i}{\alpha_i}$ *and* $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{1-\lambda_i}$ *and* $\varphi_i \in \widetilde{W}_0,\ i = 1,2$. *For the weighted Hardy type inequality*

$$\left\| x_1^{\alpha_1-1} x_2^{\alpha_2-1} \varphi_1(x_1)\varphi_2(x_2) \int\limits_0^{x_1}\int\limits_0^{x_2} \frac{f(t_1,t_2)}{\varphi_1(t_1)\varphi_2(t_2)}\, dt_1 dt_2 \right\|_{\overline{p},\overline{\lambda}} \leq C \|f\|_{\overline{p},\overline{\lambda}} \tag{4.15}$$

to hold, the conditions $\varphi_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i}+\frac{1}{p_i'}}(\mathbb{R}^1 plus)$ *is sufficient, and the condition* $\varphi_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i}+\frac{1}{p_i'}+\varepsilon}(\mathbb{R}^1 plus)$ *with an arbitrary* $\varepsilon > 0$, *is necessary,* $i = 1,2$.

Remark 4.6 *Our conjecture is that Theorem 4.5 is true also if the product weight* $\varphi_1(t_1)\varphi_2(t_2)$ *is replaced by a more general two-dimensional weight e.g. satisfying an Oinarov condition. The most challenging question is to try to find necessary and sufficient condition for* (4.15) *to hold in such cases.*

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C_0 - semigroups and iterates of positive linear operators: asymptotic behaviour

Ioan Rasa

Abstract. In the setting of the theory developed by F. Altomare and his school during the last two decades, iterates of positive linear operators are related to certain semigroups, and overiterates to the asymptotic behaviour of the semigroups.

For various classes of operators we establish qualitative and quantitative results concerning the convergence of the iterates and the asymptotic behaviour of the associated semigroups. In this context the relationship between finitely defined operators and Markov chains is investigated.

All the results are illustrated by several examples.

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1 Introduction

Iterates of positive linear operators are useful in several areas of Functional Analysis and Approximation Theory. In particular, they are connected with the theory of strongly continuous semigroups of operators. Indeed, consider a sequence $(L_n)_{n\geq 1}$ of positive linear operators on a Banach space Z of continuous functions. Under suitable assumptions, the limit

$$T(t)f := \lim_{n\to\infty} L_n^{k(n)} f \tag{1.1}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

exists for all $t \geq 0$, $f \in Z$, and all sequences $(k(n))_{n\geq 1}$ of positive integers satisfying

$$\lim_{n\to\infty} \frac{k(n)}{n} = t.$$

Moreover, $(T(t))_{t\geq 0}$ is a C_0-semigroup of operators on Z.

This setting was deeply investigated by F. Altomare and his school starting from 1989, the year of the first International Conference on Functional Analysis and Approximation Theory organized at Acquadredda di Maratea.

By using (1.1) it is possible to approximate the semigroup and, accordingly, to approximate the solution of the evolution equation associated to it. Qualitative properties of the operators $T(t)$ can be deduced from the corresponding properties of L_n, and the asymptotic behaviour of the semigroup can be investigated. This area of research has many connections with other areas; for details see, e.g., [2], [3], [4], [5], [21], and the references given there.

It was proved in [5] that under suitable assumptions, if

$$\lim_{n\to\infty} \frac{k(n)}{n} = \infty,$$

then

$$\lim_{t\to\infty} T(t)f = \lim_{n\to\infty} L_n^{k(n)} f, \quad f \in Z. \tag{1.2}$$

Moreover,

$$\lim_{t\to\infty} T(t)f = \lim_{n\to\infty} P_n f, \quad f \in Z, \tag{1.3}$$

where

$$P_n f := \lim_{k\to\infty} L_n^k f, \quad n \geq 1, \quad f \in Z. \tag{1.4}$$

Equations (1.1) - (1.4) provide motivation for investigating, in this setting, iterates and overiterates of the operators L_n : iterates are related to the semigroup, overiterates to its asymptotic behaviour.

In Section 2 we study the iterates of positive linear operators on $C[0,1]$ reproducing the affine functions. Qualitative and quantitative results concerning their convergence towards the canonical projection are established. These results are used in Section 3 in order to study the asymptotic behaviour of the associated semigroup.

Section 4 is devoted to similar problems for operators preserving the constant functions and a prescribed function w, and in Section 5 we consider operators reproducing only the constant functions.

In Section 6 we investigate the relationship between finitely defined operators and Markov chains.

All the results are illustrated by several examples.

Throughout the paper the space $C(X)$ of real-valued, continuous functions on a compact Hausdorff space X will be endowed with the uniform norm and the usual order.

The functions $e_i \in C[0,1],\ i = 0, 1, 2, \ldots$, will be defined by $e_i(x) = x^i$, $x \in [0,1]$.

$P : C[0,1] \to C[0,1]$ stands for the canonical projection defined as

$$Pf(x) = (1-x)f(0) + xf(1),\ f \in C[0,1],\ x \in [0,1].$$

For $f \in C[0,1]$ we shall denote by $\omega(f,\cdot)$ the modulus of continuity, i.e.,

$$\omega(f,t) = \sup\{|f(x) - f(y)| \quad : \quad x, y \in [0,1],\ |x-y| \le t\}.$$

2 Operators reproducing the affine functions

The following result was obtained in [21, Theorem 2.2].

Theorem 2.1 *Let $L : C[0,1] \longrightarrow C[0,1]$ be a positive linear operator which preserves the affine functions. The following assertions are equivalent:*

$$\lim_{m\to\infty} L^m f = Pf, \quad f \in C[0,1]; \tag{2.1}$$

$$Le_2(x) > e_2(x), \quad x \in]0,1[. \tag{2.2}$$

We shall describe rates of convergence in (2.1); having in mind applications, we shall consider sequences of operators instead of single operators.

Let $L_n : C[0,1] \to C[0,1],\ n \ge 1$, be positive linear operators reproducing the affine functions.

For each $n \ge 1$ and $s \in]0, +\infty[$ let

$$a_n(s) := \max\{x - x^2 - ns(L_n e_2(x) - e_2(x)) : x \in [0,1]\}. \tag{2.3}$$

We get immediately

$$\Big(1 - \frac{1}{ns}\Big)e_2 + \frac{1}{ns}(e_1 - a_n(s)e_0) \le L_n e_2 \le e_1,$$

and by induction,

$$\Big(1 - \frac{1}{ns}\Big)^m e_2 + \Big(1 - \Big(1 - \frac{1}{ns}\Big)^m\Big)(e_1 - a_n(s)e_0) \le L_n^m e_2 \le e_1$$

for all $m \geq 1,\ n \geq 1,\ s \in]0, +\infty[$.
This yields

$$0 \leq e_1 - L_n^m e_2 \leq \Big(1 - \frac{1}{ns}\Big)^m (e_1 - e_2) + a_n(s)e_0 \tag{2.4}$$

for all $m \geq 1, n \geq 1,\ s > 0$ with $ns \geq 1$.

Theorem 2.2 *Let $0 < \alpha < 1$. Then*

$$|L_n^m f(x) - Pf(x)| \leq 2\omega\Big(f, \sqrt{\Big(1 - \frac{1}{nm^\alpha}\Big)^m x(1-x) + a_n(m^\alpha)}\Big) \tag{2.5}$$

for all $f \in C[0,1],\ x \in [0,1],\ m \geq 1,\ n \geq 1$.

Proof. Let $t > 0$. The following inequality is a particular case of Proposition 5.1.5 in [2]:

$$|f - Pf| \leq \Big(e_0 + \frac{e_1 - e_2}{t^2}\Big)\omega(f, t).$$

Since L_n preserves the affine functions, we get

$$|L_n^m f - Pf| \leq (e_0 + t^{-2}(e_1 - L_n^m e_2))\omega(f, t).$$

Combined with (2.4), this gives

$$|L_n^m f - Pf| \leq (e_0 + t^{-2}((1 - \frac{1}{ns})^m (e_1 - e_2) + a_n(s)e_0))\omega(f, t),$$

and consequently

$$|L_n^m f(x) - P(x)| \leq 2\omega(f, \sqrt{\Big(1 - \frac{1}{ns}\Big)^m x(1-x) + a_n(s)}\Big) \tag{2.6}$$

for all $m \geq 1,\ n \geq 1,\ s > 0$ with $ns \geq 1$.
Taking $s = m^\alpha$ we conclude the proof. □

As in [21, Lemma 2.1] one proves

Theorem 2.3 *Let $L_n : C[0,1] \longrightarrow C[0,1]$ be a sequence of positive linear operators reproducing the affine functions and having the property*

$$L_n e_2(x) > e_2(x),\ x \in]0,1[,\ n \geq 1. \tag{2.7}$$

Then for each $n \geq 1$,

$$\lim_{s \to \infty} a_n(s) = 0. \tag{2.8}$$

Let us remark that Theorems 2.2 and 2.3 yield, under the hypotheses of Theorem 2.3,

$$\lim_{m\to\infty} L_n^m f = Pf,\ f \in C[0,1],\ n \geq 1. \tag{2.9}$$

This is (2.1) for $L = L_n$. But now we have rates of convergence for this process, expressed by (2.5). Of course, we need information about the speed of convergence in (2.8).

Here is an example.

Example 2.4 Consider the Meyer-König and Zeller operators $M_n : C[0,1] \to C[0,1]$ defined by

$$M_n f(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f(\frac{k}{n+k}),\ 0 \leq x < 1, \tag{2.10}$$

$$M_n f(1) = f(1).$$

It is well-known that they reproduce the affine functions; see, e.g., [7]. In order to verify (2.7), let $0 < x < 1$; then

$$M_n e_2(x) = (1-x)^{n+1} \sum_{k=1}^{\infty} \binom{n+k}{k} x^k (\frac{k}{n+k})^2 >$$

$$> (1-x)^{n+1} \sum_{k=1}^{\infty} \binom{n+k}{k} x^k \frac{k}{n+k} \frac{k-1}{n+k-1} = x^2 = e_2(x).$$

According to (2.9) we have, for each $n \geq 1$,

$$\lim_{m\to\infty} M_n^m f = Pf,\ f \in C[0,1].$$

To get a rate of convergence in this case, we need the following inequality established in [7] (see also [14] and the references therein):

$$M_n e_2(x) - e_2(x) \geq \frac{x(1-x)^2}{2(n+1)},\ n \geq 1,\ x \in [0,1].$$

According to it,

$$a_n(s) = \max\{x - x^2 - ns(M_n e_2(x) - e_2(x)) : x \in [0,1]\} \leq$$

$$\leq \max\{x - x^2 - ns\frac{x(1-x)^2}{2(n+1)} : x \in [0,1]\}.$$

By elementary calculus it is easy to prove that

$$a_n(s) < \frac{2}{s},\ n \geq 1,\ s \geq 4.$$

Thus, given $0 < \alpha < 1$, we have by (2.5),

$$|M_n^m f(x) - Pf(x)| \leq 2\omega(f, \sqrt{\Big(1 - \frac{1}{nm^\alpha}\Big)^m x(1-x) + \frac{2}{m^\alpha}})$$

for all $f \in C[0,1],\ x \in [0,1],\ n \geq 1,\ m \geq 1$ with $m^\alpha \geq 4$.

3 Asymptotic behaviour of the approximated semigroup

Consider again a sequence of positive linear operators $L_n : C[0,1] \to C[0,1]$ reproducing the affine functions. Denote

$$a(s) := \sup\{a_n(s) : n \geq 1\},\ s > 0. \tag{3.1}$$

Suppose that the following limit exists in $C[0,1]$ for all $t > 0$ and all $f \in C[0,1]$:

$$T(t)f := \lim_{n\to\infty} L_n^{[nt]} f. \tag{3.2}$$

The following result was proved in [21, Theorem 2.6].

Theorem 3.1 *If* $\lim_{s\to\infty} a(s) = 0$, *then*

$$\lim_{t\to\infty} T(t)f = Pf,\ f \in C[0,1]. \tag{3.3}$$

Here we are interested in rates of convergence for (3.3).
From (2.6) we get

$$|L_n^{[nt]} f(x) - Pf(x)| \leq 2\omega\Big(f, \sqrt{(1 - \frac{1}{ns})^{[nt]} x(1-x) + a(s)}\Big)$$

for all $f \in C[0,1],\ x \in [0,1],\ n \geq 1,\ t > 0,\ s > 0$ with $nt \geq 1$ and $ns \geq 1$.

Letting $n \longrightarrow \infty$ we obtain

$$|T(t)f(x) - Pf(x)| \leq 2\omega\Big(f, \sqrt{x(1-x)exp(-\frac{t}{s}) + a(s)}\Big).$$

This yields:

Theorem 3.2 *Let* $0 < \alpha < 1$. *Then*

$$|T(t)f(x) - Pf(x)| \leq 2\omega\Big(f, \sqrt{x(1-x)exp(-t^{1-\alpha}) + a(t^{\alpha})}\Big) \tag{3.4}$$

for all $t > 0,\ f \in C[0,1],\ x \in [0,1]$.

Obviously Theorem 3.1 can be deduced as a consequence of Theorem 3.2.

Example 3.3 Consider the Bernstein-Schnabl operators on the unit interval, defined in [4] as follows. Let $(\mu_x)_{0\leq x\leq 1}$ be a continuous selection of probability Borel measures on $[0,1]$, i.e., a family of probability Borel measures such that for every $f \in C[0,1]$ the function

$$Tf(x) := \int_0^1 f d\mu_x, \quad x \in [0,1],$$

is continuous on $[0,1]$.

Then $Te_0 = e_0$; suppose that, moreover, $Te_1 = e_1$. Set

$$\alpha(x) := \frac{1}{2}(Te_2(x) - x^2), \quad x \in [0,1]. \tag{3.5}$$

Since $e_2 \leq Te_2 \leq e_1$, we have

$$0 \leq \alpha \leq \frac{1}{2}(e_1 - e_2).$$

Suppose that

$$0 < \alpha(x) \leq \frac{1}{2}(x - x^2),\ x \in]0,1[. \tag{3.6}$$

The n-th Bernstein-Schnabl operator associated with the given selection is defined by

$$L_n f(x) := \int_0^1 \dots \int_0^1 f\Big(\frac{x_1 + \dots + x_n}{n}\Big) d\mu_x(x_1) \dots d\mu_x(x_n)$$

for all $f \in C[0,1]$ and $x \in [0,1]$.

Then $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operators such that (see [4]):

$$L_n e_0 = e_0,\ L_n e_1 = e_1,\ L_n e_2 = e_2 + \frac{2\alpha}{n},\ n \geq 1.$$

It was proved in [4] that for each $t \geq 0$ and $f \in C[0,1]$ there exists the limit

$$T(t)f := \lim_{n\to\infty} L_n^{[nt]} f,$$

and $(T(t))_{t\geq 0}$ is the Markov semigroup on $C[0,1]$ generated by the differential
operator

$$Au(x) := \begin{cases} \alpha(x)u''(x) & ,\text{if } 0 < x < 1, \\ 0 & ,\text{if } x = 0,1 \end{cases}$$

with domain

$$D(A) := \{u \in C[0,1] \cap C^2(]0,1[) : \lim_{x\to 0,1} \alpha(x)u''(x) = 0\}.$$

Moreover (see [21]),

$$\lim_{m\to\infty} L_n^m f = Pf,\ f \in C[0,1], \tag{3.7}$$

$$\lim_{t\to\infty} T(t)f = Pf,\ f \in C[0,1]. \tag{3.8}$$

From Theorems 2.2 and 3.2 we get rates of convergence in (3.7) and (3.8). Indeed, according to (2.3) and (3.1) we have in this case

$$a(s) = a_n(s) = \max\{x - x^2 - 2s\alpha(x) : x \in [0,1]\}. \tag{3.9}$$

Now (2.5) and (3.4) give us information about the speed of convergence in (3.7) and (3.8), in terms of the above function $a(s) = a_n(s)$.
A concrete example can be described as follows.

For $x \in [0,1]$ let ε_x be the probability Borel measure concentrated on x. Let

$$\mu_x := x(1-x)(x\varepsilon_1 + (1-x)\varepsilon_0) + (1 - x(1-x))\varepsilon_x,\ 0 \leq x \leq 1.$$

For the operators L_n corresponding to the selection $(\mu_x)_{0\leq x\leq 1}$ we have (see [4, Theorem 3.10])

$$\alpha(x) = \frac{1}{2}x^2(1-x)^2,\ 0 \leq x \leq 1.$$

From (3.9) we get with elementary calculus

$$a(s) = a_n(s) \leq \frac{1}{4s},\ s \geq 2,\ n \geq 1.$$

So in this case we get from (2.5) and (3.4), for sufficiently large m and t,

$$|L_n^m f(x) - Pf(x)| \leq 2\omega(f, \sqrt{(1 - \tfrac{1}{nm^\alpha})^m x(1-x) + \tfrac{1}{4m^\alpha}}),$$
$$|T(t)f(x) - Pf(x)| \leq 2\omega(f, \sqrt{x(1-x)exp(-t^{1-\alpha}) + \tfrac{1}{4t^\alpha}}).$$

Remark 3.4 The results of Sections 2 and 3 can be extended to operators defined on $C(X), X$ a metrizable Bauer simplex. Detailed proofs will appear elsewhere.

4 King-type operators

Let $w : [0,1] \to [0,1]$ be continuous, strictly increasing, with $w(0) = 0$ and $w(1) = 1$.
Let $K : C[0,1] \to C[0,1]$ be a positive linear operator such that $Ke_0 = e_0$ and $Kw = w$.
Operators of this kind were considered by J. P. King in [16], for $w = e_2$; see also [9], [11], [13], and the references therein.

Define the operator $Q : C[0,1] \to C[0,1]$ by

$$Qf := (e_0 - w)f(0) + wf(1),\ f \in C[0,1].$$

Theorem 4.1 *The following assertions are equivalent:*

$$\lim_{m\to\infty} K^m g = Qg,\ g \in C[0,1]; \tag{4.1}$$

$$K(w^2)(t) > w^2(t),\ t \in]0,1[. \tag{4.2}$$

Proof. Consider the operator $L : C[0,1] \to C[0,1]$,

$$Lf := (K(f \circ w)) \circ w^{-1},\ f \in C[0,1].$$

Then $Le_0 = e_0$ and $Le_1 = e_1$. Moreover, it is easy to prove that (4.1) is equivalent to (2.1), and (4.2) to (2.2). Since (2.1) and (2.2) are equivalent, we conclude that (4.1) and (4.2) are equivalent. □

Remark 4.2 Let $g \in C[0,1]$ and $t \in [0,1]$. We have

$$|K^m g(t) - Qg(t)| = |L^m(g \circ w^{-1})(w(t)) - P(g \circ w^{-1})(w(t)|.$$

So Theorem 2.2 provides rates of convergence in (4.1).

Example 4.3 Let $\alpha \in [0, +\infty)$ and

$$w = w_\alpha = \frac{e_2 + \alpha e_1}{1 + \alpha}.$$

A sequence of positive linear operators $B_{n,\alpha} : C[0,1] \longrightarrow C[0,1]$, $n \geq 2$, reproducing e_0 and w was constructed in [9]. In fact,

$$B_{n,\alpha} f(x) := \sum_{i=0}^{n} \binom{n}{i} r_{n,\alpha}(x)^i (1 - r_{n,\alpha}(x))^{n-i} f(\frac{i}{n}),$$

where

$$r_{n,\alpha}(x) := -\frac{n\alpha+1}{2(n-1)} + \sqrt{\Big(\frac{n\alpha+1}{2(n-1))}\Big)^2 + \frac{n(\alpha x+x^2)}{n-1}}.$$

To check (4.2), fix $t \in]0,1[$ and consider the function $g \in C[0,1]$,

$$g(x) := w^2(x) - w^2(t) - 2w(t)(w(x)-w(t)), \quad x \in [0,1].$$

Then we have

$$g(x) = (w(x)-w(t))^2,\ x \in [0,1],$$

and so $g(x) > 0$ for all $x \in [0,1],\ x \neq t$.
It follows immediately that

$$B_{n,\alpha}g(x) > 0, \quad x \in [0,1].$$

Since $B_{n,\alpha}e_0 = e_0$ and $B_{n,\alpha}w = w$, we get

$$B_{n,\alpha}(w^2)(x) - w^2(t) - 2w(t)(w(x)-w(t)) > 0, \quad x \in [0,1].$$

In particular, for $x = t$,

$$B_{n,\alpha}(w^2)(t) - w^2(t) > 0,$$

and so (4.2) is satisfied.

Now according to Theorem 4.1,

$$\lim_{m\to\infty} B_{n,\alpha}^m f = (e_0 - w_\alpha)f(0) + w_\alpha f(1),\ n \geq 2,\ f \in C[0,1].$$

In particular, for $\alpha = 0$ we have $w_0 = e_2$ and so

$$\lim_{m\to\infty} B_{n,0}^m f = (e_0 - e_2)f(0) + e_2 f(1),\ n \geq 2,\ f \in C[0,1]. \tag{4.3}$$

The result expressed in (4.3) was obtained in [11] with a method based on the Fixed Point Theory.

Example 4.4 In [11, Section 4] the authors raised the problem of finding polynomial positive linear operators on $C[0,1]$ reproducing e_0 and e_2. An answer to this problem is provided by the sequence $K_n : C[0,1] \to C[0,1]$,

$$K_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^{2k}(1-x^2)^{n-k} f\Big(\sqrt{\frac{k}{n}}\Big), \quad x \in [0,1],\ f \in C[0,1].$$

Let $w = e_2$ and $t \in]0,1[$. Then, denoting by B_n the classical Bernstein operators on $C[0,1]$, we have

$$\begin{aligned} K_n(w^2)(t) &= \sum_{k=0}^{n} \binom{n}{k} t^{2k}(1-t^2)^{n-k}(\tfrac{k}{n})^2 = \\ &= B_n e_2(t^2) > e_2(t^2) = t^4 = w^2(t). \end{aligned}$$

Consequently by Theorem 4.1,

$$\lim_{m\to\infty} K_n^m f = (e_0 - e_2)f(0) + e_2 f(1),\ f \in C[0,1],\ n \geq 1.$$

5 Operators reproducing the constant functions

Some classical positive linear operators used in Approximation Theory reproduce only the constant polynomials. The iterates of some of these operators will be considered in what follows.

Let X be a compact Hausdorff space and $y \in X$.
Let $\varphi \in C(X)$ with

$$\varphi(y) = 1; 0 \leq \varphi(x) < 1,\ x \in X, x \neq y. \tag{5.1}$$

Denote by $\mathbf{1}$ the constant function $\mathbf{1}(x) = x,\ x \in X$.

Theorem 5.1 *Let $L : C(X) \longrightarrow C(X)$ be a positive linear operator such that $L\mathbf{1} = \mathbf{1}$ and*

$$L\varphi(x) > \varphi(x),\ x \in X,\ x \neq y. \tag{5.2}$$

Then

$$\lim_{m\to\infty} L^m f = f(y)\mathbf{1},\ f \in C(X). \tag{5.3}$$

Proof. Since $0 \leq \varphi \leq 1$, we get $0 \leq L\varphi \leq \mathbf{1}$. For each $x \neq y$ we have $\varphi(x) < L\varphi(x) \leq 1$, which entails $1 = \varphi(y) \leq L\varphi(y) \leq 1$, i.e., $L\varphi(y) = 1$.

Let $b := L\varphi - \varphi$. According to the preceding remarks,

$$b(y) = 0;\ b(x) > 0,\ x \in X,\ x \neq y. \tag{5.4}$$

Now define the sequence $(b_k)_{k\geq 1}$ by

$$b_k := max\{1 - \varphi(x) - kb(x)\ :\ x \in X\}. \tag{5.5}$$

Taking $x = y$ we see that $b_k \geq 0$. Clearly the sequence is decreasing, so it has a limit $l \geq 0$. Suppose that $l > 0$ and let

$$B_k := \{x \in X : 1 - \varphi(x) - kb(x) \geq l\}.$$

Then $(B_k)_{k\geq 1}$ is a descending sequence of closed sets, and due to (5.4)

$$\bigcap_{k=1}^{\infty} B_k = \emptyset.$$

So there exists $m \geq 1$ such that $B_m = \emptyset$, which means that $b_m < l$, a contradiction. We conclude that $l = 0$, i.e.,

$$\lim_{k\to\infty} b_k = 0. \tag{5.6}$$

Since $\mathbf{1} - \varphi - kb \leq b_k\mathbf{1}$, we have

$$\frac{1}{k}(1-b_k)\mathbf{1} + \Big(1 - \frac{1}{k}\Big)\varphi \leq L\varphi \leq \mathbf{1}.$$

By induction,

$$(1-b_k)\Big(1 - \Big(1-\frac{1}{k}\Big)^m\Big)\mathbf{1} + \Big(1-\frac{1}{k}\Big)^m \varphi \leq L^m\varphi \leq 1 \tag{5.7}$$

for all $k \geq 1$ and $m \geq 1$.

Let $k \geq 1$ be fixed. From (5.7) we obtain

$$(1-b_k)\mathbf{1} \leq \liminf_{m\to\infty} L^m\varphi \leq \limsup_{m\to\infty} L^m\varphi \leq \mathbf{1}.$$

Combined with (5.6), this yields

$$\lim_{m\to\infty} L^m\varphi = \mathbf{1} = \varphi(y)\mathbf{1}.$$

Consider the operator $V : C(X) \to C(X)$,

$$Vf = f(y)\mathbf{1}, \quad f \in C(X).$$

Then

$$\lim_{m\to\infty} L^m\mathbf{1} = V\mathbf{1}; \quad \lim_{m\to\infty} L^m\varphi = V\varphi. \tag{5.8}$$

According to a Korovkin - type result (see [2, Theorem 3.4.3], from (5.8) we derive

$$\lim_{m\to\infty} L^m f = Vf = f(y)\mathbf{1}, \quad f \in C(X)$$

and this concludes the proof. □

Example 5.2 The operators $M_n^* : C[0,1] \to C[0,1]$ defined by

$$M_n^* f(x) := (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{(n+k)(n+k+1)}{n} x^k \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} f(t)dt, \ 0 \le x < 1,$$

$M_n^* f(1) := f(1)$
were introduced in [23] (see also [17]).

It is easy to verify that $M_n^* e_0 = e_0$. Moreover, since

$$\frac{(n+k)(n+k+1)}{n} \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} e_1(t)dt > e_1(\frac{k}{n+k}), \quad k \ge 0,$$

we deduce

$$M_n^* e_1(x) > M_n e_1(x) = e_1(x), \quad 0 \le x < 1,$$

where M_n are the Meyer-König and Zeller operators (see Example 2.4). So we may apply Theorem 5.1 with $\varphi = e_1$ and $y = 1$; consequently,

$$\lim_{m\to\infty} (M_n^*)^m f = f(1)e_0, \quad f \in C[0,1], \ n \ge 1.$$

Example 5.3 Consider now the operators $\widetilde{M_n} : C[0,1] \to C[0,1]$,

$$\widetilde{M_n} f(x) := (n+1)(1-x)^{n+2} \sum_{k=0}^{\infty} \binom{n+k+1}{k} x^k \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} \frac{f(t)}{(1-t)^2} dt, \quad 0 \le x < 1,$$

$\widetilde{M_n} f(1) := f(1)$.

They were introduced in [17]. It is easy to prove that $\widetilde{M_n} e_0 = \frac{n+1}{n} e_0$. Let $U_n := \frac{n}{n+1} \widetilde{M_n}$. Then $U_n e_0 = e_0$. For $0 \le x < 1$ we get

$$\begin{aligned} U_n e_1(x) &= 1 + (1-x)^{n+2} \sum_{k=0}^{\infty} \tbinom{n+k+1}{k} \Big(n \log \frac{n+k}{n+k+1} \Big) x^k > \\ &> 1 - (1-x)^{n+2} \sum_{k=0}^{\infty} \tbinom{n+k+1}{k} \frac{n+1}{n+k+1} x^k = \\ &= 1 - (1-x)^{n+2} \sum_{k=0}^{\infty} \tbinom{n+k}{k} x^k = 1 - (1-x) = x = e_1(x). \end{aligned}$$

Thus

$$U_n e_1(x) > e_1(x), \quad 0 \le x < 1,$$

and again we may apply Theorem 5.1 with $\varphi = e_1$ and $y = 1$. So we get

$$\lim_{m\to\infty} U_n^m f = f(1)e_0, \ f \in C[0,1], \ n \ge 1.$$

Example 5.4 The Bernstein-Cheney-Sharma operators $G_n : C[0,1] \to C[0,1]$ are defined by

$$G_nf(x) := (1+nt_n)^{-n}\sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x(x+kt_n)^{k-1} \times$$
$$\times(1-x+(n-k)t_n)^{n-k}, \quad f \in C[0,1],\ x \in [0,1],$$

where $(t_n)_{n\geq 1}$ is a given sequence of real numbers, $t_n > 0$. See [2, Section 5.3.2].
According to [2, p.324],

$$G_ne_0 = e_0;\ G_ne_1 \leq \frac{1}{1+t_n}e_1.$$

Take $\varphi = e_0 - e_1$ and $y = 0$; then the assumptions of Theorem 5.1 are satisfied, and therefore

$$\lim_{m\to\infty} G_n^m = f(0)e_0,\ f \in C[0,1],\ n \geq 1.$$

Example 5.5 Let $b > 0$ and $0 \leq a \leq b$. The iterates of the Stancu operators $S_n : C[0,1] \to C[0,1]$,

$$S_nf(x) = \sum_{j=0}^{n}\binom{n}{j}x^j(1-x)^{n-j}f(\frac{j+a}{n+b}),\ f \in C[0,1],\ x \in [0,1]$$

were investigated in [12] by using the eigenstructure of S_n. Here we obtain the corresponding results, for $a = 0$ or $a = b$, from Theorem 5.1; the case $0 < a < b$ will be studied in the next section.

It is easy to verify that

$$S_ne_0 = e_0,\ S_ne_1 = \frac{ne_1 + ae_0}{n+b},\ n \geq 1.$$

(i) Let $a = b$. Then Theorem 5.1 with $\varphi = e_1$ and $y = 1$ yields

$$\lim_{m\to\infty} S_n^m f = f(1)e_0,\ f \in C[0,1],\ n \geq 1.$$

(ii) If $a = 0$, from Theorem 5.1 with $\varphi = e_0 - e_1$ and $y = 0$ we get

$$\lim_{m\to\infty} S_n^m f = f(0)e_0,\ f \in C[0,1],\ n \geq 1.$$

Example 5.6 Let $p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k},\ x \in [0,1],\ 0 \le k \le n$. The operators $A_n : C[0,1] \to C[0,1]$,

$$A_nf(x) := (n+1)\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n,k-1}(t)f(t)dt + (1-x)^n f(0)$$

were introduced and investigated in [1].

By [1, Lemma 7] we have

$$A_ne_0 = e_0;\ A_ne_1 = \frac{n}{n+2}e_1,\ n \ge 1.$$

Now Theorem 5.1 with $\varphi = e_0 - e_1$ and $y = 0$ yields

$$\lim_{m\to\infty} A_n^m f = f(0)e_0,\ f \in C[0,1],\ n \ge 1.$$

Example 5.7 Let $n \ge 1,\ p = [n/2],\ 0 \le k \le p,\ 0 \le x \le 1$.
Consider the polynomials

$$w_{n,k}(x) := \frac{n+1-2p+2k}{(n+1)2^{n+1}x}\binom{n+1}{p-k}((1-x)^{p-k}(1+x)^{n+1-p+k} - (1-x)^{n+1-p+k}(1+x)^{p-k}).$$

The operators $\beta_n : C[0,1] \to C[0,1]$,

$$\beta_nf(x) := \sum_{k=0}^{p} f(\frac{n-2p+2k}{n})w_{n,k}(x)$$

were introduced in [22]; see also [19], [20].
It was proved in [19, Theorem 2.1 and (14)] that

(i) $\beta_ne_0 = e_0$,

(ii) β_ne_1 is increasing and convex,

(iii) $(\beta_ne_1)'(1) = \frac{n-1}{n}$.

It follows easily that

$$\beta_ne_1 \ge \frac{n-1}{n}e_1 + \frac{1}{n}e_0, \quad n \ge 1.$$

In particular, $\beta_ne_1(x) > e_1(x),\ 0 \le x < 1$.
Now Theorem 5.1 with $\varphi = e_1$ and $y = 1$ yields

$$\lim_{m\to\infty} \beta_n^m f = f(1)e_0, \quad f \in C[0,1],\ n \ge 1.$$

Example 5.8 Let K_p be the canonical simplex in $\mathbb{R}^p$, i.e.,

$$K_p := \{x \in \mathbb{R}^p : x_1 \geq 0, \ldots, x_p \geq 0,\ x_1 + \cdots + x_p \leq 1\}.$$

Consider the Bernstein operators $B_n : C(K_p) \to C(K_p)$ defined by

$$\begin{aligned} B_n f(x) := \quad & \sum f(\tfrac{i_1}{n}, \ldots, \tfrac{i_p}{n}) \tfrac{n!}{i_1! \ldots i_p! (n - i_1 - \cdots - i_p)!} \times \\ & \times x_1^{i_1} \ldots x_p^{i_p} (1 - x_1 - \cdots - x_p)^{n - i_1 - \cdots - i_p}, \end{aligned}$$

where the sum is extended to all the non-negative integers $i_1, \ldots, i_p$ with $i_1 + \cdots + i_p \leq n$. Let $c > 0$ be a given real number. For each $n \geq c$ and $f \in C(K_p)$ consider the function $f_n \in C(K_p)$ defined by

$$f_n(x) := f((1 - \frac{c}{n})x),\ x \in K_p.$$

Now let $W_n : C(K_p) \to C(K_p)$,

$$W_n f(x) = B_n f_n(x),\ n \geq c,\ f \in C(K_p),\ x \in K_p.$$

(More general operators have been considered in [3]; see also Example 6.5 in the next section).

Clearly $W_n \mathbf{1} = \mathbf{1}$. Let $\varphi \in C(K_p),\ \varphi(x) := 1 - x_1 - \cdots - x_p,\ y = 0$. We have $\varphi(y) = 1,\ 0 \leq \varphi(x) < 1,\ x \in K_p,\ x \neq 0$. Moreover, for $x \in K_p,\ x \neq 0$:

$$W_n \varphi(x) = B_n \varphi_n(x) = 1 - (1 - \frac{c}{n})(x_1 + \cdots + x_p) > 1 - (x_1 + \cdots + x_p) = \varphi(x).$$

Using Theorem 5.1 we obtain

$$\lim_{m \to \infty} W_n^m f = f(0)\mathbf{1},\ n \geq c,\ f \in C(K_p).$$

6 Finitely defined operators and Markov Chains

Let X be a compact Hausdorff space, $n \geq 1,\ v_0, v_1, \ldots, v_n \in C(X)$ with $v_0 + v_1 + \cdots + v_n = \mathbf{1}$, and $x_0, x_1, \ldots, x_n \in X$. Consider the positive linear operator $L : C(X) \to C(X)$ defined by

$$Lf = f(x_0)v_0 + f(x_1)v_1 + \cdots + f(x_n)v_n,\ f \in C(X). \tag{6.1}$$

According to the terminology of [2, Section 3.4], L is a finitely defined operator. Clearly $L\mathbf{1} = \mathbf{1}$.

Consider the stochastic matrix

$$M := \begin{pmatrix} v_0(x_0) & v_1(x_0) & \ldots & v_n(x_0) \\ v_0(x_1) & v_1(x_1) & \ldots & v_n(x_1) \\ \ldots & & & \\ v_0(x_n) & v_1(x_n) & \ldots & v_n(x_n) \end{pmatrix}.$$

It is easy to verify that for all $m \geq 1,\ f \in C(X)$ and $x \in X$,

$$L^m f(x) = (v_0(x), \ldots, v_n(x))M^{m-1}(f(x_0), \ldots, f(x_n))^t. \tag{6.2}$$

In [18], (6.2) was used in order to prove

Theorem 6.1 *([18]). The sequence $(L^m)_{m\geq 1}$ is convergent in the operator norm if and only if the sequence $(M^m)_{m\geq 1}$ converges with respect to a matrix norm.*

Theorem 6.1 was applied in [18] to the classical Bernstein operators on $C[0,1]$ and to some B-spline operators. The result concerning the Bernstein operators was obtained also in [6] by using spectral methods.

In order to investigate the convergence of the sequence $(M^m)_{m\geq 1}$, in [18] a Markov chain was associated to the operator L. The same idea was used, independently, in [12]. Here we shall use this idea in a more general case.

The homogeneous Markov chain associated to the operator L has state space $\{x_0, x_1, \ldots, x_n\}$ and its transition matrix is M.

Example 6.2 Suppose that the chain is regular, i.e., there exists $k \geq 1$ such that all the entries of M^k are strictly positive. Then it is known (see, e.g., [8], [10], [15]) that there exists a matrix

$$S = \begin{pmatrix} s_0 & s_1 & \ldots & s_n \\ s_0 & s_1 & \ldots & s_n \\ \ldots & & & \\ s_0 & s_1 & \ldots & s_n \end{pmatrix}$$

such that $s_0 \geq 0, \ldots, s_n \geq 0,\ s_0 + \cdots + s_n = 1$, and

$$\lim_{m\to\infty} M^m = S. \tag{6.3}$$

According to (6.2) and (6.3),

$$\lim_{m\to\infty} L^m f = (v_0, \ldots, v_n)S(f(x_0), \ldots, f(x_n))^t.$$

We deduce immediately

$$\lim_{m\to\infty} L^m f = (s_0 f(x_0) + \cdots + s_n f(x_n))\mathbf{1},\ f \in C(X). \tag{6.4}$$

Example 6.3 Suppose that the chain is absorbing. Then the transition matrix M has the canonical form

$$M = \begin{pmatrix} I_{k\times k} & 0_{k\times l} \\ R_{l\times k} & Q_{l\times l} \end{pmatrix}.$$

The states $\{x_0, x_1, \dots, x_{k-1}\}$ are absorbing $(1 \leq k \leq n)$ and the other states are transient.

It is known (see, e.g., [8], [10], [15]) that the fundamental matrix $N := (I_{l\times l} - Q_{l\times l})^{-1}$ exists, and

$$\lim_{m\to\infty} M^m = \begin{pmatrix} I_{k\times k} & 0_{k\times l} \\ (NR)_{l\times k} & 0_{l\times l} \end{pmatrix}. \tag{6.5}$$

From (6.2) and (6.5) we get, for $f \in C(X)$,

$$\lim_{m\to\infty} L^m f = (v_0, \dots, v_n) \begin{pmatrix} I & 0 \\ NR & 0 \end{pmatrix} (f(x_0), \dots, f(x_n))^t. \tag{6.6}$$

Example 6.4 Let $L = S_n$ be the operators described in Example 5.5. If $a = 0$ or $a = b$, the Markov chain associated with L is absorbing and the absorbing state is 0, respectively 1. If $0 < a < b$, the chain is regular; see also [12] and [5].

Example 6.5 As in Example 5.8, let K_p be the canonical simplex in $\mathbb{R}^p$ and $B_n : C(K_p) \to C(K_p)$ the Bernstein operators on $C(K_p)$.

Consider the real numbers a_i and c_{ij}, $i, j = 1, \dots, p$, such that

$$a_i \geq 0,\ i = 1, \dots, p, \tag{6.7}$$

$$a_i \geq c_{ij},\ i \neq j, \tag{6.8}$$

$$\sum_{i=1}^{p} c_{ij} \geq \sum_{i=1}^{p} a_i, \quad j = 1, \dots, p, \tag{6.9}$$

and let n be an integer such that

$$n \geq max\{c_{11}, \dots, c_{pp},\ a_1 + \dots + a_p\}. \tag{6.10}$$

Under these assumptions we can define the affine transformation $A_n : K_p \to K_p$, $A_n(x) = A_n(x_1, \dots, x_p) :=$
$= (x_1 + \frac{1}{n}(a_1 - \sum_{j=1}^{p} c_{1j}x_j), \dots, x_p + \frac{1}{n}(a_p - \sum_{j=1}^{p} c_{pj}x_j))$,
where $x = (x_1, \dots, x_p) \in K_p$.

For each $f \in C(K_p)$ let $f_n(x) := f(A_n(x)),\ x \in K_p$.
Now consider the operator $W_n : C(K_p) \to C(K_p)$,

$$W_n f := B_n f_n, \quad f \in C(K_p).$$

This is one instance of the operators constructed in [3]; see also Example 5.8.

Let $z_0, z_1, \ldots, z_p$ be the vertices of K_p. The Markov chain associated with W_n is absorbing if and only if there exists $k \in \{0, 1, \ldots, p\}$ such that $A_n(z_k)$ is a vertex of K_p; otherwise the chain is regular.

In particular, if $a_i = c_{ij} = 0,\ i, j = 1, \ldots, p$, the chain is absorbing and the absorbing states are the vertices of K_p. If all the inequalities in (6.7) - (6.10) are strict, the chain is regular.

Details and other examples will be presented in a forthcoming paper.

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My Italian Connections

Péter Vértesi (Budapest)

Dedicated to Professor GIUSEPPE MASTROIANNI on his 70th birthday

Abstract. During the last 22 years I wrote 20 papers with Giuseppe and his students. This work tries to select the most characteristic ones.

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Keywords and phrases. Interpolation, modulus of continuities, generalized Jacobi weights, weighted Fourier series on infinite intervals.

1 Introduction

First I met Giuseppe almost 25 years ago in a Conference hold at the Technical University, Budapest. However, our first mathematical conversations was during a Varna Conference in 1987. He was sitting at a table and Bianca (della Vecchia) acted as his interpreter. As a result of our short talk, next year I went to Italy – first time in my life. During the years I visited Giuseppe, his students and my other Italian friends about 30 times including the Maratea Conferences, l'Aquila, Lecce, Vico Equense, Rome and of course, the "magic triangle" Naples, Lauria and Potenza.
And after these years, I can say that Giuseppe Mastroianni is not only a very talented and dedicated mathematician but a colorful and exceptionally warm-hearted human being, an honest friend. I wish him many-many good years, even more students, good health.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Let me say some words on the structure of this survey. I chose 5 joint works which I considered as the most typical ones of our cooperation with Giuseppe and his mathematical school. I kept the main features of the original papers; it means sometimes I repeat definitions, denote the same notion with different symbols (in the different papers). It gives the advantage that each part can be read without referring to the other ones. Moreover, I kept the original titles of the articles.

2 Approximation by some operators using certain new moduli of continuity

Actually, this is our first joint paper; it was completed around 1990; that means the new modules of continuity is the DITZIAN - TOTIK modulus. I remember we argued, proved and disproved for long-long hours traveling in Giuseppe's car from Naples to Lauria. And here are some results of the paper.

2.1

Let $\varphi(x) = \sqrt{1-x^2},\ f \in C$ (=the set of continuous functions on $[-1,1]$),

$$\omega^{\varphi}(f,t) := \sup_{0<h\le t} \left\{ \max_{|x|\le 1} \left| f\left(x - \frac{h\varphi(x)}{2}\right) - f\left(x + \frac{h\varphi(x)}{2}\right) \right| \right\} \tag{2.1}$$

the DT modulus (if any argument of f is not in $[-1,1]$, then $|\dots| := 0$). As it was proved,

$$E_n(f) \le c\omega_{\varphi}\left(f, \frac{1}{n}\right) \left(\le c\omega\left(f, \frac{1}{n}\right)\right) \tag{2.2}$$

where $E_n(f) = \min_{P\in\mathcal{P}_n} \|f - P\|$, $\omega(f,t)$ is the usual modulus (when we replace $\varphi(x)$ by 1 in (2.1.)), $\|\cdot\|$ is the maximum-norm (on $[-1,1]$). The estimation

$$\omega^{\varphi}(f,t) \le \omega(f,t) \tag{2.3}$$

is obvious. If we define the 2π-periodic even function $g(\vartheta) = g_f(\vartheta)$ by $g(\vartheta) = f(\cos\vartheta)$ for $f \in C$, then as it is well-known,

$$E_n(f) = \tilde{E}_n(g) := \min_{T\in\mathcal{T}_n} \|g - T\|_{[0,2\pi)}.$$

We proved

Theorem 2.1 *By the above definitions*

$$M^{-1}\omega^{\varphi}(f,t) \le \omega(g_f,t) \le M\omega^{\varphi}(f,t), \quad 0 \le t \le t_0 \tag{2.4}$$

with some positive constants M *and* t_0.

2.2

The above theorem has many interesting applications. Let us mention some.

2.2.1

The classical Jackson kernel is

$$L_n(\vartheta) = \gamma_n^{-1}\left(\frac{\sin(n\vartheta/2)}{\sin(\vartheta/2)}\right)^4, \quad \int_{-\pi}^{\pi} L_n(\vartheta)d\vartheta = 1,$$

where the second relation defines γ_n. As it well-known, $L_n \in \mathcal{T}_{2n-2}$, positive, even. Further

$$\gamma_n \sim n^3, \quad \int_0^{\pi} \vartheta^k L_n(\vartheta)d\vartheta \sim n^{-k}, \quad k = 0,1,2.$$

For arbitrary $h \in \tilde{C}$ (= the set of 2π-periodic continuous functions) the Jackson operator

$$J_n(h,\vartheta) := \int_{-\pi}^{\pi} h(t)L_n(\vartheta - t)dt$$

is a trigonometric polynomial of degree $\le 2n-2$, even if h is even, further

$$\|J_n(h) - h\|_{[0,2\pi)} \le c\omega\left(h, \frac{1}{n}\right).$$

For $f \in C$ the (algebraic) Jackson operator

$$I_n(f,x) := \int_{-\pi}^{\pi} f(\cos t)L_n(\vartheta - t)dt, \quad x = \cos\vartheta, \tag{2.5}$$

is an algebraic polynomial of degree $\le 2n-2$, further it fulfills

$$\|I_n(f) - f\| \le c\omega\left(f, \frac{1}{n}\right).$$

By definition we have with $x = \cos\vartheta$

$$I_n(f,x) - f(x) = J_n(g_f,\vartheta) - g_f(\vartheta) = J_n(g_f,-\vartheta) - g_f(-\vartheta)$$

whence by (2.4) and Theorem 2.1 we get

Statement 2.2 *By the above notations, for arbitrary* $f \in C$

$$\|I_n(f) - f\| \le c\omega^{\varphi}\left(f, \frac{1}{n}\right).$$

2.2.2

In 1967 O. Kis and P. Vértesi introduced the trigonometric interpolatory operator $p_n(h,\vartheta) \in \mathcal{T}_{4n}$ defined for $h \in \tilde{C}$ by

$$p_n(h,\vartheta) := \sum_{k=-n}^{n} h(\vartheta_{kn})u_{kn}(\vartheta),$$

where

$$\vartheta_{kn} = \frac{2k\pi}{2n+1}, \quad k = 0, \pm1, \pm2, \dots ,$$

$$u_{kn}(\vartheta) = 4\ell_{kn}^3(\vartheta) - 3\ell_{kn}^4(\vartheta), \quad \ell_{kn}(\vartheta) = \frac{\sin\frac{2n+1}{2}(\vartheta - \vartheta_{kn})}{(2n+1)\sin\frac{\vartheta-\vartheta_{kn}}{2}},$$

$k = \pm1, \dots, \pm n$. The following properties were proved.

(i) $p_n(h,\vartheta_{kn}) = h(\vartheta_{kn}), \quad k = 0, \pm1, \pm2, \dots,$

(ii) $|p_n(h,\vartheta) - h(\vartheta)| \le c\omega(h, \frac{1}{n}), \quad n = 1, 2, \dots.$

Let $f \in C$ and let

$$K_n(f,x) = \sum_{k=-n}^{n} f(\cos\vartheta_{kn})u_{kn}(\arccos x).$$

Then $K_n \in \mathcal{P}_{4n}$, further

(iii) $K_n(f, \cos\vartheta_{kn}) = f(\cos\vartheta_{kn}), \quad k = 0, 1, 2, \dots,$

(iv) $|K_n(f,x) - f(x)| \le c\omega(f, \frac{1}{n}), \quad n = 1, 2, \dots.$

As above, with $x = \cos\vartheta$

$$K_n(f,x) - f(x) = p_n(g_f,\vartheta) - g_f(\vartheta) = p_n(g_f,-\vartheta) - g_f(-\vartheta),$$

whence by (ii) and Theorem 2.1 we have

Statement 2.3 *For arbitrary* $f \in C$

$$\|K_n(f) - f\| \le c\omega^{\varphi}\left(f, \frac{1}{n}\right), \quad n = 1, 2, \dots$$

Using theorems proved by S.N. Bernstein, D. Jackson and S. B. Stechkin for arbitrary $h \in C$.

$$\begin{cases} \tilde{E}_n(h) \le \omega(h, \frac{1}{n}), \\ \omega(h, t) \le ct \displaystyle\sum_{0 \le k \le 1/t} \tilde{E}_k(h) \end{cases} \tag{2.5}$$

Now let $f \in C$. By $E_n(f) = \tilde{E}_n(g_f)$, (2.5) and Theorem 2.1 we get a simple proof for the following

Statement 2.4 *For arbitrary* $f \in C$

$$\begin{cases} E_n(f) \le c\omega^{\varphi}(f, \frac{1}{n}), \\ \omega^{\varphi}(f, t) \le ct \displaystyle\sum_{0 \le k \le 1/t} E_k(f) \end{cases} \tag{2.5}$$

3 Mean convergence of Lagrange interpolation on arbitrary system of nodes

As far as I remember this work was completed during Giuseppe's visit in Budapest. He stayed in an apartment on Hegedűs Gyula utca (street), which was very close to the river Duna (Danube) and the Margit sziget (Margarete Island). Sometimes we walked on the island, and witnessed the unusual high float of the Danube – it was very spectacular.

Mean convergence of Lagrange interpolation based on roots of orthogonal polynomials (+ some additional points) has been studied for 70 years. The existing results heavily depend on properties of orthogonal systems. The aim of this paper (which appeared in 1993 in Acta Sci. Math. (Szeged)) was to get some general estimations which can serve to prove theorems on mean convergence of Lagrange interpolation based on *arbitrary* interpolatory point-systems. Throughout this paper we restrict ourselves to the interval $[-1, 1]$.

3.1

Let $X = \{x_{kn} = \cos\vartheta_{kn}\}$, $1 \le k \le n$, $n = 1, 2, \dots$ be an arbitrary interpolatory matrix with

$$x_{n+1,n} := -1 \le x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} \le 1 := x_{0n}, \quad n = 1, 2, \dots.$$

For X we define

$$\begin{cases} \omega_n(x) = \omega_n(X, x) := c_n \prod_{k=1}^{n} (x - x_{kn}), & c_n > 0,\ n = 1, 2, \dots, \\ l_{kn}(x) = l_{kn}(X, x) := \dfrac{\omega_n(x)}{\omega_n'(x_{kn})(x - x_k)}, & 1 \le k \le n, \\ L_n(f, x) = L_n(f, X, x) = \sum_{k=1}^{n} f(x_{kn}) \ell_{kn}(x), & n = 1, 2, \dots, \end{cases}$$

f is bounded.

$L_n(f, X, x)$ (a polynomial of degree $\le n - 1$) is the n–th Lagrange interpolatory polynomial based on X. As it turned out, the Lagrange interpolation, at least for the uniform metric, can be rather bad. Namely as it was proved by P. Erdős and P. Vértesi we have

Theorem A. *For every interpolatory matrix $X \subset [-1, 1]$ there is a continuous $F(x)$ such that*

$$\limsup_{n\to\infty} |L_n(F, X, x)| = \infty \quad a.e. \quad in [-1, 1].$$

On the other hand, considering L^2–convergence we have a positive result. To formulate, this we introduce some further notations.

Let $d\alpha$ be a non-negative measure generated by the non-decreasing bounded function $\alpha(x)$ supported in $[-1, 1]$ such that $\mathrm{supp}(d\alpha)$ is an infinite set and $\int_{-1}^{1} d\alpha > 0$. Let $\{x_{kn}(d\alpha)\} \subset (-1, 1)$ denote the n different roots of the corresponding orthonormal polynomial $p_n(d\alpha, x)$ of degree exactly n. For a given function f, let $L_n(f, d\alpha, x)$ denote the n–th Lagrange interpolatory polynomial of degree $\le n - 1$, based on $\{x_{kn}(d\alpha)\}$.

$u(x)$ is a weight (or weight function) iff $u(x) \ge 0$ (in $[-1, 1]$ and $0 < \int_{-1}^{1} u(x)dx < \infty$. Whenever $\alpha(x)$ is absolutely continuous then $\alpha'(x) = w(x)$ is a weight. In this case we apply the notations $p_n(w, x), x_{kn}(w)$, a.s.o.

Let u be a weight. We define

$$\|f\|_{u,p} := \begin{cases} \left(\int_{-1}^{1} \|f(x)\|^p u(x)dx\right)^{1/p} & 0 < p < \infty \\ \operatorname*{ess\,sup}_{-1\le x\le 1} \|f(x)\| & p = \infty. \end{cases}$$

We define

$$L_u^p := \{f; \|f\|_{u,p} < \infty\}, \quad 0 < p \le \infty, \quad C := \{f; f \text{ is continuous on} [-1,1]\}.$$

If $u \equiv 1$, we write $\|f\|_p$ and L^p while L_u and L stands for L_u^1 and L^1, respectively.
If $f \in C, \|f\| := \|f\|_\infty (= \|f\|_{\infty,u})$ for any u.

Finally we define

$$\|L_n(d\alpha)\|_{u,p} := \sup_{\|g\|\le 1} \|L_n(g, d\alpha, x)\|_{u,p}.$$

In 1936 P. Erdős and P. Turán proved

Theorem B. *Let w be a weight. Then*

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - L_n(f, w, x)|^2 w(x)dx = 0 \quad \textit{for every} \quad f \in C.$$

Theorem B motivates the next question raised by P. Turán : *Does there exist a weight w and $f \in C$ such that*

$$\limsup_{n\to\infty} \int_{-1}^{1} |L_n(f, w, x) - f(x)|^p w(x)dx = \infty$$

for every $p > 2$?

After many partial results P. Nevai proved

Theorem C. *Let $supp(d\alpha) = [-1, 1]$ and $|\log \alpha'(x)|/\sqrt{1-x^2} \in L$, $1 \le p_0 < \infty$ and $u(\ge 0) \in L$. Suppose that*

$$\int_{-1}^{1} \left(\alpha'(x)\sqrt{1-x^2}\right)^{-p/2} u(x)dx = \infty \tag{3.1}$$

for every $p > p_0$. Then there exists a continuous function f such that

$$\limsup_{n\to\infty} \int_{-1}^{1} |L_n(f, d\alpha, x)|^p u(x)dx = \infty \tag{3.2}$$

for every $p > p_0$.

Considering that for a wide class of measures (3.1) holds, Theorem C gives a "yes" answer even for a generalization of Turán's question.

In 1991 Y. G. Shi developing a new approach, was able to drop the condition $|\log \alpha'(x)|/\sqrt{1-x^2} \in L$. Namely, he proved

Theorem D. *Let* u *be a weight,* α *be absolutely continuous and* $2 \le p_0 < \infty$. *If (3.1) holds for every* $p > p_0$ *then there is an* $f \in C$ *such that (3.2) is true for every* $p > p_0$.

3.2

The proofs of Theorem B-C heavily use the fact that the nodes $X = \{x_{kn}\}$ are roots of orthogonal polynomial system. A natural question is to get analogous results using arbitrary interpolatory matrix X. In Theorem D, Shi obtained a lower estimation for the sum $S_n(X,x) = \sum |x - x_k||\ell_k(x)|$ for arbitrary X. Actually the technique to prove that $S_n(x)$ is "big" on a large set was developed by P. Erdős and P. Vértesi; one of the main achievements of Shi is the discovery of the fundamental importance of $S_n(x)$ in the mean convergence of Lagrange interpolation (while, $\sum |\ell_k(x)|$ plays a decisive role at the uniform convergence).

As it turns out, developing and properly modifying his ideas one can get statements analogous to Theorem C and D for arbitrary point-system X. To state the results first we give a definition.

Let w be a weight. By G. Szegő, if for $I = [A,B] \subset [-1,1]$ we have $\int_I w > 0$, then if n is sufficiently large, every polynomial $p_n(w,x)$ has at least one zero in $[a,b]$. This leads to our

Definition *The interpolatory matrix* X *is regular with respect to the weight* w *(*X *is* w*–regular, shortly) iff every interval* $I \subset [-1,1]$ *with* $\int_I > 0$ *contains at least one root of* $\omega_n(X,x)$, *whenever* n *is sufficiently large.*

Let

$$\|L_n(X)\|_{u,p} := \sup_{\|g\|=1} \|L_n(g,X,x)\|_{u,p}\,.$$

Theorem 3.1 *Let* u *and* w *be weights. Let* X *be* w*-regular. Then for every* p, $0 < p \le \infty$,

$$\|\omega_n(X,x)\|_{u,p} \le c\,\|\omega_n(X,x)\|_{w,1}\,\|L_n(X)\|_{u,p}\,, \quad n \ge 1$$

with a proper $c > 0$. Here c may depend on u, w, X and p, but does not depend on n. Further, if $0 < q_0 \le p \le \infty$ (q_0 fixed) then $c > 0$ will not depend on p. Especially

$$\|p_n(w)\|_{u,p} \le c \|L_n(w)\|_{u,p} \quad n \ge 1.$$

In many cases $\|p_n(w)\|_{u,p}$ can be expressed (or estimated) by w (say $w \in GSJ$ or when $w > 0$ a.e.), i.e. by Theorem 3.1. we can get a necessary condition for the corresponding mean convergence of $L_n(f, w, x)$. Now we give some applications.

3.3

In their papers G. Criscuolo, G. Mastroianni, P. Nevai and D. Occorsio consider — among others — the Lagrange interpolatory polynomials $L_{2n+1,r,s}(f, W, x)$ based on the roots of

$$\omega_N(x) := A_s(x) B_r(x) p_n(W, x) p_{n+1}(W, x),$$

where $N = 2n + 1 + r + s$, $A_s(x) = \prod_{i=1}^{s}(x - y_{in})$, $B_r(x) = \prod_{i=1}^{r}(x - z_{in})$, $W \in GSJ$ (generalized smooth Jacobi weight), i.e.

$$W(x) = g(x) v^{\alpha,\beta}(x) \prod_{i=1}^{M} |\tau_j - x|^{\Gamma_i}, \quad x \in [-1, 1],$$

where $v^{(\alpha,\beta)}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, r, s, M are fixed non-negative integers, $\alpha > -1$, $\beta > -1$, $\Gamma_i > -1$, $1 \le i \le M$, and $-1 < \tau_1 < \cdots < \tau_M < 1$. Further $g(x) \ge 0$, $g \in C$ and $\int_0^1 \omega(g, t) t^{-1} dt < \infty$ where $\omega(g, t)$ is the modulus of continuity of g. We suppose (if s or $r > 0$ respectively) that

$$-1 \le y_{1n} < y_{2n} < \cdots < y_{sn} < x_{n+1,n+1}(W) < x_{nn}(W) < \cdots < x_{1n}(W) < \\ < x_{1,n+1}(W) < z_{rn} < z_{r-1,n} < \cdots < z_{1n} \le 1,$$

further the relations

$$y_{in} - y_{i-1,n} \sim x_{n+1,n+1}(W) - y_{sn} \sim \frac{1}{n^2}, \quad 2 \le i \le s$$

($f_n(x) \sim g_n(x)$ iff $0 < c_1 \le f_n(x)/g_n(x) \le c_2 < \infty$, uniformly in $n \ge n_0$ and $x \in [-1, 1]$),

$$z_{in} - z_{i+1,n} \sim z_{rn} - x_{1,n+1}(W) \sim \frac{1}{n^2}, \quad 1 \le i \le r - 1.$$

We have the following.

Let $W \in GSJ$ and let $u \geq 0, u\log^+ u \in L$. Further, let $Wv^{(-r,-s)} \in L^1$. If

$$u^{1/p}W^{-1}v^{(r-1/2,s-1/2)} \in L^p, \quad 0 < p < \infty, \tag{3.3}$$

then

$$\lim_{n\to\infty} \|f - L_{2n+1,r,s}(f,W)\|_{u,p} = 0 \quad \forall f \in C.$$

Using this statement and the Banach-Steinhaus theorem we obtain that

$$\limsup_{n\to\infty} \|L_{2n+1,r,s}(W)\|_{u,p} < \infty, \quad p \geq 1, \tag{3.4}$$

whenever (3.3) holds true.

Conversely, we have

Theorem 3.2 *Let $W \in GSJ$ and let $u(\geq 0) \in L^1$. Further let $Wv^{(-r,-s)} \in L^1$ where $0 \leq r, s \leq 1$, then from (3.4) we have (3.3).*

Analogous converse statements (using similar proofs) hold true for the other Lagrange processes $L_{2m+1,r,s}(f,W,W_1)$ and $L_{2n+1,r,s}(f,W,W)$.

3.4

A. Máthé, P. Nevai and V. Totik proved the following general statement.

Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\alpha' > 0$ a.e. in $[-1,1]$ and suppose $0 < p \leq \infty$. If g is Lebesgue measurable then

$$\left\| g/(\alpha'\sqrt{1-x^2})^{1/2} \right\|_p \leq c \liminf_{n\to\infty} \|gp_n(d\alpha)\|_p . \tag{3.5}$$

Let $d\alpha$ be a measure, $\omega_N(x) = p_n(d\alpha,x)(1-x)^r(1+x)^s$ where $0 \leq r, s \leq 1$, are fixed, $N = n+r+s$. $L_{n,r,s}(f,d\alpha,x)$ stands for the Lagrange polynomial of degree $n+r+s-1$ based on the roots of $\omega_N(x)$. By Theorem 3.1 and (3.5) we can prove

Theorem 3.3 *Let $\sup p(d\alpha) = [-1,1]$, $\alpha' > 0$ a.e. in $[-1,1]$, u be an arbitrary weight and $1 \leq p_0 \leq \infty$. Suppose that*

$$\int_{-1}^{1} \left| \frac{(1-x)^r(1+x)^s}{\left(\alpha'(x)\sqrt{1-x^2}\right)^{1/2}} \right|^p u(x)dx = \infty \quad \text{for every} \quad p_0 < p \leq \infty.$$

Then there exist an $f \in C$ such that

$$\limsup_{n\to\infty} \int_{-1}^{1} |L_{n,r,s}(f,d\alpha,x)|^p \, u(x)dx = \infty \quad \text{for every} \quad p_0 < p \leq \infty.$$

3.5

Theorem 3.1 is sharp in the following sense.

Statement 3.4 *There exist weight functions u and w and the w-regular interpolatory matrix X such that for every $p, 0 < p < \infty$,*

$$\|\omega_n(X,x)\|_{w,1}\,\|L_n(X)\|_{u,p} \le c_p\,\|\omega_n(X,x)\|_{u,p}\,, \quad n \ge 1.$$

4 Some application of generalized Jacobi weights

This paper (completed about 15 years ago) is one of my favorite joint works with Giuseppe. It summarizes and generalizes many well-known results and conjectures related to quite general Jacobi type orthogonal polynomials.

4.1 Introduction. Notations. Preliminaries

4.1.1

In this paper we deal with orthogonal polynomials where the corresponding weight is constructed from moduli of continuity.

4.1.2

As before C is the complex plane, $\mathbf{R} = (-\infty, \infty)$, $\mathbf{N} = \{1, 2, \dots\}$, $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$.

$L^p[a,b]$ denotes the set of functions F such that

$$\begin{cases} \|F\|_{L^p[a,b]} := \left\{ \displaystyle\int_b^a |F(t)|^p\,dt \right\}^{1/p} & \text{if } 0 < p < \infty \\ \|F\|_\infty := \operatorname*{ess\,sup}_{a \le t \le b} |F(t)| & \text{if } p = \infty \end{cases}$$

is finite. (If $p \ge 1$, this is a norm; for $0 < p < 1$ its pth power defines a metric in $L^p[a,b]$.)

By a *modulus of continuity* we mean a nondecreasing, continuous semi-additive function $\omega(\delta)$ on $[0,\infty]$ with $\omega(0) = 0$ ($\omega \in MC$, shortly). If, in addition,

$$\omega(\delta) + \omega(\eta) \le 2\omega(\delta/2 + \eta/2) \quad \text{for any} \quad \delta, \eta \ge 0,$$

then $\omega(\delta)$ is a *concave* modulus continuity. In the latter case $\delta/\omega(\delta)$ is nondecreasing for $\delta \geq 0$. We define the *modulus of continuity of f in L^p* (where L^p stands for $L^p[0, 2\pi]$) as

$$\omega(f, \delta)_p = \sup_{\|\lambda\| \leq \delta} \|f(\lambda + \cdot) - f(\cdot)\|_p .$$

4.1.3

We call a function $\mathcal{W}(\vartheta)$ *a trigonometric weight* ($\mathcal{W} \in TW$) if it is 2π-periodic, measurable, $\mathcal{W}(\vartheta) \geq 0$ and $\mathcal{W}(\vartheta) \not\equiv 0 (\vartheta \in R)$. If, in addition, $\mathcal{W} \in L^1$, we call the functions $\{\Phi_n(\mathcal{W}, z)\}_{n=0}^{\infty}$ the (unique) system of orthogonormal polynomials (OPN) on the unit circle line $\Gamma_1 := \partial D$ with respect to $\mathcal{W}$ if

$$\begin{cases} \Phi_n(\mathcal{W}, z) = \kappa_n(\mathcal{W}) z^n + \text{lower degree terms}, & \kappa_n(\mathcal{W}) > 0,\ n \in \mathbf{Z}_+, \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \Phi_n(\mathcal{W}, e^{i\vartheta}) \overline{\Phi_m(\mathcal{W}, e^{i\vartheta})} \mathcal{W}(\vartheta) d\vartheta = \delta_{nm} & n, m \in \mathbf{Z}_+. \end{cases}$$

We say that a point ϑ_0 is *regular* for $\mathcal{W} \in TW$ if for some $\varepsilon > 0$ the essential suprema of $\mathcal{W}$ and $1/\mathcal{W}$ on $[\vartheta_0 - \varepsilon, \vartheta_0 + \varepsilon]$ are finite. Otherwise, $\mathcal{W}$ has a *singularity* at ϑ_0.

Similarly, $w(x) \in AW$ (algebraic weight) if w is measurable in $[-1, 1]$, $w(x) \geq 0, w(x) \not\equiv 0 (x \in [-1, 1])$. If, in addition, $w \in L_1$ (L_1 stands for $L^1[-1, 1]$), the corresponding ONP $\{p_n(w, x)\}_{n=1}^{\infty}$ on $[-1, 1]$ are uniquely defined by

$$\begin{cases} p_n(w, x) = \gamma_n(w) x^n + \text{lower degree terms}, & \gamma_n(w) > 0,\ n \in \mathbf{Z}_+, \\ \displaystyle\int_{-1}^{1} p_n(w, x) p_m(w, x) w(x) dx = \delta_{nm} & n, m \in \mathbf{Z}_+. \end{cases}$$

We define the singular and regular points of w as we did for $\mathcal{W}$.

G. Szegő established a close connection between certain $\{\Phi_n\}$ and $\{p_n\}$. Let $w \in AW \cap L_1$. If the (even) $\mathcal{W} \in TW$ is defined by

$$\mathcal{W}(\vartheta) = w(\cos\vartheta) |\sin\vartheta|, \quad x = \cos\vartheta,\ x \in [-1, 1],$$

then with $\varphi(x) = \sqrt{1 - x^2}$, we have $a_{2n-1}(\mathcal{W}) := \kappa_{2n}^{-1}(\mathcal{W}) \Phi_{2n}(\mathcal{W}, 0)$, and

$$\begin{cases} p_n\left(w, \frac{1}{2}\left(z+\frac{1}{z}\right)\right) = \dfrac{z^{-n}\Phi_{2n}(\mathcal{W},z) + z^n\Phi_{2n}(\mathcal{W},z^{-1})}{\sqrt{2\pi}\left\{1+a_{2n-1}(\mathcal{W})\right\}^{1/2}}, \\ p_{n-1}\left(w\varphi, \frac{1}{2}\left(z+\frac{1}{z}\right)\right) = \dfrac{z^{-n}\Phi_{2n}(\mathcal{W},z) - z^n\Phi_{2n}(\mathcal{W},z^{-1})}{\sqrt{2\pi}\left\{1-a_{2n-1}(\mathcal{W})\right\}^{1/2}\frac{1}{2}\left(z-\frac{1}{z}\right)}, \\ \qquad\qquad n \in \mathbf{Z}_+, z \in C. \end{cases}$$

Note that $2^{-1}(z+z^{-1}) = \cos\vartheta = x \in [-1,1]$ iff $z = e^{i\vartheta}$; at the same time $2^{-1}(z-z^{-1}) = i\sin\vartheta$.

4.1.4

If $X = \{x_{kn}; k = 1,2,\ldots;n,\ n \in \mathbf{N}\} \subset [-1,1]$ is an *interpolatory* pointsystem (i.e. $x_{kn} = x_{jn}$ iff $k = j$) then we can consider the Lagrange interpolatory polynomials

$$L_n(f,X,x) = \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(X,x), \quad n \in N,$$

where $\ell_{kn} \in \mathcal{P}_{n-1}\backslash\mathcal{P}_{n-2}$ are the fundamental polynomials of Lagrange interpolation (i.e. they satisfy $\ell_{kn}(x_{jn}) = \delta_{kj}, 1 \le k,j \le n, n \in N$). As it was proved by G. Faber, for arbitrary fixed $X, \overline{\lim}_{n\to\infty}\|L_nf\| = \infty$ with a proper $f \in C$ (= f is continuous on $[-1,1]$l $\mathcal{P}_m$ stands for the set of polynomials of degree $\le m$); $\|\cdot\| = \max_{-1\le x\le 1}|\cdot|$).

However, as it proved by P. Erdős and P. Turán

$$\lim_{n\to\infty}\left\|(L_n(f,w)-f)\sqrt{w}\right\|_2 = 0 \text{ for all } f \in C$$

for arbitrary $w \in AW \cap L_1$. Here and later $\|\cdot\|_p$ stands for $\|\cdot\|_{L_p} = \|\cdot\|_{L_p[-1,1]}$; moreover, $L_n(f,w,x) = \sum_{k=1}^{n} f\left(x_{kn}(w)\right)\ell_{kn}(w,x)$, i.e. the Lagrange interpolation based on the n roots of $p_n(w,x)$. A natural question was to investigate the expressions $\|(L_n(f,w_1)-f)\,w_2^r\|_p$ where $w_1, w_2 \in AW\cap L_1$. If w_1 and w_2 are general Jacobi weights of the form $h(x)\prod_{i=0}^{m+1}|x-t_i|^{\gamma_i}$, $-1 \equiv t_{m+1} < t_m < \cdots < t_1 < t_0 \equiv 1$, $\gamma_i > -1$, $\int_0^1 \omega(h,t)t^{-1}dt < \infty$, the corresponding problems have been settled in a series of papers. It turned out that Marcinkiewicz-type inequalities can play a fundamental role.

4.1.5

The final aim of this work is to consider the above problem for weights w_1 and w_2 constructed by moduli of continuity.

Definition *The trigonometric weight $\mathcal{W}(\vartheta)$ is a generalized trigonometric Jacobi weight ($\mathcal{W} \in GTJ$ shortly), iff*

$$\mathcal{W}(\vartheta) = h(\vartheta) \prod_{r=0}^{m+1} w_r \left(\sin \frac{|\vartheta - \tau_r|}{2} \right), \quad \vartheta \in \mathbf{R},$$

where

$$\begin{aligned} &-\pi < \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} \le \pi, \\ &\tau_{-1} = \tau_{m+1} - 2\pi, \quad \tau_{m+2} = \tau_0 + 2\pi, \end{aligned} \tag{4.1}$$

$$w_r(\delta) = \prod_{s=1}^{\ell_r} \{\omega_{rs}(\delta)\}^{\alpha(r,s)}, \tag{4.2}$$

$m, \ell_r \in \mathbf{N}$, $\alpha(r,s) \in \mathbf{R}$, $\omega_{r,s}(\delta)$ are moduli of continuity ($s = 1, 2, \ldots, \ell_r$, $r = 0, 1, \ldots, m+1$) and the function h satisfies

$$h(\vartheta) \ge 0, \quad h \text{ and } 1/h \in L^\infty. \tag{4.3}$$

(i) If $\mathcal{W} \in GTJ$ and all ω_{rs} are concave moduli of continuity then we say that $\mathcal{W} \in GTJ1$.

(ii) We say that $\mathcal{W} \in GTJ2$ if $\mathcal{W} \in GTJ$ and

$$\int_0^\delta w_r(\tau) d\tau = O\left(\delta w_r(\delta)\right), \quad \delta \to +0 \ \ (r = 0, 1, \ldots, m+1).$$

(iii) If $\mathcal{W} \in GTJ$ and

$$\omega(h,\delta)_\infty \delta^{-1} \in L^1[0,1] \quad \text{or} \quad \omega(h,\delta)_2 = O(\sqrt{\delta}),\ \delta \to 0,$$

then $\mathcal{W} \in GTJ3$.

Remark Generally we combine the above properties. E.g., we write that $\mathcal{W} \in GTJ13$ iff $\mathcal{W} \in GTJ1 \cap GTJ3$.

Definition *Let $w(x) \in AW$ be defined by*

$$w(x) = H(x)\left(\sqrt{1-x^2}\right)^{-1} w_0\left(\sqrt{1-x}\right) w_{m+1}\left(\sqrt{1+x}\right) \prod_{r=1}^{m} w_r\left(|x-t_r|\right). \tag{4.4}$$

If with $h(\vartheta) := H(\cos\vartheta)$, $t_r = \cos\tau_r \in (-1,1)(1 \le r \le m)$ the corresponding expressions $h(\vartheta), \{\tau_r\}$ and $\{w_r(\delta)\}$ satisfy (4.1)–(4.3), then $w \in GJ$ (generalized Jacobi weight). The definition of $GJ1, GJ2$, etc., are similar.

Definition *Let $w \in GJ$. Then $\mathcal{W}(w,\vartheta) := \mathcal{W}(\vartheta) = w(\cos\vartheta)|\sin\vartheta|$, $\vartheta \in \mathbf{R}$, is called a generalized even trigonometric Jacobi weight ($\mathcal{W} \in GETJ$). The definitions of $GETJ1$, etc. are analogous.*

Remarks 1. It is easy to see that the classical weights $(1-x)^\alpha(1+x)^\beta$, $(1-x)^\alpha(1+x)^\beta|x|^\gamma$, $\alpha,\beta,\gamma > -1$ are from $GJ123$. Also, $(1+|x|)(1-x)^\alpha \in GJ123$ and $([x]+2)(1+x)^\beta \in GJ123$ $(\alpha,\beta > -1)$.

2. Further it is not difficult to verify the relation

$$t^\Gamma\left(\log\frac{1}{t}\right)^\gamma \in GMC2, \quad \Gamma > -1, \ \ \gamma \in \mathbf{R}, \ \ 0 \le t \le t_0,$$

where GMC means that our modulus is of form (4.2).

Using induction one can get (with $\log_k = \overset{1}{\smile}\log, \overset{2}{\smile}\log \ldots \overset{k}{\smile}\log$) the relation

$$t^\Gamma\left(\log_k\frac{1}{t}\right)^\gamma \in GMC2 \quad (\Gamma > -1, \ \gamma \in \mathbf{R}, \ \ 0 \le t \le t_0).$$

Similar argument shows the relations

$$t^\Gamma\left(\log\frac{1}{t}\right)\left(\log\log\frac{1}{t}\right)^\gamma \in GMC2 \quad (\Gamma > -1, \ \ \gamma \in \mathbf{R}, \ \ 0 \le t \le t_0).$$

Investigation of other combinations are left to the interested reader.

3. We saw $\left(\frac{1}{\left(\log\frac{1}{x}\right)^\gamma}\right)' = \frac{\alpha}{x\left(\log\frac{1}{x}\right)^{\alpha+1}} := \alpha u(x)$, whence, if $\alpha > 0$,

$$\int_0^\delta \frac{1}{x\left(\log\frac{1}{x}\right)^{\alpha+1}}dx = \frac{1}{\alpha}\frac{1}{\left(\log\frac{1}{\delta}\right)^\alpha} \ne O\left(\delta u(\delta)\right)$$

i.e. if $w(x) = u(e/x) = e|x|^{-1}\left(\log\frac{e}{|x|}\right)^{-\alpha-1}$, then $w \in L_1$ but $w \notin GJ2$.

Definition *Let $\mathcal{W} \in TW$ and $w \in AW$. The (generalized) p-th Christoffel functions are as follows:*

$$\omega_n(\mathcal{W}, p, z) := \min_{P\in\mathcal{P}_{n-1}} \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|P(e^{ir})\right|^p}{|P(z)|^p}\mathcal{W}(\tau)d\tau, \quad z \in \mathbf{C},\ p > 0,$$

Definition *To state the next theorem we introduce the function class $GTJ4$. We say that $\mathcal{V} \in GTJ4$ if $\mathcal{V} = \mathcal{U}/\mathcal{W}$, where $\mathcal{U}, \mathcal{W} \in GTJ$, moreover both $\mathcal{U}$ and $\mathcal{W}$ have only positive exponents (cf. (4.2), further, if $\mathcal{W}$ is defined by (4.2), then*

$$\int_0^\delta \frac{1}{w_r(\tau)}d\tau = O\left(\frac{\delta}{w_r(\delta)}\right), \quad \delta \to +0,\ r = 0, 1, \ldots, m+1.$$

If $\mathcal{U}$ and $\mathcal{W} \in GETJ$, too, then we say $\mathcal{V} \in GETJ4$. Let $v \in GJ$ be of the form (4.4) (with $v_0, v_1, \ldots, v_{m+1}$). Then $|\sin\vartheta|v(\cos\vartheta) = \mathcal{V}(v)(\vartheta) \in GETJ$. If $\mathcal{V}(v) \in GETJ4$, too, then by definition, $v \in GJ4$.

Example Let $\omega(\delta) = |\log\delta|^{-\alpha}$, $\eta(\delta) = \delta^\beta$ ($\alpha \in \mathbf{R}, 0 < \beta < 1, \delta > 0$). Then $\mathcal{V}(\vartheta) = \omega\left(\sin\frac{|\vartheta-\pi|}{2}\right)/\eta\left(\sin\frac{|\vartheta|}{2}\right) \in GTJ4$. On the other hand, $V(\vartheta) = \sin\left(\frac{|\sin\vartheta|}{2}\right)/\left(\sin\frac{|\vartheta|}{2}\right) \in GTJ2\backslash GTJ4$.

4.2 New results

Throughout this part we consider $w \in GJ$ and the corresponding $\mathcal{W} = \mathcal{W}(w) \in GETJ$.

Theorem 4.1 *Let $w \in GJ2$. Then for any fixed $p > 0$*

$$\lambda_n(w, p, x) \sim \omega_n\left(\mathcal{W}(w), p, e^{i\vartheta}\right) \sim \frac{\mathcal{W}(w)(n, \vartheta)}{n}, \quad \vartheta \in \mathbf{R},\ x = \cos\vartheta,\ n \in \mathbf{N},$$

uniformly in x and n.

Remarks 1. One can prove that

$$\lambda_n(w,p,x)$$
$$\sim \frac{1}{n} w_0\left(\sqrt{1-x}+\frac{1}{n}\right) w_{m+1}\left(\sqrt{1+x}+\frac{1}{n}\right)\prod_{r=1}^{m} w_r\left(|x-t_r|+\frac{1}{n}\right)$$
$$= \frac{w(n,x)\varphi(n,x)}{n}.$$

2. The proof of the previous theorem shows that

$$\omega_n(\mathcal{W},p,e^{i\vartheta}) \sim \frac{\mathcal{W}(n,\vartheta)}{n}$$

for arbitrary $\mathcal{W} \in GTJ2$, too.

Theorem 4.2 *Let $w \in GJ2$. Then for the roots $x_{kn}(w) = \cos\vartheta_{kn}(w)$ $(1 \le k \le n)$ of $p_n(w)$ we have*

$$\vartheta_{k+1,n}(w) - \vartheta_{kn}(w) \sim \frac{1}{n}, \quad 0 \le k \le n,\ n \in \mathbf{N},$$

uniformly in k, where $\vartheta_{0n}(w) = 0$ and $\vartheta_{n+1,n}(w) = \pi$.

If $|x - x_{jn}(w)| = \min_{1\le\le n} |x - x_{kn}(w)|$, $j = j(n,x)$, then

Theorem 4.3 *Let $w \in GJ123$. Then, for $n \in \mathbf{N}$*

$$\left|p_n'(w, x_{kn}(w))\right| \sim \frac{n}{\varphi(x_{kn}(w))} \frac{1}{\sqrt{w(n,x_{kn}(w))\varphi(n,x_{kn}(w))}},$$
$$0 \le k \le n+1,$$

$$|p_n(w,x)| \sim \left|p_n'(x_{jn}(w))\right| |x - x_{jn}(w)|, \quad 1 \le j \le n,\ x \in [-1,1],$$

$$\vartheta_{k+1,n}(p_n(w)) - \vartheta_{k,n-1}\left(p_n'(w)\right)$$

$$\sim \vartheta_{k,n-1}\left(p_n'(w)\right) - \vartheta_{kn}\left(p_n(w)\right) \sim \frac{1}{n}, \quad 1 \le k \le n-1.$$

The estimations are uniform in x, k, j and n, respectively.

The next theorem is a very general quadrature inequality.

Theorem 4.4 *Let $w \in GJ2$ and $v \in GJ24$. Then for any fixed positive integer ℓ and $1 \le p < \infty$*

$$\sum_{k=1}^{n} \lambda_n \left(v, x_{kn}(w)\right) \left|P\left(x_{kn}(w)\right)\right|^p \le c \int_{-1}^{1} |P(x)|^p \, v(x) dx, \quad n \in \mathbf{N},$$

whenever $P \in \mathcal{P}_{\ell n}$. Here c does not depend on n and P.

The converse inequality comes form

Theorem 4.5 *Let $w \in GJ123$ further v, $V^{-p} \in GJ2$, finally $V^q \in GJ24$. Here $V = \sqrt{w\varphi} v^{-1/p}$, $1/p + 1/q = 1$, $1 < p < \infty$.*

Then for any $P \in \mathcal{P}_{n-1}$

$$\int_{-1}^{1} |P(x)|^p \, v(x) dx \le c \sum_{k=1}^{n} \lambda_n(v, x_{kn}(w)) \left|P(x_{kn}(w))\right|^p, \quad n \in \mathbf{N}.$$

Final remarks 1. Let $v(x) = H^{-1}(x) w(x)$ a.e. where

$$w(x) = H(x) \varphi^{-1}(x) \prod_{r=0}^{m+1} \log^{\beta r} \frac{e}{|x - t)r|}, \quad \beta_r \in \mathbf{R}, \ 0 \le r \le m+1$$

(t_r are as above), where H is defined so that $w \in G123$.

By Theorem 4.5 it is easy to conclude the L_p convergence of the corresponding weighted Lagrange interpolation for every $f \in C[-1, 1] (0 < p < \infty$. If $H(x) = 1, \beta_r = 0$ (i.e. $v = w = \varphi^{-1}$, Chebyshev weight), the result was proved by P. Erdős and E. Feldheim.

2. The interested reader may find many other theorems and references in the original quite long paper.

5 One-sided convergence conditions of Lagrange interpolation based on Jacobi-type weights

This is a three-author paper written by Bianca Della Veccia, Giuseppe and me in 2001.

5.1

Let $X = (x_{in} \in I = [-1,1] : i = 1,\dots,n; n \in N)$ be an infinite matrix where each row is a set of distinct point in I. Let us define for $f \in C(I)$ the Lagrange interpolation polynomial $L_n(f,X,x)$ by

$$L_n(f,X,x_{kn}) = f(x_{kn}), \quad k = 1,\dots,n.$$

Let

$$x_{kn}^{(\alpha,\beta)} = \cos\theta_{kn}^{(\alpha,\beta)}, \quad k = 1,\dots,n,\ \alpha,\beta > -1, \tag{5.1}$$

be the n distinct zeros (in decreasing order) of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ orthogonal on $(-1,1)$ with respect to the weight

$$p^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1.$$

Then if the corresponding Lagrange interpolation polynomial based on the knots (5.1) is denoted by $L_n^{(\alpha,\beta)}(f)$, it is well-known that generally $L_n^{(\alpha,\beta)}(f)$ does not converge uniformly to $f \in C(I)$. However for a subclass of $f \in C(I)$ the following statement holds true.

THEOREM A. *Let* $-1 < \gamma = \max(\alpha,\beta) < \frac{1}{2}$ *be fixed,* $f \in C(I)$ *and*

$$\delta(f;t) \;=\; \begin{cases} o(|\log t|^{-1}), & \text{if } -1 < \gamma \le -\dfrac{1}{2} \\ o(t^{\gamma+\frac{1}{2}}), & \text{if } -\dfrac{1}{2} < \gamma < \dfrac{1}{2}. \end{cases}$$

Then

$$\lim_{n\to\infty} \left\| L_n^{(\alpha,\beta)}(f) - f \right\| = 0. \tag{5.2}$$

Here $\|f\|$ is the usual supremum norm of f on I; moreover $\delta(f;t)$ denotes the so called *one-sided modulus of continuity of* f defined by

$$\delta(f;t) = \sup_{\substack{0\le h\le t \\ -1\le x\le x+h\le 1}} \left(f(x+h) - f(x)\right), \quad t \ge 0$$

As two corollaries, we have that $L_n^{(\alpha,\beta)}(f;x)$ converges to $f(x)$, if

1. f is of bounded variation, or
2. $-1 < \gamma \le -\frac{1}{2}$ and f satisfies the Dini-Lipschitz condition, or
3. $-\frac{1}{2} < \gamma < \frac{1}{2}$ and $f \in \operatorname{Lip}\left(\gamma + \frac{1}{2} + \varepsilon\right)$, for $0 < \varepsilon \le \frac{1}{2} - \gamma$.

Let us remark that the bounds for γ are, in a certain sense, sharp.

The aim of this paper was to generalize the above theorem using certain Jacobi type weights having "singularities" inside and to get the analogue of the following fundamental asymptotic formula proved by Gábor Szegő.

With $x = \cos\vartheta$,

$$p_n^{(\alpha,\beta)}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos(N\vartheta + \gamma) + O(\frac{1}{n \sin\vartheta})}{p^{(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4})}(x)}, \quad n \in \mathbf{N},\ n \sin\vartheta \geq c, \tag{5.3}$$

where $c > 0$ is arbitrary but fixed, $N = N(n, \alpha, \beta) = n + \frac{\alpha+\beta+1}{2}$ and $\gamma = \gamma(\alpha) = -(\alpha + \frac{1}{2})\frac{\pi}{2}$. (Generally, the order of the remainder term, $R_n^{(\alpha,\beta)}(\cos\vartheta) = (n \sin\vartheta)^{-1}$, is the best possible. The symbol "O" depends only on α and β.)

5.2

First we give the following

Definition *$F \in GJ$ (generalized Jacobi weights) iff for $x \in [-1, 1]$*

$$F(x) \equiv r(\Gamma, \mathbf{t}; x) := (1-x)^{\Gamma_0}(1+x)^{\Gamma_{v+1}} \prod_{k=1}^{v} |x - t_k|^{\Gamma_k};$$

here

$$-1 \equiv t_{v+1} < t_v < \cdots < t_1 < t_0 \equiv 1, \quad \Gamma_0, \Gamma_1 \ldots, \Gamma_{v+1} > -1.$$

If $\Gamma_k = 0$ when $1 \leq k \leq v$ (no inner singularity!), then we use our previous notation

$$p^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1,$$

for the classical Jacobi weights J.

Using $F(x)$ we introduced another weight, $r_{[\mathbf{m}]}(x)$ and the corresponding ONP, $r_n(r_{[\mathbf{m}]}, x)$. Instead of the exact quite technical definitions we list some properties.

$$\int_{-1}^{1} r_{[\mathbf{m}]} = 1. \tag{a}$$

The "singularities" and exponents of $r_{[\mathbf{m}]}$ are $t_0 = 1, \Gamma_0$; $t_{v+1} = -1, \Gamma_{v+1}$; t_k, Γ_k $(1 \le k \le v)$. Let $n = m_o + m_{v+1} + 2\sum_{k=1}^{v} m_k$. We suppose that $n \to \infty$ and $m_k \sim n$ $(0 \le k \le v+1)$. Then

$$r_{[\mathbf{m}]} > 0, \quad r_{[\mathbf{m}]}(t_k) \sim \left(\frac{1}{n}\right)^{\Gamma_k}, \quad 0 \le k \le v+1, \tag{b}$$

and, uniformly in x and n,

$$r_{[\mathbf{m}]}(x) \sim \left(1 - x + \frac{1}{n}\right)^{\Gamma_0} \left(1 + x + \frac{1}{n}\right)^{\Gamma_{v+1}} \prod_{k=1}^{v} \left(|x - t_k| + \frac{1}{n}\right)^{\Gamma_k}. \tag{c}$$

We have the following generalization of (5.3).

$$r_n^{(s)}\left(r_{[\mathbf{m}]}, x\right) = \sqrt{\frac{2}{\pi}} \left(\frac{-n}{\sin\vartheta}\right)^s \frac{\cos(n\vartheta + \varepsilon_l\left(\vartheta\right) + s\frac{\pi}{2}) + \frac{O(l)}{n|\vartheta - \tau_l||\vartheta - \tau_{l+1}|}}{\left(\varphi(\vartheta)\right)^{1/2}} \tag{5.4}$$

uniformly in n, l and $\vartheta \in E_{ln}(c) (0 \le l \le v)$. Here $c > 0$ and $s \ge 0$ are fixed, moreover

$$\varphi(\vartheta) = |\sin\vartheta| \prod_{k=0}^{v+1} |\cos\vartheta - \cos\tau_k|^{\Gamma_k},$$

$$\varepsilon_l(\vartheta) = \left(1 + \sum_{k=0}^{v+1} \Gamma_k\right) \frac{\vartheta}{2} - \left(\frac{1}{2} + \sum_{k=0}^{l} \Gamma_k\right) \frac{\pi}{2}$$

and

$$E_{\ell n}(c) = \left[\tau_\ell + \frac{c}{n}, \tau_{\ell+1} - \frac{c}{n}\right] \quad (0 \le \ell \le v).$$

By (5.4) we can obtain some fairly precise formulae for the roots $x_{kn} = \cos\vartheta_{kn}$, $1 \le k \le n$, of $r_n(r_{[\mathbf{m}]})$. Let

$$N = n + \frac{1 + \sum_{k=0}^{v+1} \Gamma_k}{2},$$

$$\gamma(\ell) = \left(-\frac{1}{2} - \sum_{k=0}^{\ell} \Gamma_k\right) \frac{\pi}{2}, \quad 0 \le \ell \le v,$$

and let $\vartheta_{kn} \in E_{\ell n}(c_0)$ $(0 \le \ell \le v,\ n \ge 1)$. Then with $K = k + A(\ell, n)$, $|A(\ell, n)| \le c$,

$$\vartheta_{kn} = \Theta_{Kn}(\ell) + \delta_{k\ell n},$$

where

$$\Theta_{un}(\ell) = \frac{2u-1}{2N}\pi - \frac{\gamma(\ell)}{N}$$

and for $\delta_{k\ell n}$, we have

$$|\delta_{k\ell n}| = O\left(\frac{1}{n^2|\Theta_{kn} - \tau_\ell||\Theta_{kn} - \tau_{\ell+1}|}\right).$$

Analogous formulae can be obtained for the roots of $r_n(r_{[\mathbf{m}]})^{(s)},\ s > 0$.

5.3

Let $L_n(r_{[\mathbf{m}]}, f, x)$ denote the Lagrange interpolation polynomial based on the roots

$$x_{kn} = x_k = \cos\vartheta_{kn} = \cos\vartheta_k, \quad 1 \le k \le n$$

of $r_n(r_{[\mathbf{m}]})$. Our main statement, a generalization of Theorem A, is the following.

Theorem 5.1 *Let*

$$\begin{cases} -1 < \Gamma_0, \Gamma_{v+1} < \dfrac{1}{2}, \quad -1 < \Gamma_k < 2, \quad 1 \le k \le v, \\ |\Gamma_i - \Gamma_j| < 2, \quad 1 \le i, j \le v. \end{cases}$$

Put

$$\rho = \max\left\{\Gamma_0 + \frac{1}{2}, \Gamma_{v+1} + \frac{1}{2}, \frac{\Gamma_k}{2}\ 1 \le k \le v\right\}.$$

Letting

$$\delta(f;t) = \begin{cases} o\left(|\log t|^{-1}\right), & \text{if } \rho \le 0 \\ o(t^\rho), & \text{if } \rho > 0 \end{cases}$$

we have

$$\lim_{n\to\infty} \left\|L_n(r_{[\mathbf{m}]}, f) - f\right\| = 0, \quad \forall f \in C(I).$$

Remarks. 1. The assumption on $\Gamma_i, 0 \le i \le v+1$, are, in a certain sense, sharp.

2. During the course of the proofs we obtained that the corresponding Lebesgue constants $\Lambda_n = \max_{|x|\le 1} \sum_{k=1}^n |l_{n,k}(x)|$ have the order

$$\Lambda_n \sim n^\Gamma + \log n,$$

where $\Gamma = \max(\Gamma_0 + \frac{1}{2}, \Gamma_{v+1} + \frac{1}{2}, \frac{\Gamma_k}{2},\ 1 \le k \le v)$. Here the only condition is $\Gamma_k > -1$, $0 \le k \le v+1$. This estimate is new and sharp.

6 Fourier Sums and Lagrange Interpolation on $(0, +\infty)$ and $(-\infty, +\infty)$

This is a joint work was dedicated to the memory of Professor Ambikeshwar Sharma; it appeared in the Memorial Volume, in 2007.

This paper is a very clear example of a nice idea of Giuseppe: using and generalizing an old observation of Muckenhoupt, instead of the *whole* MRS. (Mhaskar – Rahmanov – Saff) interval, I_n, one may use only a *part* of it, θI_n $(0 < \theta < 1)$. It turns out that this modification dramatically simplifies all the estimates involved but at the same time it preserves the order of the approximation.

– . –

Let me take the opportunity to mention another fruitful method of Giuseppe: he methodically and successfully applied the so-called one-sided approximation to get theorems in those cases when other methods failed.

– . –

6.1

This paper concerns the approximation of functions on unbounded intervals by means of orthogonal polynomials: Laguerre polynomials on $(0, \infty)$ and Freud polynomials on $(-\infty, +\infty)$.

We will modify the Fourier Sum and the interpolating polynomials. Our results have many applications and are analogous to results on finite intervals.

More precisely, starting from well known polynomial inequalities, we propose to approximate a "finite section" of the function on a special finite interval (see Proposition 2.1). This procedure s simple and is convenient in many situations, especially in the numerical treatment of integral equations.

6.2 Fourier Sums and Lagrange Interpolation in $(0, +\infty)$. Some preliminary results

Let $L^p(a, b), 0 \leq a < b \leq +\infty$ the set of all measurable functions f such that $\|f\|_{L^p(a,b)} = \left(\int_a^b |f|^p(x)dx\right)^{\frac{1}{p}} < +\infty, 1 \leq p < +\infty$.If $a = 0$ and $b = +\infty$ we write $\|f\|_p$ instead of $\|f\|_{L^p(0,+\infty)}$, and $L^p = L^p(0, +\infty)$. With

$u(x) = x^\gamma e^{-\frac{x}{2}}, \gamma > -\frac{1}{p}, x > 0$, a Laguerre weight, we set $f \in L_u^p$ if and only if $fu \in L^p,\ 1 \le p < +\infty$. When $p = +\infty$ and $\gamma > 0$ we define

$$L_u^\infty = \left\{ f \in C^0(\mathbb{R}^+) : \lim_{x\to\infty}(fu)(x) = 0 = \lim_{x\to+\infty}(fu)(x) \right\},$$

where $\mathbb{R}^+ = (0, +\infty)$ and $C^0(\mathbb{R}^+)$ is the set of all continuous functions in $\mathbb{R}^+$. In the case $p = +\infty$ and $\gamma = 0$, L_u^p is the set of all continuous functions in $[0, +\infty)$ with the condition $\lim_{x\to+\infty}(fu)(x) = 0$.

We recall the following polynomial inequality:

$$\left(\int_{4m(1+\delta)}^{\infty} |P_m u|^p(x)dx \right)^{\frac{1}{p}} \le \mathcal{C} e^{-Am} \left(\int_0^{+\infty} |P_m y|^p(x)dx \right)^{\frac{1}{p}}, \qquad (6.1)$$

holding for $1 \le p \le +\infty$, where $\delta > 0$ is fixed, P_m is an arbitrary polynomial of degree m ($P_m \in \mathbb{P}_m$), the positive constants $\mathcal{C}$ and A depend on δ and are independent on P_m.

Now we establish a proposition to be used later. To this end let $\theta \in (0, 1)$ fixed and let m be a positive integer sufficiently large (say $m > m_0$). We denote by $\Delta_\theta = \Delta_{\theta,m}$ the characteristic function of the interval $[0, 4\theta m]$ and let $M = \left[\frac{\theta}{1+\theta} m\right]$. Obviously, $M \sim m$ if $m \ge m_0(\theta)$.

Proposition 6.1 *Let $f \in L_u^p$ and $1 \le p \le +\infty$. Then, for $m > m_0$, we have*

$$\|f(1 - \Delta_\theta)u\|_p \le E_M(f)_{u,p} + \mathcal{C} e^{-Am} \|fu\|_p \quad \text{and} \qquad (6.2)$$

$$\|fu\|_p \le \mathcal{C} \left(\|f\Delta_\theta u\|_p + E_m(f)_{u,p} \right), \qquad (6.3)$$

where $E_M(f)_{u,p} = \inf_{P\in\mathbb{P}_M} \|(f - P)u\|_p$ and the constants $\mathcal{C}$ and A are independent on m and f.

The verification is quite simple, so we include it: For all $P_M \in \mathbb{P}_M$ and $1 \le p \le +\infty$ we have

$$\|f(1 - \Delta_\theta)u\|_p \le \|(f - P_M)u\|_p + \|P_M(1 - \Delta_\theta)u\|_p .$$

By (6.2)

$$\|P_M(1 - \Delta_\theta)u\|_p \le \mathcal{C} e^{-Am} \|P_M u\|_p$$

and making the infimum on P_M (6.2) easily follows. Now (6.3) follows from the indentity $f = \Delta_\theta f + (1 - \Delta_\theta)f$ and (6.2).

From Proposition 6.1 follows that, if $\{\Gamma_m : L^p_u \to \mathbb{P}_m\}_m$ is a sequence of linear operators, we can approximate a function $f \in L^p_u$ by the sequence $\{\Delta_{\theta,m}\Gamma_m(\Delta_{\theta,m}f)\}_m$. Indeed, since

$$\begin{aligned}\|[f - \Delta_{\theta,m}\Gamma_m(\Delta_{\theta,m}f)]\, u\|_p &\le \|[f - \Gamma_m(\Delta_{\theta,m}f)]\, \Delta_{\theta,m}u\|_p \\ &+ \|(1 - \Delta_{\theta,m})fu\|_p,\end{aligned}$$

the second term on the right, by (6.2), is small. Consequently it is sufficient to consider only the first term or, in other words, approximate f by the sequence of polynomials $\{\Gamma_m(\Delta_{\theta,m}f)\}_m$ on the interval $[0, 4\theta m]$. We will show that such procedure is much simpler and more convenient than the ordinary polynomial approximation.

Fourier sums

Let $w_\alpha(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x > 0$, be a Laguerre weight and $\{p_m(w_\alpha)\}_m$ the correspondent sequence of orthogonal polynomials with positive leading coefficient γ_m. With $f \in L^p_u$ we consider the Fourier sum

$$S_m(w_\alpha, f) = \sum_{k=0}^{m-1} c_k p_k(w_\alpha), \quad c_k = \int_0^{+\infty} f p_k(w_\alpha) w_\alpha.$$

If $v^\beta(x) = x^\beta$, the following theorem holds.

Theorem 6.2 *For any $f \in L^p_u$ and $1 < p < +\infty$, we have*

$$\|S_m(w_\alpha, \Delta_{\theta,m}f)\Delta_{\theta,m}u\|_p \le \mathcal{C}\, \|f\Delta_{\theta,m}u\|_p, \quad \mathcal{C} \ne \mathcal{C}(m, f)$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha\varphi}} \in L^p(0,1) \quad \text{and} \quad \frac{v^\alpha}{v^\gamma},\ \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma} \in L^q(0,1), \tag{6.4}$$

where $\varphi(x) = \sqrt{x}$ and $q = \frac{p}{p-1}$.

The previous theorem is the best possible in the following sense.

Theorem 6.3 *The following equivalences are true*

$$\|S_m(w_\alpha, f)u\|_p \le \mathcal{C}\|fu\|_p \Leftrightarrow \text{ (6.4) holds with} \frac{3}{4} < p < 4$$

$$\|S_m(w_\alpha,f)u\|_p \quad \leq \quad \mathcal{C}\begin{cases} m^{\frac{1}{3}}\|fu\|_p \\ or \\ \|fu(1+\cdot)^{\frac{1}{3}}\|_p \end{cases}$$

$$\Leftrightarrow \quad \text{(6.4) holds with } p\in(1,\infty)\setminus\left(\frac{4}{3},4\right)$$

$$\|S_m(w_\alpha,\Delta_{\theta,m}f)u\|_p \leq \mathcal{C}\|fu\|_p \Leftrightarrow \text{ (6.4) holds with } 1<p<4$$

$$\|S_m(w_\alpha,f)\Delta_{\theta,m}u\|_p \leq \mathcal{C}\|fu\|_p \Leftrightarrow \text{ (6.4) holds with } p>\frac{4}{3}.$$

The constants $\mathcal{C}>0$ are independent on m and f.

Let as mention that the estimations in Theorems 6.2 and 6.3 are analogue to the results on $[-1,1]$ with Jacobi weights; moreover, we do have the good convergence behaviour. Namely.

Theorem 6.4 *Under the assumptions of Theorem 6.2, for any $f\in L_u^p$ and $1<p<+\infty$, we have*

$$\|[f-\Delta_{\theta,m}S_m(w_\alpha,\Delta_{\theta,m}f)]\,u\|_p \leq \mathcal{C}\left[E_M(f)_{u,p}+e^{-Am}\|fu\|_p\right]$$

where $\mathcal{C}$ and A are positive constants independent on m and f.

If the parameters α and γ of the weights satisfy (6.4), then

$$\|[f-S_m(w_\alpha,\Delta_{\theta,m}f)]\,u\|_p \leq \mathcal{C}\left[E_M(f)_{u,p}+e^{-Am}\|fu\|_p\right]$$

with $1<p<4$ and

$$\|[f-S_m(w_\alpha,f)u]\,u\|_p \leq \mathcal{C}E_m(f)_{u,p}$$

with $\frac{4}{3}<p<4$. If, in the last estimate, we put $u=\sqrt{w_\alpha}$, we obtain an old result of Askey and Wanger.

Lagrange interpolation.

We denote by $L_m(w_\alpha,f)$, $f\in\mathcal{C}^0(\mathbb{R}^+)$, the Lagrange polynomial that interpolates the function f on the zeros $x_1<\cdots<x_m$, $(x_k=x_{km})$, of $p_m(w_\alpha)$. Using the Gaussian rule, we can write

$$L_m(w_\alpha,f,x)=mb_m\sum_{k=1}^{m}p_m(w_\alpha,x)\lambda_k(w_\alpha)\frac{F_m(x_k)}{x-x_k}f(x_k) \qquad (6.5)$$

where $\lambda_k(w_\alpha) = \lambda_{km}(w_\alpha) = \frac{1}{K_m(w_\alpha, x_{km})}$ is the k-th Christoffel number, $F_m(x) = p_{m+1}(w_\alpha, x) - p_{m-1}(w_\alpha, x)$ and $b_m \sim 1$. We define the integer $j = j(m)$ by

$$x_j = \min_k (x_k \geq 4\theta m), \quad \theta \in (0,1) \text{ fixed.}$$

Let $\psi \in C^\infty(\mathbb{R})$, nondecreasing; moreover $\psi(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x \geq 1 \end{cases}$ and $\psi_j(x) = \psi\left(\frac{x - x_j}{x_{j+1} - x_j}\right)$. For any $f \in C^0(\mathbb{R}^+)$, we define $f_j = (1 - \psi_j)f$. Obviously $f_j = f$ in $(0, x_j]$ and $f_j = 0$ in $[x_{j+1}, +\infty)$. Then we have

$$L_m(w_\alpha, f, x) = \sum_{k=1}^{j} \lambda_k(w_\alpha) \frac{p_m(w_\alpha, x)}{x - x_k} F_m(x_k) f(x_k)$$

by removing in (6.5) $[cm](0 < c < 1)$ terms. Finally with $\theta_1 \in (\theta, 1)$,fixed and with $\Delta_{\theta_1} = \Delta_{\theta_1, m}$ (the characteristic function of $[0, 4\theta_1 m]$), we introduce the sequence $\{\Delta_{\theta_1,m} L_m(w_\alpha, f_j)\}_m$ to approximate the function $f \in C^0(\mathbb{R}^+)$ in L^p_u norm with $u(x) = x^\gamma e^{-x/2}$. In order to study the behaviour of the above mentioned sequence, we state a lemma that will be useful in the sequel.

Lemma 6.5 *Let $0 < \theta < \theta_1 < 1$ and let $1 \leq p < +\infty$. Then, for an arbitrary polynomial $P \in \mathbb{P}_{tm}$ (with l fixed integer), we have*

$$\left(\sum_{k=1}^{j} \Delta x_k |Pu|^p(x_k) \right)^{\frac{1}{p}} \leq \mathcal{C} \left(\int_{x_1}^{4\theta_1 m} |Pu|^p(x) dx \right)^{\frac{1}{p}}$$

where $\Delta x_k = x_{k+1} - x_k$, $\mathcal{C}$ is a positive constant independent on m, P and p.

We can now state the following theorem, which is the main result of this section. Letting $v^\beta(x) = x^\beta$, we have a "Jacobi-type estimation.

Theorem 6.6 *Let $1 < p < +\infty$. Then, for all function $f \in C^0(\mathbb{R}^+)$, we have*

$$\|L_m(w_\alpha, f_j) u \Delta_{\theta_1}\|_p \leq \mathcal{C} \left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k) \right)^{\frac{1}{p}}. \quad \mathcal{C} \neq \mathcal{C}(m, f)$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha, \varphi}} \in L^p(0,1) \text{ and } \frac{\sqrt{v^\alpha \varphi}}{v^\gamma} \in L^q(0,1), \quad \varphi(x) = \sqrt{x},\ q = \frac{p}{p-1}. \quad (6.6)$$

However, if we fix w_α, u and p then the inequality

$$\|L_m(w_\alpha, f)u\|_p \le \mathcal{C}\left(\sum_{i=1}^{m} \Delta x_i |fu|^p(x_i)\right)^{\frac{1}{p}} \quad (x_{m+1} = 4m)$$

is not true for a proper $f \in C^0(\mathbb{R}^+)$ where $\mathcal{C} \neq \mathcal{C}(m, f)$.

We define

$$\Omega^r_\varphi(f,t)_{u,p} = \sum_{0<h\le t} \left\|u\Delta^r_{h\varphi} f\right\|_{I_{rh}}, \quad r \ge 1,\ \varphi(x) = \sqrt{x}$$

where $I_{rh} = \left[\mathcal{C}(2rh)^2, \mathcal{C}/h^2\right], \mathcal{C}$ arbitrary fixed constant and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + \frac{h}{2}\sqrt{x}(r-2i)\right),$$

t sufficiently small; i.e., $t < t_0$. Now we can state

Theorem 6.7 *For all continuous function in $\mathbb{R}^+$ and $p \in (1, +\infty)$ we have:*

$$\begin{aligned} \|[f - \Delta_{\theta_1} L)m(w_\alpha, f_j)]\, u\|_p \quad &\le \mathcal{C}\left(\frac{1}{(\sqrt{m})^{\frac{1}{p}}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}} dt \right. \\ &\left. + e^{-Am}\|fu\|_p\right) \end{aligned} \tag{6.7}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$ if and only if relations (6.6) are true.

In particular if we define the Zygmund space

$$Z^p_s = Z^p_s(u) = \left\{ f \in L^p_u : \sup_{t>0} \frac{\Omega^r_\varphi(t,f)_{u,p}}{t^s} < +\infty, \quad r > s \right\},$$

where $1 < p < +\infty$, $s \in \mathbb{R}^+$ and we introduce the usual norm

$$\|f\|_{Z^p_s} = \|fu\|_p + \sup_{t>0} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^s},$$

then (6.7) becomes

$$\|[f - \Delta_{\theta_1} L_m(w_\alpha, f_j)]\, u\|_p \le \frac{\mathcal{C}}{m^s}\|f\|_{Z^p_s}, \quad s > \frac{1}{p},\ 1 < p < +\infty,$$

that has the same order of the error of best approximation of function in Z^p_s, $s > \frac{1}{p}$. Similar estimates about the interpolation in $(-1, 1)$ can be found.

6.3 Fourier Sums and Lagrange Interpolation in $\mathbb{R}$. Some preliminary results

In this section we consider Freud weight of the form $w(x) = e^{-Q(x)}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and continuous, $Q' > 0$ in $(0, +\infty)$, Q'' is continuous in $(0, +\infty)$ and, for some $\hat{A}, \hat{B} > 1$ it results

$$\hat{A} \leq \frac{(xQ'(x))}{Q'(x)} \leq \hat{B}, \quad x \in (0, +\infty).$$

Denote by $\{p_m(w)\}_m$ the sequence of the polynomials that are orthonormal in $\mathbb{R}$ and whose leading coefficient $\gamma_m := \gamma_m(w)$ is positive. The MRS number, a_m, related to the weight $\sqrt{w}$ is uniquely defined by

$$u = \frac{1}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}}.$$

This class of weight has been considered by S.W.Jha and S.D. Lubinsky. Now we recall the well-known polynomial inequality

$$\int_{\Gamma_m} \left|P_m \sqrt{w}\right|^p (x)dx \leq \mathcal{C} e^{-Am} \int_{\mathbb{R}} \left|P_m \sqrt{w}\right|^p (x)dx, \quad 1 \leq p \leq +\infty,$$

where $\Gamma_m = \{x \in \mathbb{R} : |x| > (1+\delta)a_m\}$, $\delta \in (0,1)$, $\mathcal{C}$ and A are positive constants depending on δ and independent of m. As in the Laguerre case, with $\theta \in (0,1)$, we define $\Delta_{\theta,m}$ the characteristic function of the interval $[-\theta a_m, \theta a_m]$ and $M = \left[\frac{\theta}{1+\theta} m\right]$. As before $M \sim m$. For all functions $f \in L^p_{\sqrt{w}}$,

$$\left\|(1 - \Delta_{\theta,m}) f\sqrt{w}\right\|_p \leq \mathcal{C} \left[E_m(f)_{\sqrt{w},p} + e^{-Am} \left\|f\sqrt{w}\right\|_p\right],$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$, $E_M(f)_{\sqrt{w},p} = \inf_{P \in \mathbb{P}_M} \|(f-P)\sqrt{w}\|_p$ and $1 \leq p \leq +\infty$ (see (6.2)).

Fourier Sums

For all $f \in L^p_{\sqrt{w}}$, let

$$S_m(w, f) = \sum_{k=0}^{m-1} c_k p_k(w), \quad c_k = c_k(f) = \int_{\mathbb{R}} f p_k(w) w,$$

be its m-th Fourier Sum with the respect to the system $\{p_m(w)\}_m$.

Theorem 6.8 *For all $f \in L^p_{\sqrt{w}}, 1 < p < +\infty$, we have*

$$\left\|S_m(w, \Delta_{\theta,m} f)\Delta_{\theta,m}\sqrt{w}\right\|_p \le \mathcal{C} \left\|f\Delta_{\theta,m}\sqrt{w}\right\|_p$$

and

$$\left\|\left[f - \Delta_{\theta,m} S_m(w, \Delta_{\theta,m} f)\right]\sqrt{w}\right\|_p \;\le\; \mathcal{C}\left[E_m(f)_{\sqrt{w},p} + e^{-Am}\left\|f\sqrt{w}\right\|_\infty\right]$$

where $\mathcal{C}$ and A are independent of m and f.

In order to complete Theorem 6.8, we can state

Theorem 6.9 *For all $f \in L^p_{\sqrt{w}}$, we have*

$$\left\|S_m(w, \Delta_{\theta,m} f)\sqrt{w}\right\|_p \le \mathcal{C} \left\|f\sqrt{w}\Delta_{\theta,m}\right\|_p, \quad \mathcal{C} \ne \mathcal{C}(m, f),$$

if and only if $1 < p < 4$.

Lagrange Interpolation

For convenience we introduce the following notations. We denote by $x_k (= x_{km})$ the nonnegative zeros of $p_m(w)$ ($x_0 = 0$ if m is odd) and we set $x_{-k} = -x_k$, $0 \le k \le \left[\frac{m}{2}\right]$. With $f \in C^0(\mathbb{R})$, let

$$L_m(w, f, x) = \sum_{k=-\left[\frac{m}{2}\right]}^{\left[\frac{m}{2}\right]} l_k(x) f(x_k), \quad l_k(x) = \frac{p_m(w, x)}{p'_m(w, x_k)(x - x_k)}$$

be the Lagrange polynomial interpolating the function f on the knots c_k.

With $\theta \in (0, 1)$, let $j = j(m)$ be defined as $x_j = \min\{x_k \ge \theta a_m\}$, let Ψ_j be the characteristic function of the interval $[-x_j, x_j]$ and set $f_j = \Psi_j f$. Then

$$L_m(w, f_j, x) = \sum_{k=-j}^{j} l_k(x) f(x_k)$$

we want to study the behaviour of the sequences

$$\{\Psi_j L_m(w, f)\}_m \quad \text{and} \quad \{L_m(w, f_j)\}_m$$

in $L^p_{\sqrt{w}}$ assuming $f \in C^0(\mathbb{R})$. Let $t^* = t^*(t)$ be uniquely defined by $tQ'(t^*) = 1$ and define the following modulus of continuity

$$\Omega^k(f, t)_{\sqrt{w},p} = \sup_{0<h\le t} \left\|(\Delta_h^k f)\sqrt{w}\right\|_{L^p(-Ch^*, Ch^*)}.$$

We can prove

Theorem 6.10 *For all continuous function in* $\mathbb{R}$*, we have*

$$\left\|[f - L_m(w, f_j)\Psi_j]\sqrt{w}\right\|_p \;\le\; \mathcal{C}\left\{\left(\frac{a_m}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\alpha_m}{m}} \frac{\Omega^k(f,t)_{\sqrt{w},p}}{t^{1+1/p}} + e^{-Am}\left\|f\sqrt{w}\right\|_p\right\},$$

with $1 < p < +\infty$*. Moreover, if* $1 < p < 4$ *then*

$$\left\|[f - L_m(w, f_j)]\sqrt{w}\right\|_p \;\le\; \mathcal{C}\left\{\left(\frac{a_m}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\alpha_m}{m}} \frac{\Omega^k(f,t)_{\sqrt{w},p}}{t^{1+1/p}} + e^{-Am}\left\|f\sqrt{w}\right\|_p\right\},$$

Here the constants $\mathcal{C}$ *are independent of* m *and* f*.*

Especially, if $f^{(r-1)}$ absolutely continuous on $\mathbb{R}$ and $\left\|f^{(r)}\sqrt{w}\right\|_p < +\infty$, then

$$\Omega^k_\varphi(f,t)_{\sqrt{w},p} \le \mathcal{C}\left\|f^{(r)}\sqrt{w}\right\|_p t^r, \quad k > r,$$

and the right-hand sides are dominated by $\left(\frac{a_m}{m}\right)^r \left(\|f\sqrt{w}\|_p + \left\|f^{(r)}\sqrt{w}\right\|_p\right)$.

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First we list the papers of Parts 2–6, then the other joint papers.

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PART II
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Simultaneous approximation by Altomare operators

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Abstract. In the present paper we study the asymptotic simultaneous approximation behaviour of a sequence of operators introduced by Altomare and Amiar.

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1 Introduction

The sequence (B_n) of classical Bernstein polynomials, defined for $f \in C[0,1]$ by

$$(B_n f)(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} f\left(\frac{\nu}{n}\right) \qquad (x \in [0,1]),$$

was introduced by S. N. Bernstein [20] in order to have a constructive proof of Weierstrass's theorem for algebraic polynomials. It converges uniformly to f on $[0,1]$.

One of the most celebrated asymptotic formulas for sequences of linear approximation processes was established in 1932 by Voronovskaja [28]: For the sequence (B_n) of Bernstein polynomials there holds

$$\lim_{n\to\infty} n\left[(B_n f)(x) - f(x)\right] = \frac{1}{2} x(1-x) f''(x), \tag{1.1}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

provided that the second derivative $f''(x)$ exists at $x \in [0,1]$. If $f \in C^2[0,1]$, the convergence is uniform. Most likely this is the first asymptotic expansion for a sequence of polynomial operators.

In that same year, S. N. Bernstein [21] showed that for each $x \in (0,1)$ for which the derivative $f^{(2q)}(x)$ exists with $q \in \mathbb{N}$, there holds

$$(B_n f)(x) = f(x) + \sum_{k=0}^{2q} \frac{T_{n,k}(x)}{k! n^k} f^{(k)}(x) + o\left(n^{-q}\right) \qquad (n \to \infty), \quad (1.2)$$

where

$$T_{n,k}(x) = \sum_{\nu=0}^{n} (\nu - nx)^k \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \qquad (k \in \mathbb{N}_0).$$

Whereas (1.2) can be regarded as an asymptotic expansion of order q (for the Bernstein polynomials), formula (1.1) is an asymptotic expansion of order 1 (as $(B_n f)(x) = f(x) + x(1-x) f''(x)/(2n) + o\left(n^{-1}\right)$) and the constructive Weierstrass theorem is one of order 0 (as $(B_n f)(x) = f(x) + o(1)$). The complete asymptotic expansion for the sequence of Bernstein polynomials with explicitly given coefficients can be found in [1].

Asymptotic formulae play an important role in the investigation of saturation classes of sequences of positive linear operators.

Furthermore, asymptotic formulae appear to be useful in connection with the representation of C_0-semigroups of operators in terms of positive linear operators and, hence, in the study of qualitative properties of the solutions of the corresponding evolution equations (see also [7] and [12] for some surveys on the topic). This is a consequence of the pioneer work of F. Altomare ([3], [4], [5], [6])) and of the subsequent results of Altomare and Attalienti [10], Altomare and Carbone [13], Altomare and Mangino [14], Attalienti [18], Attalienti and Campiti [19], Campiti and Metafune [22], Campiti, Metafune and Pallara [23] and others.

For this purpose it is desirable to construct a sequence of positive linear operators which satisfies an asymptotic formula (with respect to a given weighted norm) whose limit operator is a second order elliptic differential operator of the form

$$Lu = \alpha u'' + \beta u' + \gamma u$$

defined on a suitable domain $D(L)$ of some weighted space of continuous functions E on a real interval, endowed with the weighted norm. Here α, β, γ are given functions. In this paper we go not in more detail of semigroup theory.

Altomare and Amiar [8], [9] obtained such a sequence (M_n) by modifying a "known" positive approximation process on E which satisfies some additional conditions involving the function α.

In this paper we study simultaneous approximation by the operators (2.1) when L_n are operators of exponential-type. Finally, we present the complete asymptotic expansion of a sequence of operators defined by Altomare and Milella.

2 Notation

Let e_r denote the monomials defined by $e_r(x) = x^r$ $(r = 0, 1, 2, \ldots)$. For $x \in \mathbb{R}$, put

$$\psi_x = e_1 - x.$$

Let I be an arbitrary interval of $\mathbb{R}$ and set $r_1 := \inf I \in \mathbb{R} \cup \{-\infty\}$ and $r_2 := \sup I \in \mathbb{R} \cup \{+\infty\}$. By $F(I)$ we denote the space of all real-valued functions defined on I and by $C(I)$ and $C_b(I)$ the spaces of all real-valued continuous, respectively, continuous and bounded functions on I. The space $C_b(I)$ will be endowed with the supnorm $\|.\|_\infty$.

If I is not compact put $C_0(I) = \{f \in C(I) : \lim_{x \to r_i} f(x) = 0$ whenever $r_i \notin I \ (i = 1, 2)\}$ and $C_*(I) = \{f \in C(I) : \lim_{x \to r_i} f(x) \in \mathbb{R}$ whenever $r_i \notin I \ (i = 1, 2)\}$. If I is compact, let $C_0(I) := C(I)$ and $C_*(I) := C(I)$. Note that $C_0(I)$ and $C_*(I)$ are closed subspaces of $C_b(I)$.

The symbols $C^2(I)$ and $UC_b^2(I)$ will denote the space of all twice continuously differentiable functions on I and the space of all functions in $C^2(I)$ whose second derivatives are uniformly continuous and bounded.

For a bounded continuous weight w on I (i.e. $w(x) > 0$ for every $x \in I$) we set

$$C_b^w(I) := \{f \in C(I) \mid wf \in C_b(I)\}$$

endowed with the Banach norm

$$\|f\|_w := \|wf\|_\infty = \sup_{x \in I} w(x) |f(x)|.$$

In 2005 Altomare and Amiar [8, Eq. (3.11) and Eq. (3.12), respectively] constructed positive linear operators M_n with a prescribed asymptotic behaviour as n tends to infinity. More precisely, for given functions $\alpha, \beta, \gamma \in C(I)$, one desires a sequence of operators $M_n : C(I) \to C(I)$ such that

$$(M_n f)(x) = f(x) + \frac{1}{n}\left[\alpha(x) f''(x) + \beta(x) f'(x) + \gamma(x) f(x)\right] + o\left(n^{-1}\right)$$

as $n \to \infty$. The starting point is a sequence of operators $L_n : C(I) \to C(I)$ with the property

$$(L_n f)(x) = f(x) + \frac{1}{n}\alpha(x) f''(x) + o\left(n^{-1}\right) \qquad (n \to \infty).$$

Altomare and Amiar [8, Eq. (3.11) and Eq. (3.12), respectively] introduced the new operators M_n given by

$$M_n f = L_n\left(\left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right)\right). \tag{2.1}$$

They proved that $\lim_{n\to\infty} M_n f = f$ for each $f \in C_0^w(I)$, where

$$C_0^w(I) := \{f \in C(I) \mid wf \in C_0(I)\},$$

(see [8, Lemma 3.1]) and studied the properties of the sequence (M_n). The main motivation for the definition of the operators (2.1) is the asymptotic behavior of the sequence (M_n) as n tends to infinity. Under several assumptions, Altomare and Amiar [8, Theorem 4.1] showed that, for every $f \in UC_b^2(I)$

$$\lim_{n\to\infty} w\left[n(M_n f - f) - \left(\alpha f'' + \beta f' + \gamma f\right)\right] = 0$$

uniformly on I, where α is such that

$$\lim_{n\to\infty} w(x)\left[n\left(L_n \psi_x^2\right)(x) - \alpha(x)\right] = 0$$

uniformly on I. Moreover, there holds

$$\lim_{n\to\infty} n(M_n f - f) = \alpha f'' + \beta f' + \gamma f. \tag{2.2}$$

uniformly on compact subsets of I.

3 Asymptotic properties of the operators M_n

The operators M_n are worth a deeper investigation in order to give more insight into their relation to L_n. In [2] the authors studied this problem and the asymptotic behavior of the sequence (M_n) in greater detail. In particular, the number of conditions in [8, Theorem 4.1] was diminished.

Generally, let us assume that the underlying sequence (L_n) satisfies an asymptotic relation

$$L_n f \sim \sum_{k=0}^{\infty} c_k(f)\, n^{-k} \qquad (n \to \infty), \tag{3.1}$$

for sufficiently smooth functions f, with certain coefficients $c_k(f)$ which are functions defined on I. In the following we suppose that all c_k are linear operators of the form

$$c_k(f) = \sum_{j=0}^{J_k} c_{k,j} f^{(j)} \tag{3.2}$$

with certain $J_k \in \mathbb{N}$ and functions $c_{k,j} \in C(I)$ $(j = 0, \ldots, J_k; k \in \mathbb{N})$. Formula (3.1) means that, for each $q \in \mathbb{N}$,

$$L_n f = \sum_{k=0}^{q} c_k(f)\, n^{-k} + o\left(n^{-q}\right) \qquad (n \to \infty). \tag{3.3}$$

For convenience, we also shall write $c_k(f, x) := (c_k f)(x)$. The following proposition reveals the relation of M_n to L_n.

Proposition 3.1 *[2] Let I be a real interval and $q \in \mathbb{N}$. Assume that the sequence of operators $L_n : C(I) \to C(I)$ possesses, for sufficiently smooth functions f, an asymptotic relation (3.3) with linear operators c_k of the form (3.2) where $J_k \in \mathbb{N}$ and $c_{k,j} \in C(I)$ $(j = 0, \ldots, J_k; k \in \mathbb{N})$. Then the sequence of operators M_n defined by (2.1), where β, γ are functions such that L_n is applicable to $\left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right)$, can be written in the form*

$$\begin{aligned}(M_n f)(x) &= \sum_{r=0}^{q} \frac{1}{r! n^r} \left(L_n\left(f^{(r)} \beta^r\right)\right)(x) \\ &\quad + \sum_{r=1}^{q} \frac{1}{(r-1)! n^r} \left(L_n\left(f^{(r-1)} \beta^{r-1} \gamma\right)\right)(x) + o\left(n^{-q}\right)\end{aligned} \tag{3.4}$$

as $n \to \infty$.

Note that Eq. (3.4) holds pointwise. Here and in the following we tacitly assume that the operators L_n are applicable to all functions $f^{(r)}\beta^r$ and $f^{(r)}\beta^r\gamma$ occurring in Proposition 3.1.

Formula (3.4) is not unexpected. Heuristically, by Taylor expansion,

$$\sum_{r=0}^{q} \frac{f^{(r)}(t)}{r!} \left(\frac{\beta(t)}{n}\right)^r = f\left(t + \frac{\beta(t)}{n}\right) + R_1$$

and

$$\sum_{r=1}^{q} \frac{f^{(r-1)}(t)}{(r-1)!} \left(\frac{\beta(t)}{n}\right)^{r-1} \frac{\gamma(t)}{n} = \frac{\gamma(t)}{n} f\left(t + \frac{\beta(t)}{n}\right) + R_2$$

with certain remainder terms R_1 and R_2. Thus, Formula (3.4) gives approximatively back the definition (Eq. (2.1))

$$(M_n f)(x) \approx L_n\left(\left(1+\frac{\gamma}{n}\right) f\left(e_1+\frac{\beta}{n}\right)\right)(x).$$

Proposition 3.2 *[2] Under the conditions of Proposition 3.1 the sequence* (M_n) *satisfies the asymptotic relation*

$$M_n f = \sum_{k=0}^{q} \hat{c}_k(f)\, n^{-k} \qquad (n\to\infty)$$

with coefficients

$$\hat{c}_k(f,x) = c_k(f,x) + \sum_{r=1}^{k} \frac{1}{r!} c_{k-r}\left(f^{(r)}\beta^r + r f^{(r-1)}\beta^{r-1}\gamma, x\right). \tag{3.5}$$

Remark 3.3 *In the special case* $q=1$, *for* $f\in C(I)$, *there holds the Voronovskaja-type formula*

$$(M_n f)(x) = c_0(f,x) + \frac{1}{n}\left[c_1(f,x) + c_0\left(f'\beta + f\gamma, x\right)\right] + o\left(n^{-1}\right) \quad (n\to\infty).$$

Remark 3.4 *In particular, if* $c_0(f) = f$ *and* $c_1(f) = \alpha f^{(2)}$ *we obtain*

$$(M_n f)(x) = f(x) + \frac{1}{n}\left[\alpha(x) f''(x) + \beta(x) f'(x) + \gamma(x) f(x)\right] + o\left(n^{-1}\right)$$

as $n\to\infty$, *which is the (pointwise) Formula* (2.2) *by Altomare and Amiar.*

4 Simultaneous approximation

We proceed with simultaneous approximation by the operators (2.1). We study the local rate of convergence of the sequence $(M_n f)'(x)$, provided that the derivative exists. For this purpose we assume that $L_n : C(I) \to C(I)$ are (positive) operators of exponential-type, i.e.,

$$(L_n f)(x) = \int_{-\infty}^{+\infty} W_n(x,t) f(t)\,\mathrm{d}t,$$

where the kernel $W_n(x,t) \geq 0$ satisfies the homogeneous partial differential equation

$$\frac{\partial}{\partial x} W_n(x,t) = \frac{n}{\varphi^2(x)} \psi_x(t) W_n(x,t)$$

and the normalization condition

$$\int_{-\infty}^{+\infty} W_n(x,t)\,\mathrm{d}t = 1,$$

which guarantees the preservation of constant functions. It is well-known that operators of exponential-type preserve linear functions and are approximation processes in appropriate function spaces. In our considerations we need only the fact that operators of exponential-type satisfy on I the relation

$$\varphi^2(x)\frac{\mathrm{d}}{\mathrm{d}x}(L_n f)(x) = n\left(L_n(\psi_x f)\right)(x). \tag{4.1}$$

Such operators were studied by C. P. May ([25] and [26]; see also [24]) who pointed out that condition (4.1) implies that $\left(L_n\psi_x^2\right)(x) = \varphi^2(x)/n$ which controls the rate of convergence of the sequence (L_n). Many well-known operators are of exponential-type, for example the Bernstein polynomials with $\varphi^2(x) = x(1-x)$ on $I=[0,1]$, the Szász operators with $\varphi^2(x) = x$ on $I=[0,\infty)$, the Baskakov operators with $\varphi^2(x) = x(1+x)$ on $I=[0,\infty)$, and the Weierstraß operator with $\varphi^2(x) = 1$ on $I=(-\infty,\infty)$.

We prove the following result on the first derivative $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)(M_n f)(x)$.

Theorem 4.1 *Fix $x \in I$ such that $\varphi^2(x) \neq 0$. For $n \in \mathbb{N}$, let $L_n : C(I) \to C(I)$ be operators of exponential-type satisfying Relation* (4.1) *such that the first derivative $(L_n f)'(x)$ exists. Assume that $((L_n f)(x))_{n\in\mathbb{N}}$ possesses an asymptotic expansion* (3.1) *with $c_0(f) = f$. Then there holds:*
1. The sequence $\left((L_n f)'(x)\right)_{n\in\mathbb{N}}$ has an asymptotic expansion

$$(L_n f)'(x) \sim \sum_{k=0}^{\infty} c_k^{[1]}(f,x)\,n^{-k} \qquad (n\to\infty) \tag{4.2}$$

with coefficients $c_k^{[1]}(f,x)$ satisfying

$$c_k^{[1]}(f,x) = \varphi^{-2}(x)\,c_{k+1}(\psi_x f, x). \tag{4.3}$$

2. The first derivative $(M_n f)'(x)$ of the operators (2.1) *exists and we have the asymptotic expansion*

$$(M_n f)'(x) \sim \sum_{k=0}^{\infty} \hat{c}_k^{[1]}(f,x)\,n^{-k} \qquad (n\to\infty) \tag{4.4}$$

with coefficients

$$\hat{c}_k^{[1]}(f,x) = c_k^{[1]}(f,x) + \sum_{r=1}^{k} c_{k-r}^{[1]}\left(\frac{1}{r!}f^{(r)}\beta^r + \frac{1}{(r-1)!}f^{(r-1)}\beta^{r-1}\gamma, x\right). \tag{4.5}$$

Remark 4.2 *The coefficients* $\hat{c}_k^{[1]}(f,x)$ *can be obtained from* $\hat{c}_k(f,x)$ *in* (3.5) *by replacing each* c_k *with* $c_k^{[1]}$.

Remark 4.3 *The assumption* $c_0(f) = f$ *in the latter theorem is not really restrictive. Recall that operators of exponential-type are usually approximation operators. This is generally the case for compact intervals* I*, and for unbounded intervals under certain growth conditions on the function* f *(see [24, Theorem 4.2]).*

5 The operator by Altomare and Milella

Recall that the positive linear operators M_n constructed by Altomare and Amiar in their paper [8, Eq. (3.11) and Eq. (3.12), respectively] possess the prescribed asymptotic behaviour

$$(M_n f)(x) = f(x) + \frac{1}{n}\left[\alpha(x) f''(x) + \beta(x) f'(x) + \gamma(x) f(x)\right] + o\left(n^{-1}\right)$$

as n tends to infinity. The functions $\beta, \gamma \in C(I)$ can be chosen arbitrarily, while the function α depends on the sequence of operators $L_n : C(I) \to C(I)$ which satisfy

$$(L_n f)(x) = f(x) + \frac{1}{n}\alpha(x) f''(x) + o\left(n^{-1}\right) \qquad (n \to \infty).$$

It is desirable to have operators with asymptotic behaviour as above with all three functions $\alpha, \beta, \gamma \in C(I)$ chosen arbitrarily.

To this end Altomare and Milella introduced in their recent paper [15, Eq. (2.5)], for $f \in \mathbb{E}(\mathbb{R})$ and $x \in \mathbb{R}$, the new operators G_n defined by

$$(G_n f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(\sqrt{\frac{2\alpha(x)}{n}}y + x + \frac{\beta(x)}{n}\right) e^{-y^2/2}\, \mathrm{d}y, \tag{5.1}$$

where $\alpha, \beta \in C(\mathbb{R})$ are two fixed functions such that $\alpha(x) \geq 0$ for every $x \in \mathbb{R}$. Here $\mathbb{E}(\mathbb{R})$ denotes the space of all functions $f \in C(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} |f(ay+b)|\, e^{-y^2/2}\, \mathrm{d}y < +\infty$$

for every $a \geq 0$ and $b \in \mathbb{R}$.

Note that, if $\alpha(x) \equiv 1$ and $\beta(x) \equiv 0$, the operator G_n turns into the nth Gauss-Weierstrass convolution operator (see, e.g., [12, Section 5.2.9]).

By a change of variable we can rewrite $(G_n f)(x)$ in the alternative form

$$(G_n f)(x) = \sqrt{\frac{n}{4\pi\alpha(x)}} \int_{-\infty}^{\infty} f(t) \exp\left(-\frac{n}{4\alpha(x)}\left(t - x - \frac{\beta(x)}{n}\right)^2\right) \mathrm{d}t, \tag{5.2}$$

provided that $\alpha(x) > 0$.

Altomare and Milella [15, Theorem 3.1] proved that $\lim_{n\to\infty} G_n f = f$ in $(C_0^w(I), \|.\|_w)$, for each $f \in C_0^w(\mathbb{R})$, and studied the properties of the sequence (G_n). They [15, Theorem 3.5] showed that under several assumptions, for every $f \in UC_b^2(I)$ with uniformly continuous and bounded second derivative, there holds

$$\lim_{n\to\infty} w\left[n(G_n f - f) - \left(\alpha f'' + \beta f'\right)\right] = 0$$

uniformly on $\mathbb{R}$. Moreover, there holds

$$\lim_{n\to\infty} n(G_n f - f) = \alpha f'' + \beta f'. \tag{5.3}$$

uniformly on compact subsets of $\mathbb{R}$. The missing term γf comes into play by considering

$$G_n^*(f) = G_n\left(\left(1 + \frac{\gamma}{n}\right) f\right) \qquad (f \in \mathbb{E}(\mathbb{R})).$$

In this section we study the asymptotic behavior of the sequence (G_n) in greater detail.

Theorem 5.1 *Let $x \in \mathbb{R}$ and $f \in \mathbb{E}(\mathbb{R})$. Assume that f possesses a derivative of order $2q$ at x. Then there holds the asymptotic expansion*

$$(G_n f)(x) = \sum_{k=0}^{q} \frac{1}{n^k} c_k(f, x) + o\left(n^{-q}\right)$$

as $n \to \infty$ with coefficients

$$c_k(f, x) = \sum_{s=0}^{k} \binom{k}{s} f^{(k+s)}(x)\, \alpha^s(x) \beta^{k-s}(x).$$

Remark 5.2 *In the special case $q = 1$, we have*

$$(G_n f)(x) = f(x) + \frac{1}{n}\left[\alpha(x) f^{(2)}(x) + \beta(x) f^{(1)}(x)\right] + o\left(n^{-1}\right) \quad (n \to \infty),$$

which is the Formula (5.3) *by Altomare and Milella.*

Remark 5.3 *For sufficiently smooth functions f there holds the complete asymptotic expansion*

$$(G_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{1}{n^k} c_k(f, x) \qquad (n \to \infty).$$

6 Auxiliary results and proofs

Proof of Theorem 4.1: The first part of the theorem is obvious because relation (4.1) implies that

$$\begin{aligned} \varphi^2(x) \frac{\mathrm{d}}{\mathrm{d}x}(L_n f)(x) &= n(L_n(\psi_x f))(x) \sim \sum_{k=0}^{\infty} \frac{1}{n^{k-1}} c_k(\psi_x f, x) \\ &= \sum_{k=0}^{\infty} \frac{1}{n^k} c_{k+1}(\psi_x f, x), \end{aligned}$$

where we used that $c_0(\psi_x f, x) = \psi_x(x) f(x) = 0$.
In this proof we indicate the dependence of the operator M_n on the functions β and γ by writing

$$M_n(f; \beta, \gamma) = L_n\left(\left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right)\right).$$

Application of relation (4.1) to the definition (2.1) yields

$$\varphi^2(x) \frac{\mathrm{d}}{\mathrm{d}x} M_n(f; \beta, \gamma)(x) = n L_n\left(\psi_x \cdot \left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right)\right)(x).$$

Using the identity

$$\begin{aligned} \left(1 + \frac{\gamma}{n}\right)\psi_x &= \left(1 + \frac{\gamma}{n}\right)\left[\psi_x \circ \left(e_1 + \frac{\beta}{n}\right) - \frac{\beta}{n}\right] \\ &= \left(1 + \frac{\gamma}{n}\right)\left[\psi_x \circ \left(e_1 + \frac{\beta}{n}\right) - e_0\right] \\ &\quad + \left(e_0 + \frac{e_0 - \beta + \gamma}{n}\right) - \frac{1}{n}\left(e_0 + \frac{\beta\gamma}{n}\right) \end{aligned}$$

which can easily be verified we conclude that

$$\begin{aligned} & \varphi^2(x)\frac{\mathrm{d}}{\mathrm{d}x}M_n(f;\beta,\gamma)(x) \\ = \; & nM_n(\psi_x f - f;\beta,\gamma)(x) + nM_n(f;\beta,e_0-\beta+\gamma)(x) \\ & -M_n(f;\beta,\beta\gamma)(x). \end{aligned}$$

By Proposition 3.2 applied with $q+1$ in place of q it follows that

$$\begin{aligned} & \varphi^2(x)\frac{\mathrm{d}}{\mathrm{d}x}M_n(f;\beta,\gamma)(x) \\ = \; & \sum_{k=0}^{q}\frac{1}{n^k}\left[\hat{c}_{k+1}(\psi_x f - f;\beta,\gamma;x) + \hat{c}_{k+1}(f;\beta,e_0-\beta+\gamma;x)\right] \\ & -\sum_{k=0}^{q}\frac{1}{n^k}\hat{c}_k(f;\beta,\beta\gamma;x) + o\left(n^{-q}\right) \qquad (n\to\infty) \end{aligned}$$

because $\hat{c}_0(\psi_x f - f;\beta,\gamma;x) + \hat{c}_0(f;\beta,e_0-\beta+\gamma;x) = c_0(\psi_x f - f,x) + c_0(f,x) = 0$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}x}M_n(f;\beta,\gamma)(x) = \sum_{k=0}^{q}\frac{1}{n^k}\hat{c}_k^{[1]}(f,x) + o\left(n^{-q}\right) \qquad (n\to\infty)$$

with

$$\begin{aligned} & \varphi^2(x)\hat{c}_k^{[1]}(f,x) \\ = \; & \hat{c}_{k+1}(\psi_x f - f;\beta,\gamma;x) + \hat{c}_{k+1}(f;\beta,e_0-\beta+\gamma;x) - \hat{c}_k(f;\beta,\beta\gamma;x). \end{aligned}$$

Application of Formula (3.5) yields

$$\begin{aligned} & \varphi^2(x)\hat{c}_k^{[1]}(f,x) \\ = \; & c_{k+1}(\psi_x f - f,x) + \sum_{r=1}^{k+1}c_{k+1-r}\Big(\frac{1}{r!}(\psi_x f - f)^{(r)}\beta^r \\ & +\frac{1}{(r-1)!}(\psi_x f - f)^{(r-1)}\beta^{r-1}\gamma,x\Big) \\ & +c_{k+1}(f,x) + \sum_{r=1}^{k+1}c_{k+1-r}\Big(\frac{1}{r!}f^{(r)}\beta^r \\ & +\frac{1}{(r-1)!}f^{(r-1)}\left(\beta^{r-1}-\beta^r+\beta^{r-1}\gamma\right),x\Big) \\ & -c_k(f,x) - \sum_{r=1}^{k}c_{k-r}\left(\frac{1}{r!}f^{(r)}\beta^r + \frac{1}{(r-1)!}f^{(r-1)}\beta^r\gamma,x\right). \end{aligned}$$

Noting that $(\psi_x f)^{(r)} = \psi_x f^{(r)} + r f^{(r-1)}$ and simplifying we obtain

$$\begin{aligned} & \varphi^2(x)\,\hat{c}_k^{[1]}(f,x) \\ = \; & c_{k+1}(\psi_x f, x) + \sum_{r=0}^{k} c_{k-r}\left(\frac{1}{(r+1)!}\psi_x f^{(r)}\beta^{r+1} + \frac{1}{r!}\psi_x f^{(r)}\beta^r\gamma, x\right). \end{aligned}$$

By Eq. (4.3) and noting that, by (3.2), for each function h, there holds $c_0(\psi_x h, x) = 0$ we conclude that

$$\begin{aligned} & \varphi^2(x)\,\hat{c}_k^{[1]}(f,x) \\ = \; & \varphi^2(x)\left(c_k^{[1]}(f,x) + \sum_{r=0}^{k-1} c_{k-r-1}^{[1]}\left(\frac{1}{(r+1)!}f^{(r)}\beta^{r+1} + \frac{1}{r!}f^{(r)}\beta^r\gamma, x\right)\right) \\ = \; & \varphi^2(x)\left(c_k^{[1]}(f,x) + \sum_{r=1}^{k} c_{k-r}^{[1]}\left(\frac{1}{r!}f^{(r-1)}\beta^r + \frac{1}{(r-1)!}f^{(r-1)}\beta^{r-1}\gamma, x\right)\right) \end{aligned}$$

which proves Eq. (4.5) for all $x \in I$ with $\varphi^2(x) \neq 0$. □

In order to derive the complete asymptotic expansion for the operators (5.1) by Altomare and Milella we use a general approximation theorem for positive linear operators due to Sikkema [27, Theorem 3].

Lemma 6.1 *For $q \in N$ and fixed $x \in I$, let $A_n\colon L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property*

$$(A_n\psi_x^s)(x) = O\left(n^{-\lfloor (s+1)/2\rfloor}\right) \quad (n\to\infty) \quad \text{for } s = 0,1,\ldots,2q+2.$$

Then, we have for each $f \in L_\infty(I)$ which is $2q$ times differentiable at x the asymptotic relation

$$(A_n f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!}(A_n\psi_x^s)(x) + o\left(n^{-q}\right) \qquad (n\to\infty). \tag{6.1}$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ in (6.1) can be replaced by $O\left(n^{-(q+1)}\right)$.

We start by deriving an exact expression for the central moments $G_n\psi_x^s$ of the operators G_n.

Lemma 6.2 *The central moments $G_n\psi_x^s$ $(s = 0, 1, 2, \ldots)$ of the operators G_n are given by*

$$\left(G_n\psi_x^s\right)(x) = s! \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} \frac{1}{k!n^k} \binom{k}{s-k} \alpha^{s-k}(x)\, \beta^{2k-s}(x).$$

Proof of Lemma 6.2: By Eq. (5.1) and the change of variable $y \to \sqrt{2}y$, the central moments $G_n\psi_x^s$ $(s = 0, 1, 2, \ldots)$ can be written in the form

$$\left(G_n\psi_x^s\right)(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{4\alpha(x)}{n}}\, y + \frac{\beta(x)}{n} \right)^s e^{-y^2}\, \mathrm{d}y.$$

Using the well-known formula

$$\int_{-\infty}^{\infty} y^m e^{-y^2}\, \mathrm{d}y = \begin{cases} 0 & \text{if} \quad m = 1, 3, 5, \ldots, \\ \Gamma\left(\frac{m+1}{2}\right) & \text{if} \quad m = 2, 4, 6, \ldots, \end{cases}$$

we obtain

$$\left(G_n\psi_x^s\right)(x) = \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\lfloor s/2 \rfloor} \binom{s}{2r} \left(\sqrt{\frac{4\alpha(x)}{n}} \right)^{2r} \left(\frac{\beta(x)}{n} \right)^{s-2r} \Gamma\left(\frac{2r+1}{2} \right).$$

Substitution of the identity

$$\frac{\Gamma\left(\frac{2r+1}{2}\right)}{\sqrt{\pi}} = \frac{\Gamma\left(r + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{(2r)!}{4^r r!}$$

yields

$$\left(G_n\psi_x^s\right)(x) = \sum_{r=0}^{\lfloor s/2 \rfloor} \left(\frac{1}{n}\right)^{s-r} \frac{s!}{(s-2r)!r!} \alpha^r(x)\, \beta^{s-2r}(x).$$

and the desired formula follows by changing the order of summation, i.e., by setting $k = s - r$. □

Remark 6.3 *Note that Lemma 6.2 implies that, for all $s \in \mathbb{N}$,*

$$\left(G_n\psi_x^s\right)(x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty).$$

If the functions α and β are bounded, the estimate holds uniformly for $x \in \mathbb{R}$.

Proof of Theorem 5.1: By Remark 6.3 the assumption of Sikkema's result Lemma 6.1 is satisfied for the operators G_n. Therefore, application of Lemma 6.2 yields

$$\begin{aligned}
&(G_n f)(x) \\
&= \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (G_n \psi_x^s)(x) + o\left(n^{-q}\right) \\
&= \sum_{s=0}^{2q} f^{(s)}(x) \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} \frac{1}{k! n^k} \binom{k}{s-k} \alpha^{s-k}(x) \beta^{2k-s}(x) + o\left(n^{-q}\right) \\
&= \sum_{k=0}^{q} \frac{1}{k! n^k} \sum_{s=k}^{2k} \binom{k}{s-k} f^{(s)}(x) \alpha^{s-k}(x) \beta^{2k-s}(x) + o\left(n^{-q}\right) \\
&= \sum_{k=0}^{q} \frac{1}{n^k} c_k(f,x) + o\left(n^{-q}\right)
\end{aligned}$$

as $n \to \infty$ with coefficients

$$c_k(f,x) = \sum_{s=0}^{k} \binom{k}{s} f^{(k+s)}(x) \alpha^s(x) \beta^{k-s}(x).$$

This completes the proof of Theorem 5.1. □

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Asymptotic formulae for Baskakov-Mastroianni operators based on q-integers

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Abstract. We establish an asymptotic formula for a general sequence of positive linear operators of discrete type. This class represents a generalization in q-Calculus of the operators introduced by G. Mastroianni, the construction taking its origin in a paper of Baskakov. We also mark out Voronovskaja-type formulas for two particular cases which are q-extensions of the Szász-Mirakjan operator and the ordinary Baskakov operator.

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1 Introduction

The roots of the paper are in connection with a general class of operators introduced by Baskakov [6] and developed by Mastroianni [8]. For comparing, [3, p. 344, p. 351] can also be consulted. A q-analogue of these operators has been introduced in [9]. The author investigated their weighted statistical approximation properties. For the sake of completeness, we recall: a *q-analogue* or a *q-extension* of a mathematical object X is a family of objects $X(q)$ (usually, $0 < q < 1$) such that $\lim_{q\to 1^-} X(q) = X$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Deeping the researches on the mentioned q-operators, in [1] all their moments were explicitly expressed with the help of a new q-analogue of the Stirling numbers of the second kind. At the same time, the rate of convergence was established in two cases: for bounded functions and for functions having a polynomial growth.

The aim of this note is to present an asymptotic formula of Voronovskaja-type for q-Baskakov-Mastroianni operators. As applications, we deduce asymptotic formulae for two special cases which are q-extensions of the Szász-Mirakjan operators and the ordinary Baskakov operators, respectively.

The paper is organized as follows. Section 2 includes elements of q-Calculus, the form of the announced class of operators and some results obtained previously. Section 3 comprises the statement of the main result and two particular asymptotic formulae. The last Section is devoted to the proofs of our results.

2 The $T_{n,q}$ operators

First of all we collect some useful formulas in q-Calculus, see, e.g., [4], [7].

Let $q > 0$. For any $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are respectively defined by

$$[n]_q = \sum_{j=0}^{n-1} q^j, \qquad [n]_q! = \prod_{j=1}^{n} [j]_q, \quad n \in \mathbb{N},$$

and $[0]_q = 0$, $[0]_q! = 1$. The q-binomial coefficients are denoted by $\left[\begin{array}{c} n \\ k \end{array} \right]_q$ and are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k = 0, 1, \ldots, n.$$

For $q = 1$ one has $[n]_1 = n$, $[n]_1! = n!$ and $\left[\begin{array}{c} n \\ k \end{array} \right]_1$ turns into the ordinary binomial coefficient $\left(\begin{array}{c} n \\ k \end{array} \right)$.

We also recall the significance of the notation $(x-a)_q^r$.

$$(x-a)_q^r = \prod_{s=0}^{r-1} (x - q^s a) = \sum_{k=0}^{r} \left[\begin{array}{c} r \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} x^{r-k} (-a)^k.$$

In the sequel we always will assume that $q \in (0,1)$.

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad D_q f(0) := \lim_{x \to 0} D_q f(x),$$

and the high q-derivatives $D_q^0 f := f$, $D_q^n f := D_q\left(D_q^{n-1} f\right)$, $n = 1, 2, \ldots$.

A function f is q-differentiable on a real interval I if for any $q \in (0,1)$ the q-derivative of f exists and is finite in every $x \in I$. Also, we recall the product rule

$$D_q\left(f(x)g(x)\right) = g(x)D_q f(x) + f(qx)D_q g(x).$$

For each $(m,r) \in \mathbb{N}_0 \times \mathbb{N}_0$, a q-analogue of Stirling numbers of the second kind can be considered the following

$$\sigma_q(m,r) = \frac{1}{[r]_q!} \sum_{j=0}^{r} (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}}. \tag{2.1}$$

Lemma 2.1 ([1, Lemma 2]) *The numbers $\sigma_q(m,r)$, $(m,r) \in \mathbb{N}_0 \times \mathbb{N}_0$, given by (2.1) enjoy the following properties.*

$$\sigma_q(m,0) = 0, \quad m \in \mathbb{N}, \quad \text{and} \quad \sigma_q(0,0) = 1, \tag{2.2}$$

$$q^r \sigma_q(m+1,r) = [r]_q \sigma_q(m,r) + \sigma_q(m,r-1), m \in \mathbb{N}_0, \quad r \in \mathbb{N}, \tag{2.3}$$

$$\sigma_q(m,r) = 0, \quad r > m. \tag{2.4}$$

It is known that a q-analogue is not unique. In this direction it is worth to be mentioned that others q-Stirling numbers denoted by $S_q(m,r)$ have been used by Ali Aral [5] in studying a q-generalization of Szász-Mirakjan operators.

Now we are in position to present $T_{n,q}$, $n \in \mathbb{N}$, $q \in (0,1)$, operators [1, Section 3]. Let $(\phi_n)_{n \geq 1}$ be a sequence of real valued functions defined on $\mathbb{R}_+$, continuously infinitely q-differentiable on $\mathbb{R}_+$ and satisfying the following conditions.

$(P1)$ $$\phi_n(0) = 1, \quad n \in \mathbb{N}, \tag{2.5}$$

$(P2)$ $$(-1)^k D_q^k \phi_n(x) \geq 0, \quad n \in \mathbb{N}, k \in \mathbb{N}_0, x \geq 0, \tag{2.6}$$

$(P3)$ For every $(n,k) \in \mathbb{N} \times \mathbb{N}_0$ there exists a positive integer i_k, $0 \leq i_k \leq k$, and a real function $\beta_{n,k,i_k,q} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$D_q^{k+1} \phi_n(x) = (-1)^{i_k+1} D_q^{k-i_k} \phi_n(q^{i_k+1}x) \beta_{n,k,i_k,q}(x), \tag{2.7}$$

where

$$\lim_n \frac{\beta_{n,k,i_k,q}(0)}{[n]_q^{i_k+1}\, q^{k-i_k}} = 1. \tag{2.8}$$

We define the operators

$$(T_{n,q}f)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} D_q^k \phi_n(x) f\left(\frac{[k]_q}{[n]_q\, q^{k-1}}\right), \quad x \geq 0, \tag{2.9}$$

where $f \in \mathcal{F}(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{R}, \text{ the series in (2.9) is absolutely convergent for all } n \in \mathbb{N}\}$.

Clearly, for each $n \in \mathbb{N}$, $T_{n,q}$ is linear and positive operator. With the help of q-Stirling numbers defined by (2.1), we are able to indicate all moments of our operators.

Lemma 2.2 ([1, Lemma 4]) *Let $T_{n,q}$, $n \in \mathbb{N}$, be defined by (2.9). One has*

$$(T_{n,q}e_m)(x) = \sum_{r=0}^{m} \frac{(-x)^r}{[n]_q^m} q^m D_q^r \phi_n(0) \sigma_q(m,r), \quad x \geq 0, \tag{2.10}$$

where e_m stands for the monomial of m degree, $m \in \mathbb{N}_0$.

We easily deduce

$$(T_{n,q}e_0)(x) = 1, \tag{2.11}$$

$$(T_{n,q}e_1)(x) = -x\frac{D_q\phi_n(0)}{[n]_q}, \tag{2.12}$$

$$(T_{n,q}e_2)(x) = x^2\frac{D_q^2\phi_n(0)}{q[n]_q^2} - x\frac{D_q\phi_n(0)}{[n]_q^2}. \tag{2.13}$$

These three particular moments have been obtained by a straightforward calculation in [9, Lemma 1], without involving q-Stirling numbers $\sigma_q(m,r)$.

Since $\lim_n [n]_q = (1-q)^{-1}$, relation (2.13) implies

$$\lim_n (T_{n,q}e_2)(x) = q^{-i_1}x^2 + (1-q)x \neq e_2(x),$$

where the index i_1 can be 0 or 1, see relation (2.7). Clearly, $(T_{n,q})_n$ does not form an approximation process.

Remark 2.3 *To become $(T_{n,q})_n$ an approximation process, for each $n \in \mathbb{N}$, the constant q will be replaced by a number $q_n \in (0,1)$ such that $\lim_n q_n = 1$. Under this assumption, for any compact $\mathcal{K} \subset \mathbb{R}_+$ and for each $f \in \mathcal{F}(\mathbb{R}_+)$ which is continuous and bounded, one has*

$$\lim_n (T_{n,q_n} f)(x) = f(x), \text{ uniformly in } x \in \mathcal{K}. \tag{2.14}$$

The fastest motivation of the above statement can be made by using a more general result obtained by Altomare [2] in the setting of locally compact metric space. It reads as follows.

Theorem 2.4 ([2, Theorem 3.5]) *Let J be a real interval and consider a lattice subspace E of the space $F(J)$ of all real-valued functions on J, containing the functions $e_0(x) = 1$, $e_1(x) = x$ and $e_2(x) = x^2$ $(x \in J)$. Consider a sequence $(L_n)\, n \geq 1$ of positive linear operators from E into $F(J)$ and assume that for every $k = 0, 1, 2$*

$$\lim_{n\to\infty} L_n(e_k) = e_k \text{ uniformly on compact subsets of } J.$$

Then

$$\lim_{n\to\infty} L_n(f) = f \text{ uniformly on compact subsets of } J$$

for every $f \in E \cap C_b(J)$ ($C_b(J)$ denoting the space of all continuous and bounded functions on J).

Choosing $J = \mathbb{R}_+$, $E = \mathcal{F}(\mathbb{R}_+)$, and taking into account relations (2.11), (2.12), (2.13), Theorem 2.4 leads us to the identity (2.14).

3 Results

Examining (2.6) and (2.7), for the sake of brevity we denote

$$d_{n,k} = D_q^k \phi_n(0), \quad n \in \mathbb{N}, \quad k \in \mathbb{N}_0. \tag{3.1}$$

For each $x \geq 0$, let φ_x be the function defined by $\varphi_x(t) = t - x$, $t \geq 0$. $T_{n,q}\varphi_x^s$ represents the s-th order central moment of the operator $T_{n,q}$, where $s \in \mathbb{N}_0$.

On the basis of relations (2.5), (2.7) and (2.8) one has $d_{n,k} \neq 0$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Also $\sigma_q(s,0) = 0$, $s \in \mathbb{N}$, see (2.2). By inspecting relation (2.10), we conclude

Remark 3.1 *For a given $s \in \mathbb{N}$, both $T_{n,q}e_s$ and $T_{n,q}\varphi_x^s$ are polynomials in x of s degree having the constant term null.*

Consequently, we can write

$$\left(T_{n,q}\varphi_x^s\right)(x) = \sum_{k=1}^{s} m_{n,s,k}(q)x^k, \quad x \geq 0. \tag{3.2}$$

Lemma 3.2 *Let $T_{n,q}$, $n \in \mathbb{N}$, be defined by (2.9). For $s = 4$, the coefficients of the polynomial introduced by (3.2) have the following values.*

$$m_{n,4,1}(q) = -\frac{d_{n,1}}{[n]_q^4}, \qquad m_{n,4,2}(q) = \frac{1+3q+3q^2}{q^3}\frac{d_{n,2}}{[n]_q^4} + 4\frac{d_{n,1}}{[n]_q^3}, \tag{3.3}$$

$$m_{n,4,3}(q) = -\frac{1+2q+3q^2}{q^5}\frac{d_{n,3}}{[n]_q^4} - \frac{4(1+2q)}{q^2}\frac{d_{n,2}}{[n]_q^3} - 6\frac{d_{n,1}}{[n]_q^2}, \tag{3.4}$$

$$m_{n,4,4}(q) = \frac{d_{n,4}}{q^6[n]_q^4} + 4\frac{d_{n,3}}{q^3[n]_q^3} + 6\frac{d_{n,2}}{q[n]_q^2} + 4\frac{d_{n,1}}{[n]_q} + 1. \tag{3.5}$$

Lemma 3.3 *Let $(q_n)_{n\geq 1}$, $0 < q_n < 1$, be a sequence such that $\lim\limits_n q_n = 1$. Supposing that the properties $(P1)$, $(P2)$, $(P3)$ take place for $q = q_n$, one has*

$$\lim_n \frac{d_{n,k}}{[n]_{q_n}^k} = (-1)^k, \quad k \in \mathbb{N}_0, \tag{3.6}$$

where $d_{n,k}$ is defined as in (3.1).

Lemma 3.4 *Let $(q_n)_{n\geq 1}$, $0 < q_n < 1$, and $(\lambda_n)_{n\geq 1}$, $\lambda_n > 0$, be sequences such that $\lim\limits_n q_n = 1$, $\lim\limits_n \lambda_n = \infty$ and $\lim\limits_n \frac{\lambda_n}{[n]_{q_n}}$ is finite. Let T_{n,q_n}, $n \in \mathbb{N}$, be defined as in (2.9). If the sequences*

$$\left(\lambda_n^2 m_{n,4,3}(q_n)\right)_n \quad \text{and} \quad \left(\lambda_n^2 m_{n,4,4}(q_n)\right)_n \quad \text{are bounded}, \tag{3.7}$$

then $\left(\lambda_n^2\left(T_{n,q_n}\varphi_x^4\right)(x)\right)_n$ is bounded with respect to n. In the above, $m_{n,4,3}$ and $m_{n,4,4}$ are given by (3.4) and (3.5), respectively.

As usual, $C^2(\mathbb{R}_+)$ denotes the space of all real-valued continuous functions on $\mathbb{R}_+$ which are twice continuously differentiable in $\mathbb{R}_+$.

Returning at the sequence $(\lambda_n)_{n\geq 1}$ of strictly positive real numbers with the property $\lim\limits_n \lambda_n = \infty$, we suppose that the real numbers τ_1, τ_2, τ_3 exist

verifying the following relations

$$\begin{cases} \lim\limits_n \lambda_n \left(\frac{d_{n,1}}{[n]_{q_n}} + 1 \right) = \tau_1, \\ \lim\limits_n \frac{\lambda_n}{[n]_{q_n}} = \tau_2, \\ \lim\limits_n \lambda_n \left(1 + 2\frac{d_{n,1}}{[n]_{q_n}} + \frac{d_{n,2}}{q_n [n]_{q_n}^2} \right) = \tau_3, \end{cases} \tag{3.8}$$

where $d_{n,1}, d_{n,2}$ are defined as in (3.1) with $q = q_n$. The sequence $(\lambda_n)_n$ will play a crucial role in establishing the Voronovskaja-type theorem.

Theorem 3.5 *Let $(q_n)_{n\geq 1}$, $0 < q_n < 1$, and $(\lambda_n)_{n\geq 1}$, $\lambda_n > 0$, be sequences such that $\lim\limits_n q_n = 1$, $\lim\limits_n \lambda_n = \infty$ and the conditions (3.8) are fulfilled. Let T_{n,q_n}, $n \in \mathbb{N}$, be defined as in (2.9). If (3.7) takes place, then for any function $f \in \mathcal{F}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$ with f'' bounded, one has*

$$\lim_{n\to\infty} \lambda_n \left(\left(T_{n,q_n} f \right)(x) - f(x) \right) = -\tau_1 f'(x) + \frac{1}{2} \left(\tau_2 x + \tau_3 x^2 \right) f''(x), \tag{3.9}$$

for any $x > 0$.

The theorem shows that $\left(T_{n,q_n} f \right)(x) - f(x)$ is of order not better then $1/\lambda_n$, if $f'(x)$ and $f''(x)$ are not simultaneous null.

Two special cases of T_{n,q_n}, $n \in \mathbb{N}$, operators have been exhibited in [9, Section 5]. In order to obtain an asymptotic formula of Voronovskaja-type for these classes of q-operators, we will check the possibility to apply Theorem 3.5.

Application 1. We choose $\phi_n(x) := E_{q_n}\left(-[n]_{q_n} x \right)$, $x \geq 0$, $n \in \mathbb{N}$. Here E_{q_n} is the known expansion in q-Calculus of the exponential function being defined as follows

$$E_{q_n}(x) = \sum_{k=0}^{\infty} q_n^{k(k-1)/2} \frac{x^k}{[k]_{q_n}!}, \quad x \in \mathbb{R},$$

see, e.g., [7, p. 31]. T_{n,q_n} operators turn into S^*_{n,q_n}, a q-analogue of Szász-Mirakjan operators. For all $(n,k) \in \mathbb{N} \times \mathbb{N}_0$, we have $\phi_n(0) = 1$,

$$D^k_{q_n} \phi_n(x) = (-1)^k [n]^k_{q_n} q_n^{\frac{k(k-1)}{2}} E_{q_n}\left(-[n]_{q_n} q_n^k x \right), \quad x \geq 0,$$

and

$$D^{k+1}_{q_n} \phi_n(x) = -D^k_{q_n} \phi_n(q_n x) \beta_{n,k,0,q_n}(x).$$

Consequently, relations (2.5)-(2.8) hold, where, for every $(n,k)\in\mathbb{N}\times\mathbb{N}_0$, $i_k=0$ and $\beta_{n,k,0,q_n}(x)=[n]_{q_n}q_n^k$ is a constant function. Taking $\lambda_n=[n]_{q_n}$, all we have to do is to verify conditions (3.8) and (3.7).

Since $d_{n,k}=(-1)^k[n]_{q_n}^k q_n^{\frac{k(k-1)}{2}}$, we get $\tau_1=0$, $\tau_2=1$ and $\tau_3=0$. Further on, by using (3.4) and (3.5), we easily deduce

$$m_{n,4,3}(q_n)=\frac{(1-q_n)^2}{q_n^2[n]_{q_n}},\quad m_{n,4,4}(q_n)=0,$$

and (3.7) takes place. The asymptotic formula (3.9) for S^*_{n,q_n} will be read as follows

$$\lim_{n\to\infty}[n]_{q_n}\left(\left(S^*_{n,q_n}f\right)(x)-f(x)\right)=\frac{x}{2}f''(x),\quad x>0, \tag{3.10}$$

where $f\in\mathcal{F}(\mathbb{R}_+)\cap C^2(\mathbb{R}_+)$ with f'' bounded.

Application 2. Choosing $\phi_n(x):=(1+q_n^nx)_{q_n}^{-n}$, $x\geq 0$, $n\in\mathbb{N}$, T_{n,q_n} operators become a q-analogue of the ordinary Baskakov operators, say V^*_{n,q_n}. With the help of the known formulas $[-n]_q=-[n]_q\,q^{-n}$, $n\in\mathbb{N}$, and $D_q(1+ax)_q^\alpha=[\alpha]_q\,a\,(1+aqx)_q^{\alpha-1}$ for any real numbers a,α, we deduce

$$D^k_{q_n}\phi_n(x)=(-1)^k\left(\prod_{j=0}^{k-1}[n+j]_{q_n}\right)\left(1+q_n^{n+k}x\right)_{q_n}^{-n-k},\quad x\geq 0.$$

Conditions (2.5) and (2.6) are satisfied. Choosing $i_k=k$ and $\beta_{n,k,k,q_n}(x)=\left(\prod_{j=0}^{k}[n+j]_{q_n}\right)\left(1+q_n^{k+1}x\right)_{q_n}^{-k-1}$, $n\in\mathbb{N}$, $k\in\mathbb{N}_0$, (2.7) and (2.8) are also fulfilled. In the above we used the formula $(1+y)_q^\alpha(1+q^\alpha y)_q^\beta=(1+y)_q^{\alpha+\beta}$ for $q=q_n$, $\alpha=-n$, $\beta=-k-1$ and $y=q_n^{n+k+1}x$.

Again, we choose $\lambda_n=[n]_{q_n}$. This time, $d_{n,k}=(-1)^k[n]_{q_n}\cdot\ldots\cdot[n+k-1]_{q_n}$ and, in accordance with (3.8), one gets $\tau_1=0$, $\tau_2=1$, $\tau_3=1$. With a little more effort we find

$$\begin{aligned}\lambda_n^2m_{n,4,3}(q_n) &= \frac{(q_n-1)^2}{q_n^2}[n]_{q_n}-\frac{(1+2q_n)(q_n^2-2q_n-1)}{q_n^4}\\ &+ \frac{(1+q_n)(1+2q_n+3q_n^2)}{q_n^5[n]_{q_n}}\end{aligned}$$

and

$$\lambda_n^2m_{n,4,4}(q_n)=\frac{(q_n-1)^2}{q_n^3}[n]_{q_n}-\frac{q_n^3-3q_n-1}{q_n^5}+\frac{(1+q_n)(1+q_n+q_n^2)}{q_n^6[n]_{q_n}}.$$

We used the identities $[n+k]_{q_n} = 1+q_n+\ldots+q_n^k[n]_{q_n}$ for $k=1,2,3$. Since $q_n \in (0,1)$ and $q_n \to 1$, the requirement (3.7) is fulfilled. Actually, both sequences are convergent.

Applying Theorem 3.5, we obtain the following asymptotic formula.

$$\lim_n [n]_{q_n}\left(\left(V^*_{n,q_n}f\right)(x)-f(x)\right)=\frac{x(x+1)}{2}f''(x),\quad x>0, \tag{3.11}$$

where $f \in \mathcal{F}(\mathbb{R}_+)\cap C^2(\mathbb{R}_+)$ with f'' bounded.

Remark 3.6 *For $q_n = 1$, $S^*_{n,1}$ and $V^*_{n,1}$ turn into the classical Szász-Mirakjan and Baskakov operators, respectively. Our formulae (3.10), (3.11) become the known Voronovskaja-type identities verified by these discrete operators. In these two special cases the order of approximation is $1/n$.*

4 Proofs

Proof of Lemma 3.2

First of all, examining relations (2.2)-(2.4), we deduce $\sigma_q(m,m) = q^{-\frac{m(m+1)}{2}}$, $m \in \mathbb{N}_0$.

To calculate all required coefficients, we need to know some particular values of q-Stirling numbers $\sigma_q(m,r)$ described by (2.1). The below table may be useful.

$m \setminus r$	0	1	2	3	4
0	1	0	0	0	0
1	0	q^{-1}	0	0	0
2	0	q^{-2}	q^{-3}	0	0
3	0	q^{-3}	$(1+2q)q^{-5}$	q^{-6}	0
4	0	q^{-4}	$(1+3q+3q^2)q^{-7}$	$(1+2q+3q^2)q^{-9}$	q^{-10}

The identities (3.3)-(3.5) result after a boring calculation based on (2.10) and taking in view that

$$\left(T_{n,q}\varphi_x^4\right)(x)=\sum_{m=0}^{4}(-1)^m\begin{pmatrix}4\\ m\end{pmatrix}x^m\left(T_{n,q}e_m\right)(x).$$

□

Proof of Lemma 3.3

For $k = 0$ the conclusion is evident, see (2.5). Further on, choosing $x = 0$ in (2.7), we can write

$$\frac{d_{n,k+1}}{[n]_{q_n}^{k+1}} = (-1)^{i_k+1} q_n^{k-i_k} \frac{\beta_{n,k,i_k,q_n}(0)}{[n]_{q_n}^{i_k+1} q_n^{k-i_k}} \frac{d_{n,k-i_k}}{[n]_{q_n}^{k-i_k}}, \tag{4.1}$$

for a certain index $i_k \in \{0, 1, \ldots, k\}$.

The proof runs by mathematical induction with respect to k. Assuming $\lim\limits_n \frac{d_{n,j}}{[n]_{q_n}^j} = (-1)^j$ for $j = \overline{0,k}$, relations (4.1) and (2.8) imply $\lim\limits_n \frac{d_{n,k+1}}{[n]_{q_n}^{k+1}} = (-1)^{k+1}$. Consequently, relation (3.6) holds.

□

Proof of Lemma 3.4

Since $\lim\limits_n \frac{1}{[n]_{q_n}} = 0$, setting $\lim\limits_n \frac{\lambda_n}{[n]_{q_n}} = \tau_2 \in \mathbb{R}$, we get $\lim\limits_n \lambda_n^2 m_{n,4,1}(q_n) = 0$ and $\lim\limits_n \lambda_n^2 m_{n,4,2}(q_n) = 7\tau_2^2 - 4\tau_2$, see (3.3). Taking into account (3.2), the assumptions' lemma guarantee the achievement of the statement.

□

Proof of Theorem 3.5

Let $x > 0$ be fixed. For any $f \in \mathcal{F}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$ with f'' bounded, we define

$$\phi_f(x;t) = \begin{cases} \frac{f(t)-f(x)-\varphi_x(t)f'(x)-2^{-1}\varphi_x^2(t)f''(x)}{(t-x)^2}, & \text{if } t \neq x, \\ 0, & \text{if } t = x, \end{cases}$$

where $t \in \mathbb{R}_+$. We get $\lim\limits_{t\to x} \phi_f(x;t) = 0 = \phi_f(x;x)$, consequently $\phi_f(x;\cdot) \in C(\mathbb{R}_+)$. Moreover, φ_x^2 and $\varphi_x^2\phi_f(x;\cdot)$ belong to $\mathcal{F}(\mathbb{R}_+)$. For the function f, we can write the Lagrange form of the Taylor formula

$$f(t) = f(x) + \varphi_x(t)f'(x) + \frac{1}{2}\varphi_x^2(t)f''(x) + \varphi_x^2(t)\phi_f(x;t).$$

Applying T_{n,q_n} and using (2.11), we obtain

$$\left(T_{n,q_n}f\right)(x) - f(x)$$

$$= \left(T_{n,q_n}\varphi_x\right)(x)f'(x) + \frac{1}{2}\left(T_{n,q_n}\varphi_x^2\right)(x)f''(x) + T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x).$$

Taking into account relations (2.12)-(2.13), we get

$$\begin{aligned}\left(T_{n,q_n}\varphi_x\right)(x) &= -\left(\frac{d_{n,1}}{[n]_{q_n}}+1\right)x,\\ \left(T_{n,q_n}\varphi_x^2\right)(x) &= \left(\frac{d_{n,2}}{q_n[n]_{q_n}^2}+2\frac{d_{n,1}}{[n]_{q_n}}+1\right)x^2-\frac{d_{n,1}}{[n]_{q_n}^2}x,\end{aligned}$$

and hence

$$\begin{aligned}&\lambda_n\left(\left(T_{n,q_n}f\right)(x)-f(x)\right)=-\lambda_n\left(\frac{d_{n,1}}{[n]_{q_n}}+1\right)xf'(x)\\ &+\frac{1}{2}\left(\lambda_n\left(\frac{d_{n,2}}{q_n[n]_{q_n}^2}+2\frac{d_{n,1}}{[n]_{q_n}}+1\right)x^2-\frac{\lambda_n d_{n,1}}{[n]_{q_n}^2}x\right)f''(x)\\ &+\lambda_n T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x).\end{aligned}\tag{4.2}$$

Since $\varphi_x^2\phi_f(x,\cdot)\in\mathcal{F}(\mathbb{R}_+)$ and the series in (2.9) is absolutely convergent, we deduce $\varphi_x^2\left|\phi_f(x,\cdot)\right|\in\mathcal{F}(\mathbb{R}_+)$. Further on, by applying Cauchy inequality for the last term of (4.2), one has

$$\begin{aligned}0\le\lambda_n\left|T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x)\right|&\le\lambda_n T_{n,q_n}\left(\varphi_x^2\left|\phi_f(x;\cdot)\right|,x\right)\\ &\le\left\{\lambda_n^2\left(T_{n,q_n}\varphi_x^4\right)(x)\right\}^{1/2}\left\{\left(T_{n,q_n}\phi_f^2(x;\cdot)\right)(x)\right\}^{1/2}.\end{aligned}$$

Clearly, $\phi_f^2(x;\cdot)$ is continuous on $\mathbb{R}_+$. Under the assumption made on the function f, we get $\phi_f^2(x,\cdot)\in C_b(\mathbb{R}_+)\subset\mathcal{F}(\mathbb{R}_+)$, and in harmony with Remark 2.3, we have $\lim\limits_n\left(T_{n,q_n}\phi_f^2(x;\cdot)\right)(x)=\phi_f^2(x;x)=0$. On the other hand, since (3.7) holds, Lemma 3.4 guarantees that a constant $k(x)$ independent of n exists, such that $\lambda_n^2\left(T_{n,q_n}\varphi_x^4\right)(x)\le k(x)$ for each $n\in\mathbb{N}$.

Consequently, $\lim\limits_n\lambda_n\left|T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x)\right|=0$. Returning at (4.2), for n tending to infinity, on the basis of (3.8) and (3.6) with $k=1,2$, we obtain the desired pointwise convergence.

□

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Partial Inner Product Spaces and Operators on Them

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Abstract. Many families of function spaces play a central role in analysis, in particular in signal processing (e.g. wavelet or Gabor analysis). Such are L^p spaces, Besov spaces, amalgam spaces or modulation spaces. In all such cases, the parameter indexing the family measures the behavior (regularity, decay properties) of particular functions or operators. It turns out that all these space families are, or contain, scales or lattices of Banach spaces, which are special cases of *partial inner product spaces (*PIP*-spaces)*. In this context, it is often said that such families should be taken as a whole and operators, bases, frames on them should be defined globally, for the whole family, instead of individual spaces.

In this paper, we shall give an overview of PIP-spaces and operators on them, illustrating the results by space families of interest in mathematical physics and signal analysis. The interesting fact is that they allow a global definition of operators, and various operator classes on them have been defined.

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1 Motivation

Hilbert space is in general the first infinite dimensional topological vector space one encounters. On the mathematical side, it is the one that more

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closely resembles the familiar Euclidean space and thus it offers a smooth introduction into functional analysis. On the physics side, the fact is simply that Hilbert space is the daily language of quantum theory, so that mastering it is an essential tool for the quantum physicist.

However, the tool in question is insufficient. A pure Hilbert space formulation of quantum mechanics is both inconvenient and foreign to the daily behavior of most physicists, who stick to the more suggestive version of Dirac, which lacks a rigorous formulation. On the other hand, the physically interesting solutions of most partial differential equations are seldom smooth or square integrable, they rather correspond to changes of regime, which mean discontinuities and/or distributions. Thus it is no coincidence that many textbooks on PDEs begin with a thorough study of distribution theory [20, 22].

All this naturally leads to the introduction of Rigged Hilbert Spaces (RHS) [15]. In a nutshell, a RHS is a triplet

$$\Phi \hookrightarrow \mathcal{H} \hookrightarrow \Phi^{\times}, \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space, Φ is a dense subspace of the $\mathcal{H}$, equipped with a locally convex topology, finer than the norm topology inherited from $\mathcal{H}$, and $\Phi^{\times}$ is the space of continuous conjugate linear functionals on Φ, endowed with the strong dual topology. By duality, each space in (1.1) is dense in the next one and all embeddings are linear and continuous. In addition, the space Φ is in general required to be reflexive and nuclear. Standard examples of rigged Hilbert spaces are the Schwartz distribution spaces over $\mathbb{R}$ or $\mathbb{R}^n$, namely $\mathcal{S} \subset L^2 \subset \mathcal{S}^{\times}$ or $\mathcal{D} \subset L^2 \subset \mathcal{D}^{\times}$ [15, 24, 27].

The problem with the RHS (1.1) is that, besides the Hilbert space vectors, it contains only two types of elements, "very good" functions in Φ and "very bad" ones in $\Phi^{\times}$. If one wants a fine control on the behavior of individual elements, one has to interpolate somehow between the two extreme spaces. In the case of the Schwartz triplet, $\mathcal{S} \subset L^2 \subset \mathcal{S}^{\times}$, a well-known solution is given by a chain of Hilbert spaces, the so-called Hermite representation of tempered distributions [28].

In fact, this is not at all an isolated case. Indeed many function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, ...). The typical structure is a *chain of Hilbert or (reflexive) Banach spaces*. Let us give two familiar examples.

(i) The Lebesgue L^p spaces on a finite interval, e.g. $\mathcal{I} = \{L^p([0,1], dx)$,

$1 \leqslant p \leqslant \infty\}$:

$$L^\infty \subset \ldots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \ldots \subset L^2 \subset \ldots \subset L^r \subset L^q \subset \ldots \subset L^1, \tag{1.2}$$

where $1 < q < r < 2$. Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q+1/\overline{q} = 1)$, and similarly $L^r, L^{\overline{r}}$ $(1/r + 1/\overline{r} = 1)$. By the Hölder inequality, the (L^2) inner product

$$\langle f|g\rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx \tag{1.3}$$

is well-defined if $f \in L^q$, $g \in L^{\overline{q}}$. However, it is *not* well-defined for two arbitrary functions $f, g \in L^1$. Take for instance, $f(x) = g(x) = x^{-1/2}$: $f \in L^1$, but $fg = f^2 \notin L^1$. Thus, on L^1, (1.3) defines only a *partial* inner product. The same result holds for any compact subset of $\mathbb{R}$ instead of [0,1].

(ii) The scale of Hilbert spaces[1] built on the powers of a positive self-adjoint operator $A \geqslant 1$ in a Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual):

$$\mathcal{H}_\infty(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \ldots$$
$$\ldots \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \ldots \subset \mathcal{H}_{-\infty}(A) := \bigcup_n \mathcal{H}_n. \tag{1.4}$$

Note that here the index n could also be taken as real, the link between the two cases being established by the spectral theorem for self-adjoint operators. Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{-n}$, but on $\mathcal{H}_{-\infty}(A)$ it yields only a partial inner product. The following examples are standard:

- $(A_{\rm p} f)(x) = (1 + x^2) f(x)$ in $L^2(\mathbb{R}, dx)$.
- $(A_{\rm m} f)(x) = (1 - \frac{d^2}{dx^2}) f(x)$ in $L^2(\mathbb{R}, dx)$: Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$ or $\mathbb{R}$.
- $(A_{\rm osc} f)(x) = (1 + x^2 - \frac{d^2}{dx^2}) f(x)$ in $L^2(\mathbb{R}, dx)$.

[1] A discrete chain of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is called a *scale* if there exists a self-adjoint operator $B \geqslant 1$ such that $\mathcal{H}_n = D(B^n)$, $\forall\, n \in \mathbb{Z}$, with the graph norm $\|f\|_n = \|B^n f\|$. A similar definition holds for a continuous chain $\{\mathcal{H}_\alpha\}_{\alpha\in\mathbb{R}}$.

(The notation is suggested by the operators of position, momentum and harmonic oscillator energy in quantum mechanics, respectively). Note that both $\mathcal{H}_\infty(A_\mathrm{p}) \cap \mathcal{H}_\infty(A_\mathrm{m})$ and $\mathcal{H}_\infty(A_\mathrm{osc})$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathcal{H}_{-\infty}(A_\mathrm{osc})$ with the space $\mathcal{S}^\times(\mathbb{R})$ of tempered distributions.

However, a moment's reflection shows that the total order relation inherent in a chain is in fact an unnecessary restriction, partially ordered structures are sufficient, and indeed necessary in practice. For instance, in order to get a better control on the behavior of individual functions, one may consider the lattice built on the powers of A_p and A_m simultaneously. Then the extreme spaces are still $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^\times(\mathbb{R})$. Similarly, in the case of several variables, controlling the behavior of a function in each variable separately requires a nonordered set of spaces. This is in fact a statement about tensor products (remember that $L^2(X \times Y) \simeq L^2(X) \otimes L^2(Y)$). Indeed the tensor product of two chains of Hilbert spaces, $\{\mathcal{H}_n\} \otimes \{\mathcal{K}_m\}$ is naturally a lattice $\{\mathcal{H}_n \otimes \mathcal{K}_m\}$ of Hilbert spaces. For instance, in the example above, for two variables x, y, that would mean considering intermediate Hilbert spaces corresponding to the product of two operators, $\big(A_\mathrm{m}(x)\big)^n \big(A_\mathrm{m}(y)\big)^m$.

Thus the structure to analyze is that of *lattices of Hilbert or Banach spaces*, interpolating between the extreme spaces of a RHS, as in (1.1). Many examples can be given, for instance the lattice generated by the spaces $L^p(\mathbb{R}, dx)$, the amalgam spaces $W(L^p, \ell^q)$, the mixed norm spaces $L^{p,q}_m(\mathbb{R}, dx)$, and many more. In all these cases, which contain most families of function spaces of interest in analysis and in signal processing, a common structure emerges for the "large" space V, defined as the union of all individual spaces. There is a lattice of Hilbert or reflexive Banach spaces V_r, with an (order-reversing) involution $V_r \leftrightarrow V_{\overline{r}}$, where $V_{\overline{r}} = V_r^\times$ (the space of continuous conjugate linear functionals on V_r), a central Hilbert space $V_o \simeq V_{\overline{o}}$, and a partial inner product on V that extends the inner product of V_o to pairs of dual spaces $V_r, V_{\overline{r}}$.

Moreover, many operators should be considered globally, for the whole scale or lattice, instead of on individual spaces. In the case of the spaces $L^p(\mathbb{R})$, such are, for instance, operators implementing translations ($x \mapsto x-y$) or dilations ($x \mapsto x/a$), convolution operators, Fourier transform, etc. In the same spirit, it is often useful to have a *common* basis for the whole family of spaces, such as the Haar basis for the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. Thus we need a notion of operator and basis defined globally for the scale or lattice itself.

This state of affairs prompted A.Grossmann and one of us (JPA) to sys-

tematize this approach, and this led to the concept of *partial inner product space* or PIP-*space* [3, 5]. Many papers followed, with various collaborators. Then, in recent years, the topic has found a new interest, in particular through developments of signal processing and the underlying mathematics. Thus it appeared to the us that the time had come for reviewing the whole subject, so that we finally made a thorough analysis in a recent monograph [8], to which we refer for further information. The aim of this paper is to present the formalism of PIP-spaces, which indeed answers the questions raised above. In a first part, the structure of PIP-space is derived systematically from the abstract notion of compatibility and then particularized to the examples listed above. In a second part, operators on PIP-spaces are introduced and illustrated by several operators commonly used in Gabor or wavelet analysis. Of course, the treatment is sketchy, for lack of space. For a complete information, we refer the reader to our monograph [8].

2 Partial Inner Product Spaces

2.1 Basic Definitions

The basic question is how to generate PIP-spaces in a systematic fashion. In order to answer it, we may reformulate it as follows: given a vector space V and two vectors $f, g \in V$, when does their inner product make sense? A way of formalizing the answer is given by the idea of *compatibility*.

Definition 2.1 *A* linear compatibility relation *on a vector space V is a symmetric binary relation $f\#g$ which preserves linearity:*

$$\begin{aligned} f\#g &\iff g\#f, \ \forall\, f, g \in V, \\ f\#g, f\#h &\implies f\#(\alpha g + \beta h), \ \forall\, f, g, h \in V, \forall\, \alpha, \beta \in \mathbb{C}. \end{aligned}$$

As a consequence, for every subset $S \subset V$, the set $S^{\#} = \{g \in V : g\#f, \forall\, f \in S\}$ is a vector subspace of V and one has

$$S^{\#\#} = (S^{\#})^{\#} \supseteq S, \ S^{\#\#\#} = S^{\#}.$$

Thus one gets the following equivalences:

$$\begin{aligned} f\#g &\iff f \in \{g\}^{\#} \iff \{f\}^{\#\#} \subseteq \{g\}^{\#} \\ &\iff g \in \{f\}^{\#} \iff \{g\}^{\#\#} \subseteq \{f\}^{\#}. \end{aligned} \tag{2.1}$$

From now on, we will call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V,\#)$ the family of all assaying subsets of V, ordered by inclusion. Let F be the isomorphy class of $\mathcal{F}$, that is, $\mathcal{F}$ considered as an abstract partially ordered set. Elements of F will be denoted by $r, q, \ldots$, and the corresponding assaying subsets $V_r, V_q, \ldots$. By definition, $q \leqslant r$ if and only if $V_q \subseteq V_r$. We also write $V_{\overline{r}} = V_r^{\#}$, $r \in F$. Thus the relations (2.1) mean that $f \# g$ if and only if there is an index $r \in F$ such that $f \in V_r$, $g \in V_{\overline{r}}$. In other words, vectors should not be considered individually, but only in terms of assaying subspaces, which are the building blocks of the whole structure.

It is easy to see that the map $S \mapsto S^{\#\#}$ is a closure, in the sense of universal algebra, so that the assaying subspaces are precisely the "closed" subsets. Therefore one has the following standard result.

Theorem 2.2 *The family $\mathcal{F}(V,\#) \equiv \{V_r, r \in F\}$, ordered by inclusion, is a complete involutive lattice, i.e., it is stable under the following operations, arbitrarily iterated:*

$$\begin{array}{lrcll} \text{. } \textit{involution:} & V_r & \leftrightarrow & V_{\overline{r}} = (V_r)^{\#}, & \\ \text{. } \textit{infimum:} & V_{p\wedge q} & \equiv & V_p \wedge V_q = V_p \cap V_q, & (p,q,r \in F) \\ \text{. } \textit{supremum:} & V_{p\vee q} & \equiv & V_p \vee V_q = (V_p + V_q)^{\#\#}. & \end{array}$$

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. By definition, the index set F is also a complete involutive lattice; for instance,

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

Definition 2.3 *A* partial inner product *on $(V,\#)$ is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. A* partial inner product space *(*PIP*-space) is a vector space V equipped with a linear compatibility and a partial inner product.*

Note that the partial inner product is not required to be positive definite.

The partial inner product clearly defines a notion of *orthogonality:* $f \perp g$ if and only if $f \# g$ and $\langle f|g\rangle = 0$.

Definition 2.4 *The* PIP*-space $(V,\#,\langle\cdot|\cdot\rangle)$ is* nondegenerate *if $(V^{\#})^{\perp} = \{0\}$, that is, if $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$.*

From now on, we will assume that our PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is nondegenerate. As a consequence, $(V^\#, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in F$, are a dual pair in the sense of topological vector spaces [21]. We also assume that the partial inner product is positive definite.

Now, one wants the topological structure to match the algebraic structure, in particular, the topology τ_r on V_r should be such that its conjugate dual be $V_{\overline{r}} : (V_r[\tau_r])^\times = V_{\overline{r}}$, $\forall r \in F$. This implies that the topology τ_r must be finer than the weak topology $\sigma(V_r, V_{\overline{r}})$ and coarser than the Mackey topology $\tau(V_r, V_{\overline{r}})$:

$$\sigma(V_r, V_{\overline{r}}) \preceq \tau_r \preceq \tau(V_r, V_{\overline{r}}).$$

From here on, we will assume that every V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$. This choice has two interesting consequences. First, if $V_r[\tau_r]$ is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology. Next, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^\#$ is dense in every V_r.

2.2 Examples

Let us give some simple examples of PIP-spaces.

(i) Sequence spaces

Let V be the space ω of *all* complex sequences $x = (x_n)$ and define on it (i) a compatibility relation by $x\#y \Leftrightarrow \sum_{n=1}^\infty |x_n\, y_n| < \infty$; (ii) a partial inner product $\langle x|y\rangle = \sum_{n=1}^\infty \overline{x_n}\, y_n$.

Then $\omega^\# = \varphi$, the space of finite sequences, and the complete lattice $\mathcal{F}(\omega, \#)$ consists of Köthe's perfect sequence spaces [21, § 30]. Among these, typical assaying subspaces are the weighted Hilbert spaces

$$\ell^2(r) = \{(x_n) : \sum_{n=1}^\infty |x_n|^2\, r_n^{-2} < \infty\}, \tag{2.2}$$

where $r = (r_n)$, $r_n > 0$, is a sequence of positive numbers. The involution is $\ell^2(r) \leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^\times$, where $\overline{r}_n = 1/r_n$. In addition, there is a central, self-dual Hilbert space, namely, $\ell^2(1) = \ell^2(\overline{1}) = \ell^2$, where $1 = (1)$.

(ii) Spaces of locally integrable functions

Let now V be $L^1_{\rm loc}(\mathbb{R}, dx)$, the space of Lebesgue measurable functions, integrable over compact subsets, and define a compatibility relation on

it by $f\#g \Leftrightarrow \int_{\mathbb{R}} |f(x)g(x)|\,dx < \infty$ and a partial inner product $\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)}g(x)\,dx$.

Then $V^{\#} = L^{\infty}_{\rm c}(\mathbb{R}, dx)$, the space of bounded measurable functions of compact support. The complete lattice $\mathcal{F}(L^1_{\rm loc}, \#)$ consists of Köthe function spaces [11, 16]. Here again, typical assaying subspaces are weighted Hilbert spaces

$$L^2(r) = \{f \in L^1_{\rm loc}(\mathbb{R}, dx) : \int_{\mathbb{R}} |f(x)|^2\, r(x)^{-2}\, dx < \infty\}, \tag{2.3}$$

with $r, r^{-1} \in L^2_{\rm loc}(\mathbb{R}, dx), r(x) > 0$ a.e. The involution is $L^2(r) \leftrightarrow L^2(\overline{r})$, with $\overline{r} = r^{-1}$, and the central, self-dual Hilbert space is $L^2(\mathbb{R}, dx)$.

(iii) Nested Hilbert Spaces

This is the original construction of Grossmann [18] for finding an "easy" substitute to distributions, and actually one of the motivations for introducing PIP-spaces. And indeed the two are closely related, see [8, Sec.2.4.1].

(iv) Rigged Hilbert Spaces

This is the simplest example of PIP-space, but it is a rather poor one. Indeed, in the RHS (1.1), two elements are compatible if both belong to $\mathcal{H}$, or one of them belongs to Φ. Thus the three defining spaces are the only assaying subspaces. The partial inner product is, of course, simply that of $\mathcal{H}$ (provided the sesquilinear form that puts Φ and $\Phi^{\times}$ in duality has been correctly normalized).

3 Lattices of Hilbert or Banach Spaces

From the previous examples, we learn that $\mathcal{F}(V, \#)$ is a huge lattice (it is complete!) and that assaying subspaces may be complicated, such as Fréchet spaces, nonmetrizable spaces, etc. This situation suggests to choose an involutive sublattice $\mathcal{I} \subset \mathcal{F}$, indexed by I, such that

(i) $\mathcal{I}$ is generating:

$$f\#g \;\Leftrightarrow\; \exists\, r \in I \text{ such that } f \in V_r, g \in V_{\overline{r}}\,; \tag{3.1}$$

(ii) every $V_r, r \in I$, is a Hilbert space or a reflexive Banach space;

(iii) there is a unique self-dual, Hilbert, assaying subspace $V_o = V_{\overline{o}}$.

In that case, the structure $V_I := (V, \mathcal{I}, \langle\cdot|\cdot\rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). Both types are

particular cases of the so-called indexed PIP-spaces [8]. Note that $V^\#, V$ themselves usually do *not* belong to the family $\{V_r,\, r \in I\}$, but they can be recovered as

$$V^\# = \bigcap_{r\in I} V_r, \quad V = \sum_{r\in I} V_r.$$

In the LBS case, the lattice structure takes the following form

- $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm

$$\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q\,;$$

- $V_{p\vee q} = V_p + V_q$, with the *inductive* norm

$$\|f\|_{p\vee q} = \inf_{f=g+h} \left(\|g\|_p + \|h\|_q\right), g \in V_p,\, h \in V_q.$$

These norms are usual in interpolation theory [9]. In the LHS case, one takes similar definitions with squared norms, in order to get Hilbert norms throughout.

The definition given above leads to the proper notion of comparison betwen two linear compatibilies on the same vector space. Namely, we shall say that a compatibilty $\#_1$ is *finer* than $\#_2$, or that $\#_2$ is *coarser* that $\#_1$, which we note $\#_2 \trianglelefteq \#_1$, if $\mathcal{F}(V, \#_2)$ is an involutive cofinal sublattice of $\mathcal{F}(V, \#_1)$ (Given a partially ordered set F, a subset $K \subset F$ is *cofinal* to F if, for any element $x \in F$, there is an element $k \in K$ such that $x \leqslant k$.).

Now, suppose a linear compatibility $\#$ is given on V. Then, every involutive cofinal sublattice of $\mathcal{F}(V, \#)$ defines a coarser PIP-space, and *vice versa*. Thus coarsening is always possible, and will ultimately lead to a minimal PIP-space, consisting of $V^\#$ and V only, that is, the situation of distribution spaces. However, the operation of refining is not always possible, in particular there is no canonical solution, *a fortiori* no unique maximal solution. There are exceptions, however. In the case of a LHS, refining is possible, with infinitely many solutions, by use of interpolation methods or the spectral theorem for self-adjoint operators. In particular, one may always refine a *discrete* scale of Hilbert spaces into a (nonunique) *continuous* one, and even into a continuous LHS [8, Chap. 5].

4 Examples of LBSs

In this section, we will list a series of concrete examples of LBSs, some of which are of particular interest in signal processing. For simplicity, we will restrict ourselves to one dimension, although most spaces may be defined on $\mathbb{R}^n$, $n > 1$, as well.

4.1 Chains of Hilbert or Banach spaces

Typical are the two examples described in Section 1.

(1) The chain of Lebesgue spaces on a finite interval $\mathcal{I} = \{L^p([0,1], dx), 1 < p < \infty\}$.

The chain (1.2) is a (totally ordered) lattice. The corresponding lattice completion is obtained by adding "nonstandard" spaces , such as

$$L^{p-} = \bigcap_{1<q<p} L^q \text{ (non-normable Fréchet)}, \quad L^{p+} = \bigcup_{p<q<\infty} L^q \text{ (nonmetrizable)}.$$

(2) The scale (1.4) of Hilbert spaces $\{\mathcal{H}_n, n \in \mathbb{Z}\}$ built on powers of $A = A^* \geqslant 1$. The lattice completion is similar to the previous one, introducing analogous "nonstandard" spaces [8, Sec.5.1].

4.2 Sequence spaces

In ω, we may take the lattice $\mathcal{I} = \{\ell^2(r)\}$ of the weighted Hilbert spaces defined in (2.2), with

- infimum: $\ell^2(p \wedge q) = \ell^2(p) \wedge \ell^2(q) = \ell^2(r)$, $r_n = \min(p_n, q_n)$;
- supremum: $\ell^2(p \vee q) = \ell^2(p) \vee \ell^2(q) = \ell^2(s)$, $s_n = \max(p_n, q_n)$;
- duality: $\ell^2(p \wedge q) \leftrightarrow \ell^2(\overline{p} \vee \overline{q})$, $\ell^2(p \vee q) \leftrightarrow \ell^2(\overline{p} \wedge \overline{q})$

(the norms above are equivalent to the projective and inductive norms, respectively).

4.3 Spaces of locally integrable functions

In $L^1_{\rm loc}(\mathbb{R}, dx)$, we may take the lattice $\mathcal{I} = \{L^2(r)\}$ of the weighted Hilbert spaces defined in (2.3), with

- infimum: $L^2(p \wedge q) = L^2(p) \wedge L^2(q) = L^2(r)$, $r(x) = \min(p(x), q(x))$;
- supremum: $L^2(p \vee q) = L^2(p) \vee L^2(q) = L^2(s)$, $s(x) = \max(p(x), q(x))$;
- duality: $L^2(p \wedge q) \leftrightarrow L^2(\overline{p} \vee \overline{q})$, $L^2(p \vee q) \leftrightarrow L^2(\overline{p} \wedge \overline{q})$.

(here too, these norms are equivalent to the projective and inductive norms, respectively).

4.4 The spaces $L^p(\mathbb{R}, dx)$, $1 < p < \infty$

The spaces $L^p(\mathbb{R}, dx)$, $1 < p < \infty$ do not constitute a scale, since one has only the inclusions $L^p \cap L^q \subset L^s$, $p < s < q$. Thus one has to consider the

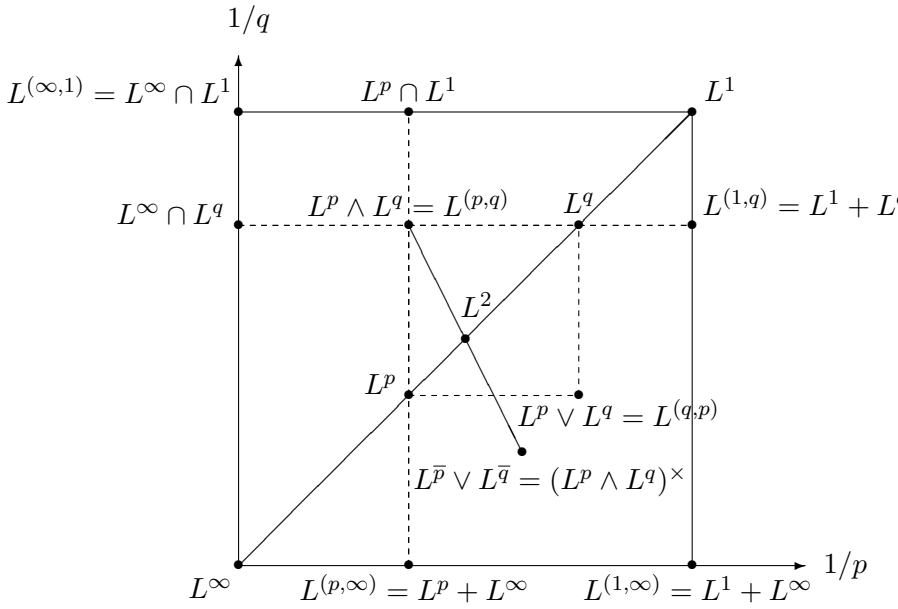


Figure 1: The unit square describing the lattice J.

lattice they generate, with the following lattice operations:

- $L^p \wedge L^q = L^p \cap L^q$, with projective norm,
- $L^p \vee L^q = L^p + L^q$, with inductive norm.

For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive Banach spaces and their conjugate duals are, respectively, $(L^p \wedge L^q)^\times = L^{\overline{p}} \vee L^{\overline{q}}$ and $(L^p \vee L^q)^\times = L^{\overline{p}} \wedge L^{\overline{q}}$.

It is convenient to introduce the following unified notation :

$$L^{(p,q)} = \begin{cases} L^p \wedge L^q, & \text{if } p \geqslant q, \\ L^p \vee L^q, & \text{if } p \leqslant q. \end{cases}$$

Then, for $1 < p, q < \infty$, $L^{(p,q)}$ is a reflexive Banach space, with dual $L^{(\overline{p},\overline{q})}$.

Next, if we represent (p, q) by the point of coordinates $(1/p, 1/q)$, we may associate all the spaces $L^{(p,q)}$ $(1 \leqslant p, q \leqslant \infty)$ in a one-to-one fashion with the points of a unit square $\mathrm{J} = [0, 1] \times [0, 1]$ (see Figure 1). Thus, in this picture, the spaces L^p are on the main diagonal, intersections $L^p \cap L^q$ above it and sums $L^p + L^q$ below.

The space $L^{(p,q)}$ is contained in $L^{(p',q')}$ if (p, q) is on the left and/or

above (p', q'). Thus the smallest space is

$$V_{\rm J}^{\#} = L^{(\infty,1)} = L^\infty \cap L^1$$

and it corresponds to the upper left corner, the largest one is

$$V_{\rm J} = L^{(1,\infty)} = L^1 + L^\infty,$$

corresponding to the lower right corner. Inside the square, duality corresponds to (geometrical) symmetry with respect to the center (1/2,1/2) of the square, which represents the space L^2. The ordering of the spaces corresponds to the following rule:

$$L^{(p,q)} \subset L^{(p',q')} \iff (p,q) \leqslant (p',q') \iff p' \geqslant p \ \text{ and } q' \leqslant q. \tag{4.1}$$

With respect to this ordering, J is an involutive lattice with the operations

$$\begin{aligned} (p,q) \wedge (p',q') &= (p \vee p', q \wedge q'), \\ (p,q) \vee (p',q') &= (p \wedge p', q \vee q'), \\ \overline{(p,q)} &= (\overline{p}, \overline{q}), \end{aligned}$$

where $p \wedge p' = \min\{p, p'\}$, $p \vee p' = \max\{p, p'\}$. It is remarkable that the lattice $\mathcal{J}$ generated by $\mathcal{I} = \{L^p\}$ is obtained at the first "generation". One has, for instance, $L^{(r,s)} \wedge L^{(a,b)} = L^{(r\vee a, s\wedge b)}$, both as sets and as topological vector spaces.

4.5 Mixed norm Lebesgue spaces $L^{p,q}_m(\mathbb{R}^2)$

Mixed norm spaces are close relatives to the Lebesgue L^p spaces . The definition runs as follows [17, Sec.11.1] (for the general case, one replaces $\mathbb{R}^d \times \mathbb{R}^d$ by the product $(X,\mu) \times (Y,\nu)$ of two σ-finite measure spaces):
$L^{p,q}_m(\mathbb{R}^{2d}) = \{f \text{ Lebesgue measurable on } \mathbb{R}^{2d} : \|f\|^{p,q}_m < \infty\}$, $1 \leqslant p, q < \infty$, where

$$\|f\|^{p,q}_m = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,\omega)|^p \, m(x,\omega)^p dx \right)^{q/p} d\omega \right)^{1/q}.$$

Here m is a weight function, that is, a non-negative locally integrable function on $\mathbb{R}^{2d}$. In addition, m is assumed to be v-moderate, i.e., $m(z_1 + z_2) \leqslant v(z_1)m(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$, with v a submultiplicative weight function, that is, $v(z_1 + z_2) \leqslant v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$. The typical weights are of polynomial growth: $v_s(z) = (1 + |z|)^s$, $s \geqslant 0$.

The space $L^{p,q}_m(\mathbb{R}^{2d})$ is a Banach space for the norm $\|\cdot\|^{p,q}_m$. The duality property is, as expected, $(L^{p,q}_m)^\times = L^{\overline{p},\overline{q}}_{1/m}$. Of course, things simplify when $p = q$: $L^{p,p}_m(\mathbb{R}^{2d}) = L^p_m(\mathbb{R}^{2d})$, a weighted L^p space.

Concerning lattice properties of the family of $L^{p,q}_m$ spaces, we cannot expect more than for the L^p spaces. Two $L^{p,q}_m$ spaces are never comparable, even for the same weight m, so one has to take the lattice generated by intersection and duality.

A different type of mixed-norm spaces is obtained if one takes $X = Y = \mathbb{Z}^d$, with the counting measure. Thus one gets the space $\ell^{p,q}_m(\mathbb{Z}^{2d})$, which consists of all sequences $a = (a_{kn})$, $k, n \in \mathbb{Z}^d$, for which the following norm is finite:

$$\|a\|_{\ell^{p,q}_m} := \left(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} |a_{kn}|^p \, m(k,n)^p \Big)^{q/p} \right)^{1/q},$$

where $m(\cdot,\cdot)$ is a weight function on $\mathbb{Z}^{2d}$. Contrary to the continuous case, here we do have inclusion relations: if $p_1 \leqslant p_2, q_1 \leqslant q_2$ and $m_2 \leqslant Cm_1$, then $\ell^{p_1,q_1}_{m_1} \subseteq \ell^{p_2,q_2}_{m_2}$.

Discrete mixed-norm spaces have been used extensively in functional analysis and signal processing. For instance, they are key to the proof that certain operators are bounded between two given function spaces, such as modulation spaces (see below) or ℓ^p spaces. In general, a mixed-norm space will prove useful whenever one has a signal consisting of sequences labeled by two indices that play different roles. An obvious example is time-frequency or time-scale analysis: a Gabor or wavelet basis (or frame) is written as $\{\psi_{j,k},\, j,k \in \mathbb{Z}\}$, where j indexes the scale or frequency and k the time. More generally, this applies whenever signals are expanded with respect to a dictionary with two indices.

4.6 Amalgam spaces

A situation intermediate between the mixed-norm spaces $L^{p,q}(\mathbb{R}^{2d})$ (for $m \equiv 1$) and the spaces $\ell^{p,q}(\mathbb{Z}^{2d})$ is that of the so-called *amalgam spaces*. They were introduced specifically to overcome the inability of the L^p norms to distinguish between the local and the global behavior of functions. The simplest ones are the spaces $W(L^p, \ell^q)$ of N. Wiener [14], which consist of functions on $\mathbb{R}$ which are locally in L^p with ℓ^q behavior at infinity, in the sense that the L^p norms over the intervals $(n, n+1)$ form an ℓ^q sequence.

It is a Banach space for the norm

$$\|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p dx \right]^{q/p} \right\}^{1/q}, \ 1 \leqslant p, q < \infty.$$

The following inclusion relations, with all embeddings continuous, are more or less obvious from those of the L^p and the ℓ^q spaces:

- If $q_1 \leqslant q_2$, then $W(L^p, \ell^{q_1}) \subset W(L^p, \ell^{q_2})$.
- If $p_1 \leqslant p_2$, then $W(L^{p_2}, \ell^q) \subset W(L^{p_1}, \ell^q)$.

It follows that the smallest space is $W(L^\infty, \ell^1)$ and the largest space is $W(L^1, \ell^\infty)$.

As for duality, one has $W(L^p, \ell^q)^\times = W(L^{\overline{p}}, \ell^{\overline{q}})$, for $1 \leqslant p, q < \infty$.

The interesting fact is that, for $1 \leqslant p, q \leqslant \infty$, the set $\mathcal{J}$ of all amalgam spaces $\{W(L^p, \ell^q)\}$ may be represented by the points (p, q) of the *same* unit square J as in the example of the L^p spaces, with the *same* order structure. However, $\mathcal{J}$ is not a lattice with respect to the order (4.1). One has indeed

$$W(L^p, \ell^q) \wedge W(L^{p'}, \ell^{q'}) \supset W(L^{p \vee p'}, \ell^{q \wedge q'}),$$
$$W(L^p, \ell^q) \vee W(L^{p'}, \ell^{q'}) \subset W(L^{p \wedge p'}, \ell^{q \vee q'}),$$

where again $\wedge$ means intersection with projective norm and $\vee$ means vector sum with inductive norm, but equality is not obtained. Thus, as in the previous case, one gets chains by varying either p or q, but not both.

Among the amalgam spaces, especially useful is the family $W(\mathcal{F}L^p, \ell^q)$, $1 \leqslant p, q \leqslant \infty$, where $\mathcal{F}L^p$ denotes the set of Fourier transforms of L^p functions (one may even add weights on both spaces). These spaces have, for instance, nice inclusion and convolution properties.

Among these, the most interesting one is $\mathcal{S}_0 = W(\mathcal{F}L^1, \ell^1)$, called the *Feichtinger algebra* [12, 13]. The space $\mathcal{S}_0$ has many interesting properties, for instance:

(i) $\mathcal{S}_0$ is a Banach space for the norm $\|f\|_{\mathcal{S}_0} = \|V_{g_0} f\|_1$, and $\mathcal{S} \hookrightarrow \mathcal{S}_0 \hookrightarrow L^2$, with all embeddings continuous with dense range. Here g_0 is the Gaussian and $V_g f$ denotes the Short Time Fourier (or Gabor) Transform (STFT) of $f \in L^2(\mathbb{R}^d)$, namely,

$$(V_g f)(x, \omega) := \int_{\mathbb{R}^d} \overline{g(y - x)}\, f(y)\, e^{-2\pi i y\omega}\, dy, \quad x, \omega \in \mathbb{R}^d, \tag{4.2}$$

(ii) $\mathcal{S}_0$ is a Banach algebra with respect to pointwise multiplication and

convolution.

(iii) Time-frequency shifts $T_x M_\omega$ are isometric on $\mathcal{S}_0$: $\|T_x M_\omega f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$, where, as usual, $(T_x g)(y) = g(y-x)$ (translation) and $(M_\omega h)(y) = e^{2\pi i y\omega} h(y)$ (modulation). $\mathcal{S}_0$ is continuously embedded in any Banach space with the same property and containing g_0, thus it is the smallest Banach space with this property.

(iv) The Fourier transform is an isometry on $\mathcal{S}_0 : \|\mathcal{F} f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.

Next we turn to the (conjugate) dual $\mathcal{S}_0^\times$ of $\mathcal{S}_0$. It is a Banach space with norm $\|f\|_{\mathcal{S}_0^\times} = \|V_g f\|_\infty$. The space $\mathcal{S}_0^\times$ contains both the δ function and the pure frequency $\chi_\omega(x) = e^{-2\pi i x\omega}$.

In virtue of (i) above, we have

$$\mathcal{S} \hookrightarrow \mathcal{S}_0 \hookrightarrow L^2 \hookrightarrow \mathcal{S}_0^\times \hookrightarrow \mathcal{S}^\times, \tag{4.3}$$

where all embeddings are continuous and have dense range. It turns out that the triplet

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}_0^\times(\mathbb{R}^d) \tag{4.4}$$

is the prototype of a *Banach Gel'fand triple*, that is a RHS (or LBS) in which the extreme spaces are (nonreflexive) Banach spaces. This is often a very convenient substitute for Schwartz' RHS and it is widely used in signal processing.

4.7 Other function spaces

Many other families of function spaces of interest in analysis or signal processing are, or contain, lattices of Banach spaces. To quote a few:

(i) Köthe perfect sequence spaces, mentioned in Section 2.2 (ii);

(ii) Köthe function spaces (which contain all types of Lebesgue spaces);

(iii) Modulation spaces $M^{p,q}_m(\mathbb{R}^d)$, well adapted to Gabor analysis. Indeed, given a v-moderate weight function $m(x,\omega)$, the modulation space $M^{p,q}_m$ is defined in terms of a mixed-norm of a STFT:

$$M^{p,q}_m(\mathbb{R}^d) = \{f \in \mathcal{S}^\times(\mathbb{R}^d) : V_g f \in L^{p,q}_m(\mathbb{R}^{2d})\}, \quad 1 \leqslant p, q \leqslant \infty.$$

For $p = q$, one writes $M^p_m \equiv M^{p,p}_m$. The space $M^{p,q}_m$ is a Banach space for the norm

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m}.$$

In particular, the Feichtinger algebra and its dual are simple modulation spaces, namely, $\mathcal{S}_0 = M^1$ and $\mathcal{S}_0^\times = M^\infty$.

(iv) Inhomogeneous Besov spaces $B^s_{pq}(\mathbb{R}^d)$, well adapted to wavelet analysis;

(v) α-modulation spaces, which are intermediate between modulation spaces and Besov spaces;

(vi) Coorbit spaces, which contain many of the previous cases.

For further information about those spaces, we refer to our monograph [8, Chap.4 and Chap.8].

5 Operators on PIP-spaces

5.1 General definitions

As already mentioned, the basic idea of (indexed) PIP-spaces is that vectors should not be considered individually, but only in terms of the subspaces V_r ($r \in F$ or $r \in I$), the building blocks of the structure, see (3.1). Correspondingly, an operator on a PIP-space should be defined in terms of assaying subspaces only, with the proviso that only bounded operators between Hilbert or Banach spaces are allowed. Thus an operator is a *coherent collection* of bounded operators. More precisely,

Definition 5.1 *Given a LHS or LBS $V_I = \{V_r, r \in I\}$, an* operator *on V_I is a map $A : \mathcal{D}(A) \to V$, where*

(i) $\mathcal{D}(A) = \bigcup_{q \in \mathsf{d}(A)} V_q$, with $\mathsf{d}(A)$ a nonempty subset of I;

(ii) for every $q \in \mathsf{d}(A)$, there exists a $p \in I$ such that the restriction of A to V_q is linear and continuous into V_p (we denote this restriction by A_{pq});

(iii) A has no proper extension satisfying (i) and (ii).

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $\mathsf{j}(A) = \{(q,p) \in I \times I : A_{pq} \text{ exists}\}$. Thus the operator A may be identified with the collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q,p) \in \mathsf{j}(A)\}.$$

By condition (iii), the set $\mathsf{d}(A)$ is obtained by projecting $\mathsf{j}(A)$ on the "first coordinate" axis. The projection $\mathsf{i}(A)$ on the "second coordinate" axis

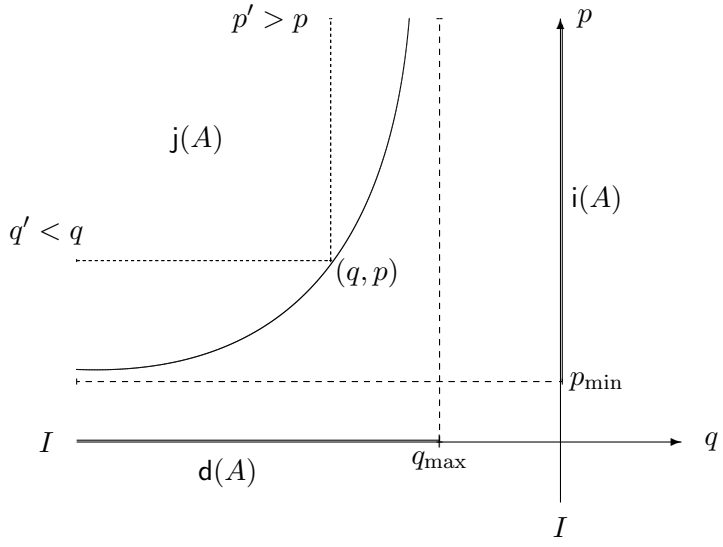


Figure 2: Characterization of the operator A, in the case of a chain.

plays, in a sense, the role of the range of A. More precisely,

$$\mathsf{d}(A) = \{q \in I : \text{there is a } p \text{ such that } A_{pq} \text{ exists}\},$$
$$\mathsf{i}(A) = \{p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\}.,$$

The following properties are immediate:

- $\mathsf{d}(A)$ is an initial subset of I: if $q \in \mathsf{d}(A)$ and $q' < q$, then $q' \in \mathsf{d}(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator (this is what we mean by a 'coherent' collection).
- $\mathsf{i}(A)$ is a final subset of I: if $p \in \mathsf{i}(A)$ and $p' > p$, then $p' \in \mathsf{i}(A)$ and $A_{p'q} = E_{p'p}A_{pq}$.
- $\mathsf{j}(A) \subset \mathsf{d}(A) \times \mathsf{i}(A)$, with strict inclusion in general.

In order to get a feeling about this machinery of representatives of operators, we present in Figure 2 the case of a chain, i.e., $I \sim \mathbb{R}$ (or $\mathbb{Z}$). Given a pair $(q,p) \in \mathsf{j}(A)$, the quarter plane above and on the left of that point (q,p) contains all successors of (q,p) with respect to the partial order $\leqslant$ defined in (4.1). Thus $\mathsf{j}(A)$ is a union of such quarter planes. As for its projections, either $\mathsf{d}(A) = I \sim \mathbb{R}$ or the initial set $\mathsf{d}(A)$ has a maximal

element $q_{\max} = \vee_{q\in\mathsf{d}(A)}\, q$. Similarly, either $\mathsf{i}(A) = I \sim \mathbb{R}$ or the final set $\mathsf{i}(A)$ has a minimal element $p_{\min} = \wedge_{p\in\mathsf{i}(A)}\, p$. When both $q_{\max}$ and $p_{\min}$ exist (or are 'finite'), then $\mathsf{j}(A)$ is contained in the quarter plane delimited by $q \leqslant q_{\max}$ and $p \geqslant p_{\min}$. In this way one sees that the larger $\mathsf{j}(A)$, the 'better' the operator A (think, for instance, of the Sobolev spaces).

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. Of course, a similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs and the corresponding set $\mathrm{Op}(V_I, Y_K)$.

Since $V^\#$ is dense in V_r, for every $r \in I$, an operator may be identified with a separately continous sesquilinear form on $V^\# \times V^\#$. Indeed, the restriction of any representative A_{pq} to $V^\# \times V^\#$ is such a form, and all these restrictions coincide. Equivalently, an operator may be identified with a continuous linear map from $V^\#$ into V (continuity with respect to the respective Mackey topologies).

But the idea behind the notion of operator is to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint :* every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^\times \in \mathrm{Op}(V_I)$, defined by the relation

$$\langle A^\times x|y\rangle = \langle x|Ay\rangle, \text{ for } y \in V_r,\ r \in \mathsf{d}(A), \text{ and } x \in V_{\overline{s}},\ s \in \mathsf{i}(A),$$

that is, $(A^\times)_{\overline{r}\overline{s}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint).

It follows that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of Definition 5.1.

(ii) *Partial multiplication:* AB is defined if and only if there is a $q \in \mathsf{i}(B) \cap \mathsf{d}(A)$, that is, if and only if there is a continuous factorization through some V_q :

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{i.e.,} \quad (AB)_{sr} = A_{sq}B_{qr}\,.$$

It is worth noting that, for a LHS/LBS, the domain $\mathcal{D}(A)$ is always a vector subspace of V (this is not true for a general PIP-space). Therefore, $\mathrm{Op}(V_I)$ is a vector space and a *partial *-algebra* [6].

The concept of PIP-space operator is very simple, yet it is a far reaching generalization of bounded operators. It allows indeed to treat on the same footing all kinds of operators, from bounded ones to very singular ones. By this, we mean the following, loosely speaking. Take

$$V_r \subset V_o \simeq V_{\overline{o}} \subset V_s \quad (V_o = \text{Hilbert space}).$$

Three cases may arise:

. if A_{oo} exists, then A corresponds to a bounded operator $V_o \to V_o$;

. if A_{oo} does not exist, but only $A_{or} : V_r \to V_o$, with $r < o$, then A corresponds to an unbounded operator, with domain $D(A) \supset V_r$;

. if no A_{or} exists, but only $A_{sr} : V_r \to V_s$, with $r < o < s$, then A corresponds to a singular operator, with Hilbert space domain possibly reduced to $\{0\}$.

5.2 Special Classes of Operators on PIP-spaces

Exactly as for Hilbert or Banach spaces, one may define various types of operators between PIP-spaces, in particular LBS/LHS. We discuss briefly the most important classes, namely, regular operators, homomorphisms and isomorphisms, unitary operators, symmetric operators, and orthogonal projections. Further details may be found in the monograph [8].

5.2.1 Regular and totally regular operators

An operator A on a nondegenerate, positive definite PIP-space, resp. a LBS/LHS V_I, is called *regular* if $\mathsf{d}(A) = \mathsf{i}(A) = I$ or, equivalently, if $A : V^{\#} \to V^{\#}$ and $A : V \to V$ continuously for the respective Mackey topologies. This notion depends only on the pair $(V^{\#}, V)$, not on the particular compatibility $\#$. The set of all regular operators $V_I \to V_I$ is denoted by $\mathrm{Reg}(V_I)$. Thus a regular operator may be multiplied both on the left and on the right by an arbitrary operator. Clearly, the set $\mathrm{Reg}(V_I)$ is a *-algebra and can often be identified with an O*-algebra [6, 23].

We give two examples.

(1) If $V = \omega, V^{\#} = \varphi$, then $\mathrm{Op}(\omega)$ consists of arbitrary infinite matrices and $\mathrm{Reg}(\omega)$ of infinite matrices with finite rows and finite columns.

(2) If $V = \mathcal{S}^{\times}, V^{\#} = \mathcal{S}$, then $\mathrm{Op}(\mathcal{S}^{\times})$ consists of arbitrary tempered kernels, while $\mathrm{Reg}(\mathcal{S}^{\times})$ contains those kernels that can be extended to $\mathcal{S}^{\times}$ and map $\mathcal{S}$ into itself. A nice example is the Fourier transform.

An operator A is called *totally regular* if $\mathsf{j}(A)$ contains the diagonal of $I \times I$, i.e., A_{rr} exists for every $r \in I$ or A maps every V_r into itself continuously.

5.2.2 Homomorphisms

Let V_I, Y_K be two LHSs or LBSs. With an obvious modification of the definition of the set $\mathsf{j}(A)$, given in Section 5.1, an operator $A \in \mathrm{Op}(V_I, Y_K)$

is called a *homomorphism* if

(i) $\mathsf{j}(A) = I \times K$ and $\mathsf{j}(A^\times) = K \times I$;

(ii) $f \#_I g$ implies $Af \#_K Ag$.

We denote by $\mathrm{Hom}(V_I, Y_K)$ the set of all homomorphisms from V_I into Y_K. The following properties are easy to prove:

(i) $A \in \mathrm{Hom}(V_I, Y_K)$ if and only if $A^\times \in \mathrm{Hom}(Y_K, V_I)$.

(ii) If $A \in \mathrm{Hom}(V_I, Y_K)$, then $\mathsf{j}(A^\times A)$ contains the diagonal of $I \times I$ and $\mathsf{j}(AA^\times)$ contains the diagonal of $K \times K$.

The homomorphism $A \in \mathrm{Hom}(V_I, Y_K)$ is an *isomorphism* if there exists a homomorphism $B \in \mathrm{Hom}(Y_K, V_I)$ such that $BA = 1\!\!1_V, AB = 1\!\!1_Y$, where $1\!\!1_V, 1\!\!1_Y$ denote the identity operators on V_I, Y_K, respectively.

For instance, Fourier transform is an isomorphism of the Schwartz RHS $\mathcal{S} \subset L^2 \subset \mathcal{S}^\times$ onto itself and, similarly, of the Feichtinger triplet (4.4) onto itself. In both cases, the property extends to several scales of lattices interpolating between the two extreme spaces, for instance, the Hilbert chain of the Hermite representation of tempered distributions.

5.2.3 Unitary operators and group representations

The operator $U \in \mathrm{Op}(V_I, Y_K)$ is *unitary* if $U^\times U$ and $UU^\times$ are defined and $U^\times U = 1\!\!1_V, UU^\times = 1\!\!1_Y$. We emphasize that unitary operators need *not* be homomorphisms! In fact, unitarity is a rather weak property and it is insufficient for group representations..

Thus a unitary representation of group G into a PIP-space V_I is defined as a homomorphism of G into the *unitary isomorphisms* of V_I. Given such a unitary representation U of G into V_I, where the latter has the central Hilbert space $\mathcal{H}_0$, consider the representative $U_{00}(g)$ of $U(g)$ in $\mathcal{H}_0$ Then $g \mapsto U_{00}(g)$ is a unitary representation of G into $\mathcal{H}_0$, in the usual sense.

To give an example, let V_I be the scale built on the powers of the operator (Hamiltonian) $H = -\Delta + v(\mathbf{r})$ on $L^2(\mathbb{R}^3, d\mathbf{x})$, where Δ is the Laplacian on $\mathbb{R}^3$ and v is a (nice) rotation invariant potential. The system admits as symmetry group[2] $G = \mathrm{SO}(3)$ and the representation U_{00} is the natural representation of $\mathrm{SO}(3)$ in $L^2(\mathbb{R}^3)$:

$$[U_{00}(\rho)\psi](\mathbf{x}) = \psi\left(\rho^{-1}\mathbf{x}\right), \ \rho \in \mathrm{SO}(3).$$

[2]The full symmetry group might be larger; for instance, the Coulomb potential admits SO(4) as symmetry group for its bound states.

Then U_{00} extends to a unitary representation U by totally regular isomorphisms of V_I and angular momentum decompositions, corresponding to irreducible representations of SO(3), extend to V_I as well. In addition, this is a good setting also for representations of the Lie algebra $\mathfrak{so}(3)$. Notice that the representation U is totally regular, but this need not be the case. For instance, if the potential v is not rotation invariant, U will no longer be totally regular, although it is still an isomorphism.

5.2.4 Symmetric operators

An operator $A \in \mathrm{Op}(V_I)$ is *symmetric* if $A^\times = A$. Since one has $A^{\times\times} = A$, no extension is allowed, by the maximality condition. Thus symmetric operators behave essentially like bounded self-adjoint operators in a Hilbert space. Yet, they can be very singular, as indicated above, for a chain

$$V^\# \subset \ldots \subset V_r \subset V_o = V_{\overline{o}} \subset V_s \subset \ldots \subset V.$$

In this case, the question is whether a symmetric operator $A \in \mathrm{Op}(V_I)$ has a self-adjoint restriction to the central Hilbert space V_o. In a Hilbert space context, an answer is given by the celebrated KLMN theorem.[3] Actually, this classical result already has a distinct PIP-space flavor. Thus is not surprising that the KLMN theorem has a natural generalization to a LHS or a positive definite PIP-space with central Hilbert space $V_o = V_{\overline{o}}$, and a quadratic form version as well.

An interesting application is a correct description of very singular Hamiltonians in quantum mechanics, typically with δ or δ' interactions. For instance, one can treat in this way the Kronig-Penney crystal model, which consists of a δ potential at each node of a lattice, in one, two or three dimensions [19].

5.2.5 Orthogonal projections, bases, frames

An operator P on a nondegenerate PIP-space V, resp. a LBS/LHS V_I, is an *orthogonal projection* if $P \in \mathrm{Hom}(V_I)$ and $P^2 = P = P^\times$. It follows immediately from the definition that an orthogonal projection is totally regular, i.e., $\mathrm{j}(P)$ contains the diagonal $I \times I$, or still that P leaves every assaying subspace invariant. Equivalently, P is an orthogonal projection if P is an idempotent operator (that is, $P^2 = P$) such that $\{Pf\}^\# \supseteq \{f\}^\#$ for every $f \in V$ and $\langle g|Pf\rangle = \langle Pg|f\rangle$ whenever $f \# g$. We denote by $\mathrm{Proj}(V)$ the set of all orthogonal projections in V and similarly for $\mathrm{Proj}(V_I)$.

[3] KLMN stands for Kato, Lax, Lions, Milgram, Nelson.

These projection operators enjoy several properties similar to those of Hilbert space projectors. Two of them are of special interest in the present context.

(i) Given a nondegenerate PIP-space V, there is a natural notion of subspace, called *orthocomplemented*, which guarantees that such a subspace W of V is again a nondegenerate PIP-space with the induced compatibility relation and the restriction of the partial inner product. There are equivalent topological conditions, so that orthocomplemented subspaces yield the proper notion of PIP-subspace [4]. Then the basic theorem about projections states that a subspace W of V is orthocomplemented if and only if W is the range of an orthogonal projection $P \in \mathrm{Proj}(V)$, i.e., $W = PV$. Then $V = W \oplus Z$, where Z is another orthocomplemented subspace.

(ii) An orthogonal projection P is of finite rank if and only if $W = \mathrm{Ran}\, P \subset V^{\#}$ and $W \cap W^{\perp} = \{0\}$ (this condition is, of course, superfluous when the partial inner product is positive definite).

There is a natural partial order on the set of projections:

$$P_W \leqslant P_Y \text{ if and only if } W \subseteq Y,$$

but lattice properties are unknown.

Property (ii) has important consequences for the structure of bases. First we recall that a sequence $\{e_n, n = 1, 2, \ldots\}$ of vectors in a Banach space E is a *Schauder basis* if, for every $f \in E$, there exists a unique sequence of scalar coefficients $\{c_k, k = 1, 2, \ldots\}$ such that

$$\lim_{m\to\infty} \|f - \sum_{k=1}^{m} c_k e_k\| = 0.$$

Then one may write

$$f = \sum_{k=1}^{\infty} c_k e_k. \tag{5.1}$$

The basis is *unconditional* if the series (5.1) converges unconditionally in E (i.e., it keeps converging after an arbitrary permutation of its terms).

A standard problem is to find, for instance, a sequence of functions that is an unconditional basis for *all* the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. In the PIP-space language, this statement means that the basis vectors must belong to $V^{\#} = \cap_{1<p<\infty} L^p(\mathbb{R}) = L_c^{\infty}(\mathbb{R})$. Also, since (5.1) means that f may be approximated by finite sums, the property (ii) of orthogonal projections implies that all the basis vectors must belong to $V^{\#}$. A typical example is the well-known Haar basis.

Actually, in the context of signal processing, orthogonal (in the Hilbert sense) bases are not enough, one needs also biorthogonal bases and, more generally, *frames*. We recall that a countable family of vectors $\{\psi_n\}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there are two positive constants m,M, with $0 < \mathsf{m} \leqslant \mathsf{M} < \infty$, such that

$$\mathsf{m}\,\|f\|^2 \;\leqslant\; \sum_{n=1}^{\infty} |\langle\psi_n|f\rangle|^2 \;\leqslant\; \mathsf{M}\,\|f\|^2,\ \forall\, f \in \mathcal{H}. \tag{5.2}$$

The two constants m, M are called *frame bounds*. If $\mathsf{m} = \mathsf{M}$, the frame is said to be *tight*. Consider the analysis operator $C : \mathcal{H} \to \ell^2$ defined by $C : f \mapsto \{\langle\psi_n|f\rangle\}$ and the frame operator $S = C^*C$. Then the vectors $\widetilde{\psi}_n = S^{-1}\psi_n$ also constitute a frame, called the *dual frame*, and one has the (strongly converging) expansions

$$f = \sum_{n=1}^{\infty} \langle\psi_n|f\rangle\widetilde{\psi}_n = \sum_{n=1}^{\infty} \langle\widetilde{\psi}_n|f\rangle\,\psi_n.$$

Then the considerations made above for bases should apply to frame vectors as well, i.e., the vectors $\psi_n, \widetilde{\psi}_n$ should also belong to $V^{\#}$.

6 Application of PIP-spaces

6.1 Mathematical physics

PIP-spaces have found many applications in mathematical physics. Most of what follows is described in detail in [8].

(1) Rigorous formulations of Dirac formalism in quantum mechanics: as it is well-known, the Dirac formalism of quantum mechanics cannot be given a rigorous formulation in a pure Hilbert space context. This had prompted several authors [2, 10, 26] to propose a rigorous version in terms of a RHS (generated from the observables of the system at hand), combined with the generalized spectral theorem of Gel'fand-Maurin [15, 24]. However, the construction is such that a PIP-space version is easily obtained.

(2) Symmetries, singular interactions in quantum mechanics: several other topics in quantum mechanics can be advantageously formulated in a RHS or PIP-space language. For instance, the implementation of symmetries, with the two dual points of view, the active one and the passive one [7]. Another problem is a correct definition of a Hamiltonian with a singular interaction,

already mentioned in Sec. 5.2.4 [19]. Here the PIP-space approach is a convenient substitute to the usual formulation in terms of von Neumann's theory of self-adjoint extensions of symmetric operators [1].

(3) Quantum scattering theory: the Weinberg-van Winter (WVW) formulation of scattering theory [29] has a very natural interpretation in terms of a LHS, whose components, including the extreme ones, are Hilbert spaces consisting of functions analytic in a sector; thus the indexing parameter is the opening angle of that sector. With this technique it has been possible to show that the WVW formalism is a particular case of the Complex Scaling Method [8, Sec. 7.2.3], a result hitherto unknown.

(4) Quantum field theory: various rigorous formulations of QFT rely heavily on a RHS or a PIP-space approach, for instance, Wightman's axiomatic formulation, the construction via the Borchers algebra, Nelson's Euclidean field theory, or the precise treatment of unsmeared fields (fields at a point). See [8, Sec.7.3] for a detailed presentation.

(5) Representations of Lie groups and Lie algebras: the proper framework for this topic is Nelson's theory of analytic vectors. Let G be a Lie group and U_{00} a unitary representation of G in a Hilbert space $\mathcal{H}_0$. Let $\overline{\Delta}$ be the closure of the Nelson operator $\Delta := \sum_{j=1}^{n} X_j^2$, where $\{X_j, j = 1, \dots, n\}$ are the representatives under U_{00} of the elements of a basis of the Lie algebra $\mathfrak{g}$ of G. $\overline{\Delta}$ is self-adjoint and $\overline{\Delta} \geqslant 0$. Define $V_I := \{\mathcal{H}_n, n \in \mathbb{Z}\}$ as the canonical scale of Hilbert spaces generated by the operator $\overline{\Delta} + 1$. Then U_{00} extends to a unitary representation U in the LHS V_I, in the sense of Section 5.2.3. In addition, both the Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ have nice representations in V_I. See [8, Sec.7.] for further details.

6.2 Signal Processing

Most families of function spaces used in Gabor/wavelet analysis come in scales or lattices and, in fact, are or contain PIP-spaces. The (lattice) indices defining the (partial) order characterize the properties of the corresponding functions or distributions: smoothness, local integrability, decay at infinity, etc. Such are, for instance, amalgam spaces, modulation spaces, Besov spaces or coorbit spaces, mentioned above. Thus it seems natural to formulate the properties of various operators globally, using the theory of PIP-space operators, in particular the set $\mathfrak{j}(A)$ of an operator encodes its properties in a very convenient and visual fashion. In addition, it is often possible to determine uniquely whether a function belongs to one of

those spaces simply by estimating the (asymptotic) behavior of its Gabor or wavelet coefficients, a real breaktrough in functional analysis [25].

We may therefore expect that the PIP-space formalism will play a significant role in Gabor/wavelet analysis, as well as in mathematical physics.

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Quasi *-algebras arising from extensions of positive linear functionals

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Abstract. In this paper we show how some known quasi *-algebras can also be obtained through the construction of slight extensions of nonclosable positive linear functionals defined on dense *-subalgebras $\mathfrak{A}_0$ of given topological *-algebras. Moreover, we consider also, for each of the functionals we present, their absolutely convergent extensions and the GNS *-representations of the quasi *-algebras arising in the process of extension.

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1 Introduction

In [3] two techniques to extend nonclosable positive linear functionals have been shown. Under suitable hypotheses, one of them permitted to construct an absolutely convergent extension, the other one permitted to construct slight extensions. In most cases, the domain of the extension (constructed by means of a positive sesquilinear form associated to the functional) turns out to be a quasi *-algebra. In this paper, we want to show, in full detail, that known examples of quasi *-algebras can arise in the process of extension of nonclosable positive linear functionals. The first quasi *-algebra we take into consideration is $(L^2(\tau), \mathcal{J}_1)$, where $L^2(\tau)$ is the noncommutative L^2-space constructed on a von Neumann algebra $\mathfrak{M}$, starting from a normal faithful semifinite trace τ. The second is $(L^2(X), C(X))$ with $X = [0, 1]$.

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It comes out by considering the extension of the Riemann integral on the space of the continuous functions. The last case we examine is that of $(\ell_2, \mathfrak{A}_0)$ (with ℓ_2 the space of square-summable sequences and $\mathfrak{A}_0$ the *-algebra of the finite sequences [1]), arising from the extension to infinite sum of the sum of the elements of finite sequences.

Before going forward, we want to overview the main definitions and results.

A *quasi*-algebra* is a couple $(\mathfrak{A}, \mathfrak{A}_0)$, where $\mathfrak{A}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{A}$ and $\mathfrak{A}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$ [1, 8].

Throughout this paper we will deal with a topological *-algebra $\mathfrak{A}[\tau]$ and a dense *-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$, where a positive linear functional ω is defined.
Such a functional ω is called

- *closable* if
$$x_\alpha \xrightarrow{\tau} 0 \text{ and } \omega(x_\alpha) \to \ell, \text{ then } \ell = 0.$$
- *admissible* if, for every $b \in \mathfrak{A}_0$, there exists $\gamma_b > 0$ such that
$$|\omega(a^*ba)| \leq \gamma_b \omega(a^*a), \qquad \forall a \in \mathfrak{A}_0.$$
- *trace on* $\mathfrak{A}_0$ if, $\omega(ab) = \omega(ba)$ for every $a, b \in \mathfrak{A}_0$.

Let us define

$$G_\omega = \{(x, \omega(x)) \in \mathfrak{A}_0 \times \mathbb{C};\ x \in \mathfrak{A}_0\}.$$

If ω is nonclosable, $\overline{G_\omega}$ contains couples $(0, \ell)$ with $\ell \neq 0$. To every H, subspace of $\mathfrak{A} \times \mathbb{C}$, such that

(g1) $G_\omega \subseteq H \subset \overline{G_\omega}$;

(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

there corresponds a so-called *slight* extension ω_H of ω, defined on

$$D(\omega_H) = \{x \in \mathfrak{A} : (x, \ell) \in H\}$$

[1] The space of all sequences and the *-algebra of finite sequences are usually indicated by ω and φ respectively, according to Köthe [6]. However, here this notation would be confusing.

by

$$\omega_H(x) = \ell,$$

where ℓ is the unique complex number such that $(x, \ell) \in H$, see [4].

Remark 1.1 If ω is nonclosable, then ω admits infinitely many (maximal) slight extensions (see Proposition 2.6 in [4]).

The set $\mathfrak{A}^+ = \left\{\sum_{j=1}^n a_j^* a_j;\ a_j \in \mathfrak{A}\right\}$ is a convex cone in $\mathfrak{A}$, whose elements may be regarded as the *positive* elements of $\mathfrak{A}$. Let $\mathfrak{A}_h = \{x \in \mathfrak{A} : x = x^*\}$ denote the real vector space of all hermitian elements of $\mathfrak{A}$. If $x, y \in \mathfrak{A}_h$, we write $x \leq y$ if $y - x \in \mathfrak{A}^+$. We say that $\mathfrak{A}$ has the property (D) if, for every $x \in \mathfrak{A}_h$, there exists a unique pair $(x_+, x_-) \in \mathfrak{A}^+$ such that $x_+ x_- = x_- x_+ = 0$ and

(D1) $x = x_+ - x_-$;

(D2) $(x+y)_+ \leq x_+ + y_+, \quad \forall x, y \in \mathfrak{A}_h$ with $xy = yx$;

(D3) $(\lambda x)_+ = \lambda x_+, \quad \forall x \in \mathfrak{A}_h, \lambda \in \mathbb{R}^+$;

(D4) if $x \in \mathfrak{A}_h \cap \mathfrak{A}_0$, then $x_+ \in \mathfrak{A}_0^+$.

Definition 1.2 Let $\mathfrak{A}$ satisfy condition (D). A positive slight extension $\widetilde{\omega}$ of ω is called *absolutely convergent* if

$$x = x^* \in D(\widetilde{\omega}) \Leftrightarrow x_+, x_- \in D(\widetilde{\omega}).$$

Definition 1.3 A slight extension $\tilde{\omega}$ of a positive linear functional ω is said to be *positively regular* if

$$\tilde{\omega}(a) = \sup\{\omega(b) :\ 0 \leq b \leq a;\ b \in \mathfrak{A}_0\}, \quad \forall a \in D(\tilde{\omega}) \cap \mathfrak{A}^+.$$

Let us recall the technique to construct a slight extension of a nonclosable positive linear functional by means of a sesquilinear form associated to it, discussed in [3]. Even if ω is nonclosable, it may happen that the associated sesquilinear form φ_ω on $\mathfrak{A}_0 \times \mathfrak{A}_0$ defined by

$$\varphi_\omega(a, b) = \omega(b^* a), \qquad a, b \in \mathfrak{A}_0$$

is closable. For sesquilinear forms, closability can be formulated as follows:

$$a_n \xrightarrow{\tau} 0 \text{ and } \varphi_\omega(a_n - a_m, a_n - a_m) \to 0 \text{ as } n, m \to \infty \Rightarrow \varphi_\omega(a_n, a_n) \to 0.$$

If the topology of $\mathfrak{A}$ is nonmetrizable, one should consider nets, instead of sequences, in order to define the closability of φ_ω. However, this choice

would produce a much stronger (and unnecessary) condition than the one used here.
If φ_ω is closable, one can put

$$D(\overline{\varphi_\omega}) = \{a \in \mathfrak{A} \ : \ \exists a_n \xrightarrow{\tau} a,\ a_n \in \mathfrak{A}_0 \text{ and } \varphi_\omega(a_n - a_m, a_n - a_m) \to 0 \text{ as } n, m \to \infty\}$$

and

$$\overline{\varphi_\omega}(a, a) = \lim_{n\to\infty} \varphi_\omega(a_n, a_n)$$

The closability condition implies that $\overline{\varphi_\omega}$ is well defined. Since $D(\overline{\varphi_\omega})$ is a vector space, $\overline{\varphi_\omega}$ can be extended to $D(\overline{\varphi_\omega}) \times D(\overline{\varphi_\omega})$ by polarization; i.e.,

$$\overline{\varphi_\omega}(a, b) = \frac{1}{4} \sum_{k=0}^{3} i^k \overline{\varphi_\omega}(a + i^k b, a + i^k b), \quad a, b \in D(\overline{\varphi_\omega}).$$

The form $\overline{\varphi_\omega}$ is closed in $D(\overline{\varphi_\omega})$.

Once $\overline{\varphi_\omega}$ has been defined, if $\mathfrak{A}_0$ has a unit e we can extend ω to the whole $D(\overline{\varphi_\omega})$ as follows

$$\tilde{\omega}(x) = \overline{\varphi_\omega}(x, e), \quad x \in D(\overline{\varphi_\omega}). \tag{1.1}$$

By the definition itself, $\tilde{\omega}$ is a *slight* extension of ω.

Remark 1.4 Proposition 4.4. in [3] guarantees that if ω is an admissible trace on $\mathfrak{A}_0$, then $D(\overline{\varphi_\omega})$ is a quasi *-algebra over $\mathfrak{A}_0$.

Now let us recall the main definitions about *-representations of quasi *-algebras [1]. Let $\mathcal{D}$ be a dense subspace of the Hilbert space $\mathcal{H}$, we put

$$\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{X \text{ closable linear operator in } \mathcal{H} : \mathcal{D}(X) = \mathcal{D} \text{ and } \mathcal{D}(X^*) \supset \mathcal{D}\}.$$

The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra w. r. to the usual operations of addition, multiplication by scalar, the involution $X \mapsto X^\dagger = X^* \restriction \mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\square\, X_2 = {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, iff $X_2\mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ and $X_1^\dagger \mathcal{D} \subset \mathcal{D}(X_2^*)$.

Definition 1.5 A *-representation of a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-homomorphism π of $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$, for some pair $(\mathcal{D}_\pi, \mathcal{H}_\pi)$, where $\mathcal{D}_\pi$ is a dense subspace of the Hilbert space $\mathcal{H}_\pi$, i.e. a linear map $\pi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ such that:

i) $\pi(x^*) = \pi(x)^\dagger, \quad \forall x \in \mathfrak{A}$,

ii) if $x \in \mathfrak{A}$, $a \in \mathfrak{A}_0$, then $\pi(x)$ is a weak left multiplier of $\pi(a)$ and $\pi(x) \square \pi(a) = \pi(xa)$.

If $(\mathfrak{A}, \mathfrak{A}_0)$ has unit e, we assume $\pi(e) = \mathbb{I}$, the identity operator of $\mathcal{D}_\pi$.

The functional ω defined on a quasi *-algebra $\mathfrak{A}$, with distinguished $\mathfrak{A}_0$, is said *representable* in the sense of [5] if

(L.1) $\omega(a^*a) \geq 0, \quad \forall a \in \mathfrak{A}_0$;

(L.2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}, \quad \forall a, b \in \mathfrak{A}_0, \forall x \in \mathfrak{A}$;

(L.3) $\forall x \in \mathfrak{A}$, there exists $\gamma_x > 0$ such that

$$|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}, \quad \forall a \in \mathfrak{A}_0.$$

Indeed, as proved in Theorem 3.5 [5], conditions (L.1), (L.2) and (L.3) allow a generalized GNS construction for the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$.

2 Extensions of traces on von Neumann algebras

In this section, we show that the quasi *-algebra $L^2(\tau)$ over $\mathcal{J}_1$ (where $L^2(\tau)$ is the space of all measurable operators A such that $\mu(A^*A) < \infty$ with μ defined in (2.1) below and $\mathcal{J}_1$ the *-ideal of the elements of the von Neumann algebra whose modulus has semifinite trace), is the domain of an extension of a normal faithful finite trace τ, extension obtained by means of a technique we introduced in [3, Section 4]. We also construct the absolutely convergent extension of τ, and another extension of it by means of an increasing net of central projections converging to the identity operator (in alternative to a construction given by Bongiorno et al. in [4], by means of an approximate identity) and, in the end, we show as the *-representation of $(L^2(\tau), \mathcal{J}_1)$ defined by τ works.

Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and let τ be a normal faithful semifinite trace defined on $\mathfrak{M}_+$.

We recall for the reader's convenience the definition of the topology of convergence *in measure* on $\mathfrak{M}$. For $\epsilon, \delta > 0$, let $N(\epsilon, \delta) = \{A \in \mathfrak{M} : \text{for some projection } P \in \mathfrak{M}, \|AP\| \leq \varepsilon \text{ and } \tau(P^{\perp}) \leq \delta\}$. We endow $\mathfrak{M}$ with the translation-invariant topology $\mathtt{t}$ generated by the system $\{N(\epsilon, \delta); \epsilon, \delta > 0\}$ of neighborhoods of 0.

If we complete $\mathfrak{M}$ w.r. to $\mathtt{t}$ we obtain $\mathfrak{A} = \widetilde{\mathfrak{M}}$ the *-algebra of measurable operators. It should be noticed that $\widetilde{\mathfrak{M}}$ consists of operators affiliated with $\mathfrak{M}$. From this fact it follows that, if $X \in \mathfrak{A}_h$, then the operators X_+, X_- and $|X| = (X^*X)^{1/2}$ are in $\mathfrak{A}$.
We recall that the mappings

(i) $A \mapsto A^*$ of $\mathfrak{M} \to \mathfrak{M}$;

(ii) $(A, B) \mapsto A + B$ of $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$;

(iii) $(A, B) \mapsto AB$ of $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$,

have unique continuous extensions as mappings of $\mathfrak{A} \to \mathfrak{A}$ and $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. The mappings (i), (ii) and their extensions, are uniformly continuous. The map (iii) is uniformly continuous on products of sets which are bounded in measure [9].

Let

$$\mathcal{J}_1 = \{X \in \mathfrak{M} : \tau(|X|) < \infty\}.$$

Then, $\mathcal{J}_1$ is a *-ideal of $\mathfrak{M}$. Following Nelson [7], we denote with $L^1(\tau)$ the Banach space completion of $\mathcal{J}_1$ with respect to the norm

$$\|X\|_1 := \tau(|X|), \quad X \in \mathcal{J}_1.$$

We can extend τ to $L^1(\tau)$ and obtain its unique absolutely convergent extension.

To do this, let us consider $\mu : \mathfrak{A}^+ \to [0, +\infty]$ defined by

$$\mu(A) = \sup\{\tau(X);\ 0 \leq X \leq A,\ X \in \mathcal{J}_1\}. \tag{2.1}$$

Then μ verifies properties (AP_1) and (AP_2) in [4], hence, by Proposition 2.21 in [4], μ is a positively regular absolutely convergent extension of τ. Put $D(\mu) = \{A \in \mathfrak{A} : \mu(|A|) < \infty\}$. Then, using the notation of [3], the "integral" of an element $A \in L^1(\tau) = D(\mu)$ is defined, $\mu = \dot{\tau}$ with $\dot{\tau} : L^1(\tau) \to [0, +\infty]$ and $\dot{\tau}(A) = \dot{\tau}(B) - i\dot{\tau}(C) = \dot{\tau}(B_+) - \dot{\tau}(B_-) + i[\dot{\tau}(C_+) - \dot{\tau}(C_-)]$, taking into account that any $A \in \mathfrak{A}$ can be decomposed as $A = B_+ - B_- + iC_+ - iC_-$, where $B = \frac{A+A^*}{2}$ and

$C = \frac{A-A^*}{2i}$.

Now, we want to extend further the trace τ to a larger set of measurable operators. Let us suppose that $\mathfrak{M}$ is *finite* (i.e. $\tau(\mathbb{I}) < \infty \Leftrightarrow \mathcal{J}_1 = \mathfrak{M}$). In [4] Bongiorno et al. constructed a slight extension of τ by means of an approximate identity of $\mathcal{J}_1$. Here we want to construct a slight extension by using an increasing net of central projections converging to the identity operator.

Let $\{P_\alpha\}$ with $P_\alpha = P_\alpha^2 = P_\alpha^*$, $P_\alpha \in \mathfrak{M} \cap \mathfrak{M}'$, with $\mathfrak{M}'$ the commutant of $\mathfrak{M}$, such that $P_\alpha \uparrow \mathbb{I}$.

Recall that the following property holds (see [7]):

$$X \in L^1(\tau),\ A \in \mathfrak{M} \Rightarrow\ AX \in L^1(\tau) \text{ and } |\tau(AX)| \leq \|A\|\tau(|X|). \qquad (2.2)$$

Put

$$D(\mathfrak{M}, \{P_\alpha\}) = \{X \in \mathfrak{A} : P_\alpha X \in L^1(\tau) \text{ and } \exists \lim_\alpha |\tau(P_\alpha X)| < \infty\},$$

depending on the net $\{P_\alpha\}$ and

$$\hat{\tau}(X) = \lim_\alpha \tau(P_\alpha X).$$

The extension $\hat{\tau}$ is slight and $L^1(\tau) \subseteq D(\mathfrak{M}, \{P_\alpha\})$.

Remark 2.1 In general the inclusion of $L^1(\tau)$ in $D(\mathfrak{M}, \{P_\alpha\})$ is proper. By (2.2) it follows that $L^1(\tau) \subseteq D(\mathfrak{M}, \{P_\alpha\})$. Let us consider the interval $I = [0, 1]$, $f \in L^\infty(I)$ and the operator of multiplication by f

$$M_f : \varphi \in L^2(I) \to f\varphi \in L^2(I).$$

The set $\mathfrak{M} = \{M_f : f \in L^\infty(I)\}$ is a von Neumann algebra. The functional $\tau(M_f) = \int_0^1 f(x)dx$, with $f \geq 0$, is a normal faithful finite trace on $\mathfrak{M}$. Let us consider the intervals $]\alpha, 1]$ with $\alpha > 0$ and the operator P_α of multiplication by the characteristic functions $\chi_{]\alpha,1]}$. The function $f_0(x) = \frac{1}{x}\sin\frac{1}{x^3}$ is not Lebesgue integrable, hence $M_{f_0} \notin L^1(\tau)$, however $\tau(P_\alpha M_{f_0}) = \int_0^1 \chi_{]\alpha,1]}(x) f_0(x)dx = \int_\alpha^1 \frac{1}{x}\sin\frac{1}{x^3}dx = \int_{\alpha^{-1}}^1 t(-\frac{1}{t^2})\sin t dt < \infty$, and, as $\alpha \to 0$, we obtain the well-known converging generalized integral $\int_1^\infty \frac{\sin t}{t}dt$. Thus, $M_{f_0} \in D(\mathfrak{M}, \{P_\alpha\})$.

Remark 2.2 We could endow $D(\mathfrak{M}, \{P_\alpha\})$ of a norm strictly weaker than that of $L^1(\tau)$:

$$\|X\|_{\tilde{1}} = \lim_\alpha |\tau(P_\alpha X)|.$$

If $X \in L^1(\tau)$ then, by (2.2), $|\tau(P_\alpha X)| \leq \|P_\alpha\| \|X\|_1$. Hence $\|X\|_{\tilde{1}} \leq \|X\|_1$. The example discussed in Remark 2.1 also shows that the two norms are not equivalent.

Now let us come back to case where τ is semifinite and consider the sesquilinear form φ_τ on $\mathcal{J}_1 \times \mathcal{J}_1$, associated to τ, defined by

$$\varphi_\tau(A, B) = \tau(B^* A), \qquad A, B \in \mathcal{J}_1. \tag{2.3}$$

Proposition 2.3 *The sesquilinear form φ_τ defined in (2.3) is closable.*

Proof – Let us consider a sequence $\{A_n\} \subseteq \mathcal{J}_1$ such that $A_n \overset{\mathtt{t}}{\to} 0$ and $\varphi_\tau(A_n - A_m, A_n - A_m) \to 0$ as $n, m \to \infty$. The former condition implies that $\|A_n\|_1 = \tau(|A_n|) \to 0$, the latter implies that

$$\varphi_\tau(A_n - A_m, A_n - A_m) = \tau(|A_n - A_m|^2) \to 0.$$

This last condition implies that $\{A_n\}$ is a Cauchy sequence in the Banach space $L^2(\tau)$ (the completion of $\{X \in \mathfrak{M} : \tau(|X|^2) < \infty\}$ in the norm defined by $\|X\|_2 := \tau(|X|^2)^{1/2}$) then there exists $B \in L^2(\tau)$ such that $A_n \overset{\|\cdot\|_2}{\to} B$ hence w. r. to $\|\cdot\|_1$; it follows $B = 0$ since $A_n \overset{\|\cdot\|_1}{\to} 0$.

$$\varphi_\tau(A_n, A_n) = \tau(A_n^* A_n) = \tau(|A_n|^2) = \|A_n\|_2^2 \to 0.$$

□

The domain of the closure of φ_τ is

$$D(\overline{\varphi_\tau}) = \{A \in \mathfrak{A} \;:\; \exists A_n \overset{\|\cdot\|_1}{\to} A,\ A_n \in \mathcal{J}_1 \text{ and } \|A_n - A_m\|_2^2 \to 0 \text{ as } n, m \to \infty\} = L^2(\tau).$$

Then we put

$$\overline{\varphi_\tau}(A, A) = \lim_{n\to\infty} \varphi_\tau(A_n, A_n) = \lim_{n\to\infty} \|A_n\|_2^2$$

The closability condition implies that $\overline{\varphi_\tau}$ is well defined, it can be extended to $L^2(\tau) \times L^2(\tau)$ by polarization and it is closed in $L^2(\tau)$.

By means of the technique defined in [3, Section 4], if $\mathcal{J}_1$ has the identity operator $\mathbb{I}$ (i.e. if $\mathcal{J}_1 = \mathfrak{M}$, or equivalently if τ is finite), we extend τ to $L^2(\tau)$ by putting

$$\tilde{\tau}(X) = \overline{\varphi_\tau}(X, \mathbb{I}), \quad X \in L^2(\tau), \tag{2.4}$$

and look for properties of $\tilde{\tau}$. By the definition itself, $\tilde{\tau}$ is a slight extension of τ.

The trace τ is admissible. Indeed, for every $B \in \mathcal{J}_1$, there exists $\gamma_B > 0$ such that

$$|\tau(A^*BA)| \leq \gamma_B \tau(A^*A), \qquad \forall A \in \mathcal{J}_1,$$

in fact, by (2.2)

$$A \in L^1(\tau),\ X \in \mathfrak{M} \Rightarrow\ XA \in L^1(\tau) \text{ and } |\tau(XA)| \leq \|X\|\tau((A^*A)^{1/2}).$$

In particular for $X = A^*B$, with $A, B \in \mathcal{J}_1$,

$$|\tau(A^*BA)| = |\tau(BAA^*)| \leq \|B\|\tau(AA^*) = \|B\|\tau(A^*A).$$

Since τ is an admissible trace, Proposition 4.4 in [3] forecasts that $(D(\overline{\varphi_\tau}), \mathcal{J}_1) = (L^2(\tau), \mathcal{J}_1)$ is a quasi-*algebra (as can be checked directly).

The functional $\tilde{\tau}$ enjoys the properties (L.1), (L.2) and (L.3). They are relevant, since they allow a generalized GNS construction for the quasi *-algebra $(L^2(\tau), \mathcal{J}_1)$.

As for the *-representation of $\tilde{\tau}$, the Hilbert space on which we construct it, is equal to $L^2(\tau)$, because of the fact that the subset $\{A \in L^2(\tau) : \tilde{\tau}(A^*A) = \tau(|A|^2) = 0\}$ is the trivial one.

Now put $\mathcal{D} = \mathcal{D}_{\tilde{\tau}} = \mathcal{J}_1$ and if $A \in L^2(\tau)$

$$\pi_{\tilde{\tau}}(A)B = AB,\ \ \forall B \in \mathcal{J}_1.$$

Thus, $\pi_{\tilde{\tau}}(A)$ is well-defined and maps $\mathcal{J}_1$ into $\mathcal{J}_1$, moreover $\pi_{\tilde{\tau}}(A) \in \mathcal{L}^\dagger(\mathcal{J}_1, L^2(\tau))$ and $\pi_{\tilde{\tau}}(A)^\dagger = \pi_{\tilde{\tau}}(A^*)$. Also, $\pi_{\tilde{\tau}}(A) \,\square\, \pi_{\tilde{\tau}}(B)$ is well-defined and $\pi_{\tilde{\tau}}(AB) = \pi_{\tilde{\tau}}(A) \,\square\, \pi_{\tilde{\tau}}(B)$, $\forall A \in L^2(\tau), B \in \mathcal{J}_1$. Thus, $\pi_{\tilde{\tau}}$ is a *-representation mapping $\mathcal{J}_1$ in $L^2(\tau)$, hence $\pi_{\tilde{\tau}}$ is a *bounded* *-representation.

3 Extensions of the Riemann integral

The possibility of extending the Riemann integral (a classical problem that, starting from Lebesgue, has allowed a series of different "new integrals") can also be cast in this general framework of extensions of positive linear functionals defined on a *-algebra. In this case, the starting point is the *-algebra $C(X)$ of continuous functions, with $X = [0, 1]$, which can be viewed

as a dense subspace of the *-algebra of Lebesgue measurable functions.

So, let $\mathfrak{A}$ be the *-algebra of Lebesgue measurable functions on X, τ the topology of convergence in measure, $C(X)$ the C*-algebra of all continuous functions on X and

$$\omega(f) = \int_0^1 f(x)dx. \tag{3.1}$$

The functional ω is nonclosable. To see this, it suffices to consider the sequence $\{f_n(x)\}$ with $f_n(x) = nx(1-x^2)^n$, $\forall n \in \mathbb{N}$, which tends pointwise to zero, while the sequence of integrals has a nonzero limit.

Thus ω admits infinitely many extensions; one of them is, for instance, the improper integral defined on the space of the locally integrable functions but there are many others. All of them are not absolutely convergent, with the relevant exception of the Lebesgue integral.

Now we follow the path outlined in Section 1 to construct the extension by the positive sesquilinear form φ_ω on $C(X) \times C(X)$, defined by

$$\varphi_\omega(f,g) = \int_0^1 f(x)\overline{g(x)}dx.$$

As shown in [3], φ_ω is closable.
We put

$$D(\overline{\varphi_\omega}) = \{f \in \mathfrak{A} \quad : \quad \exists f_n \xrightarrow{\tau} f,\ f_n \in C(X) \text{ and } \varphi_\omega(f_n - f_m, f_n - f_m) \to 0 \text{ as } n,m \to \infty\}$$

and

$$\overline{\varphi_\omega}(f,f) = \lim_{n\to\infty} \int_0^1 |f_n(x)|^2 dx.$$

Then we have the following

Proposition 3.1 *$D(\overline{\varphi_\omega})$ is equal to $L^2(X)$.*

Proof – We just sketch the proof we gave in [2]. If $f \in D(\overline{\varphi_\omega})$, then, by definition, there exists a sequence of continuous functions $\{f_n\}$, converging in measure to f, which is Cauchy in $L^2(X)$. Hence $\{f_n\}$ converges to some g w.r. to the L^2-norm, then there exists a subsequence $\{f_{n_k}\}$ converging a.e. to g. Since the convergence a.e. implies the convergence in measure, it follows that $f = g \in L^2(X)$.
Viceversa, if $f \in L^2(X)$, because of the density of $C(X)$ in $L^2(X)$, there

exists a sequence $\{f_n\} \subset C(X)$ converging to f in measure. Moreover, $\{f_n\}$ is Cauchy in $L^2(X)$, hence $f \in D(\overline{\varphi_\omega})$. □

Since $L^2(X)$ is a vector space, $\overline{\varphi_\omega}$ can be extended to $L^2(X) \times L^2(X)$ by polarization and through it we obtain the Lebesgue integral, the absolutely convergent extension of ω i.e.

$$\tilde{\omega}(f) = \overline{\varphi_\omega}(f, \mathbb{I}) = \int_0^1 f(x)dx, \quad f \in L^2(X).$$

Since $\tilde{\omega}$ is an admissible trace, the couple $(D(\overline{\varphi_\omega}), C(X))$ is a quasi *-algebra. In this process of extension, we obtained the announced quasi *-algebra with unit $(L^2(X), C(X))$.

The functional $\tilde{\omega}$ satisfies the conditions (L.1), (L.2) and (L.3), hence it is representable in the sense of [5].
Let us construct the generalized GNS construction for the quasi *-algebra $(L^2(X), C(X))$.
Since the subset $\{f \in L^2(X) | \, \tilde{\omega}(f^*f) = \int_0^1 |f(x)|^2 dx = 0\}$ is the null subspace, the Hilbert space on which we construct the *-representation of $\tilde{\omega}$ is equal to $L^2(X)$.
Put $\mathcal{D} = \mathcal{D}_{\tilde{\omega}} = C(X)$. If $f \in L^2(X)$, we put

$$\pi_{\tilde{\omega}}(f)g = fg \qquad \forall g \in C(X).$$

Then, $\pi_{\tilde{\omega}}(f)$ is well-defined and maps $C(X)$ into $L^2(X)$, moreover $\pi_{\tilde{\omega}}(f) \in \mathcal{L}^\dagger(C(X), L^2(X))$ and $\pi_{\tilde{\omega}}(f)^\dagger = \pi_{\tilde{\omega}}(f^*)$.

Also, $\pi_{\tilde{\omega}}(f) \,\square\, \pi_{\tilde{\omega}}(g)$ is well-defined and $\pi_{\tilde{\omega}}(fg) = \pi_{\tilde{\omega}}(f) \,\square\, \pi_{\tilde{\omega}}(g)$, $\forall f \in L^2(X), g \in C(X)$. Thus, $\pi_{\tilde{\omega}}$ is a *-representation of the quasi *-algebra $(L^2(X), C(X))$. By Theorem 3.5 in [5], there exists a closed cyclic *-representation $\tilde{\pi}_\omega$ of $(L^2(X), C(X))$, with cyclic vector $\mathbb{I}$, such that $\tilde{\omega}(f) = \langle \tilde{\pi}_\omega(f)\mathbb{I} | \mathbb{I} \rangle_2$, $\forall f \in L^2(X)$. This representation is unique up to unitary equivalence.

As we already observed in [2], the representation $\tilde{\pi}_\omega$ is not bounded. Were it so, every $\tilde{\pi}_\omega(f)$ should be bounded $\forall f \in L^2(X)$, hence we could extend $\tilde{\pi}_\omega$ from $C(X)$ to the whole $L^2(X)$. But this would imply that the product of two elements in $L^2(X)$ is in $L^2(X)$, which is known to be not true if, as in our case, X is an interval of the real line.

4 Sequence spaces and infinite sums

In this section we will show that the quasi *-algebra ℓ_2 over $\mathfrak{A}_0$, the space of the finite complex sequences, can be obtained by extending a nonclosable

functional defined on $\mathfrak{A}_0$. We show also its absolutely convergent extension and the bounded *-representation of the quasi *-algebra $(\ell_2, \mathfrak{A}_0)$, by means of infinite series on ℓ_2.

Let us consider the space $\mathfrak{A}$ of all complex sequences[2]:

$$\mathfrak{A} := \{(a_n) : a_n \in \mathbb{C}, \forall n \in \mathbb{N}\}.$$

Then, $\mathfrak{A}$ is a *-algebra with respect to the usual operations. Let τ be the locally convex Hausdorff topology on $\mathfrak{A}$ defined by means of the separating family of seminorms

$$p_F((a_n)) := \sup_{n\in F} |a_n|,$$

with $F \subset \mathbb{N}$ a finite subset.
Let us consider now the *-algebra $\mathfrak{A}_0$ of all finite sequences, i.e.:

$$\mathfrak{A}_0 := \{(a_n) : \exists N \in \mathbb{N} \ a_n = 0 \ \forall n > N\}.$$

Then, as it can be easily checked, $\mathfrak{A}_0$ is a τ-dense *-subalgebra of $\mathfrak{A}$. Let us define a positive linear functional ω on $\mathfrak{A}_0$ by

$$\omega((a_n)) = \sum_{n=1}^{+\infty} a_n, \qquad (a_n) \in \mathfrak{A}_0. \tag{4.1}$$

Then ω is an admissible trace on $\mathfrak{A}_0$.
In [3] we proved that the functional ω is nonclosable (it suffices to consider the sequence $(\mathbf{b}_k) \subseteq \mathfrak{A}_0$, with $\mathbf{b}_k = (b_{k,n})$ and $b_{k,n} = n\left(\frac{1}{k+1}\right)^{n-1}$ if $n \leq k$ and $b_{k,n} = 0$ if $n > k$, for $n \geq 1$), and that there exist infinitely many possible extensions of ω, which reduce to the ordinary sum for finite sequences. This fact corresponds to the old problem of defining "infinite sums". The absolutely convergent extension $\dot{\omega}$ of ω has domain coinciding with the sequence space ℓ_1 (see [3]) and is defined by

$$\dot{\omega}((a_n)) = \sum_{n=1}^{+\infty} a_n, \qquad (a_n) \in \ell_1.$$

Following the technique presented in [3, Section 4], we can further extend ω to a complex subspace of $\mathfrak{A}$, larger than $\mathfrak{A}_0$, by means of the positive sesquilinear form associated to ω:

$$\varphi_\omega(\mathbf{x}, \mathbf{y}) := \sum_{n=1}^{+\infty} \overline{y_n} x_n. \tag{4.2}$$

[2]See footnote at page 236.

for any two finite sequences $\mathbf{x}, \mathbf{y}$, with $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$.
The following proposition holds.

Proposition 4.1 *The sesquilinear form φ_ω defined in (4.2) is closable.*

Proof – We just outline the proof we gave in [2]. Let us consider a "sequence" of finite sequences $(\mathbf{x}_k)_{k\in\mathbb{N}}$, with $\mathbf{x}_k = (x_{n,k})_{n\in\mathbb{N}} \in \mathfrak{A}_0$ such that

i) $\mathbf{x}_k \xrightarrow{\tau} 0$

ii) $\varphi_\omega(\mathbf{x}_k - \mathbf{x}_h, \mathbf{x}_k - \mathbf{x}_h) := \sum_{n=1}^{+\infty} |x_{n,k} - x_{n,h}|^2 \to 0 \quad \text{as } k, h \to +\infty.$

Property *i)* implies that, $\forall n \in \mathbb{N}$, $\forall \epsilon > 0$, there exixts $\bar{k} \in \mathbb{N}$ such that $\forall k > \bar{k}$, $|x_{n,k}| < \epsilon$. From *ii)* it follows that $(\mathbf{x}_k)$ is Cauchy in ℓ_2, hence it converges in ℓ_2, then, in particular, $\forall n \in \mathbb{N}$, $\forall \epsilon > 0$ there exists $\bar{k}_2 \in \mathbb{N}$ such that $\forall k > \bar{k}_2$, $|x_{n,k} - x_n| < \epsilon$.
We can conclude that, $\forall n \in \mathbb{N}$, $\forall \epsilon > 0$ and $\forall k \geq \max\{\bar{k}, \bar{k}_2\}$

$$0 \leq |x_n| \leq |x_{n,k}| + |x_{n,k} - x_n| < 2\epsilon,$$

hence the sequence $\mathbf{x} = \{x_n\} \in \ell_2$ is such that $x_n = 0$, $\forall n \in \mathbb{N}$ and,

$$\varphi_\omega(\mathbf{x}_k, \mathbf{x}_k) := \sum_{n=1}^{+\infty} |x_{n,k}|^2 \to 0 \text{ as } k \to +\infty.$$

□

We put

$$\begin{aligned} D(\overline{\varphi_\omega}) \quad = \quad & \{\mathbf{x} = (x_n) \in \mathfrak{A} : \exists \mathbf{x}_k \xrightarrow{\tau} \mathbf{x},\ \mathbf{x}_k = (x_{n,k})_{n\in\mathbb{N}} \in \mathfrak{A}_0 \text{ and} \\ & \varphi_\omega(\mathbf{x}_k - \mathbf{x}_h, \mathbf{x}_k - \mathbf{x}_h) \to 0 \text{ as } k, h \to \infty\} \end{aligned}$$

and

$$\overline{\varphi_\omega}(\mathbf{x}, \mathbf{x}) = \lim_{k\to\infty} \sum_{n=1}^{+\infty} |x_{n,k}|^2.$$

The closability condition implies that $\overline{\varphi_\omega}$ is well defined. Since $D(\overline{\varphi_\omega})$ is a vector space, $\overline{\varphi_\omega}$ can be extended to $D(\overline{\varphi_\omega}) \times D(\overline{\varphi_\omega})$ by polarization. The form $\overline{\varphi_\omega}$ is closed in $D(\overline{\varphi_\omega})$.

By Proposition 4.4 in [3] we already know that $(D(\overline{\varphi_\omega}), \mathfrak{A}_0)$ is a quasi *-algebra. This, on the other hand, could be checked directly by the following proposition.

Proposition 4.2 *The set $D(\overline{\varphi_\omega})$ coincides with ℓ_2.*

Proof – We just give an idea of the proof we gave in [2]. Fix $\mathbf{x} = \{x_n\} \in D(\overline{\varphi_\omega})$, then there exists $(\mathbf{x}_k)_{k\in\mathbb{N}} \subset \mathfrak{A}_0$ with $\mathbf{x}_k = \{x_{n,k}\}_{n\in\mathbb{N}}$ such that

i) $\mathbf{x}_k \xrightarrow{\tau} \mathbf{x}$

ii) $\varphi_\omega(\mathbf{x}_k - \mathbf{x}_h, \mathbf{x}_k - \mathbf{x}_h) := \sum_{n=1}^{+\infty} |x_{n,k} - x_{n,h}|^2 \to 0 \quad \text{as } k, h \to +\infty.$

Then, from *i)* it follows that $\forall n \in \mathbb{N}$, $\forall \epsilon > 0$, there exixts $\bar{k} \in \mathbb{N}$ such that $\forall k > \bar{k}$, $|x_{n,k} - x_n| < \epsilon$ and from *ii)*, $(\mathbf{x}_k)$ is Cauchy in ℓ_2, then it converges in ℓ_2. Hence, it follows that, $\forall n \in \mathbb{N}$, $\forall \epsilon > 0$ there exists $\bar{k}_2 \in \mathbb{N}$ such that $\forall k > \bar{k}_2$, $|x_{n,k} - y_n| < \epsilon$.
We can conclude that, $\forall n \in \mathbb{N}$, $\forall \epsilon > 0$ and $\forall k \geq \max\{\bar{k}, \bar{k}_2\}$

$$0 \leq |x_n - y_n| \leq |x_{n,k} - x_n| + |x_{n,k} - y_n| < 2\epsilon,$$

hence $\mathbf{y} = \mathbf{x} \in \ell_2$.

To prove the viceversa, considered any $\mathbf{x} = (x_n) \in \ell_2$, we take the suitable sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ with $\mathbf{x}_k = (x_{n,k})_{n\in\mathbb{N}}$ with $x_{n,k} = x_n$, if $n \leq k$ and $x_{n,k} = 0$, if $n > k$; if we put $l_{h,k} = \min\{h, k\}$ and $L_{h,k} = \max\{h, k\}$, then $\forall \epsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that $\forall s \geq \bar{n}$

$$\varphi_\omega(\mathbf{x}_k - \mathbf{x}_h, \mathbf{x}_k - \mathbf{x}_h) = \sum_{n=1}^{+\infty} |x_{n,k} - x_{n,h}|^2 = \sum_{s=l_{h,k}+1}^{L_{h,k}} |x_s|^2 < \epsilon;$$

because of the fact that the sequence (x_n) is Cauchy in ℓ_2. □

We extend ω to $D(\overline{\varphi_\omega}) = \ell_2$ by putting

$$\tilde{\omega}(\mathbf{x}) = \overline{\varphi_\omega}(\mathbf{x}, e), \quad \mathbf{x} \in \ell_2. \tag{4.3}$$

By the definition itself $\tilde{\omega}$ is a slight extension of ω.

Since $\tilde{\omega}$ satisfies conditions (L.1),(L.2) and (L.3), it is representable. The subset $\{\mathbf{x} = (x_n) \in \ell_2 : \tilde{\omega}(\mathbf{x}^*\mathbf{x}) = \sum_{n=1}^{+\infty} |x_n|^2 = 0\}$ is the null subspace, then the Hilbert space on which we construct the *-representation of $\tilde{\omega}$ is exactly ℓ_2. Hence, we put $\mathcal{D} = \mathcal{D}_{\tilde{\omega}} = \mathfrak{A}_0$ and, if $\mathbf{x} \in \ell_2$,

$$\pi_{\tilde{\omega}}(\mathbf{x})\mathbf{y} = \mathbf{x} \cdot \mathbf{y} = (x_n y_n)_{n\in\mathbb{N}}, \ \ \forall \mathbf{y} \in \mathfrak{A}_0.$$

Thus, $\pi_{\tilde{\omega}}(\mathbf{x})$ is well-defined and maps $\mathfrak{A}_0$ into $\mathfrak{A}_0$, moreover $\pi_{\tilde{\omega}}(\mathbf{x}) \in \mathcal{L}^\dagger(\mathfrak{A}_0, \ell_2)$ and $\pi_{\tilde{\omega}}(\mathbf{x})^\dagger = \pi_{\tilde{\omega}}(\mathbf{x}^*)$. Also, $\pi_{\tilde{\omega}}(\mathbf{x}) \,\square\, \pi_{\tilde{\omega}}(\mathbf{y})$ is well-defined and $\pi_{\tilde{\omega}}(\mathbf{xy}) = \pi_{\tilde{\omega}}(\mathbf{x}) \,\square\, \pi_{\tilde{\omega}}(\mathbf{y})$, $\forall \mathbf{x} \in \ell_2, \mathbf{y} \in \mathfrak{A}_0$. Thus, $\pi_{\tilde{\omega}}$ is a *-representation mapping $\mathfrak{A}_0$ in ℓ_2, hence $\pi_{\tilde{\omega}}$ is a *bounded* *-representation.

5 Conclusions

The research of an absolutely convergent extension of a functional τ is important, but it is known in many examples. Even more interesting, however, appears to us the research, among the infinitely many non absolutely convergent extensions, of those extensions that enjoy a certain set of prefixed properties. The widespread interest for the non absolutely convergent extensions of the Lebesgue integral, would motivate work in this direction.

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On a fixed point in the metric space of normalized Hausdorff moment sequences

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Abstract. We show that the transformation $(x_n)_{n\geq 1} \to (1/(1+x_1+\ldots+x_n))_{n\geq 1}$ of the compact set of sequences $(x_n)_{n\geq 1}$ of numbers from the unit interval $[0,1]$ has a unique fixed point, which is attractive. The fixed point turns out to be a Hausdorff moment sequence studied in [3].

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1 Introduction

Let $\mathcal{K} = [0,1]^{\mathbb{N}}$ denote the product space of sequences $(x_n) = (x_n)_{n\geq 1}$ of numbers from the unit interval $[0,1]$.

We consider a transformation $T : \mathcal{K} \to \mathcal{K}$ defined by

$$(T(x_n))_n = \frac{1}{1+x_1+\ldots+x_n}, \quad n \geq 1. \tag{1.1}$$

Since T is a continuous transformation of the compact convex set $\mathcal{K}$ in the space $\mathbb{R}^{\mathbb{N}}$ of real sequences equipped with the product topology, it has a fixed point (m_n) by Tychonoff's extension of Brouwer's fixed point theorem. Furthermore, it is clear by (1.1) that the fixed point (m_n) is uniquely determined by the equations

$$(1+m_1+\ldots+m_n)m_n = 1, \quad n \geq 1. \tag{1.2}$$

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Therefore

$$m_{n+1}^2 + \frac{m_{n+1}}{m_n} - 1 = 0, \tag{1.3}$$

giving

$$m_1 = \frac{-1+\sqrt{5}}{2}, \quad m_2 = \frac{\sqrt{22+2\sqrt{5}}-\sqrt{5}-1}{4}, \ldots .$$

Berg and Durán studied this fixed point in [3], and it was proved that $m_0 = 1, m_1, m_2, \ldots$ is a normalized Hausdorff moment sequence, i.e., of the form

$$m_n = \int_0^1 x^n \, d\tau(x), \tag{1.4}$$

where τ is a probability measure on the interval $[0,1]$. For details about the measure τ, see [3] and [4]. We mention that τ has an increasing and convex density with respect to Lebesgue measure.

There is no reason a priori that the fixed point (m_n) should be a Hausdorff moment sequence, but the motivation for the study of T came from the theory of moment sequences because of the following theorem from [2]:

Theorem 1.1 *Let $(a_n)_{n\geq 0}$ be a Hausdorff moment sequence of a measure $\mu \neq 0$ on $[0,1]$. Then the sequence $(b_n)_{n\geq 0}$ defined by $b_n = 1/(a_0 + \ldots + a_n)$ is again a Hausdorff moment sequence, and its associated measure $\nu = \widehat{T}(\mu)$ has the properties $\nu(\{0\}) = 0$ and*

$$\int_0^1 \frac{1-t^{z+1}}{1-t}\, d\mu(t) \int_0^1 t^z \, d\nu(t) = 1 \quad \textit{for} \quad \Re z \geq 0. \tag{1.5}$$

Let $\mathcal{H}$ denote the set of normalized Hausdorff moment sequences, where we throw away the zero'th moment which is always 1, i.e.,

$$\mathcal{H} = \{(a_n)_{n\geq 1} \mid a_n = \int_0^1 x^n \, d\mu(x), \ \mu([0,1]) = 1\}. \tag{1.6}$$

Clearly, $\mathcal{H} \subset \mathcal{K}$ and by Theorem 1.1 we have $T(\mathcal{H}) \subseteq \mathcal{H}$. It is easy to see that $\mathcal{H}$ is a compact convex subset of $\mathcal{K}$, e.g., by using Hausdorff's 1921 characterization of Hausdorff moment sequences as completely monotonic sequences, i.e., sequences $(a_n)_{n\geq 0}$ satisfying

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{n+k} \geq 0 \quad \text{for} \quad m, n \geq 0. \tag{1.7}$$

See [1] for details.

It was proved in Theorem 2.3 in [3] that (m_n) is an attractive fixed point of the restriction of T to $\mathcal{H}$. This proof was a direct proof not building on any classical results on attracting fixed points. For classical fixed point theory see [5].

The purpose of the present paper is to prove the following extension of this:

Theorem 1.2 *The unique fixed point (m_n) of the transformation $T : \mathcal{K} \to \mathcal{K}$ given by* (1.1) *is attractive.*

2 Proofs and complements

The product topology on the vector space $\mathbb{R}^{\mathbb{N}}$ is induced by the metric

$$d((a_n),(b_n)) = \sum_{n=1}^{\infty} 2^{-n} \min\{|a_n - b_n|, 1\} \quad \text{for} \quad (a_n),(b_n) \in \mathbb{R}^{\mathbb{N}},$$

which makes it a Fréchet space.

On the compact subset $\mathcal{K} = [0,1]^{\mathbb{N}}$ the expression for the metric is simplified to

$$d((a_n),(b_n)) = \sum_{n=1}^{\infty} 2^{-n} |a_n - b_n| \quad \text{for} \quad (a_n),(b_n) \in \mathcal{K}. \tag{2.1}$$

Before we find the best Lipschitz constant for T, let us introduce some notation. For $0 \le a \le 1$ let $\underline{a} = (a^n)_{n\ge 1}$, so $\underline{0} = 0, 0, \ldots$ and $\underline{1} = 1, 1, \ldots$. Clearly, $T(\underline{0}) = \underline{1}$, $T(\underline{1}) = 1/2, 1/3, \ldots$, while

$$\left(T\left(\frac{1}{n+1}\right)\right)_n = \frac{1}{H_{n+1}}, \quad \text{where} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

The numbers H_n are called the harmonic numbers. In [2] it is proved that

$$\frac{1}{H_{n+1}} = \int_0^1 x^n \left(\sum_{p=0}^{\infty} \alpha_p x^{-\xi_p}\right) dx,$$

where $0 = \xi_0 > \xi_1 > \xi_2 > \ldots$ satisfy $-p-1 < \xi_p < -p$ for $p = 1, 2, \ldots$ and $\alpha_p > 0, p = 0, 1, \ldots$. More precisely, it is proved that ξ_p is the unique solution $x \in \,]-p-1, -p[$ of the equation $\Psi(1+x) = -\gamma$, and $\alpha_p = 1/\Psi'(1+\xi_p)$. Here $\Psi(x) = \Gamma'(x)/\Gamma(x)$ and γ is Euler's constant.

In [3] it is proved directly that $T^n(\underline{0})$ converges to the fixed point (m_n), and this was used to derive that the same holds independent of where in $\mathcal{H}$ the iteration starts.

We cannot apply Banach's fixed point theorem directly because of the following Lemma.

Lemma 2.1 *The best Lipschitz constant c in*

$$d(T(a_n), T(b_n)) \le c\, d((a_n), (b_n)) \quad \textit{for} \quad (a_n), (b_n) \in \mathcal{K}$$

is $c = 2$.

Proof. For $(a_n), (b_n)$ we find

$$\begin{aligned} & d(T(a_n), T(b_n)) \\ & \quad = \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\sum_{j=1}^{k}(b_j - a_j)\right|}{(1 + a_1 + \ldots + a_k)(1 + b_1 + \ldots + b_k)} \\ & \quad \le \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} |a_j - b_j| = \sum_{j=1}^{\infty} |a_j - b_j| \sum_{k=j}^{\infty} 2^{-k} \\ & \quad = 2d((a_n), (b_n)), \end{aligned} \tag{2.2}$$

which shows that T is Lipschitz with constant $c = 2$.

Assume next that T satisfies a Lipschitz condition with constant c.

We note that

$$d(\underline{0}, \underline{a}) = \sum_{n=1}^{\infty} (a/2)^n = \frac{a}{2 - a}.$$

Furthermore, $T(\underline{0}) = \underline{1}$ and for $0 \le a < 1$ we have

$$(T(\underline{a}))_n = \frac{1 - a}{1 - a^{n+1}},$$

so finally

$$d(T(\underline{0}), T(\underline{a})) = a \sum_{n=1}^{\infty} 2^{-n} \frac{1 - a^n}{1 - a^{n+1}}.$$

This gives

$$\sum_{n=1}^{\infty} 2^{-n} \frac{1 - a^n}{1 - a^{n+1}} \le c \frac{1}{2 - a}, \quad 0 < a \le 1,$$

and letting $a \to 0$ we find $c \ge 2$. □

Proof of Theorem 1.2.
Let us now introduce the set

$$\mathcal{C} = \left\{(a_n) \in \mathcal{K} \mid a_1 \geq \tfrac{1}{2}\right\},$$

which is a compact convex subset of $\mathcal{K}$. We first note that $T(\mathcal{K}) \subseteq \mathcal{C}$ because for any $(a_n) \in \mathcal{K}$ we have $a_1 \leq 1$, hence

$$(T(a_n))_1 = \frac{1}{1+a_1} \geq \frac{1}{2}.$$

By (2.2) we always have

$$\begin{aligned} &d(T(a_n), T(b_n)) \\ &\quad = \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\sum_{j=1}^{k}(b_j - a_j)\right|}{(1+a_1+\ldots+a_k)(1+b_1+\ldots+b_k)} \\ &\quad \leq \frac{1}{(1+a_1)(1+b_1)} \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} |a_j - b_j| \\ &\quad = \frac{1}{(1+a_1)(1+b_1)} \sum_{j=1}^{\infty} |a_j - b_j| \sum_{k=j}^{\infty} 2^{-k} \\ &\quad = \frac{2}{(1+a_1)(1+b_1)} d((a_n), (b_n)). \end{aligned} \tag{2.3}$$

If now $(a_n), (b_n) \in \mathcal{C}$, we have $a_1, b_1 \geq \frac{1}{2}$, and hence $\frac{2}{(1+a_1)(1+b_1)} \leq \frac{8}{9}$ so that

$$d(T(a_n), T(b_n)) \leq \frac{8}{9} d((a_n), (b_n)) \quad \text{for} \quad (a_n), (b_n) \in \mathcal{C},$$

showing that T is a contraction on $\mathcal{C}$. Since T maps $\mathcal{K}$ into $\mathcal{C}$, any fixed point of T on $\mathcal{K}$ must belong to $\mathcal{C}$, and $T : \mathcal{C} \to \mathcal{C}$ has a unique fixed point by Banach's fixed point theorem, and this must be the sequence (m_n) determined by (1.3). We also see that for any $\xi = (a_n) \in \mathcal{K}$ the iterates $T^{n+1}(\xi) = T^n(T(\xi))$ converge to the fixed point (m_n), which finishes the proof of Theorem 1.2. □

Proposition 2.2 *The compact set $\mathcal{H}$ defined in* (1.6) *has diameter*

$$\operatorname{diam}(\mathcal{H}) = 1,$$

and the only two points $a, b \in \mathcal{H}$ for which $d(a,b) = 1$ are $\{a,b\} = \{\underline{0}, \underline{1}\}$.

Proof. It is easy to see that

$$1 = d(\underline{0}, \underline{1}) \leq \operatorname{diam}(\mathcal{H}) \leq 1,$$

so we only have to prove that if $d(a, b) = 1$ for some $a = (a_n), b = (b_n)$ from $\mathcal{H}$, then $\{a, b\} = \{\underline{0}, \underline{1}\}$. Supposing that

$$\sum_{n=1}^{\infty} 2^{-n} |a_n - b_n| = 1,$$

then necessarily $|a_n - b_n| = 1$ for all $n \geq 1$. In particular, for each n necessarily a_n and b_n are either 0 and 1 or these numbers reversed. Assume now that $a_1 = 0, b_1 = 1$. Since $a_n = \int_0^1 x^n \, d\mu(x)$ for a probability measure μ, the condition $a_1 = 0$ forces μ to be the Dirac measure δ_0 with unit mass concentrated at 0, and hence $a_n = 0$ for all $n \geq 1$. This shows that $a = \underline{0}$ and $b = \underline{1}$. □

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On Best Approximation Theorem in Nonlinear Analysis

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Abstract. In this paper we discuss Ky Fan's best approximation theorem and derive results in fixed point theory, approximation theory and variational inequalities.

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1 Introduction

Most of the fixed point theorems are given for self maps, that is, for a function with domain and range the same. A function with domain and range the same is called a selfmap. In case a function does not have the same domain and range, then we need a boundary condition to guarantee the existence of a fixed point. The importance of the approximation theorem presented here is in establishing the fixed point theorem for mappings with different domain and range. Ky Fan (see Theorem 2 [8]) proved an important theorem, now it is called the Fan's best approximation theorem, has applications in fixed point theory, approximation theory, variational inequalities, minimax theory, and mathematical economics.

This paper is in final form and no version of it will be submitted for publication elsewhere.

2 Finite dimensional case

The following result, given in R^n, is known as the Fan's best approximation theorem (see Theorem 2 [8]).

Theorem 2.1. *Let C be a closed bounded convex subset of R^n and $f : C \to R^n$ a continuous function. Then there is a $y \in C$ such that*

$$\|y - f(y)\| = d(f(y), C), \qquad (*)$$

where

$$d(x, C) = inf\{\|x - y\| : y \in C\}, \quad x \in R^n, x \notin C.$$

Proof: We note that y is a solution of (*) if and only if y is a fixed point of $P \circ f$, where $P : R^n \to C$ is the metric projection onto C and P is a continuous function.

Thus, $P \circ f : C \to C$ is a continuous function and has a fixed point in C by Brouwer's fixed point theorem.

Let $P \circ f(y) = y, \quad y \in C$. Then $\|y - f(y)\| = d(f(y), C)$ follows.

Theorem 2.1 has an application in fixed point theory [1, 2].

The following is the well known Brouwer fixed point theorem.

Theorem 2.2. *Let $f : C \to C$ be a continuous function, where C is a closed bounded convex subset of R^n. Then f has a fixed point.*

In this case $d(f(y), C) = 0$ and $f(y) = y$.

Remark 2.1. In case $f : C \to R^n$ is a continuous function and C is a closed bounded convex subset of R^n, then f has a fixed point provided one of the following conditions is satisfied:

i) If $f(y) \neq y$, then the line segment $[y, f(y)]$ has at least two points of C. or

ii) If $f : C \to R^n$, with $f(\partial C) \subset C$, (∂C is the boundary of C) [1].

Remark 2.2. If B in R^n is a ball of radius r with center 0 and $f : B \to R^n$ is a continuous function, then there is a $y \in B$ such that $\|y - f(y)\| = d(f(y), B)$.

We derive the following fixed point results for continuous function $f : B \to R^n$ under one of the following conditions for $y \in \partial B$.

i) $\|f(y)\| \leq \|y\|$

ii) $< y, f(y) > \leq \|y\|^2$
iii) $\|f(y)\| \leq \|y - f(y)\|$
iv) If $f(y) = ky$, for some $k \in R$, then $k \leq 1$
v) $\|f(y)\|^2 \leq \|y\|^2 + \|f(y) - y\|^2$.

For example, if i) holds, then $\|f(y)\| \leq r$, and $f(y) \in B$.
Thus $d(f(y), B) = 0$ and f has a fixed point.

In case iv) holds, then for $y \in \partial B$, $\|y\| = r$.
If $f(y) = ky$, then $\frac{\|f(y)\|}{\|y\|} = k$. Since $k \leq 1$ so $f(y) \in B$ and f has a fixed point.

Theorem 2.1 has an application in approximation theory.

Theorem 2.3. *A closed convex bounded subset C of R^n is a set of existence, that is, for each $x \in R^n$, $x \notin C$, there is a $y \in C$ such that $\|y - x\| = d(x, C)$.*

Proof: If we define $f : C \to R^n$ by $f(y) = x$ for all $y \in C$, then f is continuous and by Theorem 2.1, there is a $y \in C$ such that $\|y - f(y)\| = d(f(y), C)$, that is, $\|y - x\| = d(x, C)$ for $y \in C$.

Remark 2.3. $\|y - x\| = d(x, C)$ if and only if

$$< x - y, y - z > \geq 0$$

for all $z \in C$.

Theorem 2.1 has an application in variational inequalities [2, 9].

Theorem 2.4. *If C is a closed bounded and convex subset of R^n, and $f : C \to R^n$ is a continuous function, then there is a $y \in C$ such that*

$$< f(y), x - y > \geq 0, \quad \textit{for all } x \in C \qquad (VIP)$$

Proof: Let $g = I - f$, where I is the identity function. Then $g : C \to R^n$ is a continuous function and, by Theorem 2.1, $\|y - g(y)\| = d(g(y), C)$.
Thus $< g(y) - y, y - x > \geq 0$ for all $x \in C$, that is, $< f(y), x - y > \geq 0$ for all $x \in C$.

Remark 2.4. The variational inequality problem (VIP) has a solution if and only if $P(I - f)$ has a fixed point.

Now an application of Theorem 2.1 is given to find a fixed point and in turn to find a zero.

Theorem 2.5. *Let C be a closed, bounded, convex subset of R^n and $f : C \to R^n$ a continuous function.*

Let $x - rf(x) \in C$ for all $x \in C$ and some $r > 0$.

Then $f(x) = 0$ has a solution, that is, there is a $z \in C$ such that $fz = 0$.

Proof: Let $g(x) = x - rf(x)$ for all $x \in C$. Then $g : C \to R^n$ is a continuous function and there is a $z \in C$ such that $\|z - g(z)\| = d(g(z), C)$.

Since for all $x \in C$, $g(x) \in C$, therefore g has a fixed point, say $g(z) = z$ and consequently, $z - rf(z) = z$, that is, $f(z) = 0$ since $r > 0$.

3 Results in Hilbert spaces

Theorem 2.1 does not hold true if the space is infinite dimensional. If we take a Hilbert space and a continuous function f, then the result is not true.

For example, if B is a unit closed ball in a Hilbert space H and $f : B \to B$ is a continuous function, then f does not have a fixed point.

If Theorem 2.1 holds, then the self map must have a fixed point.

However, if the function is Lipschitz continuous, say for example a nonexpansive map, then the result holds.

We need a few preliminaries.

Definition 3.1. *Let $f : H \to H$ be a map, where H is a Hilbert space. Then f is said to be nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in H$.*

Remark 3.1. A nonexpansive map is Lipschitz continuous.

Definition 3.2. In case $\|f(x) - f(y)\| \leq k\|x - y\|$, where $0 < k < 1$, then f is said to be a contraction map.

Definition 3.3. Let X be a normed linear space and C a nonempty subset of X. Let $x \in X$. An element $y \in C$ is called an element of best approximation to x by elements of C, if

$$\|x - y\| = d(x, C) = inf\{\|x - z\| : z \in C\}.$$

Let $P(x)$ denote the set of all points in C closest to x, that is,

$$P(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

The set C is said to be proximinal if $P(x)$ is nonempty for all $x \in X$.

If $P(x)$ is a singleton for each $x \in X$, then C is said to be a Chebyshev set.

If C is a Chebyshev set, then P is a single valued map.

In case $P : X \to 2^C$ is a multivalued map, it is called the metric projection or the best approximation operator.

Let C be a closed convex subset of a real Hilbert space H. Then for each $x \in H$, $x \notin C$, there is a unique $y \in C$ nearest to x, that is.

$$\|x - y\| = d(x, C) = inf\{\|x - z\| : z \in C\}.$$

In this case the set C is a Chebyshev set and

$$P(x) = \{y \in C : \|x - y\| = d(x, C)\}$$

is the best approximation operator or the proximity map on H.

Remark 3.2. It is easy to see that the proximity map staisfies the following properties.

i) $< x - P(x), P(x) - P(y) > \geq 0$ *for all* $x, y \in H$*;*

ii) $\|P(x) - P(y)\| \leq \|x - y\|$ *for all* $x, y \in H$*;*

iii) $\|x - P(x)\|^2 + \|P(x) - y\|^2 \leq \|x - y\|^2, \;\; x \notin C, y \in C.$

Indeed, for $0 \leq t \leq 1$,

$$\|x - P(x)\|^2 \leq \|x - tP(y) - (1-t)P(x)\|^2 = \|(x - P(x)) + t(P(x) - P(y))\|^2$$

$$= \|x - P(x)\|^2 + t^2\|P(x) - P(y)\|^2 + 2t < x - P(x), P(x) - P(y) > .$$

Thus,

$$t^2\|P(x) - P(y)\|^2 + 2t < x - P(x), P(x) - P(y) > \geq 0.$$

This will be violated for small t, unless

$$< x - P(x), P(x) - P(y) > \geq 0.$$

Now,

$$< x, P(x) - P(y) > \geq < P(x), P(x) - P(y) > \tag{1}$$

from i).

Similarly,

$$< y - P(y), P(y) - P(x) > \geq 0$$

implies that

$$< y, P(y) - P(x) > \geq < P(y), P(y) - P(x) >,$$

that is,

$$< -y, P(x) - P(y) > \geq < -P(y), P(x) - P(y) > \tag{2}$$

Summing (1) with (2) we get

$$< P(x) - P(y), x - y > \geq < P(x) - P(y), P(x) - P(y) > .$$

This implies that

$$\|P(x) - P(y)\| \leq \|x - y\|.$$

iii) For $x \notin C$ and $y \in C$,

$$\|x - P(x)\|^2 + \|P(x) - y\|^2$$

$$\leq \|x - P(x)\|^2 + \|P(x) - y\|^2 + 2 < x - P(x), P(x) - y >$$

$$= \|x - P(x) + P(x) - y\|^2 = \|x - y\|^2.$$

Definition 3.4. Let $f : H \to H$. Then f is said to be monotone if

$$< f(x) - f(y), x - y > \geq 0$$

for all $x, y \in H$.

Remark 3.3. If H is a real Hilbert space and $f : H \to H$ is a nonexpansive map, then $I - f$ is a monotone map.

Indeed, since f is nonexpansive,

$$\|f(x) - f(y)\| \leq \|x - y\|$$

for all $x, y \in H$.

Let $g = I - f$. Then

$$\|f(x) - f(y)\|^2 \leq \|x - y\|^2 + \|g(x) - g(y)\|^2.$$

Now,

$$\|(I - g)(x) - (I - g(y)\|^2 = \|x - y - (g(x) - g(y))\|^2$$

$$= \|x - y\|^2 + \|g(x) - g(y)\|^2 - 2 < g(x) - g(y), x - y >$$

$$\leq \|x-y\|^2 + \|g(x)-g(y)\|^2,$$

only if $< g(x)-g(y), x-y >\geq 0$. Thus $g = I - f$ is monotone.

We state the following result (see [26]).

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H and $f : C \to H$ a nonexpansive map with $f(C)$ bounded.*

Then there exists a $y \in C$ such that $\|y - f(y)\| = d(f(y), C)$.

Proof: Let $P : H \to C$ be a metric projection. Then P is a nonexpansive map.

It is easy to see that $P \circ f : C \to C$. Since P is nonexpansive and f is nonexpansive map, then $P \circ f$ is also a nonexpansive map. Hence $P \circ f$ has a fixed point $y \in C$ (see [26]).

Let $P \circ f(y) = y$. Then we get $\|y - f(y)\| = d(f(y), C)$..

Remark 3.4. If $f(C) \subset C$ in Theorem 3.1, then f has a fixed point.

The following theorem, proved independently by Browder in [3], Gohde in [11] and Kirk in [15], is derived as a corollary from Theorem 3.1.

Theorem 3.2. *If C is a closed bounded convex subset of H and $f : C \to C$ is a nonexpansive map, then f has a fixed point.*

The following examples illustrate that Theorem 3.1 is more general.

Example 3.1: Let $f : [-1, 1] \to R$ be given by $f(x) = 1 - x \ \ (x \in [-1, 1])$. Then $f(-1) = 2 \notin [-1, 1]$, that is, f is not a self map but f does have a fixed point $x = \frac{1}{2} = f(\frac{1}{2})$.

Example 3.2: Let $f : [0, \infty) \to R$ be given by $f(x) = \frac{1}{1+x}$. Then f is a nonexpansive map and $f(y) = y = \frac{\sqrt{5}-1}{2}$.

But $[0, \infty)$ is not bounded.

In case we take

$$B_r = \{x \in H : \|x\| \leq r\},$$

a ball of radius r and center 0, in place of C in Theorem 3.1, then the following result holds (see [1]).

Let $f : B_r \to H$ be a nonexpansive map. Then there is a $y \in B_r$ such that $\|y - f(y)\| = d(f(y), B)$.

If $f : B_r \to B_r$, then f has a fixed point.

Hartman and Stampacchia in [13] proved the following result about variational inequalities, which is useful in mathematical, physical and economic problems.

Theorem 3.3. *Let C be a closed bounded convex subset of a Hilbert space H and $f : C \to H$ a monotone, continuous map. Then there is a $y \in C$ such that*

$$< f(y), x - y > \geq 0 \quad \textit{for all} \quad x \in C. \qquad (VIP)$$

Remark 3.5. We know that the (VIP) has a solution if and only if $P(I - f) : C \to C$ has a fixed point.

An application of Theorem 3.3 allows to prove the following result which is contained in [12].

Theorem 3.4. *Let C be a closed bounded convex subset of a Hilbert space H and $f : C \to C$ a nonexpansive map. Then f has a fixed point.*

Proof: Since f is a nonexpansive map, it is continuous. Consider $g : C \to H$ defined by $g = I - f$, then g is a continuous map.

It is easy to see that g is monotone, that is, $< g(x) - g(y), x - y > \geq 0$.

As g is continuous and monotone, by Theorem 3.3 there exists a $y \in C$ such that

$$< g(y), x - y > \geq 0$$

for all $x \in C$, that is,

$$< y - f(y), x - y > \geq 0 \quad \text{for all} \quad x \in C.$$

By taking $x = f(y)$ we get that

$$< y - f(y), f(y) - y > \geq 0,$$

that is $< y - f(y), f(y) - y > \leq 0$. But $\|y - f(y)\|^2 \geq 0$ hence $y = f(y)$.

The following result is an application of Theorem 3.3 to approximation theory [12].

Theorem 3.5. *A closed bounded convex subset of a Hilbert space H is a set of existence.*

Recall that if C is closed convex subset of a Hilbert space H and $y \notin C$, then there exists $x \in C$ such that $< y - x, x - z > \geq 0$ for all $z \in C$, that is, x is a nearest point to y.

Proof: Let $f : C \to H$ be defined by $f(z) = z - y$ for all $z \in C$.

Then f is a continuous and monotone function. Hence, by Theorem 3.3 there is an $x \in C$ such that $< f(x), w - x > \geq 0$ for all $w \in C$.

Thus, $< x - y, w - x > \geq 0$ for all $w \in C$, that is, $< y - x, x - w > \geq 0$ for all $w \in C$ and x is nearest to y.

It follows easily that x is unique.

4 Results dealing with approximating sequences

The following is an interesting question treated in [3].

Let C be a closed convex subset of a Banach space X and $f : C \to C$ a nonexpansive map.

Define $f_k(x) = kf(x) + (1-k)p$ for some $p \in C$ and $k \in (0,1)$.

Then each f_k is a contraction map with Lipschitz constant k. For k sufficiently close to 1, f_k is a contractive approximation to f. Each f_k has a unique fixed point by the Banach contraction principle, say

$$x_k = f_k(x_k) = kf(x_k) + (1-k)p.$$

It is natural to ask if the sequence $\{x_k\}$ converges to a fixed point of f.

One can not expect, in general, an affirmative answer to this question as there are nonexpansive mappings which do not have fixed points.

For example, a translation map is nonexpansive and does not have any fixed point.

However, in Hilbert spaces the following theorem holds.

Theorem 4.1. *Let C be a closed bounded convex subset of a Hilbert space H and $f : C \to C$ a nonexpansive map.*

Define $f_k(x) = kf(x) + (1-k)p$, where $p \in C$ and $k \in (0,1)$. Let $f_k(x_k) = x_k$. Then $\{x_k\}$ converges strongly to y, where y is a fixed point of f closest to p.

Definition 4.1. A mapping $f : C \to X$ is said to be demiclosed if for each sequence $\{x_k\}$ weakly converging to x_0 and such that $\{f(x_k)\}$ strongly converges to y_0, then $f(x_0) = y_0$.

If C is a closed convex subset of a Hilbert space H and $f : C \to H$ is a nonexpansive map, then $I - f$ is a demiclosed map (see [3]).

The following is a general result whose proof is based on approximation techniques (see [27]).

Theorem 4.2. *Let H be a real Hilbert space and C a closed convex subset of H. Let $f : C \to H$ be a nonexpansive map with $f(C)$ bounded and $f(\partial C) \subseteq C$.*

Let us assume that $0 \in C$. Define

$$f_k(x) = kf(x) + (1-k)x_0$$

for some $x_0 \in C$ and $0 < k < 1$.

Let x_k be a fixed point for f_k, that is $f_k x_k = x_k$.

Then x_k strongly converges to y_0, as $k \to 1$, where y_0 is the fixed point of f closest to x_0.

Proof: Let us denote the fixed point set of f by $F(f)$. We know that the set $F(f)$ is nonempty closed and convex. There exists a unique closest point $y_0 = f(y_0)$ to x_0.

For the sake of convenience take $x_0 = 0$.

Now

$$\|\frac{x_k}{k} - y_0\|^2 = \|f(x_k) - y_0\|^2 = \|f(x_k) - f(y_0)\|^2 \leq \|x_k - y_0\|^2.$$

By simplifying we get

$$\|x_k\|^2 \leq \frac{2k}{1+k} < x_k, y_0 > \leq < x_k, y_0 > .$$

So, $\|x_k\| \leq \|y_0\|$, and $\{x_k\}$ is bounded.

The bounded sequence $\{x_k\}$ has a subsequence $\{x_{k_i}\}$ weakly converging to x.

Now, denoting by $x_i = x_{k_i}$,

$$\|x_i - f(x_i)\| = \|k_i f(x_i) - f(x_i)\| = (k_i - 1)\|f(x_i)\| \to 0$$

as $i \to \infty$.

Thus $x_i - f(x_i) \to 0$.

Since $I - f$ is demiclosed, therefore $(I - f)(x) = 0$, that is, $f(x) = x$.

Since $\|x_i\| \leq \|y_0\|$, therefore $\|x\| \leq \|y_0\|$. Since y_0 is closest to 0, $\|x\| = \|y_0\|$ and $\{x_i\}$ weakly converges to y_0.

Now

$$\|y_0\|^2 \geq \|x_i\|^2 = \|x_i - y_0 + y_0\|^2$$

$$= \|x_i - y_0\|^2 + \|y_0\|^2 + 2 < x_i - y_0, y_0 > .$$

By passing to the limit as $i \to \infty$, the right hand side goes to 0 giving $\|x_i - y_0\| \to 0$.

Thus x_i strongly converges to y_0. Since x_i is an arbitrary subsequence of x_k, the sequence x_k strongly converges to y_0.

The following result deals with the convergence of sequence of iterates (see [17]).

Theorem 4.3. *Let C be a closed convex subset of a Hilbert space H and $f : C \to H$ a continuous function such that $I - \lambda f$ is a contraction function.*

Then there exists a unique solution $u \in C$ *of* $< f(u), v - u > \geq 0$ *for all* $v \in C$ *and* $u = \lim u_n$, *where* $u_{n+1} = P(I - \lambda f)(u_n), \quad u_0 \in C$.

It is easy to see that if f is Lipschitz continuous and strongly monotone, then $I - \lambda f$ is a contraction for a suitable $\lambda \in (0, 1)$.

For example, if $f : C \to H$ satisfies

$$\|f(x) - f(y)\| \leq k\|x - y\|$$

for all $x, y \in C, k > 0$, and f is strongly monotone, that is,

$$\alpha\|x - y\|^2 \leq < f(x) - f(y), x - y >$$

for all $x, y \in C, \alpha > 0$.

Then $I - \lambda f$ is a contraction map for $0 < \lambda < \frac{2\alpha}{k^2}$.

5 Two Function theorems and Coincidence Points

In this section Prolla's extension of Fan's best approximation theorem is discussed. Recently Prolla proved a theorem of Ky Fan type by introducing a second function g. If $g = I$, then one gets Fan's theorem. Prolla used fixed point theorem for multifunctions to prove this result. Its applications in fixed point theory and coincidence theory have been given by a few.

Definition 5.1. Let C be a convex subset of X. Then $f : C \to X$ is said to be almost-affine if it satisfies

$$\|f(\lambda x + (1 - \lambda)y) - z\| \leq \lambda\|f(x) - z\| + (1 - \lambda)\|f(y) - z\|$$

for all $x, y \in C$, $z \in X$ and $0 < \lambda < 1$.

f is said to be an affine map if

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y),$$

$\lambda \in (0, 1)$.

Definition 5.2.. A sequence $\{x_n\}$ in C, where C is a subset of a normed linear space X, is called a minimizing sequence for $y \in X$ if $\|y - x_n\| \to d(y, C)$.

Definition 5.3. A set C is called approximatively compact if for each $y \in X$, each minimizing sequence for y has a subsequence converging to a point of C.

A metric projection P onto an approximatively compact set is upper semicontinuous.

A compact set is approximatively compact. The converse is not true. For example, the closed unit ball of an infinite dimensional uniformly convex Banach space X is approximatively compact but not compact.

The following is due to Prolla (see [21]).

Theorem 5.1. *Let C be a nonempty compact convex subset of a normed linear space X, $g : C \to C$ a continuous almost affine and surjective function.*

If $f : C \to X$ is a continuous map, then there is a $y \in C$ such that

$$\|g(y) - f(y)\| = d(f(y), C).$$

Remark 5.1. In case g is the identity map, we get Ky Fan's theorem. If $f : C \to C$, then we get $g(y) = f(y)$, which is a so-called a coincidence result.

The following theorem contained in [24] extends the result of Prolla (see Theorem 5.1 in [21]).

Theorem 5.2. *Let C be a nonempty convex subset of a normed linear space X and $P : X \to 2^C$ the metric projection satisfying the following assumptions.*

i) $P(x)$ is nonempty for each $x \in X$;

ii) P maps compact subsets of X into compact subsets of C;

iii) $g : C \to C$ a continuous, almost affine and into mapping such that: for any compact set D in C, $g^{-1}(D)$ is compact.

Then for any continuous map $f : C \to X$ such that $f(C)$ is relatively compact, there exists a $y \in C$ for which

$$\|g(y) - f(y)\| = d(f(y), C).$$

Carbone in [4] introduced the following definition.

Definition 5.4. *The function $f : C \to X$ is said to be almost quasi convex if*

$$\|f(\lambda x + (1 - \lambda)y) - z\| \leq max(\|f(x) - z\|, \|f(y) - z\|),$$

for any $x, y \in C, z \in X$ and $0 < \lambda < 1$.

He proved the following where only the image set of C is compact [4].

Theorem 5.3. *Let C be a nonempty convex subset of a normed linear space X and $P : X \to 2^C$ the metric projection.*

Assume that

i) $P(x) \neq \emptyset$ for each $x \in X$;

ii) P maps compact subset of X into compact subsets of C.

Let $g : C \to C$ be a continuous almost quasi convex onto and proper map.

Then for each continuous function $f : C \to X$ such that the closure of $f(C)$ is compact, there is a $y \in C$ such that $\|g(y) - f(y)\| = d(f(y), C)$.

Definition 5.5. Recall that if $g : C \to C$ is a function, then g is said to be proper if $g^{-1}(A)$ is compact provided A is compact too.

6 Multivalued analogue of Ky Fan's Theorem

We need a few preliminaries concerning with multivalued maps.

Let X and Y be two sets in a normed linear space E. A multivalued map $F : X \to Y$ is a subset $F \subseteq X \times Y$. The inverse of F, denoted by $F^{-1} : Y \to X$, is defined by $(y, x) \in F$ if and only if $(x, y) \in F$.

The values of F are the sets

$$F(x) = \{y \in Y : (x, y) \in F\}, \ (x \in X),$$

the fibres of F are the sets

$$F^{-1}(y) = \{x \in X : (x, y) \in F\}$$

for $y \in Y$.

For $A \subseteq X$, the set $F(A) = \cup\{F(x) : x \in A\}$ is called the image of A under F, for $B \subseteq Y$, the set

$$F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\},$$

which is the image of B under F^{-1}, is called the inverse image of B under F.

A multifunction (or multivalued map) $F : X \to Y$ has a fixed point if $x \in F(x)$ for some $x \in X$.

Let X and Y be two topological spaces. A multifunction $F : X \to Y$ is called upper semicontinuous (usc in short) if the inverse image of any closed set is closed, and it is called lower semicontinuous (lsc) if the inverse image of any open set is open. A multifunction is said to be continuous if it is both upper and lower semicontinuous.

First, we list following fixed point theorems used in the proof of Ky Fan type theorems for multivalued maps.

Theorem 6.1. *Let C be a nonempty compact convex subset of a normed linear space X and $F : C \to 2^C$ an upper semicontinuous multifunction such that $F(x)$ is nonempty, closed and convex for each $x \in C$.*

Then F has a fixed point, that is there exists $x \in C$ such that $x \in F(x)$ (see [8]).

The following is due to Himmelberg [14].

Theorem 6.2. *Let C be a nonempty convex subset of a normed linear space X and $F : C \to 2^C$ an usc multifunction where $F(x)$ nonempty, closed convex for each $x \in C$. If $F(C)$ is contained in a compact subset of C, then F has a fixed point.*

Definition 6.1. If $F(x)$ is compact for $x \in C$, then $F(x)$ is point compact. If $F(x)$ is convex for $x \in C$, then $F(x)$ is point convex.

The following multivalued analogue is due to Reich [23].

Theorem 6.3. *Let C be a compact convex subset of a locally convex Hausdorff topological vector space E, and $F : C \multimap 2^C$ a continuous function with point compact, point convex, and consider a continuous semi norm p on E, then there exists an $x \in C$ such that $d(x, F(x)) = d(F(x), C)$.*

Recall that

$$d(A, B) = inf\{d(a, b) : a \in A \quad \text{and} \quad b \in B\}.$$

Sehgal and Singh proved the following result in [25].

Theorem 6.4. *Let C be a nonempty, approximatively compact convex subset of a locally convex Haurdorff topological vector space E and $F : C \to 2^C$ a continuous multifunction with nonempty closed convex values such that $F(C)$ is relatively compact.*

Then there is an $x \in C$ such that

$$d(x, F(x)) = d(F(x), C).$$

Continuity can not be replaced by upper semicontinuity as the following example illustrates (see [25]).

Example 6.1. Consider $X = R^2$ endowed with the Euclidean norm and $C = [0, 1] \times \{0\}$. Then C is a compact and convex set.

Define $F : C \to 2^{R^2}$ by

$$F(x,0) = \begin{cases} (0,1) & \text{if} \quad x \neq 0 \\ L = \text{the line segment}[(0,\ 1),\ (1,\ 0)] & \text{if} \quad x = 0 \end{cases}$$

F is a compact and convex valued map.

For any $A \subset R^2$

$$F^{-1}(A) = \begin{cases} \emptyset & \text{if} \quad A \cap L = \emptyset \\ C & if \quad (0,1) \in A \\ (0,0) & \text{if} \quad (0,1) \notin A, A \cap L \neq \emptyset \end{cases}$$

Thus F is an upper semicontinuous multifunction but it is not lower semicontinuous and $F(C)$ is compact set.

However, for any $(x, 0)$,

$d((x,0), F(x,0)) > 1 = d(F(x,0), C)$ if $x \neq 0$,

$d((0,0), F(0,0)) = \frac{\sqrt{2}}{2} \neq d(F(0,0), C) = 0$ if $x = 0$.

Thus the result doesn't hold true if F is not continuous.

The main application of Ky Fan's best approximation theorem allows to deduce fixed point theorems. However, for the multivalued case, there are some fixed point theorems for usc maps, while as Ky Fan's theorem requires a continuous map.

The following result improves earlier ones and only requires upper semicontinuity [19].

A subset K of X is called σ-compact if K is the countable union of compact sets. A nonempty topological space is acyclic if all of its reduced Cech homology groups over rationals vanish (or is empty) [19].

We say that $F : X \to Y$ is a Kakutani map, in symbols $F \in K(X, Y)$, if X is a topological space, Y a convex space and F is an upper semicontinuous multifunction with nonempty compact convex values.

F is an acyclic map, written $F \in V_c(X, Y)$ if F is upper semicontinuos with compact acyclic values. We write that $F \in V^+(X, Y)$ if, for any σ-compact subset K of X, there is a $G : K \to Y$, $G \in V(K, Y)$ such that $G(x) \subseteq F(x)$ for each $x \in K$.

$F \in V_c^+(X, Y)$ if, for any σ-compact subset K of X, there is a $G \in V_c(K, Y)$ such that $G(x) \subseteq F(x)$ for each $x \in K$.

The following result, due to Park, Singh and Watson is stated below (see [19]).

Lemma 6.1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $F \in V_c^+(X, X)$.*

If F is compact, then F has a fixed point.

Theorem 6.5. *Let C be a nonempty approximatively p-compact convex subset of a locally convex Hausdorff topological vector space E, and let $F \in V_c^+(C, E)$ be a compact multifunction. Then for each continuous semi norm p on E, there exists $(x_0, y_0) \in F$ such that $p(x_0 - y_0) \leq p(x - y_0)$ for all $x \in C$* [19].

The following example illustrates the theorem [19].

Example 6.2. Let $E \to R^2$ and $C = [0, 1] \times \{0\}$. Suppose $F \in V(C, E)$ is given by

$F(a, 0) = (0, 1)$, if $a \neq 0$, and

$F(a, 0) = [(0, 1), (1, 0)] \cup [(-1, 0), (0, 1)]$, if $a = 0$,

where $[a, b]$ stands for the line segment joining a and b in the plane.

Here F is not lower semicontinuous, but $x_0 = (0, 0)$ and $y_0 = (\frac{1}{2}, \frac{1}{2})$ satisfy the conclusion of the theorem.

One derives fixed point theorems for upper semicontinuous multifunctions.

7 Maximal elements in economics

In the end, we give the following.

The existence of maximal elements in mathematical economics is very useful in the study of fixed point theory, variational inequalities, approximation theory and complementarity problems.

Definition 7.1. Let X be a nonempty subset of a topological space and $F : X \to 2^X$ a multivalued map defined by $F(x) = \{y \in X | (y, x) \in F\}$.

If there exists $z \in X$ such that $F(z) = \emptyset$, then z is said to be a maximal element.

The following result is due to Ky Fan (see [8]).

Theorem 7.1. *Let C be a nonempty compact convex subset of R^n and $F : C \to 2^C$ a multifunction such that $x \notin F(x)$, $F(x)$ is convex for each x, and F has an open graph. Then F has a maximal element.*

The following result illustrates an application of maximal elements to approximation theory.

Theorem 7.2. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function.*

Then there exists $x_0 \in C$ such that

$$\|f(x_0) - x_0\| = d(f(x_0), C) \leq \|f(x_0) - y\| \quad \textit{for all} \quad y \in C.$$

Proof: Define F on C by letting $y \in F(x)$ if and only if $\|f(x) - y\| < \|f(x) - x\|$ for each $y \in C$.

Then, by Theorem 7.1, F has a maximal element since $x \notin F(x)$, F has an open graph and $F(x)$ is convex for each $x \in C$.

Hence $F(x_0) = \emptyset$ for some $x_0 \in C$, that is,

$$\|x_0 - f(x_0)\| \leq \|f(x_0) - y\| \quad \text{for all} \quad y \in C.$$

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A quadrature formula for integrals of highly oscillatory functions

Maria Carmela De Bonis and Pietro Pastore

Dedicated to our Professor Giuseppe Mastroianni for his 70th birthday

Abstract. In this paper, the authors introduce a quadrature rule to approximate integrals of highly oscillatory functions. They prove that it is stable and convergent. Moreover they give applications to the approximation of integrals of singular highly oscillatory functions on bounded and unbounded intervals. Finally they show some numerical tests in which they compare their results with the ones obtained with some of the methods available in literature.

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1 Introduction

Let us consider the integrals

$$I = \int_a^b F(x)g(\omega x)dx, \tag{1.1}$$

where $F(x)$ is a Riemann integrable function in $[a, b] \subset \mathbb{R}$, $g(\omega x)$ is an analytic highly oscillatory function and $\omega \in \mathbb{R}$ is the parameter of the oscillation.

This paper is in final form and no version of it will be submitted for publication elsewhere.

The numerical evaluation of the above integrals has interested several authors and actually it is a special chapter of the numerical integration. The interested reader can consult for example [3, 4, 6] and the references there in.

The following example is useful to point out the difficulties that appear in the computation of (1.1). We consider the following integral

$$I = \int_{-1}^{1} e^x \sin(5000x) dx. \tag{1.2}$$

If we apply a Gauss-Legendre quadrature rule we get

$$I = \sum_{k=1}^{m} G(x_k)\lambda_k + e_m(G) := \Phi_m(G) + e_m(G), \tag{1.3}$$

where $G(x) = e^x \sin(5000x)$, $x_k, k = 1, \ldots, m$, are the zeros of the m-th Legendre polynomial and $\lambda_k, k = 1, \ldots, m$, are the corresponding Christoffel numbers.
Since G is differentiable, we can write the error as (see [7, p. 333])

$$e_m(G) = \frac{G^{(2m)}(\xi_x)}{\gamma_m^2 (2m)!}, \tag{1.4}$$

where $\xi_x \in [-1, 1]$ and γ_m is the leading coefficient of the $m-$th orthonormal Legendre polynomial, and it satisfies the following estimate (see the proof of Theorem 3.4)

$$|e_m(G)| = \mathcal{O}\left[\left(\frac{3399}{m}\right)^{2m}\right].$$

Therefore, we can obtain some exact digits using a Gauss-Legendre quadrature rule based on a number of points greater than 3399. But it is not realistic.
On the other hand, looking at the graph of G in Figure 1, it seems that, working in finite arithmetics, a "small" error in the computation of x_k can produce a "big" error in the evaluation of $G(x_k)$, with an eventually change of sign. In conclusion, the Gaussian rules cannot give a reasonable approximation of the integrals (1.1). This fact is confirmed by the numerical test in double arithmetics shown in Table 1.

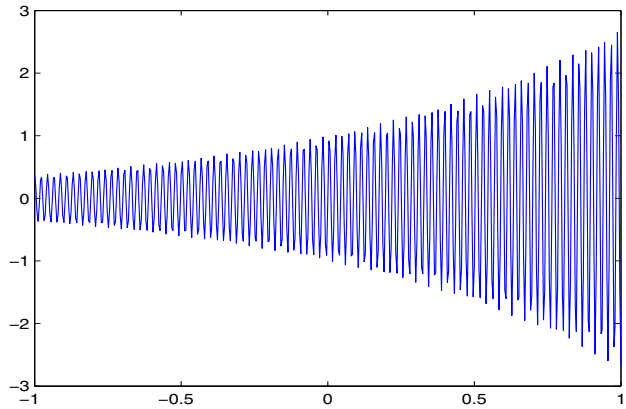


Figure 1: Graph of $G(x) = e^x \sin(5000x)$

m	$\Phi_m(G)$
8	0.428672207843679
16	−0.192123222625601
32	0.024304986655065
64	0.042348837355750
128	0.078391819539537
256	0.170674839281539
512	−0.013440192353402

Table 1: $G(x) = e^x \sin(5000x)$

In this paper we propose a new procedure to approximate the integrals (1.1). It is essentially based on the idea of Professor G. Mastroianni of dilating the integration interval, in order to reduce the oscillation of the function g, of writing the new integral as the sum of a certain number of integrals and of approximating each of them using a Gaussian rule.

We prove that such procedure is stable and convergent and we show how it can be also used to approximate integrals of singular highly oscillatory functions on bounded and unbounded intervals.

The paper is organized as follows: in Section 2, we give some definitions and preliminary result; in Section 3, we describe the quadrature rule to approximate the integrals (1.1) and state the theorems dealing with its stability and convergence; in sections 4 and 5, we show its application to approximate singular highly oscillatory integrals on bounded intervals; in Section 6, we consider the case of highly oscillatory integrals on unbounded intervals; Section 7 is dedicated to the proofs; in Section 8 we show some

numerical tests.

2 Preliminaries

We will denote by $C^0(A)$ the space of all continuous functions in an interval $A \subseteq \mathbb{R}$. For all $f \in C^0(A)$ we will write

$$||f||_A = \max_{x \in A} |f(x)|.$$

We will also consider functions belonging to the following Sobolev type space of index $r \in \mathbb{N}$, $r \geq 1$

$$W_r([a,b]) = \left\{ f \in C^0([a,b]) : f^{(r-1)} \in AC(a,b) \text{ and } ||f^{(r)}\varphi^r||_{[a,b]} < \infty \right\},$$

where $AC(a,b)$ denotes the set of all absolutely continuous functions in the open interval (a,b) and $\varphi(x) = \sqrt{(x-a)(b-x)}$.
We equip it with the following norm

$$||f||_{W_r([a,b])} = ||f||_{[a,b]} + ||f^{(r)}\varphi^r||_{[a,b]}.$$

We recall that for the remainder term of the gaussian rule (1.3) the following estimate holds true (see [7, p. 337])

$$|e_m(f)| \leq 4E_{2m-1}(f)_{[-1,1]}, \quad f \in C^0([-1,1]), \tag{2.1}$$

where

$$E_n(f)_A = \inf_{P \in \mathbb{P}_n} \|f - P\|_A$$

is the error of best approximation of a function f in the interval $A \subseteq \mathbb{R}$ by means of polynomials of degree at most n.
We recall that for $f \in W_r([a,b])$ the following estimate (see [7, p.172])

$$E_n(f)_{[a,b]} \leq \frac{\mathcal{C}}{n^r} ||f^{(r)}\varphi^r||_{[a,b]} \tag{2.2}$$

holds true, where $\mathcal{C}$ is independent of f, n and r.
In the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$.

3 The Quadrature Rule

As announced in the introduction, the procedure to obtain our quadrature rule consists of the following steps: first, we dilate the integration interval in order to reduce the oscillation of the function $g(\omega x)$; second, we choose a suitable partition of the new interval; third we transform each interval of the partition into $[-1,1]$; fourth we use a Gauss-Legendre quadrature rule to approximate the new integrals.

Letting $y=\omega x$ and choosing a suitable parameter $d\in\mathbb{R}$, we can write

$$I=\frac{1}{\omega}\int_{\omega a}^{\omega b}F\Big(\frac{y}{\omega}\Big)g(y)dy. \tag{3.1}$$

Note that the approximation of the integral (3.1) by means of a gaussian rule is not convenient because, greater is the interval $[\omega a,\omega b]$ and higher is the number of knots and weights that are necessary in order to achieve the required accuracy. Moreover, for large values of ω, the gaussian rule cannot be implemented.

Then we consider the following partition of the integration interval into subintervals of size d

$$I=\frac{1}{\omega}\sum_{j=0}^{S-1}\Big[\int_{\omega a+jd}^{\omega a+(j+1)d}F\Big(\frac{y}{\omega}\Big)g(y)dy\Big]+\frac{1}{\omega}\int_{\omega a+Sd}^{\omega b}F\Big(\frac{y}{\omega}\Big)g(y)dy,$$

where $S=\Big\lfloor\dfrac{\omega(b-a)}{d}\Big\rfloor$ ($\lfloor a\rfloor$ denotes the greatest integer smaller than a). Transforming the intervals $A_j=[\omega a+jd,\omega a+(j+1)d], j=0,\dots,S-1$, and $B=[\omega a+Sd,\omega b]$ into $[-1,1]$ we get

$$I=\frac{d}{2\omega}\sum_{j=0}^{S-1}\Big[\int_{-1}^{1}F_j(x)g_j(x)dx\Big]+\frac{\omega(b-a)-Sd}{2\omega}\int_{-1}^{1}\widetilde{F}(x)\widetilde{g}(x)dx, \tag{3.2}$$

where

$$F_j(x)=F\Big(\frac{\gamma_j(x)}{\omega}\Big),\ g_j(x)=g(\gamma_j(x)),\ \widetilde{F}(x)=F\Big(\frac{\delta(x)}{\omega}\Big),\ \widetilde{g}(x)=g(\delta(x)),$$
$$\gamma_j(x)=\frac{d}{2}\ (x+1)+\omega a+jd,\ \delta(x)=\Big(\frac{\omega(b-a)-Sd}{2}\Big)(x+1)+\omega a+Sd.$$

Now, all the integrals in (3.2) are well approximated by the gaussian rule (1.3) and we get

$$I=\Psi_m^{a,b}(F)+r_m^{a,b}(F), \tag{3.3}$$

where

$$\Psi_m^{a,b}(F) = \frac{d}{2\omega}\sum_{j=0}^{S-1}\Phi_m(F_j g_j) + \frac{\omega(b-a)-Sd}{2\omega}\ \Phi_m(\tilde{F}\tilde{g}) \tag{3.4}$$

and

$$r_m^{a,b}(F) = \frac{d}{2\omega}\sum_{j=0}^{S-1} e_m(F_j g_j) + \frac{\omega(b-a)-Sd}{2\omega}\ e_m(\tilde{F}\tilde{g}). \tag{3.5}$$

In conclusion, we approximate the integral I by the sum of $S+1$ gaussian rules. Both the theoretical error and the numerical one are the sum of $S+1$ errors. In particular, in order to reduce the global numerical error, it could be convenient to sum the $S+1$ terms in ascending order of magnitude. Moreover in order to avoid the numerical cancellation it is better that the first two terms of the sum have the same sign.

The following results establish the stability and the convergence of the quadrature rule $\Psi_m^{a,b}$:

Theorem 3.1. *If $F \in C^0([a,b])$ and $g \in C^0([\omega a, \omega b])$, then*

$$|\Psi_m^{a,b}(F)| \le (b-a)\max_{x\in[a,b]}|F(x)g(\omega x)|. \tag{3.6}$$

Proposition 3.2. *If $F \in C^0([a,b])$ and $g \in C^0([\omega a, \omega b])$, we have*

$$\begin{aligned} |r_m^{a,b}(F)| \ &\le\ \frac{2d}{\omega}\sum_{j=0}^{S-1}\Big[||F||_{A'_j}E_{m-1}(g)_{A_j} + 2||g||_{A_j}E_{m-1}(F)_{A'_j}\Big] \\ &+\ \frac{2d}{\omega}\Big[||F||_{B'}E_{m-1}(g)_B + 2||g||_B E_{m-1}(F)_{B'}\Big], \end{aligned} \tag{3.7}$$

where $A'_j = \big[a + j\frac{d}{\omega}, a + (j+1)\frac{d}{\omega}\big]$, $j = 0,\dots,S-1$, and $B' = \big[a + S\frac{d}{\omega}, b\big]$.

The following theorems give estimates of the error (3.5) for special classes of functions F.

Theorem 3.3. *If $F \in W_r([a,b]), r \ge 1$, and $g \in C^{2m}([\omega a, \omega b])$ then* (3.7) *becomes*

$$|r_m^{a,b}(F)| \le \frac{\mathcal{C}}{m^r}4\left((b-a)+\frac{d}{\omega}\right)||F||_{W_r([a,b])}||g||_{W_r([\omega a,\omega b])}, \tag{3.8}$$

where $\mathcal{C} \ne \mathcal{C}(m,F,g,d)$.

Theorem 3.4. *If $F \in C^{2m}([a,b])$ and $g \in C^{2m}([\omega a, \omega b])$ we have*

$$|r_m^{a,b}(F)| \leq$$
$$\leq \left(\frac{(b-a)}{2} + \frac{d}{2\omega}\right) \sup_{k=0,\ldots,2m} \left\{ \|F^{(k)}\|_{[a,b]} \|g^{(k)}\|_{[\omega a, \omega b]} \right\} \left(\frac{e\left(\frac{d}{2} + \frac{d}{2\omega}\right)}{4m} \right)^{2m}. \tag{3.9}$$

3.1 The choice of the parameter d

In the implementation of the quadrature rule $\Psi_m^{a,b}$ a crucial role is played by the parameter d used for partitioning the dilated integration interval. Smaller is the value chosen for d and higher is the number S of the intervals of the partition and then higher is the computational cost (Sm) of the quadrature rule. On the other hand, since when we transform the intervals of the partition into $[-1,1]$ the oscillating parameter of the function g becomes $\frac{d}{2}$, as shown in theorems 3.3 and 3.4, greater we take d and slower is the convergence of the error of the Gauss-Legendre rule used to approximate each integral of the partition.

For example, if F and g satisfy the hypotheses of Theorem 3.3, in order to compute the integral I with an accuracy of the order of 10^{-k}, a crude estimate of the parameter d could be obtained solving the inequality

$$\frac{d}{2\omega} \frac{1}{m^r} < 10^{-k}$$

for any fixed m (less than 500). However, looking at the numerical tests shown in Section 8, it results very pessimistic.

Therefore the optimal choice of the parameter d is still an open problem.

In next sections we will give some application of the above introduced quadrature rule.

4 A special class of Hilbert Transforms

We consider the following kind of integrals

$$H(f,t) = \int_a^b \frac{f(x)}{x-t}\, g(\omega x) dx, \tag{4.1}$$

where the integral is understood in the Cauchy principal value, f is a Hölder continuous function in $[a,b]$, $a<t<b$ and

$$g(\omega x)=\left\{\begin{array}{l}\sin(\omega x)\\ \cos(\omega x)\end{array}\right. , \quad \omega\in\mathbb{R}.$$

In order to approximate the integrals (4.1), we consider three different cases: $a<t\le a+\frac{1}{\omega}$, $a+\frac{1}{\omega}<t<b-\frac{1}{\omega}$ and $b-\frac{1}{\omega}\le t<b$. We start from the second case. We use the following decomposition of the integral (4.1):

$$\begin{aligned} H(f,t) &= \left\{\int_a^{t-\frac{1}{\omega}}+\int_{t-\frac{1}{\omega}}^{t+\frac{1}{\omega}}+\int_{t+\frac{1}{\omega}}^{b}\right\}\frac{f(x)}{x-t}\,g(\omega x)dx\\ &=: I_1+I_2+I_3. \end{aligned} \tag{4.2}$$

We note that only I_2 is a Cauchy singular integral. Therefore, to approximate the integrals I_1 and I_3 we can apply the quadrature formula described in the previous section. We obtain

$$I_1+I_3 = \Psi_m^{a,t-\frac{1}{\omega}}(F)+\Psi_m^{t+\frac{1}{\omega},b}(F)+r_m^{a,t-\frac{1}{\omega}}(F)+r_m^{t+\frac{1}{\omega},b}(F), \tag{4.3}$$

where

$$F(x)=\frac{f(x)}{x-t}. \tag{4.4}$$

Concerning I_2, making the change of variable $y=\omega x$, we get

$$I_2=\int_{\omega t-1}^{\omega t+1}\frac{f\left(\frac{y}{\omega}\right)}{y-\omega t}g(y)dy.$$

Transforming the integration interval into $[-1,1]$ and adding and subtracting $f(t)$ we have

$$\begin{aligned} I_2 &= \int_{-1}^{1}\frac{f\left(\frac{y}{\omega}+t\right)}{y}g(y+\omega t)dy=\int_{-1}^{1}\frac{f\left(\frac{y}{\omega}+t\right)-f(t)}{y}g(y+\omega t)dy\\ &+ \begin{cases}2f(t)\cos(\omega t)Si(1) & \text{if } g(x)=\sin(x)\\ -2f(t)\sin(\omega t)Si(1) & \text{if } g(x)=\cos(x),\end{cases} \end{aligned}$$

where

$$Si(x)=\int_0^x\frac{\sin(y)}{y}dy$$

is the special function sine integral. Approximating the integral by (1.3) based on an even number of knots, i.e. $m = 2n$, we obtain

$$I_2 = \Phi_{2n}(L_1) + \begin{cases} 2f(t)\cos(\omega t)Si(1) & \text{if } g(x) = \sin(x) \\ -2f(t)\sin(\omega t)Si(1) & \text{if } g(x) = \cos(x) \end{cases} + e_{2n}(L_1), \quad (4.5)$$

where

$$L_1(y) = \frac{f\left(\frac{y}{\omega} + t\right) - f(t)}{y}\, g(y + \omega t).$$

Therefore, when $a + \dfrac{1}{\omega} < t < b - \dfrac{1}{\omega}$, combining (4.3) and (4.5) with (4.2), we get

$$H(f,t) = \Theta_{2n}^{a,b}(f,t) + R_{2n}^{a,b}(f,t),$$

where

$$\begin{aligned} \Theta_{2n}^{a,b}(f,t) &= \Psi_{2n}^{a,t-\frac{1}{\omega}}(F) + \Psi_{2n}^{t+\frac{1}{\omega},b}(F) + \Phi_{2n}(L_1) \\ &+ \begin{cases} 2f(t)\cos(\omega t)Si(1) & \text{if } g(x) = \sin(x) \\ -2f(t)\sin(\omega t)Si(1) & \text{if } g(x) = \cos(x) \end{cases} \end{aligned} \quad (4.6)$$

and

$$R_{2n}^{a,b}(f,t) = r_{2n}^{a,t-\frac{1}{\omega}}(F) + r_{2n}^{t+\frac{1}{\omega},b}(F) + e_{2n}(L_1). \quad (4.7)$$

When $a < t \le a + \frac{1}{\omega}$ we decompose the integral (4.1) as follows:

$$\begin{aligned} H(f,t) &= \left\{ \int_a^{2t-a} + \int_{2t-a}^b \right\} \frac{f(x)}{x-t} g(\omega x)dx \\ &=: I_1' + I_2'. \end{aligned} \quad (4.8)$$

Now only I_1' is a Cauchy singular integral. Then we approximate the integral I_2' by the quadrature rule (3.3) and obtain:

$$I_2' = \Psi_m^{2t-a,b}(F) + r_m^{2t-a,b}(F), \quad (4.9)$$

where F is defined in (4.4).
Concerning the integral I_1', making the change of variable $y = \omega x$, we get

$$I_1' = \int_{\omega a}^{2\omega t - \omega a} \frac{f\left(\frac{y}{\omega}\right)}{y - \omega t} g(y)dy.$$

Transforming the integration interval into $[-1, 1]$ we obtain

$$I_1' = \int_{-1}^{1} L_2(y)dy + \begin{cases} 2f(t)\cos(\omega t)Si(\omega(t-a)) & \text{if } g(x) = \sin(x) \\ -2f(t)\sin(\omega t)Si(\omega(t-a)) & \text{if } g(x) = \cos(x), \end{cases} \quad (4.10)$$

where

$$L_2(y) = \frac{f((t-a)y+t) - f(t)}{y}\ g(\omega(t-a)y + \omega t).$$

Therefore, for $a < t \leq a + \frac{1}{\omega}$, approximating the integral in (4.10) by the rule (1.3) based on an even number of knots and combining (4.10) and (4.9) with (4.8), we get

$$H(f,t) = \Theta_{2n}^{a,b}(f,t) + R_{2n}^{a,b}(f,t),$$

where

$$\Theta_{2n}^{a,b}(f,t) = \Phi_{2n}(L_2) + \Psi_{2n}^{2t-a,b}(F) + \begin{cases} 2f(t)\cos(\omega t)Si(\omega(t-a)) & \text{if } g(x) = \sin(x) \\ -2f(t)\sin(\omega t)Si(\omega(t-a)) & \text{if } g(x) = \cos(x) \end{cases}$$

and

$$R_{2n}^{a,b}(f,t) = e_{2n}(L_2) + r_{2n}^{2t-a,b}(F).$$

Finally, in the case where $b - \frac{1}{\omega} \leq t < b$ we decompose the integral as follows:

$$\begin{aligned} H(f,t) &= \left\{ \int_a^{-b+2t} + \int_{-b+2t}^{b} \right\} \frac{f(x)}{x-t} g(\omega x)dx \\ &=: I_1'' + I_2'' \end{aligned}$$

and, proceeding as done for the previous cases, we get

$$H(f,t) = \Theta_{2n}^{a,b}(f,t) + R_{2n}^{a,b}(f,t),$$

where

$$\Theta_{2n}^{a,b}(f,t) = \Phi_{2n}(L_3) + \Psi_{2n}^{a,-b+2t}(F) + \begin{cases} 2f(t)\cos(\omega t)Si(\omega(b-t)) & \text{if } g(x) = \sin(x) \\ -2f(t)\sin(\omega t)Si(\omega(b-t)) & \text{if } g(x) = \cos(x) \end{cases}$$

and

$$R_{2n}^{a,b}(f,t) = e_{2n}(L_3) + r_{2n}^{a,-b+2t}(F),$$

with F defined in (4.4) and

$$L_3(y) = \frac{f((b-t)y+t) - f(t)}{y}\ g(\omega(b-t)y + \omega t).$$

The following theorem shows that the quadrature rule $\Theta_{2n}^{a,b}$ is stable and convergent

Theorem 4.1. *If f is a Hölder continuous function in $[a,b]$, for $a<t<b$ fixed, we have*

$$|\Theta_{2n}^{a,b}(f,t)| \le \mathcal{C}\|f\|_{[a,b]}\log(2n). \tag{4.11}$$

Moreover, if $f \in W_r([a,b]), r \ge 1$, then

$$|R_{2n}^{a,b}(f,t)| = \mathcal{O}\left(\frac{\log(2n)}{(2n)^r}\right), \quad a<t<b \text{ fixed.} \tag{4.12}$$

Here the constant $\mathcal{C}$ and the constants in "$\mathcal{O}$" do not depend on f, n and t.

5 The case of integrals of weakly singular highly oscillatory functions

If we consider the integrals

$$K(f,t) = \int_a^b \frac{f(x)}{|x-t|^\mu}\, g(\omega x)dx,$$

where $f \in C^0([a,b])$, $a<t<b$, $0<\mu<1$ and

$$g(\omega x) = \begin{cases} \sin(\omega x) \\ \cos(\omega x) \end{cases}, \quad \omega \in \mathbb{R},$$

we can proceed analogously to the case of the Hilbert transform $H(f,t)$. Then we get:

1. case $a+\dfrac{1}{\omega} < t < b-\dfrac{1}{\omega}$

$$\begin{aligned} K(f,t) &= \Psi_m^{a,t-\frac{1}{\omega}}(G) + \Psi_m^{t+\frac{1}{\omega},b}(G) + r_m^{a,t-\frac{1}{\omega}}(G) + r_m^{t+\frac{1}{\omega},b}(G) \\ &+ \omega^{\mu-1}\int_{-1}^{1} \frac{f\left(\frac{y}{\omega}+t\right)}{|y|^\mu} g(y+\omega t)dy, \end{aligned}$$

where

$$G(x) = \frac{f(x)}{|x-t|^\mu}. \tag{5.1}$$

Approximating the function $h_1(y) = \omega^{\mu-1} f\left(\frac{y}{\omega}+t\right) g(y+\omega t)$ by the Lagrange polynomial based on the zeros $t_1,\dots,t_m$ of the $m-$th orthonormal Chebyshev polynomial of first kind T_m, we get

$$\begin{aligned} \int_{-1}^{1} h_1(y)|y|^{-\mu}dy &= \frac{\pi}{m}\sum_{k=1}^{m} h_1(t_k)\sum_{j=0}^{m-1} T_j(t_k)\int_{-1}^{1} T_j(x)|x|^{-\mu}dx + \bar{r}_m(h_1) \\ &=: \bar{\Phi}_m(h_1) + \bar{r}_m(h_1). \end{aligned}$$

We compute the modified moments $M_j = \int_{-1}^{1} T_j(x)|x|^{-\mu}dx$ using the three-terms recurrence relation proposed in [10]. Consequently, we obtain

$$K(f,t) = \Lambda_m^{a,b}(f,t) + \bar{R}_m^{a,b}(f,t),$$

where

$$\Lambda_m^{a,b}(f,t) = \Psi_m^{a,t-\frac{1}{\omega}}(G) + \Psi_m^{t+\frac{1}{\omega},b}(G) + \bar{\Phi}_m(h_1)$$

and

$$\bar{R}_m^{a,b}(f,t) = r_m^{a,t-\frac{1}{\omega}}(G) + r_m^{t+\frac{1}{\omega},b}(G) + \bar{r}_m(h_1);$$

2. case $a < t \leq a + \frac{1}{\omega}$

$$K(f,t) = \Lambda_m^{a,b}(f,t) + \bar{R}_m^{a,b}(f,t),$$

where

$$\Lambda_m^{a,b}(f,t) = \Psi_m^{2t-a,b}(G) + \bar{\Phi}_m(h_2)$$

and

$$\bar{R}_m^{a,b}(f,t) = r_m^{2t-a,b}(G) + \bar{r}_m(h_2),$$

with G defined in (5.1) and $h_2(y) = \omega^{\mu-1} f((t-a)y+t)g(\omega(t-a)y+\omega t)$;

3. case $b - \frac{1}{\omega} \leq t < b$

$$K(f,t) = \Lambda_m^{a,b}(f,t) + \bar{R}_m^{a,b}(f,t),$$

where

$$\Lambda_m^{a,b}(f,t) = \Psi_m^{a,-b+2t}(G) + \bar{\Phi}_m(h_3)$$

and

$$\bar{R}_m^{a,b}(f,t) = r_m^{a,-b+2t}(G) + \bar{r}_m(h_3),$$

with G defined in (5.1) and $h_3(y) = \omega^{\mu-1} f((b-t)y+t)g(\omega(b-t)y+\omega t)$.

The stability and the convergence of the quadrature rule $\Lambda_m^{a,b}$ are stated in the following theorem.

Theorem 5.1. *If $f \in C^0([a,b])$, for $a < t < b$ fixed, we have*

$$|\Lambda_m^{a,b}(f,t)| \leq \mathcal{C}\|f\|_{[a,b]}.$$

Moreover, if $f \in W_r([a,b]), r \geq 1$, then

$$|\bar{R}_m^{a,b}(f,t)| = \mathcal{O}\left(m^{-r}\right), \quad a < t < b \text{ fixed.}$$

Here the constant $\mathcal{C}$ and the constants in "$\mathcal{O}$" do not depend on f, m and t.

6 Integrals on unbounded intervals

In this section, we show how it is possible to apply the previous quadrature rules to approximate integrals of the following kind

$$\bar{I} = \int_0^{+\infty} F(x)\ g(\omega x)dx,$$

under the condition

$$\exists M > 0 \quad \text{s.t.,} \quad \text{for some } \eta > 0, \quad |F(x)| < \frac{\mathcal{C}}{x^{\eta+1}} \quad \forall x > M. \tag{6.1}$$

In fact, we can write

$$\bar{I} \;=\; \int_0^{M} F(x)g(\omega x)dx + \int_M^{+\infty} F(x)g(\omega x)dx = \bar{I}_1 + \bar{I}_2.$$

Since, under the assumption (6.1), we have

$$\left| \int_M^{+\infty} F(x)g(\omega x)dx \right| \leq \frac{\mathcal{C}}{M^{\eta}},$$

we neglect $\bar{I}_2$ and approximate only $\bar{I}_1$ using the quadrature rule $\Psi_m^{0,M}$ if the function F is smooth, the quadrature rule $\Theta_{2n}^{0,M}$ if $F(x) = \frac{f(x)}{x-t}$ and the quadrature rule $\Lambda_m^{0,M}$ if $F(x) = f(x)|x-t|^{-\mu}, 0 < \mu < 1$. Then, we get

$$\bar{I} = \begin{cases} \Psi_m^{0,M}(F) + r_m^{0,M}(F) \\ \\ \Theta_{2n}^{0,M}(f,t) + R_{2n}^{0,M}(f,t) \quad + \dfrac{1}{M^{\eta}}. \\ \\ \Lambda_m^{0,M}(f,t) + \bar{R}_m^{0,M}(f,t) \end{cases}$$

Using the same argument we can also treat the following type of integrals

$$I^* = \int_{-\infty}^{+\infty} F(x)g(\omega x)dx,$$

under the condition

$$\exists M > 0 \quad \text{s.t.,} \quad \text{for some } \eta > 0, \quad |F(x)| < \frac{\mathcal{C}}{x^{\eta+1}} \quad \forall\ |x| > M.$$

In fact, we can consider the following decomposition of I^*:

$$I^* \;=\; \int_{-\infty}^{M} F(x)g(\omega x)dx + \int_{-M}^{M} F(x)g(\omega x)dx + \int_M^{+\infty} F(x)g(\omega x)dx.$$

7 Proofs

Proof of Theorem 3.1. We have

$$\begin{aligned}|\Psi_m^{a,b}(F)| &\leq \frac{1}{\omega}\sum_{k=1}^{m}\left\{\frac{d}{2}\left[\sum_{j=0}^{S-1}\left|F\left(\frac{\gamma_j(x_k)}{\omega}\right)g(\gamma_j(x_k))\right|\right]\right.\\ &+ \left.\left(\frac{\omega(b-a)-Sd}{2\omega}\right)\left|F\left(\frac{\delta(x_k)}{\omega}\right)g(\delta(x_k))\right|\right\}\lambda_k\\ &\leq (b-a)\max_{x\in[a,b]}|F(x)g(\omega x)|.\end{aligned}$$

Then (3.6) follows. □

Proof of Proposition 3.2. By (3.5), using (2.1) and the following inequality [7, p. 384]

$$E_m(fg)_A \leq ||g||_A E_{\lfloor\frac{m}{2}\rfloor}(f)_A + 2||f||_A E_{\lfloor\frac{m}{2}\rfloor}(g)_A,$$

we get

$$\begin{aligned}|r_m^{a,b}(F)| &\leq \frac{2d}{\omega}\sum_{j=0}^{S-1}E_{2m-1}(F_jg_j)_{[-1,1]} + 4\Big(\frac{\omega(b-a)-Sd}{2\omega}\Big)E_{2m-1}(\widetilde{F}\widetilde{g})_{[-1,1]}\\ &\leq \frac{2d}{\omega}\sum_{j=0}^{S-1}\Big[||F_j||_{[-1,1]}E_{m-1}(g_j)_{[-1,1]} + 2||g_j||_{[-1,1]}E_{m-1}(F_j)_{[-1,1]}\Big]\\ &+ \frac{2d}{\omega}\Big[||\widetilde{F}||_{[-1,1]}E_{m-1}(\widetilde{g})_{[-1,1]} + 2||\widetilde{g}||_{[-1,1]}E_{m-1}(\widetilde{F})_{[-1,1]}\Big]\\ &= \frac{2d}{\omega}\sum_{j=0}^{S-1}\Big[||F||_{A_j'}E_{m-1}(g)_{A_j} + 2||g||_{A_j}E_{m-1}(F)_{A_j'}\Big]\\ &+ \frac{2d}{\omega}\Big[||F||_{B'}E_{m-1}(g)_B + 2||g||_BE_{m-1}(F)_{B'}\Big].\end{aligned}$$

□

Proof of Theorem 3.3. Estimate (3.8) easily follows by (3.7), taking into account that, under our assumptions on F and g, (2.2) holds true. □

Proof of Theorem 3.4. In order to prove (3.9) we start from (3.5) and estimate the errors $e_m(F_jg_j)$ and $e_m(\tilde{F}\tilde{g})$ using (1.4). Under our assumptions on F and g, we have

$$|[F_j(x)g_j(x)]^{(2m)}| \leq \sum_{k=0}^{2m}\binom{2m}{k}\left|\left[F\left(\frac{\gamma_j(x)}{\omega}\right)\right]^{(2m-k)}\right|\left|[g(\gamma_j(x))]^{(k)}\right|$$

$$\leq \sup_{k=0,\dots,2m} \left\{ \|F^{(k)}\|_{[a,b]} \|g^{(k)}\|_{[\omega a,\omega b]} \right\} \sum_{k=0}^{2m} \binom{2m}{k} \left(\frac{d}{2\omega}\right)^{2m-k} \left(\frac{d}{2}\right)^{k}$$

$$= \sup_{k=0,\dots,2m} \left\{ \|F^{(k)}\|_{[a,b]} \|g^{(k)}\|_{[\omega a,\omega b]} \right\} \left(\frac{d}{2\omega} + \frac{d}{2}\right)^{2m}.$$

Moreover, taking into account that [8] $\lim_m 2^{-m}\gamma_m = \frac{1}{\sqrt{\pi}}$ and using the well-known Stirling formula (see [11])

$$\left(\frac{m}{e}\right)^m \sqrt{2\pi m}\; e^{-\frac{1}{12m}} \leq m! \leq \left(\frac{m}{e}\right)^m \sqrt{2\pi m}\; e^{-\frac{1}{12m+1}}, \quad m = 1, 2, \dots,$$

we get

$$|e_m(F_j g_j)| \leq \sup_{k=0,\dots,2m} \left\{ \|F^{(k)}\|_{[a,b]} \|g^{(k)}\|_{[\omega a,\omega b]} \right\} \left[\frac{\left(\frac{d}{2} + \frac{d}{2\omega}\right) e}{4m} \right]^{2m} \frac{\sqrt{\pi} e^{\frac{1}{24m}}}{2\sqrt{m}}.$$

Since an analogous estimate holds true for $e_m(\tilde{F}\tilde{g})$, (3.9) follows. □

In order to prove Theorem 4.1 we need the following

Proposition 7.1. *Let $F \in W_r([a,b]), r \geq 1$. For $n > 1, c > 0$ and t such that $a < t + cu < b$, we set*

$$\Upsilon(F,t,c) = \int_{-1}^{1} \frac{F(t+cu) - F(t)}{u} g(u\ c\ \omega + \omega t) du$$

and

$$\Upsilon_{2n}(F,t,c) = \sum_{i=1}^{2n} \frac{F(t+cx_i) - F(t)}{x_i} g(x_i c\ \omega + \omega t) \lambda_i.$$

Then, we have

$$|\Upsilon(F,t,c) - \Upsilon_{2n}(F,t,c)| \leq \mathcal{C} \frac{\log(2n)}{(2n)^r} \|F^{(r)} \varphi^r\|_{[a,b]}, \tag{7.1}$$

where $\mathcal{C}$ is a positive constant independent of n and F.

Proof. Setting $N = 2n$ and denoting by $L_N(f,x)$ the Lagrange polynomial interpolating the function f on the zeros of $N-$th Legendre polynomial, we can write

$$\begin{aligned}\Upsilon(F,t,c)-\Upsilon_N(F,t,c)=&\\ &=\quad\Big[\int_{-1}^{1}\frac{\Delta_{cu}F(t)}{u}L_N(g(\cdot c\ \omega+\omega t),u)du\\ &-\quad\sum_{i=1}^{N}\frac{\Delta_{cx_i}F(t)}{x_i}g(x_ic\ \omega+\omega t)\lambda_i\Big]\\ &+\quad\int_{-1}^{1}\frac{\Delta_{cu}F(t)}{u}[g(uc\ \omega+\omega t)-L_N(g(\cdot c\ \omega+\omega t),u)]du\\ &:=\quad\sigma_N(F,t)+\rho_N(F,t).\end{aligned}\tag{7.2}$$

Taking into account that (see [7, p. 54])

$$|g(uc\ \omega+\omega t)-L_N(g(\cdot c\ \omega+\omega t),u)|\le\frac{\mathcal{C}}{2^NN!}\frac{1}{\sqrt[4]{1-u^2}},$$

we have

$$|\rho_N(F,t)|\quad\le\quad\frac{\mathcal{C}}{2^NN!}\int_{-1}^{1}\left|\frac{\Delta_{cu}F(t)}{u}\right|(1-u^2)^{-\frac14}du.\tag{7.3}$$

Now we prove that

$$\int_{-1}^{1}\left|\frac{\Delta_{cu}F(t)}{u}\right|(1-u^2)^{-\frac14}du\le\mathcal{C}\|F\|_{[a,b]}\log N+\frac{\mathcal{C}}{N^2}\|F'\varphi\|_{[a,b]}.\tag{7.4}$$

To this end we consider the following decomposition

$$\begin{aligned}\int_{-1}^{1}\left|\frac{\Delta_{cu}F(t)}{u}\right|\frac{1}{(1-u^2)^{\frac14}}du=&\\ &=\quad\left\{\int_{-1}^{-\frac{1}{N^2}}+\int_{\frac{1}{N^2}}^{1}+\int_{-\frac{1}{N^2}}^{\frac{1}{N^2}}\right\}\left|\frac{\Delta_{cu}F(t)}{u}\right|\frac{1}{(1-u^2)^{\frac14}}du\\ &:=\quad A+B+C.\end{aligned}\tag{7.5}$$

Concerning A, we have

$$\begin{aligned}|A|\quad&\le\quad\|F\|_{[a,b]}\left\{\int_{-1}^{-\frac12}+\int_{-\frac12}^{-\frac{1}{N^2}}\right\}\frac{1}{|u|(1+u)^{\frac14}}du\\ &\le\quad\mathcal{C}\|F\|_{[a,b]}\log N.\end{aligned}\tag{7.6}$$

Analogously, we obtain

$$|B| \leq \mathcal{C}\|F\|_{[a,b]} \log N. \tag{7.7}$$

Concerning C, we get

$$\begin{aligned} |C| &\leq \mathcal{C} \int_{-\frac{c}{N^2}}^{\frac{c}{N^2}} \left| \frac{F(t+u)-F(t)}{u} \right| du \\ &\leq \mathcal{C} \int_0^{\frac{c}{N^2}} \int_{t-v}^{t} |(F'\varphi)(u)| \frac{du}{\varphi(u)} \frac{dv}{v} + \mathcal{C} \int_0^{\frac{c}{N^2}} \int_{t}^{t+v} |(F'\varphi)(u)| \frac{du}{\varphi(u)} \frac{dv}{v} \\ &\leq \frac{\mathcal{C}}{N^2} \|F'\varphi\|_{[a,b]}. \end{aligned} \tag{7.8}$$

Combining (7.6)-(7.8) with (7.5), (7.4) follows and then, by (7.3), we have

$$|\rho_N(g,t)| \leq \frac{\mathcal{C}}{2^N N!} [\|F\|_{[a,b]} + \|F'\varphi\|_{[a,b]}] \log N. \tag{7.9}$$

Let $P \in \mathbb{P}_{N+1}$ be the polynomial of best approximation of F, since $\dfrac{\Delta_{cu}P(t)}{u} L_N(g(\cdot c\,\omega+\omega t, u)) \in \mathbb{P}_{2N-1}$ and $L_N(g(\cdot c\,\omega+\omega t, x_i) = g(x_i c\omega+\omega t)$, the Gaussian rule is exact and we can write

$$\begin{aligned} \sigma_N(F,t) &= \sigma_N(F-P,t) \\ &= \int_{-1}^{1} \frac{\Delta_{cu}(F-P)(t)}{u} L_N(g(\cdot c\,\omega+\omega t), u) du \\ &- \sum_{i=1}^{N} \frac{\Delta_{cx_i}(F-P)(t)}{x_i} g(x_i c\,\omega+\omega t)\lambda_i \\ &= \int_{-1}^{1} \frac{\Delta_{cu}(F-P)(t)}{u} g(uc\,\omega+\omega t) du - \rho_N(F-P,t) \\ &- \sum_{i=1}^{N} \frac{\Delta_{cx_i}(F-P)(t)}{x_i} g(x_i c\,\omega+\omega t)\lambda_i \\ &:= \Sigma_1 - \rho_N(F-P,t) - \Sigma_2. \end{aligned} \tag{7.10}$$

Concerning Σ_1, proceeding as in the proof of (7.4), we obtain

$$|\Sigma_1| \leq \mathcal{C} E_N(F)_{([a,b])} \log N + \frac{\mathcal{C}}{N^2} \|(F-P)'\varphi\|_{[a,b]}. \tag{7.11}$$

By (7.9), we get

$$|\rho_N(F-P,t)| \leq \frac{\mathcal{C}}{2^N N!} [E_N(F)_{([a,b])} + \|(F-P)'\varphi\|_{[a,b]}] \log N \tag{7.12}$$

Moreover, by [2, (5.16)], we have

$$|\Sigma_2| \leq \mathcal{C}E_N(F)_{([a,b])} \sum_{i=1}^{N} \frac{\lambda_i}{|x_i|} \leq \mathcal{C}E_N(F)_{([a,b])} \log N. \tag{7.13}$$

Substituting (7.11)-(7.13) into (7.10), we obtain

$$|\sigma_N(F,t)| \leq \mathcal{C}[E_N(F)_{([a,b])} + \frac{1}{N^2}\|(F-P)'\varphi\|_{[a,b]}] \log N.$$

Since by [5] the following inequality

$$\|(F-P)'\varphi\|_{[a,b]} \leq \mathcal{C}NE_N(F)_{([a,b])} + \mathcal{C}E_{N-1}(F')_{\varphi,([a,b])} \leq \mathcal{C}E_{N-1}(F')_{\varphi,([a,b])}$$

holds true, where

$$E_n(G)_{\varphi,([a,b])} = \inf_{P\in\mathbb{P}_n} \|(G-P)\varphi\|_{[a,b]}$$

and $\mathcal{C} \neq \mathcal{C}(N,F,P)$, by our assumptions on the function F and (2.2) we get

$$|\sigma_N(F,t)| \leq \frac{\mathcal{C}}{N^r}\|F^{(r)}\varphi^r\|_{[a,b]} \log N. \tag{7.14}$$

Finally, combining (7.9) and (7.14) with (7.2), (7.1) follows. □

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. We consider only the case $a + \frac{1}{\omega} < t < b - \frac{1}{\omega}$, being the proof in the other cases similar. By (4.6) we have

$$|\Theta_{2n}^{a,b}(f,t)| \leq |\Psi_{2n}^{a,t-\frac{1}{\omega}}(F)| + |\Psi_{2n}^{t+\frac{1}{\omega},b}(F)| + |\Phi_{2n}(L_1)| + \mathcal{C}\|f\|_{[a,b]}.$$

Taking into account that $F(x) = f(x)(x-t)^{-1}$ in the intervals $[a, t-\frac{1}{\omega}]$ and $[t+\frac{1}{\omega}, b]$ has the same regularity of f, using Theorem 3.1 we get

$$|\Psi_{2n}^{a,t-\frac{1}{\omega}}(F)| + |\Psi_{2n}^{t+\frac{1}{\omega},b}(F)| \leq \mathcal{C}\|f\|_{[a,b]}.$$

Moreover, since using [2, (5.16)] we have

$$|\Phi_{2n}(L_1)| \leq 2\|f\|_{[a,b]} \sum_{i=1}^{2n} \frac{\lambda_i}{|x_i|} \leq \mathcal{C}\|f\|_{[a,b]} \log(2n),$$

(4.11) follows. Concerning (4.12), by (4.7) we have

$$|R_{2n}^{a,b}(f,t)| \leq |r_{2n}^{a,t-\frac{1}{\omega}}(F)| + |r_{2n}^{t+\frac{1}{\omega},b}(F)| + |e_{2n}(L_1)|.$$

Now, using Theorem 3.3 we get

$$|r_{2n}^{a,t-\frac{1}{\omega}}(F)| + |r_{2n}^{t+\frac{1}{\omega},b}(F)| \leq \frac{\mathcal{C}}{(2n)^r}$$

and, by Proposition 7.1, we deduce

$$|e_{2n}(L_1)| = \left|\Upsilon\left(f,t,\frac{1}{\omega}\right) - \Upsilon_{2n}\left(f,t,\frac{1}{\omega}\right)\right| \leq \frac{\mathcal{C}}{(2n)^r}\log(2n).$$

Then (4.12) is proved. □

Proof of Theorem 5.1. We prove the theorem only in the case $a + \frac{1}{\omega} < t < b - \frac{1}{\omega}$. The proof in the other cases is similar. As for the proof of Theorem 4.1, taking into account theorems 3.1 and 3.3, we need only to prove that

$$|\bar{\Phi}_m(h_1)| \leq \mathcal{C}\|f\|_{[a,b]} \tag{7.15}$$

and

$$|\bar{r}_m(h_1)| \leq \frac{\mathcal{C}}{m^r}. \tag{7.16}$$

We can write

$$\bar{\Phi}_m(h_1) = \int_{-1}^{1} \frac{L_m^{-\frac{1}{2},-\frac{1}{2}}(h_1,x)}{|x|^\mu}dx,$$

where $L_m^{-\frac{1}{2},-\frac{1}{2}}(f,x)$ denotes the Lagrange polynomial of a function f based on the zeros of the Chebyshev polynomials of the first kind. Since by a result due to Nevai in [9], for $0 < \mu < 1$, we have

$$\int_{-1}^{1} \frac{|L_m^{-\frac{1}{2},-\frac{1}{2}}(f,x)|}{|x|^\mu}dx \leq \mathcal{C}\|f\|_{[-1,1]}, \tag{7.17}$$

we deduce

$$|\bar{\Phi}_m(h_1)| \leq \mathcal{C}\|h_1\|_{[-1,1]} \leq \mathcal{C}\|f\|_{[a,b]},$$

i.e. (7.15). In order to prove (7.16), denoting by $P_{m-1} \in \mathbb{P}_{m-1}$ the polynomial of best approximation of h_1, we write

$$\begin{aligned}\bar{r}_m(h_1) &= \int_{-1}^{1} \frac{h_1(x) - L_m^{-\frac{1}{2},-\frac{1}{2}}(h_1,x)}{|x|^\mu}dx \\ &= \int_{-1}^{1} \frac{h_1(x) - P_{m-1}(x)}{|x|^\mu}dx + \int_{-1}^{1} \frac{L_m^{-\frac{1}{2},-\frac{1}{2}}(h_1 - P_{m-1},x)}{|x|^\mu}dx.\end{aligned}$$

Using (7.17) again, we get

$$|\bar{r}_m(h_1)| \leq \mathcal{C}\|h_1 - P_{m-1}\|_{[-1,1]} = \mathcal{C}E_{m-1}(h_1)_{[-1,1]} \leq \mathcal{C}\frac{\|f^{(r)}\varphi^r\|_{[a,b]}}{m^r},$$

i.e. (7.16). The proof is now complete. □

8 Numerical tests

In this section we show by, some examples, that our theoretical results are confirmed by the numerical tests. As announced in Subsection 3.1, the convergence order of the error of the proposed quadrature rules depends not only on the smoothness of the functions F and g but also on the value chosen for d.

We will denote by $CC = Sm = \lfloor \frac{\omega(b-a)}{d} \rfloor m$ the computational cost of the quadrature rules in terms of the number of the evaluations of the functions F.

When we don't know the exact values of the integrals, we will think as exact their approximate values obtained for $m = 300$ and in all the tables we will report only the digits which are correct according to them.

Example 1 As first example we consider the integral

$$\int_{-1}^{1} e^x \sin(\omega x) dx = \frac{\omega \cos\omega + \sin\omega + e^2(\sin\omega - \omega\cos\omega)}{e(1+\omega^2)},$$

that is (1.2) for $\omega = 5000$. As you can see in Table 2, applying our procedure with $d = 16$ and $m = 14$, we get the machine precision in double arithmetic.

m	$r_m^{-1,1}(F) \quad d = 16$
8	$7.23e-008$
14	$1.21e-016$

Table 2: $F(x) = e^x$, $\omega = 5000$

Example 2 Given the integral

$$\int_0^{2\pi} x\cos(50x)\sin(\omega x)dx = \\ = \begin{cases} \dfrac{2\omega\pi(2500-\omega^2)\cos(2\omega\pi) + (2500+\omega^2)\sin(2\omega\pi)}{(2500-\omega^2)^2} & \omega \neq 50, \\ -\dfrac{\pi}{100} & \omega = 50, \end{cases} \tag{8.1}$$

in case when $\omega = 30$, in Table 3 we compare the error of our quadrature rule with the one obtained using the Price's method and showed in [3, p. 154]. In Table 4 we show that for $\omega = 1000$ the proposed quadrature rule gives

an approximation of (8.1) with the same precision. As announced in Subsection 3.1, increasing d we need to increase also m but the computational cost CC decreases.

$d = 4$, $CC = 613$ $r_{13}^{0,2\pi}(F)$	$d = 16$, $CC = 330$ $r_{28}^{0,2\pi}(F)$	Price's method
$1.94e-016$	$1.8e-016$	$7.67e-003$

Table 3: $F(x) = x\cos(50x)$, $\omega = 30$

m	CC	d	$r_m^{0,2\pi}(F)$
7	21992	2	$5.2e-018$
9	14138	4	$1.1e-017$
13	8169	10	$6.9e-018$
16	5891	16	$4.3e-018$

Table 4: $F(x) = x\cos(50x)$, $\omega = 1000$

Example 3 Now we consider the following integral on the the real semiaxis

$$\int_0^\infty e^{-x}\sin(\omega x)dx = \frac{\omega}{1+\omega^2}.$$

Since $F(x) = e^{-x}$ satisfies (6.1) and $|F(x)| \leq 10^{-16}$ for $x > 37$, we approximate the above integral using the rule $\Psi_m^{0,37}$. For $\omega = 100$, in Table 5 we compare our results with the ones showed in [3, p. 238-240] and obtained using the methods proposed by Gustafson and Dahlquist (GD) and by Krylov and Skoblya (KS). In Table 6 we show our results obtained for $\omega = 1000$.

$r_{16}^{0,37}(F)$ $d = 16$	GD	KS
$3.46e-018$	$9.8e-003$	$9.9e-007$

Table 5: $F(x) = e^{-x}$, $\omega = 100$

m	$r_m^{0,37}(F)$ $d = 16$
8	$1.02e-006$
15	$3.34e-017$

Table 6: $F(x) = e^{-x}$, $\omega = 1000$

Example 4 We take the following integral on the real line

$$\int_{-\infty}^\infty xe^{-2|x|}e^{\sin(\omega x)}dx.$$

In this case $|xe^{-2|x|}| \leq 10^{-16}$ for $|x| > 20$. In Table 7 we show our results obtained choosing $d = 10$.

m	$\Psi_m^{-20,20}(F) \quad d = 10$
20	$9.029e-009$
32	$9.02937116e-009$
40	$9.029371163e-009$

Table 7: $F(x) = xe^{-2|x|}$, $\omega = 1000$

Example 5 Now we consider the following Hilbert transform

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x-\frac{1}{2}} \cos(\omega x) dx.$$

For $\omega = 100$, in Table 8 we compare our results with the ones obtained using the quadrature rule I_{2n}^1 proposed in [1]. Since the function $f(x) = \sqrt{1-x^2}$ is not smooth we need to increase m to reach exact digits. In the case where $\omega = 5000$ the approximations obtained using our method are shown in Table 9.

$2n$	$\Theta_{2n}^{-1,1}(f, 0.5) \quad d = 4$	$I_{2n}^1(0.5)$
8	0.71221	0.71
16	0.71221	0.712
32	0.712213	0.712
64	0.712213	0.71221
128	0.712213598	0.71221

Table 8: $f(x) = \sqrt{1-x^2}$, $\omega = 100$

$2n$	$\Theta_{2n}^{-1,1}(f, 0.5) \quad d = 10$
8	1.768797
16	1.76879751
32	1.768797516
64	1.7687975165
128	1.76879751653
256	1.768797516538

Table 9: $f(x) = \sqrt{1-x^2}$, $\omega = 5000$

Example 6 Given the Hilbert transform

$$\int_0^{\infty} \frac{\log(1+x)\sin(\omega x)}{(1+x^2)^{\frac{7}{2}}(x-1.7)} dx,$$

it results $\left|\dfrac{\log(1+x)}{(1+x^2)^{\frac{7}{2}}(x-1.7)}\right| < 10^{-16}$ for $x > 122$. In Table 10 we show the results obtained, for different values of ω, using the quadrature rule $\Theta_{2n}^{0,122}(f, 1.7)$ choosing $d = 16$.

$2n$	$\Theta_{2n}^{0,122}(f,1.7) \quad \omega = 1000$	$\Theta_{2n}^{0,122}(f,1.7) \quad \omega = 5000$
8	-0.02477	0.01098
16	-0.02477261286973	0.01098589258860
32	-0.024772612869733	0.01098589258860
64	-0.024772612869733	0.0109858925886056

Table 10: $f(x) = \dfrac{\log(1+x)}{(1+x^2)^{\frac{7}{2}}}$, $d = 16$

Example 7 Now we take the following integral

$$\int_0^1 \frac{\sinh x}{|x-0.5|^{\frac{1}{2}}} \sin(\omega x) dx.$$

In Table 11 we show the results obtained using the rule $\Lambda_m^{0,1}$, with $d = 16$, for different values of ω.

m	$\Lambda_m^{0,1}(f,0.5) \quad \omega = 1000$	$\Lambda_m^{0,1}(f,0.5) \quad \omega = 10000$
8	-0.02029	-0.01274
16	-0.020295243694676	-0.012746276943
32	-0.0202952436946768	-0.01274627694393

Table 11: $f(x) = \sinh x$, $d = 16$

Example 8 As last example we consider

$$\int_0^\infty \frac{\sin(\omega x)}{(x^{10}+5)|x-0.0001|^{\frac{1}{3}}} dx.$$

Taking into account that $|(x^{10}+5)^{-1}|x-0.0001|^{-\frac{1}{3}}| < 10^{-16}$ for $x > 32$, we approximate the integral using $\Lambda_m^{0,32}(f, 0.0001)$. The results, for different values of ω, are presented in Table 12.

m	$\Lambda_m^{0,32}(f,0.0001) \quad \omega = 1000$	$\Lambda_m^{0,32}(f,0.0001) \quad \omega = 10000$
8	0.002976	0.0008681
16	0.0029769	0.0008681106039
32	0.0029769728	0.0008681106039240
54	0.002976972829814	0.0008681106039240

Table 12: $f(x) = (x^{10}+5)^{-1}$, $d = 8$

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Some relationship between dual vector equilibrium problem and vector equilibrium problem

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Dedicated to my father

Abstract. The purpose of this paper is to present a relationship between the solution sets of vector equilibrium problem and dual vector equilibrium problem. We also give some existence theorems for a solution of dual vector equilibrium problem.

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Keywords and phrases. Dual vector equilibrium problem, vector equilibrium problem, KKM-Fan lemma.

1 Introduction

Throughout this paper we shall make use of the following notations. X and Y are real Hasudorff topological vector spaces, K is a nonempty convex subset of X, C is a closed convex pointed cone in Y and $f : K \times K \to Y$ is a vector-valued function on $K \times K$. The scalar (that is, $Y = \mathbb{R}$) equilibrium problem, that is (for short, EP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

This problem contains as special cases for instance optimization problems, problems of Nash equilibria, complementarity problems, fixed point

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problems and variational inequalities. For details, we can refer to [5, 8, 9] and the references therein

Motivated by the extensions of variational inequalities to vector variational inequalities, many authors generalized EP to the vector case in the following ways:

find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\text{int}C, \quad \forall y \in K, \tag{1.2}$$

find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \in C, \quad \forall y \in K, \tag{1.3}$$

where int$C$ denotes the interior of C. In this case, we assume that int$C \neq \emptyset$, and $f : K \times K \to Y$.

These problems are called vector equilibrium problems (for short, VEP).

The following problem, termed as dual vector equilibrium problem (for short, DVEP),

find $\bar{x} \in K$ such that

$$f(y, \bar{x}) \in -C, \quad \forall y \in K,$$

is closely related to VEP (1.3)

We denote by S_K and S_K^D the set of all solutions of VEP (1.3) and DVEP, respectively. Recently, Aussel and Hadisavvas [2] introduced the concept of local solutions of variational inequalities. Bianchi and Pini [4] extended this concept of local solutions to EP in the setting of reflexive Banach spaces. We further extend this concept for vector equilibrium problems in the setting of topological vector spaces.

The sets of local solutions for VEP (1.3) and DVEP are denoted by $S_{K,loc}$ and $S_{K,loc}^D$, respectively. We define $S_{K,loc}$ and $S_{K,loc}^D$ as follows:

$$S_{K,loc}^D = \{x \in K : \exists V \text{ open neighborhood of } x \text{ such that } f(y, x) \in -C, \forall y \in V \cap K\},$$

$$S_{K,loc} = \{x \in K : \exists V \text{ open neighborhood of } x \text{ such that } f(x, y) \in C, \forall y \in V \cap K\}.$$

Obviously, $S_K^D \subseteq S_{K,loc}^D$ and $S_K \subseteq S_{K,loc}$.

In most of the papers appeared in the literature on the different aspects, VEP (1.2) is considered and studied. It is worth mentioning that

the weak (strong, respectively) Pareto optimal solution of a vector optimization problem can be studied by using VEP (1.2) (VEP (1.3), respectively). Our results generalize and extend the results of Bianchi and Pini [4] to the topological vector space setting.

2 Main results

In this section, we first recall some concepts and then we give main results.

The following concept of upper sign continuity is vector version of the the upper sign continuity was introduced by Bianchi and Pini [4] which extends the earlier notion in the framework of variational inequalities in [10].

Definition 2.1. Let y be an arbitrary element of K. A mapping $x \to f(x, y)$, from K into Y, is said to be *upper sign continuous* at y if for any $x \in K$, the following statement holds

$$f(u, y) \in C, \quad \forall u \in \,]x, y[\quad \Rightarrow f(x, y) \in C,$$

where $]x, y[$ denotes the line segment joining x and y except the end points x and y.

Definition 2.2. A mapping $f : K \times K \to Y$ is said to be

(i) *C-pseudomonotone* if for any $x, y \in K$,

$$f(x, y) \in Y \setminus (-C) \quad \Rightarrow \quad f(y, x) \in -C \setminus \{0\};$$

(ii) *C-quasimonotone* if for any $x, y \in K$,

$$f(x, y) \in Y \setminus (-C) \quad \Rightarrow \quad f(y, x) \in -C;$$

(iii) *C-properly quasimonotone* if, for all $\{x_1, \ldots, x_n\} \subseteq K$ and for all $x \in \mathrm{co}\{x_1, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that $f(x_i, x) \in -C$, where $\mathrm{co}\{x_1, \ldots, x_n\}$ denotes the convex hull generated by $x_1, \ldots, x_n$.

It is clear from the definition that C-pseudomonotonicity of f implies C-quasimonotonicity. But in general there is no relationship between C-properly quasimonotonicity and quasimonotonicity or pseudomonotonicity.

The next result is a vector version of a proposition given in [3] with mild assumptions which provides a link between C-pseudomonotonicity and C-proper quasimonotonicity .

Proposition 2.1. *Let $f : K \times K \to Y$ be a vector-valued function such that one of the following conditions holds:*

(i) The set $\{x \in K : f(x,y) \in Y\backslash(-C\backslash\{0\})\}$ is convex and $f(y,y) \in -C$ for all $y \in K$;

(ii) The set $\{y : f(x,y) \in -C\backslash\{0\}\}$ is convex and f is C-pseudomonotone such that $f(x,x) = 0$ for all $x \in K$.

Then, f is C-properly quasimonotone.

Proof. Suppose that (i) holds and on contrary that f is not C-properly quasimonotone. Hence, there exist $x_1, \ldots, x_n \in K$ and $\tilde{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \tilde{x}) \in Y\backslash(-C)$ for $i = 1, 2, \ldots, n$, Therefore, by (i) we have $f(\tilde{x}, \tilde{x}) \in Y\backslash(-C)$ and so $f(\tilde{x}, \tilde{x}) \notin -C$ which is impossible by (i). Then f is C-properly quasimonotone. Now let (ii) hold and the result of proposition is not true. So there exist $x_1, \ldots, x_n \in K$ and $\tilde{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \tilde{x}) \in Y\backslash(-C)$ for $i = 1, 2, \ldots, n$ and hence, because f is C-pseudomonotone, $f(\tilde{x}, x_i) \in -C\backslash\{0\}$. Then, since the set $\{y : f(x,y) \in -C\backslash\{0\}\}$ is convex, we have $f(\tilde{x}, \tilde{x}) \in -C\backslash\{0\}$ and so $f(\tilde{x}, \tilde{x}) \neq 0$. This is impossible in view of (ii). This completes the proof. □

The following lemma plays a key role in this section. Moreover, it improves Lemma 2.1 in [4].

Lemma 2.1. *Let K be a nonempty convex subset of X and $f : K \times K \to Y$ be a vector-valued function such that the following conditions hold:*

(i) $f(x,x) \in C$ for all $x \in K$;

(ii) For all $y \in K$, the mapping $x \mapsto f(x,y)$ is upper sign continuous;

(iii) If $f(x,y) \in Y\backslash C$ and $f(x,z) \in -C$, then $f(x,u) \in Y\backslash C$, for all $u \in\ ]y,z[$.

Then, $S^D_{K,loc} \subseteq S_K$.

Proof. We argue by contradiction. Then there is $z \in K$ such that $z \in S^D_{K,loc}$ and $z \notin S_K$. So there exist $y \in K$ and an open neighborhood V of z such that $f(z,y) \in Y\backslash C$ and $f(v,z) \in -C$ for all $v \in K \cap V$. Since $V - z$ is a neighborhood of 0, there exists $t_0 \in]0,1[$ such that $t(y - z) \in V - z$ for

all $0 < t \leq t_0$. Let $\bar{y} = z + t_0(y - z)$ and $y_t = (1-t)z + t\bar{y} \in [z, \bar{y}]$ for $t \in [0,1]$. It is clear from $y_t = (1-t)z + tz + t\ t_0(y-z) = z + t\ t_0(y-z)$ and $t\ t_0(y-z) \in V - z$ that $y_t \in K \cap V$ which implies $f(y_t, z) \in -C$, note $z \in S^D_{K,loc}$, and so by (i) we have $f(z,z) = 0$. We claim that $f(u, \bar{y}) \in C$ for all $u \in\ ]z, \bar{y}[$. Otherwise, there is $u \in]z, \bar{y}[$ such that $f(u, \bar{y}) \in Y \backslash C$ and so ,note that $f(u,z) \in -C$, by (iii) we conclude $f(u,v) \in Y \backslash C$ for all $v \in]z, \bar{y}[$ and in particular $f(u, v = u) = 0 \in Y \backslash C$. Therefore $0 \notin C$ which contradicts the fact that $0 \in C$. Note that C is a pointed cone. Consequently, $f(u, \bar{y}) \in C$ for all $u \in\ ]z, \bar{y}[$. Thus (ii) implies that

$$f(z, \bar{y}) \in C. \tag{2.1}$$

Relation (2.1) is impossible because the facts $f(z,z) = 0$, $f(z,y) \in Y \backslash C$ and (iii) imply that $f(z, \bar{y}) \in Y \backslash C$. This completes the proof.

The following example shows that condition (iii) of Lemma 2.1 is essential.

Example 2.1. Let $X = Y = \Re$, $K = [-1,1]$, $C = [0, \infty)$ and $f : [-1,1] \times [-1,1] \to \Re$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if} \quad (x,y) \in \{0\} \times \left[\frac{-1}{2}, \frac{1}{2}\right] \ \ or \ \ x = y\ , \\ -1 & \text{otherwise.} \end{cases}$$

It is obvious that $f(x,x) = 0$, for all $x \in K$ and if $f(u,y) \geq 0$, for all $u \in]x, y[$, then $u = 0$, for all $u \in]x, y[$, which is impossible. This shows that the mapping $x \to f(x,y)$ is upper sign continuous mapping for each fixed $y \in K$. From the relations $f(\frac{1}{4}, \frac{1}{3}) < 0, f(\frac{3}{4}, \frac{1}{3}) < 0$, one can verify that the statement $f(u, \frac{1}{3}) < 0$, does not true for all $u \in]\frac{1}{4}, \frac{3}{4}[$, for example take $u = \frac{1}{3} \in]\frac{1}{4}, \frac{3}{4}[$, and so the example all the assumptions of Lemma 2.1 except condition (iii). Also one can easily see that $0 \in S^D_{K,loc}$ and $S_K = \emptyset$.

Remark 2.1. (a) If the mapping $y \mapsto f(x,y)$ is C-convex, that is,

$$tf(x,y) + (1-t)f(x,z) - f(x, ty + (1-t)z) \in C, \quad \forall y, z \in K,\ \forall t \in [0,1], \tag{2.2}$$

then condition (ii) of Lemma 2.1 holds. To see this, let $f(x,y) \in Y \backslash C$ and $f(x,z) \in -C$. Since $Y \backslash C$ and $-C$ are cones, we have $tf(x,u) \in Y \backslash C$ and $(1-t)f(x,z) \in -C$. Also the inclusion $(Y \backslash C) - C \subseteq Y \backslash C$ implies that

$$tf(x,u) + (1-t)f(x,z) \in Y \backslash C. \tag{2.3}$$

From (2.2), (2.3) and $(Y\backslash C) - C \subseteq Y\backslash C$, we obtain

$$f(x, ty + (1-t)z) \in Y\backslash C, \quad \forall t \in [0,1].$$

This shows that condition (ii) of Lemma 2.1 holds.

(b) Lemma 2.1 improves and extends Lemma 2.4 in [11] and Lemma 2.1 in [4] to vector-valued bifunctions.

(c) It is straightforward to see that $S_K \subseteq S_K^D$ whenever the following statement holds, for all $x, y \in K$,

$$f(x,y) \in Y\backslash(-C\backslash\{0\}) \Rightarrow f(y,x) \in -C. \qquad (*)$$

Therefore, if we add $(*)$ to the assumptions of Lemma 2.1 then we obtain $S_K^D = S_{K,loc}^D = S_K$. As an application of this remark, if we take $Y = \Re$ and $C = [0, \infty)$, then we deduce Proposition 2.5 in [11].

A set-valued map (or multimap) $F : K \rightrightarrows X$ is a function from K into the power set 2^X of a set X. Let K be a convex subset of a vector space X. Then a set-valued map $F : K \rightrightarrows X$ is called a KKM map if for each nonempty finite subset A of K, the inclusion $\mathrm{co}A \subseteq F(A)$ holds, where $\mathrm{co}A$ denotes the convex hull of A and $F(A) = \bigcup\{F(x) : x \in A\}$.

We need the next result in the sequel. It is a special case of Fan-KKM principle [6].

Lemma 2.2. *Let X be a nonempty subset of a topological vector space E and $F : X \to 2^E$ a KKM mapping with closed values. Assume that there exists a nonempty compact convex subset B of X such that $\bigcap_{x\in B} F(x)$ is compact. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Now we establish an existence result for a solution of DVEP.

Theorem 2.1. *Let K be a nonempty convex subset of X and $f : K\times K \to Y$ be a vector-valued function with $f(x,x) = 0$, for each $x \in K$. Assume that the following conditions hold:*

(i) For each finite subset A of K and $x \in coA\backslash A$, there exists $y \in K$ such that $f(y,x) \in -C$;

(ii) For each $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed in K;

(iii) There exist a nonempty compact subset D and a nonempty compact convex subset B of K such that for each $y \in K \backslash D$, there exists $x \in B$ such that $f(x,y) \in Y \backslash -C$.

Then, the solution set S_K^D of DVEP is nonempty and compact.

Proof. Define $\Gamma : K \rightrightarrows K$ by

$$\Gamma(x) = \{y \in K : f(x,y) \in -C\}.$$

By (i) Γ is a KKM mapping. In fact let $A = \{x_1, ..., x_n\} \subset K$ and $x \in coA$. If $x \in A$ then $f(x,x) = 0$ and so $x \in \Gamma(x)$. Now if $x \in coA \backslash A$ then by (i) there exists $y \in K$ such that $f(y,x) \in -C$ and hence $x \in \Gamma(y)$. Consequently, $coA \subset \cup_{i=1}^n \Gamma(x_i)$. This means that Γ is a KKM mapping. The values of Γ, that is $\Gamma(x)$, for all $x \in K$. is closed and it is obvious from (iii) that $\bigcap_{x \in B} \Gamma(x) \subseteq D$. Hence Γ satisfies all the assumptions of Lemma 2.2 and $\bigcap_{x \in K} \Gamma(x)$ is nonempty. It is clear from the definition of DVEP that $S_K^D = \bigcap_{x \in K} \Gamma(x)$. Then S_K^D is nonempty and by (iii) it is a compact subset of K. □

Lemma 2.3. *If, for each $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed and convex subset of K, then either f is C-properly quasimonotone or $S_{K,loc}^D \neq \emptyset$.*

Proof. Assume that f is not C-properly quasimonotone. Hence, there exist $x_1, \ldots, x_n \in K$ and $\bar{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \bar{x}) \in Y \backslash (-C)$, for all $i = 1, 2, \ldots, n$. Then, there exists an open neighborhood V of $\bar{x}$ such that

$$f(x_i, y) \in Y \backslash (-C), \quad \forall y \in K \cap V, \ i = 1, 2, \ldots, n.$$

Hence from the $C-$ quasimonotonicity of f we have $f(y, x_i) \in Y - C$ for all $i = 1, 2, \ldots, n$ and $y \in K \cap V$ and so by our assumption, that is the set $\{y \in K : f(x,y) \in -C\}$ is convex, we get

$$f(y, \bar{x}) \in -C, \quad \forall y \in K \cap V.$$

Therefore, $\bar{x} \in S_{K,loc}^D$. This completes the proof. □

In the following we establish some existence results for a solution of VEP (1.3) with or without C-quasimonotonicity assumption.

Theorem 2.2. *Let $f : K \times K \to Y$ be a C-quasimonotone function satisfying the conditions of Lemmas 2.1, 2.3 and condition (iii) of Theorem 2.1. Then $S_K \neq \emptyset$.*

Proof. If f is C-properly quasimonotone, then the result follows from Theorem 2.1 through Lemma 2.1. Otherwise, we obtain the result from Lemmas 2.3 and 2.1. □

Theorem 2.3. *Assume that $f : K \times K \to Y$ satisfies condition (iii) of Lemma 2.1. Let $x_0 \in K$ be a local solution of VEP (1.3) in a neighborhood V of x_0. If there exists $\bar{y} \in K \cap int\, V$ such that $f(x_0, \bar{y}) \in -C$, then $x_0 \in S_K$.* □

Proof. We argue by contradiction. Then there exists $z \in K$ such that $f(x_0, z) \in Y \backslash C$. So, by condition (iii) of Lemma 2.1 we have

$$f(x, u) \in Y \backslash C, \quad \forall u \in]\bar{y}, z[.$$

Since $\bar{y} \in K \cap \text{int}V$, there is a $u \in\ ]\bar{y}, z[$ such that $f(x_0, u) \in Y \backslash C$. This contradicts the fact that x_0 is a local solution of VEP (1.3) in a neighborhood V of x_0. □

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On a positive linear operator on the semiaxis

Luisa Fermo

To my Professor Giuseppe Mastroianni

Abstract. A positive linear operator is introduced on the semiaxis and its convergence is investigated in suitable weighted spaces equipped with the uniform and the L^1-norm. As corollaries, error estimates for a special Hermite-Fejér polynomial are obtained.

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1 Introduction

In 1975 I. Joó investigated the following operator

$$P_m^{(\alpha)}(f,x)=\sum_{k=1}^{m}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)f(x_k)l_{m,k}^2(x) \tag{1.1}$$

where $x_1<x_2<...<x_m$ are the zeros of the polynomial $p_m(w_\alpha)$ which is orthonormal to the Laguerre weight $w_\alpha(x)=x^\alpha e^{-x}$, $-1<\alpha\leq 0$, and $l_{m,k}(x)=\frac{p_m(w_\alpha,x)}{p'_m(w_\alpha,x_k)(x-x_k)}$ are the fundamental Lagrange polynomials based on the nodes x_k. It is a positive linear operator and is a part of the Hermite-

This paper is in final form and no version of it will be submitted for publication elsewhere.

Fejér interpolation $F_m^{(\alpha)}$. Indeed the latter one can be written as

$$F_m^{(\alpha)}(f,x) = P_m^{(\alpha)}(f,x) - \sum_{k=1}^{m} f(x_k)(x-x_k)l_{m,k}^2(x).$$

Particularly, in [4] I. Joó proved that, under certain growth conditions on f, an error estimate in uniform norm can be given in finite intervals. In fact he got the following relation

$$\|P_m^{(\alpha)}(f) - f\| = \begin{cases} O\left(\omega\left(f, \frac{1}{m^{1/4}}\right) + \frac{\|f\|}{m^{1/4}} + \frac{\|f\|^*}{\sqrt{m}}\right), & -1 < \alpha \le -\frac{1}{2} \\ O\left(\omega\left(f, \frac{1}{m^{-\alpha/2}}\right) + \frac{\|f\|}{m^{-\alpha/2}} + \frac{\|f\|^*}{m^{-\alpha}}\right), & -\frac{1}{2} \le \alpha < 0 \end{cases}$$

where ω denotes the ordinary modulus of smoothness on the interval $[0, 2A]$ with $A > 0$ arbitrary and fixed, $\|\cdot\|$ denotes the maximum-norm of the corresponding function in the interval $[0, A]$, $\|f\|^* = 2\sup_{0\le x<\infty}\{|f(x)e^{-x}|\}$ and the O sign depends only on A.

Later in [6] I. Joó and J. Szabados investigated the weighted convergence of $P_m^{(\alpha)}$ in L^1. In particular they considered a continuous function f on $[0,\infty)$ such that $\lim_{x\to\infty} f(x)e^{-\lambda x} = 0$, $\lambda < 1$ and associated to every f of this type a function F defined as $F(x) = f(x^2)e^{-\lambda x^2}$. Thus the authors proved the following estimate

$$\int_0^\infty x^\alpha e^{-x}|f(x) - P_m^{(\alpha)}(f,x)|dx = O\left(\omega\left(F, \frac{\log^2 m}{\sqrt{m}}\right)\right)$$

where $\omega(F,t) = \sup_{0<h\le t}\|\Delta_h F\|_{\mathbb{R}}$ and the O sign denotes a constant depending only on α and λ.

In this paper we will see that, by means of a slight change of the operator $P_m^{(\alpha)}$, it is possible to approximate a wide class of functions defined on $(0,\infty)$. In order to make this modification we will proceed in the following way.

Let $x_1 < ... < x_m$ be the zeros of $p_m(w_\alpha)$ and let $x_{m+1} := 4m$. We consider the Hermite-Fejér polynomial based on these $m+1$ points defined as

$$\mathcal{F}_{m+1}^{(\alpha)}(f,x) = \sum_{k=1}^{m+1}\left(1 - \frac{q''(x_k)}{q'(x_k)}(x-x_k)\right)\tilde{l}_{m,k}^2(x)f(x_k)$$

where $q(x) = p_m(w_\alpha, x)(4m-x)$ and

$$\tilde{l}_{m,k}(x) = \left(\frac{4m-x}{4m-x_k}\right)l_{m,k}(w_\alpha,x), \quad k \le m, \quad \tilde{l}_{m,m+1}(x) = \frac{p_m(w_\alpha,x)}{p_m(w_\alpha,4m)}$$

are the fundamental Lagrange polynomials based on the zeros $x_1 < x_2 < ... < x_m < x_{m+1}$. By developing the expression of the coefficients we can write

$$\mathcal{F}_{m+1}^{(\alpha)}(f,x) = \mathcal{P}_{m+1}^{(\alpha)}(f,x) + \sum_{k=1}^{m+1} \left(-1 + \frac{2}{4m - x_k}\right)(x - x_k) f(x_k) \tilde{l}_{m,k}^2(x) \tag{1.2}$$

where

$$\mathcal{P}_{m+1}^{(\alpha)}(f,x) = \sum_{k=1}^{m+1} \left(\frac{(1+\alpha)x}{x_k} - \alpha\right) \tilde{l}_{m,k}^2(x) f(x_k). \tag{1.3}$$

Now if

$$x_j = \min_{1 \le k \le m} \{x_k : x_k \ge 2m\}$$

and

$$\Psi(x) = \begin{cases} 1, & \text{if } x \ge 1; \\ 0, & \text{if } x \le 0. \end{cases}, \quad \Psi_j(x) = \Psi\left(\frac{x - x_j}{x_{j+1} - x_j}\right),$$

for any $f \in C^{\circ}((0,\infty))$, we introduce the following function

$$f_j(x) = (1 - \Psi_j(x)) f(x) \tag{1.4}$$

and the following "truncated" Hermite-Fejér polynomial

$$\begin{aligned} \mathcal{F}_{m+1}^{*(\alpha)}(f,x) &:= \mathcal{F}_{m+1}^{(\alpha)}(f_j,x) \\ &= \mathcal{P}_{m+1}^{*(\alpha)}(f,x) + \sum_{k=1}^{j} \left(-1 + \frac{2}{4m - x_k}\right)(x - x_k) f(x_k) \tilde{l}_{m,k}^2(x) \end{aligned} \tag{1.5}$$

where

$$\mathcal{P}_{m+1}^{*(\alpha)}(f,x) := \mathcal{P}_{m+1}^{(\alpha)}(f_j,x) = \sum_{k=1}^{j} \left(\frac{(1+\alpha)x}{x_k} - \alpha\right) \tilde{l}_{m,k}^2(x) f(x_k). \tag{1.6}$$

In this way we have defined a new operator $\mathcal{P}_{m+1}^{*(\alpha)}$ which has the same expression as the operator $P_m^{(\alpha)}$ defined in (1.1) with the only difference that the sum is until j and not until m and that the fundamental Lagrange polynomials are based on $m+1$ points and not on m points.

In this paper we will investigate the operator $\mathcal{P}_{m+1}^{*(\alpha)}$. Particularly in Theorem 3.1 we will show the convergence of $\mathcal{P}_{m+1}^{*(\alpha)}(f)$ to f and we will give an error estimate in uniform norm on $(0,\infty)$ (and not on finite intervals).

We will also consider $\mathcal{P}_{m+1}^{*(\alpha)} : L_u^\infty \to L_v^1$, where L_u^∞ and L_v^1 are two suitable weighted spaces equipped with the uniform and the L^1-norm, respectively. Then in Theorem 3.2 we will prove

$$\|[f - \mathcal{P}_{m+1}^{*(\alpha)}(f)]v\|_1 \leq \mathcal{C}\left[\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u + \frac{\|fu\|_\infty}{\sqrt{m}}\right],$$

where ω_φ is the first weighted φ-modulus of smoothness in uniform norm and $\mathcal{C}$ is a positive constant independent of m and f.

As corollaries, we will obtain some error estimates for the " truncated" Hermite-Fejér polynomial based on the zeros $x_1 < x_2 < ... < x_m < x_{m+1}$.

The paper is structured as follows. In Section 2 we will define the weighted spaces we consider. In Section 3 the main results are showed while in Section 4 the proofs of the main results are given.

2 Preliminary results

First we give some notations and preliminary results. In the following we will denote by $\mathcal{C}$ a positive constant which may assume different values in different formulae and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, ...)$ to indicate that $\mathcal{C}$ is independent of the parameters $a, b, ...$ Moreover if A and B are two positive quantities depending on some parameters, we will write $A \sim B$ if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, where $\mathcal{C}$ is a positive constant independent of the above parameters.

Now let $w_\gamma(x) = x^\gamma e^{-x}$, $\gamma \geq 0$ and let us denote by $C^\circ(B)$ the set of all continuous functions on $B \subseteq [0, \infty)$.

With this notation we introduce the space $L_{w_\gamma}^\infty$

$$L_{w_\gamma}^\infty = \left\{ f \in C^\circ((0,\infty)) \ : \ \lim_{\substack{x\to 0 \\ x\to+\infty}} (fw_\gamma)(x) = 0 \right\}$$

equipped with the norm

$$\|f\|_{L_{w_\gamma}^\infty} := \|fw_\gamma\|_\infty = \sup_{x\geq 0} |(fw_\gamma)(x)|.$$

If $\gamma = 0$ then $L_{w_\gamma}^\infty$ denotes the set of all continuous functions on $[0, \infty)$ such that $\lim_{x\to+\infty}(fw_\gamma)(x) = 0$. In other words, when $\gamma > 0$, the functions $f \in L_{w_\gamma}^\infty$ could take very large values, with algebraic growth, as x approaches zero from the right, and could have an exponential growth as $x \to \infty$.

In order to characterize these type of functions we introduce the following modulus of smoothness (see, for instance, [10, p. 175])

$$\omega_\varphi(f,t)_{w_\gamma} = \Omega_\varphi(f,t)_{w_\gamma} + \inf_{\mathcal{C}} \|[f-\mathcal{C}]w_\gamma\|_{L^\infty([0,4t^2])} + \inf_{\mathcal{C}} \|[f-\mathcal{C}]w_\gamma\|_{L^\infty([\frac{1}{t^2},\infty])},$$

where

$$\Omega_\varphi(f,t)_{w_\gamma} = \sup_{0<h\le t} \sup_{x\in\left(4h^2,\frac{1}{h^2}\right)} |f(x+h\varphi(x))-f(x)|w_\gamma(x), \quad \varphi(x)=\sqrt{x}$$

and we recall the following properties (see, for instance, [10, p. 169], [13], [1])

$$\Omega_\varphi(f,\lambda\ \delta)_{w_\gamma} \le ([\lambda]+1)\ \Omega_\varphi(f,\delta)_{w_\gamma}, \quad \lambda\in\mathbb{R}^+ \tag{2.1}$$

$$\Omega_\varphi(f,\delta_1)_{w_\gamma} \le \Omega_\varphi(f,\delta_2)_{w_\gamma}, \quad \delta_1\le\delta_2 \tag{2.2}$$

$$\Omega_\varphi(f,\delta)_{w_\gamma} \le \omega_\varphi(f,\delta)_{w_\gamma}, \tag{2.3}$$

$$\frac{\Omega_\varphi(f,\delta_1)}{\delta_1} \le \frac{\Omega_\varphi(f,\delta_2)}{\delta_2}, \quad \delta_1>\delta_2 \tag{2.4}$$

where $[\lambda]$ denotes the integer part of λ.

3 Main Results

Let $w_\alpha(x)=x^\alpha e^{-x}$, $x>0$, $\alpha>-1$ be a Laguerre weight and let $\{p_m(w_\alpha)\}$ be the sequence of orthonormal Laguerre polynomials defined as

$$p_m(w_\alpha,x)=\gamma_m x^m+..., \quad \gamma_m>0,$$

$$\int_0^\infty p_m(w_\alpha,x)p_n(w_\alpha,x)w_\alpha(x)dx=\delta_{m,n}.$$

We denote by $x_k:=x_{m,k}$ the zeros of $p_m(w_\alpha)$, $m\ge1$, $x_j=\min_{1\le k\le m}\{x_k : x_k\ge 2m\}$ and for all $f\in L^\infty_{w_\gamma}$ we consider the operator $\mathcal{P}^{*(\alpha)}_{m+1}(f)$ defined in (1.6)

$$\mathcal{P}^{*(\alpha)}_{m+1}(f,x):=\mathcal{P}^{(\alpha)}_{m+1}(f_j,x)=\sum_{k=1}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}^2_{m,k}(x)f(x_k).$$

By the definition it follows that if $-1<\alpha<0$ then $\mathcal{P}^{*(\alpha)}_{m+1}$ is a positive linear operator and it does not preserve the constant functions.

Next theorem shows, that under suitable conditions on the parameter γ of the weight of the space, $\mathcal{P}^{*(\alpha)}_{m+1}(f)$ converges to f in the norm of $L^\infty_{w_\gamma}$.

Theorem 3.1. *Let* $w_\alpha(x) = x^\alpha e^{-x}$, $w_\gamma(x) = x^\gamma e^{-x}$, *such that*

$$\max\left\{0, \alpha + \frac{1}{2}\right\} \leq \gamma \leq \alpha + 1, \quad -1 < \alpha \leq 0.$$

Then for every $f \in L^\infty_{w_\gamma}$ *it results*

$$\|[f - \mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_\infty \leq \mathcal{C}\left[\sum_{i=1}^{m} \frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + \frac{\log m}{\sqrt{m}}\|fw_\gamma\|_\infty\right] \quad (3.1)$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

We underline that the right-hand side of (3.1) tends to zero. Indeed, by the properties of the modulus of smoothness we have

$$\begin{aligned}\sum_{i=1}^{m} \frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} &= \sum_{i=1}^{m} \frac{1}{i^2}\omega_\varphi\left(f, \frac{\frac{i}{\log^2 i}\log^2 i}{\sqrt{m}}\right)_{w_\gamma} \\ &\leq \omega_\varphi\left(f, \frac{\log^2 m}{\sqrt{m}}\right)_{w_\gamma} \sum_{i=1}^{m} \frac{1}{i \log^2 i} \\ &\leq \mathcal{C}\omega_\varphi\left(f, \frac{\log^2 m}{\sqrt{m}}\right)_{w_\gamma}.\end{aligned}$$

Note also that the second term on the right-hand side of (3.1) cannot be dropped, since the operator is not exact for the constant functions.

Moreover we observe that if $f \in W_1(w_\gamma)$ with

$$W_1(w_\gamma) = \left\{f \in L^\infty_{w_\gamma} \mid \|f\|_{W_1(w_\gamma)} = \|fw_\gamma\|_\infty + \|f'\varphi w_\gamma\|_\infty < \infty\right\},$$

$\varphi(x) = \sqrt{x}$, by (3.1) it results

$$\|[f - \mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_\infty \leq \ \mathcal{C}\ \frac{\log m}{\sqrt{m}}\ \|f\|_{W_1(w_\gamma)}.$$

Furthermore by Theorem 3.1 we can immediately deduce an error estimate for the "truncated" Hermite-Fejér polynomial defined in (1.5).

Corollary 3.1. *Under the assumptions of Theorem* 3.1, *for every* $f \in L^\infty_{w_\gamma}$ *it results*

$$\|[f - \mathcal{F}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_\infty \leq \mathcal{C}\left[\sum_{i=1}^{m} \frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + \frac{\log m}{\sqrt{m}}\|fw_\gamma\|_\infty\right] \quad (3.2)$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

By (3.2) and (3.1) we can deduce that a homogeneous error estimate cannot be given. Indeed, in order to approximate a function f with the Hermite-Fejér polynomial or with the operator $\mathcal{P}_{m+1}^{*(\alpha)}$ in the norm of the space $L_{w_\gamma^2}^\infty$, we have to assume $f \in L_{w_\gamma}^\infty$. On the other hand, the investigations of D.S. Lubinsky in [9], J. Szabados in [18] and S.V. Szabó in [19] showed that, if we consider the Hermite-Fejér polynomial F_m based on m nodes of the m-th orthonormal polynomial w.r.t. some Freud-type weights, error estimate of the type $\lim_m \|[f - F_m(f)]w_1\|_\infty = 0$, $\forall f \in L_{w_2}^\infty$ cannot be given with $w_1 = w_2$.

Now we consider $\mathcal{P}_{m+1}^{*(\alpha)}(f) : L_u^\infty \to L_v^1$, where $v(x) = \frac{w_\gamma(x)}{(1+x)^\lambda}$, $u(x) = v(x)(1+x)^{2\lambda} = w_\gamma(x)(1+x)^\lambda$, $\lambda \in \mathbb{R}$ and L_v^1 is the space of all measurable functions in $(0,\infty)$ such that

$$\|f\|_{L_v^1} = \|fv\|_1 = \int_0^\infty |(fv)(x)|dx < \infty.$$

The following proposition shows the boundedness of $\mathcal{P}_{m+1}^{*(\alpha)} : L_u^\infty \to L_v^1$.

Proposition 3.1. *Let $w_\gamma(x) = x^\gamma e^{-x}$, $v(x) = w_\gamma(x)(1+x)^{-\lambda}$ and $u(x) = v(x)(1+x)^{2\lambda}$ such that*

$$\lambda > 1, \quad \max\left\{0, \alpha + \frac{1}{2}\right\} \le \gamma < \alpha + 1, \quad -1 < \alpha \le 0.$$

Then for every $f \in L_u^\infty$ we have

$$\|\mathcal{P}_{m+1}^{*(\alpha)}(f)v\|_1 \le \mathcal{C}\|fu\|_\infty$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$.

The following theorem gives an error estimate for $\mathcal{P}_{m+1}^{*(\alpha)}$ in the L^1-norm.

Theorem 3.2. *Let $w_\gamma(x) = x^\gamma e^{-x}$, $v(x) = w_\gamma(x)(1+x)^{-\lambda}$ and $u(x) = v(x)(1+x)^{2\lambda}$ such that*

$$1 < \lambda < \frac{3}{2}, \quad \max\left\{0, \alpha + \frac{1}{2}\right\} \le \gamma < \alpha + 1, \quad -1 < \alpha \le 0.$$

Then for every $f \in L_u^\infty$ we have

$$\|[f - \mathcal{P}_{m+1}^{*(\alpha)}(f)]v\|_1 \le \mathcal{C}\left[\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u + \frac{\|fu\|_\infty}{\sqrt{m}}\right]$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$.

By the previous theorem we can easily deduce an error estimate for the "truncated" Hermite-Fejér polynomial (1.5) in L^1-norm as showed by the following corollary.

Corollary 3.2. *Under the assumptions of Theorem* 3.2*, for every* $f \in L^\infty_u$ *we have*

$$\|[f - \mathcal{F}^{*(\alpha)}_{m+1}(f)]v\|_1 \leq \mathcal{C}\left[\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u + \frac{\|fu\|_\infty}{\sqrt{m}}\right]$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

4 Proofs

Before proving the main results we recall some properties of the zeros of $p_m(w_\alpha)$ and some estimates of the fundamental Lagrange polynomials $l_{m,k}(x)$ based on the nodes of $p_m(w_\alpha)$.

Namely, the zeros $x_k := x_{m,k}(w_\alpha)$ of $p_m(w_\alpha)$, $m \geq 1$ are such that (see, for instance, [20], [3], [5]),

$$\frac{\mathcal{C}}{m} < x_1 < ... < x_m = 4m + 2\alpha + 2 - \mathcal{C}(4m)^{\frac{1}{3}}, \quad x_k \sim \frac{k^2}{m} \tag{4.1}$$

and $\forall k = 1, 2, ..., m-1$

$$\Delta x_k := x_{k+1} - x_k \sim \sqrt{\frac{x_k}{4m - x_k}}, \qquad \Delta x_k \sim \Delta x_{k+1} \tag{4.2}$$

where $\mathcal{C}$ and the constants involved in $\sim$ are independent of m and k.

Moreover, if x_d is a node closest to x,

$$x_j = \min_{1 \leq k \leq m} \{x_k : x_k \geq 2m\}$$

$0 \leq x \leq 4m$, $k \neq \{d, d \pm 1\}$, $k = 1, ...j$, $d \leq m$ and $x \neq x_k$, (see, for instance, [12, Lemma 5]) it results

$$|x - x_k| \geq (|d - k| + 1) \min(\Delta x_d, \Delta x_k) \geq \mathcal{C}(|d - k| + 1)\Delta x_k \tag{4.3}$$

where $\mathcal{C} \neq \mathcal{C}(m, k)$.

Furthermore, denoting by $\lambda_m(w_\alpha, x) = [\sum_{k=0}^{m-1} p_k^2(w_\alpha, x)]^{-1}$ the m-th Christoffel function, one has (see, for instance, [20] and [15])

$$l^2_{m,k}(x) = p^2_m(w_\alpha, x)\frac{x_k \lambda_m(w_\alpha, x_k)}{(x - x_k)^2}, \quad \forall k = 1, ..., m. \tag{4.4}$$

Now, since (see, for instance, [3], [5], [15]) $\forall k = 1, 2, ..., m$

$$\lambda_m(w_\alpha, x_k) \sim w_\alpha(x_k)\Delta x_k \sim w_\alpha(x_k)\sqrt{\frac{x_k}{4m - x_k}} \tag{4.5}$$

and (see, for instance, [11])

$$|p_m(w_\alpha, x)\sqrt{w_\alpha(x)}| \leq \frac{\mathcal{C}}{\sqrt[4]{x}\sqrt[4]{4m - x + \frac{4m}{m^{2/3}}}}, \quad x \in \left(\frac{\mathcal{C}}{m}, 4m\right) \tag{4.6}$$

we can deduce the following useful estimate for the fundamental Lagrange polynomials

$$l^2_{m,k}(x)w_\gamma(x) \leq \mathcal{C}x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x - x_k)^2}\,\frac{x_k w_\alpha(x_k)}{\sqrt{4m - x}}, \quad x \in \left(\frac{\mathcal{C}}{m}, 4m\right) \tag{4.7}$$

$\forall k = 1, 2, ..., m$.

The following lemma [11, Lemma 3.1-3.2] (see, also, [2], [8], [9]) will be useful for our aims.

Lemma 4.1. *Let $\{l_{m,k}\}_{k=1}^m$ be the fundamental Lagrange polynomials based on the zeros of $p_m(w_\alpha)$, let $x \in [x_1, x_m]$ and $d = d(x) \in \{1, 2, ..., m\}$ be an index of a zero of $p_m(w_\alpha)$ closest to x. Then, for some positive constants $\mathcal{C} \neq \mathcal{C}(m, x, d)$ we have*

$$\frac{1}{\mathcal{C}} \leq \frac{w_\alpha(x)l^2_{m,d}(x)}{w_\alpha(x_d)} \leq \mathcal{C}. \tag{4.8}$$

From now on, for simplicity of notation, we will write l_k instead of $l_{m,k}$.

In the sequel we also need a Remez type inequality stated in the following proposition (see, for instance, [1]).

Proposition 4.2. *Let $w_\gamma(x) = x^\gamma e^{-x}$, $\gamma \geq 0$. Then for each polynomial P_m of degree at most m and for a fixed $a > 0$, there exists a positive constant $\mathcal{C}$ that depends on a but not on m and P_m, such that the following inequality*

$$\|P_m w_\gamma\|_\infty \leq \mathcal{C}\|P_m w_\gamma\|_{L^\infty([\frac{a}{m}, 2m])}$$

holds true.

Finally, we introduce the error of best approximation by means of polynomials of degree at most n ($P_n \in \mathbb{P}_n$)

$$E_n(f)_{w_\gamma} = \inf_{P_n \in \mathbb{P}_n} \|(f - P_n)w_\gamma\|_\infty$$

and recall the following Jackson's inequality (see, for instance, [13], [1])

$$E_n(f)_{w_\gamma} \le \mathcal{C}\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_{w_\gamma}, \quad \mathcal{C} \ne \mathcal{C}(n, f). \tag{4.9}$$

In order to prove Theorem 3.1 we need the following lemma.

Lemma 4.2. *Let $w_\gamma(x) = x^\gamma e^{-x}$ and $w_\alpha(x) = x^\alpha e^{-x}$ such that*

$$\max\left\{0, \alpha + \frac{1}{2}\right\} \le \gamma \le \alpha + 1, \quad -1 < \alpha \le 0.$$

Then for every $f \in L^\infty_{w_\gamma}$ we have

$$\|[f - \mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_{L^\infty((0,x_j])} \le \mathcal{C}\left[\sum_{i=1}^{m} \frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + \frac{\log m}{\sqrt{m}}\|fw_\gamma\|_\infty\right]$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$.

Proof. Since $\mathcal{F}^{(\alpha)}_{m+1}(1, x) = 1$, by (1.2) we have

$$\begin{aligned}
&\|[f - \mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_{L^\infty((0,x_j])} \\
&= \sup_{x\in(0,x_j]} |[f(x)\mathcal{F}_{m+1}(1, x) - \mathcal{P}^{*(\alpha)}_{m+1}(f, x)]w^2_\gamma(x)| \\
&= \sup_{x\in(0,x_j]} \left\{\sum_{k=1}^{j}\left(\frac{(1+\alpha)x}{x_k} - \alpha\right)\tilde{l}^2_k(x)w^2_\gamma(x)[f(x) - f(x_k)]\right. \\
&\qquad + \sum_{k=j+1}^{m+1}\left(\frac{(1+\alpha)x}{x_k} - \alpha\right)\tilde{l}^2_k(x)f(x)w^2_\gamma(x) \\
&\qquad \left. + \sum_{k=1}^{m+1}\left(-1 + \frac{2}{4m - x_k}\right)(x - x_k)\tilde{l}^2_k(x)f(x)w^2_\gamma(x)\right\} \\
&:= \sup_{x\in(0,x_j]} \{A_1(x) + A_2(x) + A_3(x)\}.
\end{aligned} \tag{4.10}$$

We first consider $A_1(x)$. We denote by x_d a node closest to x and assume $x_{d-1} < x < x_d < x_{d+1}$, $d \ge 2$. Then we write

$$\begin{aligned}
|A_1(x)| &\le \sum_{k=1}^{d-1}\left(\frac{(1+\alpha)x}{x_k} - \alpha\right)\tilde{l}^2_k(x)w^2_\gamma(x)|f(x) - f(x_k)| \\
&\quad + \tilde{l}^2_d(x)w^2_\gamma(x)|f(x) - f(x_d)| + \sum_{k=d+1}^{j}\tilde{l}^2_k(x)w^2_\gamma(x)|f(x) - f(x_k)| \\
&:= \Sigma_1(x) + \Sigma_2(x) + \Sigma_3(x).
\end{aligned} \tag{4.11}$$

being $\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\le 1$ if $x<x_k$. We first estimate $\Sigma_2(x)$. To this end observe that being $x_{d-1}<x<x_d$ and $x\in(0,x_j]$ by (4.2) we have

$$x_d-x<x_d-x_{d-1}=\Delta x_{d-1}\le \mathcal{C}\sqrt{\frac{x_d}{m}}. \tag{4.12}$$

Therefore, setting $\varphi(y)=\sqrt{y}$, taking into account that $\sup_x w_\gamma(x)\le\mathcal{C}$ and by applying (2.1) and (4.8), we get

$$\begin{aligned}\Sigma_2(x)&=|f(x+(x_d-x))-f(x)|w_\gamma(x)\\&\le \sup_{0<h\le\frac{1}{\sqrt{2m}}}\ \sup_{y\in\left(4h^2,\frac{1}{h^2}\right)}|f(y+h\varphi(y))-f(y)|w_\gamma(y)\\&\le \mathcal{C}\ \Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}.\end{aligned} \tag{4.13}$$

Now we consider $\Sigma_1(x)$. We write

$$\begin{aligned}\Sigma_1(x)&\le -\alpha\sum_{k=1}^{d-1}\tilde{l}_k^2(x)w_\gamma^2(x)|f(x)-f(x_k)|\\&\qquad+(\alpha+1)\sum_{k=1}^{d-1}\frac{x}{x_k}\tilde{l}_k^2(x)w_\gamma^2(x)|f(x)-f(x_k)|\\&:=I_1(x)+I_2(x).\end{aligned} \tag{4.14}$$

About I_1 we observe that, by using relation (4.7) and taking into account that $\sup_x w_\gamma(x)\le\mathcal{C}$, we have

$$\begin{aligned}&I_1(x)\\&\le\mathcal{C}\sum_{k=1}^{d-1}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac12}\frac{\Delta x_k}{(x-x_k)^2}\ \frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}|f(x)-f(x_k)|w_\gamma(x_k)\\&=\mathcal{C}\sum_{k=1}^{d-1}\left(\frac{4m-x}{4m-x_k}\right)^2\sqrt{\frac{x}{4m-x}}\frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}\\&\qquad\qquad\cdot|f(x)-f(x_k)|w_\gamma(x_k).\end{aligned}$$

Therefore, since $\left(\frac{4m-x}{4m-x_k}\right)^2<2$, $\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}<1$ being $\gamma\le\alpha+1$, $\sqrt{\frac{x}{4m-x}}\Delta x_k\le\mathcal{C}\Delta^2x_d\le\mathcal{C}(x-x_d)^2$ being $\Delta x_k<\Delta x_d$, and by applying

(4.12), (2.1) and (2.3) we find

$$\begin{aligned} I_1(x) &\le \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 |f(x_k+(x-x_k))-f(x_k)|w_\gamma(x_k) \\ &\le \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k+1}{\sqrt{m}}\right)_{w_\gamma} \\ &= \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=1}^{[\frac{d}{2}]}\left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right. \\ &\qquad \left. + \sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \end{aligned} \tag{4.15}$$

If $1 \le k \le [\frac{d}{2}]$ since in general it results

$$x_i - x_r = \sum_{j=r}^{i-1}\Delta x_j > \Delta x_r(i-r), \quad \forall i > r \tag{4.16}$$

we have

$$\begin{aligned} (x-x_k) > (x_{d-1}-x_k) &= (x_{d-1}-x_{[\frac{d}{2}]}) + (x_{[\frac{d}{2}]}-x_k) \\ &> \frac{d}{2}\Delta x_{[\frac{d}{2}]} + \left(\left[\frac{d}{2}\right]-k\right)\Delta x_k \\ &> \frac{d}{2}\Delta x_{[\frac{d}{2}]} > \mathcal{C}\frac{d}{2}\Delta x_d. \end{aligned} \tag{4.17}$$

Therefore

$$\begin{aligned} \sum_{k=1}^{[\frac{d}{2}]}\left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} &\le \mathcal{C}\sum_{k=1}^{[\frac{d}{2}]}\frac{1}{d^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \\ &\le \mathcal{C}\frac{1}{d}\Omega_\varphi\left(f,\frac{d}{\sqrt{m}}\right)_{w_\gamma} \\ &\le \mathcal{C}\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma} \end{aligned} \tag{4.18}$$

in virtue of (2.1).
If $[\frac{d}{2}]+1 \le k \le d-2$, then by applying (4.16) and by using (4.2) and (4.1)

we have

$$\sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}$$
$$\leq \mathcal{C}\sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{\Delta x_d}{(d-k)\Delta x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}$$
$$\leq \mathcal{C}\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{d^2}{(d-k)^2k^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}$$
$$\leq \mathcal{C}\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}. \tag{4.19}$$

Thus by replacing (4.18) and (4.19) in (4.15) we obtain

$$I_1(x)\leq \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \tag{4.20}$$

Now we consider I_2. By using (4.7) and taking into account that $\sup_x w_\gamma(x)\leq\mathcal{C}$, we have

$$I_2(x)\leq \mathcal{C}x^{\gamma-\alpha+\frac{1}{2}}\sum_{k=1}^{d-1}\frac{\Delta x_k}{(x-x_k)^2}\frac{x_k^{\alpha-\gamma}}{\sqrt{4m-x}}w_\gamma(x_k)|f(x)-f(x_k)|$$
$$\leq \mathcal{C}x^{\gamma-\alpha+\frac{1}{2}}\sum_{k=1}^{d-1}\frac{\Delta x_k}{(x-x_k)^2}\frac{x_k^{\alpha-\gamma}}{\sqrt{4m-x}}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}$$
$$\leq \mathcal{C}\left[x^{\gamma-\alpha+\frac{1}{2}}\left\{\sum_{k=1}^{[\frac{d}{2}]}+\sum_{k=[\frac{d}{2}]+1}^{d-2}\right\}\frac{\Delta x_k}{(x-x_k)^2}\frac{x_k^{\alpha-\gamma}}{\sqrt{4m-x}}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right.$$
$$\left.+\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}\right].$$

In the first sum we observe that $\frac{1}{\sqrt{4m-x}}\leq\frac{\mathcal{C}}{\sqrt{m}}$, and in virtue of (4.1) $x_{[\frac{d}{2}]}\sim\frac{1}{4}\frac{d^2}{m}\sim\frac{1}{4}x_d$ while in the second one we use (4.2) and (4.3). Therefore

by applying (2.2) we have

$$I_2(x) \le \mathcal{C} \left[\Omega_\varphi\left(f, \frac{d}{\sqrt{m}}\right)_{w_\gamma} \frac{x_d^{\gamma-\alpha-3/2}}{\sqrt{m}} \sum_{k=1}^{[\frac{d}{2}]} \int_{x_{k-1}}^{x_k} \frac{dt}{t^{\gamma-\alpha}} + \Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} \right.$$
$$\left. + x^{\gamma-\alpha+\frac{1}{2}} \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{\Delta x_k}{(x-x_k)^2} x_k^{\alpha-\gamma-\frac{1}{2}} \sqrt{\frac{4m-x_k}{4m-x}} \sqrt{\frac{x_k}{4m-x_k}} \right]$$
$$\le \mathcal{C} \left[\Omega_\varphi\left(f, \frac{d}{\sqrt{m}}\right)_{w_\gamma} \frac{x_d^{\gamma-\alpha-3/2}}{\sqrt{m}} \int_0^{x_d/2} \frac{dt}{t^{\gamma-\alpha}} + \Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} \right.$$
$$\left. + \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{1}{(d-k)^2} \left(\frac{x}{x_k}\right)^{\gamma-\alpha+\frac{1}{2}} \Omega_\varphi\left(f, \frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \right].$$

Thus, by the assumption on γ, we get

$$I_2(x) \le \mathcal{C} \left[\Omega_\varphi\left(f, \frac{d}{\sqrt{m}}\right)_{w_\gamma} \frac{1}{\sqrt{x_d m}} + \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{1}{(d-k)^2} \Omega_\varphi\left(f, \frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \right.$$
$$\left. + \Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} \right]$$
$$\le \mathcal{C} \left[\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{1}{(d-k)^2} \Omega_\varphi\left(f, \frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \right]. \quad (4.21)$$

Consequently, by replacing (4.20) and (4.21) in (4.14), we get

$$\Sigma_1(x) \le \mathcal{C} \left[\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{1}{(d-k)^2} \Omega_\varphi\left(f, \frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \right]. \quad (4.22)$$

Finally, we consider $\Sigma_3(x)$. By using (4.7) and taking into account that

$\sup_x w_\gamma(x) \le \mathcal{C}$, we obtain

$$\begin{aligned}
&\Sigma_3(x)\\
&\le \mathcal{C}\sum_{k=d+1}^{j}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\ \frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}|f(x)-f(x_k)|w_\gamma(x_k)\\
&= \mathcal{C}\sum_{k=d+1}^{j}\left(\frac{4m-x}{4m-x_k}\right)^{3/2}\sqrt{\frac{x_k}{4m-x_k}}\frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x}{x_k}\right)^{\gamma-\frac{1}{2}-\alpha}\\
&\qquad\qquad \cdot|f(x)-f(x_k)|w_\gamma(x_k).
\end{aligned}$$

Then since $\Delta x_k \sim \sqrt{\frac{x_k}{4m-x_k}}$, $\frac{4m-x}{4m-x_k} < 2$, $w_\gamma(x_k) \le \mathcal{C} w_\gamma(x)$, $\left(\frac{x}{x_k}\right)^{-\frac{1}{2}-\alpha+\gamma} < 1$ being $\gamma > \alpha + \frac{1}{2}$,we have

$$\Sigma_3(x) \le \mathcal{C}\sum_{k=d+1}^{j}\left(\frac{\Delta x_k}{x-x_k}\right)^2 |f(x)-f(x_k)|w_\gamma(x),$$

from which by using (4.3) we get

$$\Sigma_3(x) \le \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=d+2}^{j}\frac{1}{(k-d)^2}\Omega_\varphi\left(f,\frac{k-d}{\sqrt{m}}\right)_{w_\gamma}\right]. \quad (4.23)$$

Hence by replacing (4.13), (4.22) and (4.23) in (4.11) we have

$$|A_1(x)| \le \mathcal{C}\sum_{i=1}^{j}\frac{1}{i^2}\Omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}. \quad (4.24)$$

Now we estimate A_2. Since $x < x_k$ in virtue of (4.7) we have

$$\begin{aligned}
&|A_2(x)|\\
&:= \sum_{k=j+1}^{m+1}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}^2_{m,k}(x)w^2_\gamma(x)f(x)\\
&\le (fw_\gamma)(x)\left[\tilde{l}^2_{m,m+1}(x)w_\gamma(x) + \sum_{k=j+1}^{m}\tilde{l}^2_{m,k}(x)w_\gamma(x)\right]\\
&\le (fw_\gamma)(x)\left[\frac{p^2_m(w_\alpha,x)}{p^2_m(w_\alpha,4m)}\frac{w_\gamma(x)}{w_\gamma(4m)}w_\gamma(4m)\right.\\
&\qquad\left. +\mathcal{C}x^{\gamma-\alpha-\frac{1}{2}}\sum_{k=j+1}^{m}\frac{\Delta x_k}{(x-x_k)^2}\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}\left(\frac{4m-x}{4m-x_k}\right)^2 w_\gamma(x_k)\right].
\end{aligned}$$

Consequently, being $\frac{p_m^2(w_\alpha,x)}{p_m^2(w_\alpha,4m)}\frac{w_\gamma(x)}{w_\gamma(4m)} \le \mathcal{C}$ (see, for instance [11], estimate (4.12)), $2m < x_k < 4m$, $x < 2m$, $\sqrt{\frac{x_k}{4m-x}} \le \mathcal{C}$, $x < x_k$, $\gamma > \alpha + \frac{1}{2}$ we have

$$
\begin{aligned}
&|A_2(x)| \\
&\le \mathcal{C}|(fw_\gamma)(x)|e^{-2m}\left[e^{-2m} + \sum_{k=j+1}^{m} \frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}}\left(\frac{4m-x}{4m-x_k}\right)^2\right] \\
&\le \mathcal{C}|(fw_\gamma)(x)|e^{-2m}\left[e^{-2m} + \sum_{k=j+1}^{m} \frac{\Delta x_k}{(4m-x_k)^2}\right] \\
&\le \mathcal{C}|(fw_\gamma)(x)|e^{-2m}\left[e^{-2m} + \left\{\sum_{k=j+1}^{[\frac{3}{2}j]} + \sum_{k=[\frac{3}{2}j]+1}^{m}\right\}\frac{\Delta x_k}{(4m-x_k)^2}\right].
\end{aligned}
$$

Therefore, since in virtue of (4.1), we have $x_{[\frac{3}{2}j]} \sim \frac{9}{4}\frac{j^2}{m} \sim \frac{9}{4}x_j$, we can deduce

$$
\begin{aligned}
&|A_2(x)| \\
&\le \mathcal{C}|(fw_\gamma)(x)|e^{-2m}\left[e^{-2m} + \frac{m}{(4m-x_{[\frac{3}{2}j]})^2} + \sum_{k=[\frac{3}{2}j]+1}^{m}\int_{x_{[\frac{3}{2}j]+1}}^{x_m}\frac{dt}{(4m-t)^2}\right] \\
&\le \mathcal{C}|(fw_\gamma)(x)|e^{-2m}\left[e^{-2m} + \frac{1}{m} + \frac{1}{(4m-x_m)}\right]. \qquad (4.25)
\end{aligned}
$$

It remains to estimate A_3.
By the definition and being $-1 + \frac{2}{4m-x_k} \le 1$ we can write

$$
\begin{aligned}
|A_3(x)| \le |(fw_\gamma)(x)| \Bigg[& |x-x_d|\tilde{l}_d^2(x)w_\gamma(x) + \sum_{\substack{k=1\\k\neq d}}^{j}\tilde{l}_k^2(x)|x-x_k|w_\gamma(x) \\
& + \sum_{k=j+1}^{m}|x-x_k|\tilde{l}_k^2(x)w_\gamma(x) + |x-4m|\tilde{l}_{m+1}^2(x)w_\gamma(x)\Bigg].
\end{aligned}
$$

About the first two terms we can note that by using relation (4.7) and

being $(\frac{4m-x}{4m-x_k}) \leq \mathcal{C}$ we have

$$|x-x_d|\tilde{l}_d^2(x)w_\gamma(x)+\sum_{\substack{k=1\\k\neq d}}^{j}\tilde{l}_k^2(x)|x-x_k|w_\gamma(x)$$

$$\leq |x-x_d|\frac{\tilde{l}_d^2(x)}{w_\gamma(x_d)}w_\gamma(x)w_\gamma(x_d)+x^{\gamma-\alpha-\frac{1}{2}}\sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{|x-x_k|}\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}w_\gamma(x_k)$$

$$\leq \mathcal{C}\sqrt{\frac{x_d}{m}}w_\gamma(x_d)+\frac{x^{\gamma-\alpha-\frac{1}{2}}}{\sqrt{m}}\sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{|x-x_k|}x_k^{\frac{1}{2}+\alpha-\gamma}\sqrt{x_k}w_\gamma(x_k)$$

by applying (4.8), (4.12) and taking into account that $\frac{1}{\sqrt{4m-x}} \leq \frac{\mathcal{C}}{\sqrt{m}}$. Therefore, since $\sup_{x_k}\sqrt{x_k}w_\gamma(x_k) \leq 1$ and (see, for instance, [11])

$$\sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{|x-x_k|x_k^\rho} \leq \mathcal{C}\ \frac{\log m}{x^\rho}, \quad 0 \leq \rho \leq 1, \tag{4.26}$$

by the assumptions on γ we deduce

$$|x-x_d|\tilde{l}_d^2(x)w_\gamma(x)+\sum_{\substack{k=1\\k\neq d}}^{j}\tilde{l}_k^2(x)|x-x_k|w_\gamma(x) \leq \mathcal{C}\frac{\log m}{\sqrt{m}}. \tag{4.27}$$

About the last two terms we note that being $2m < x_k < 4m$, $x < x_k$, $\gamma > \alpha + \frac{1}{2}$ and $\sqrt{\frac{x_k}{4m-x}} \leq \mathcal{C}$ we have

$$\sum_{k=j+1}^{m}|x-x_d|\tilde{l}_k^2(x)w_\gamma(x)+|x-4m|\ \tilde{l}_{m+1}^2(x)w_\gamma(x)$$

$$\leq \mathcal{C}x^{\gamma-\alpha-\frac{1}{2}}\sum_{k=j+1}^{m}\frac{\Delta x_k}{|x-x_k|}\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}\left(\frac{4m-x}{4m-x_k}\right)^2 w_\gamma(x_k)+|x-4m|e^{-4m}$$

$$\leq e^{-2m}(4m-x)\sum_{k=j+1}^{m}\frac{\Delta x_k}{(4m-x_k)^2}+|x-4m|e^{-4m}.$$

Then by estimating the sum as already done for A_2 and taking into account estimate (4.27) we get

$$|A_3(x)| \leq \mathcal{C}\frac{\log m}{\sqrt{m}}. \tag{4.28}$$

Hence by substituting (4.24), (4.25) and (4.28) in (4.10) we have

$$\|[f-\mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_{L^\infty((0,x_j])} \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\Omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}+\frac{\log m}{\sqrt{m}}\|fw_\gamma\|_\infty\right]$$

and the theorem follows by applying (2.3). □

Proof of Theorem 3.1. Let $f\in L^\infty_{w_\gamma}$ and let $\mathcal{P}^{(\alpha)}_{m+1}$ and f_j be the functions defined in (1.3) and (1.4), respectively. We write

$$\begin{aligned}\|[f-\mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_\infty &= \|[f-\mathcal{P}^{(\alpha)}_{m+1}(f_j)]w^2_\gamma\|_\infty\\ &\le \|[f-f_j]w^2_\gamma\|_\infty+\|[f_j-\mathcal{P}^{(\alpha)}_{m+1}(f_j)]w^2_\gamma\|_\infty\\ &\le \|[f-f_j]w^2_\gamma\|_\infty+\|[f-\mathcal{P}^{*(\alpha)}_{m+1}(f)]w^2_\gamma\|_{L^\infty((0,x_j])}\\ &\qquad+\|\mathcal{P}^{*(\alpha)}_{m+1}(f)w^2_\gamma\|_{L^\infty((x_j,+\infty))}.\end{aligned} \tag{4.29}$$

In order to estimate the first term on the right-hand side we denote by Q_M the near best polynomial approximation of f i.e.

$$\|[f-Q_M]w_\gamma\|_\infty \le \mathcal{C}E_M(f)_{w_\gamma},\quad M=\left[\frac{m}{3}\right]. \tag{4.30}$$

Thus, by observing that $f=f_j$ in $(0,x_j]$ and that $w_\gamma(x)\le\mathcal{C}$, we write

$$\begin{aligned}\|[f-f_j]w^2_\gamma\|_\infty &\le \mathcal{C}\|fw_\gamma\|_{L^\infty((x_j,+\infty))}\\ &\le \|[f-Q_M]w_\gamma\|_{L^\infty((x_j,+\infty))}+\|Q_Mw_\gamma\|_{L^\infty((x_j,+\infty))}\\ &\le \mathcal{C}E_M(f)_{w_\gamma}+\|Q_Mw_\gamma\|_{L^\infty([2m,+\infty))}\end{aligned} \tag{4.31}$$

being $x_j\ge 2m$. Now we recall that (see, for instance, [13]) $\forall P_m\in\mathbb{P}_m$ and $\forall\delta>0$ it results

$$\|P_mw_\gamma\|_{L^\infty([4m(1+\delta),+\infty))}\le\mathcal{C}e^{-Am}\|P_mw_\gamma\|_\infty \tag{4.32}$$

where $\mathcal{C}$ and A are positive constants independent of m and P_m.
Therefore using (4.32) to the last term of (4.31) we have

$$\begin{aligned}\|[f-f_j]w^2_\gamma\|_\infty &\le \mathcal{C}[E_M(f)_{w_\gamma}+e^{-Am}\|Q_Mw_\gamma\|_\infty]\\ &\le \mathcal{C}[E_M(f)_{w_\gamma}+e^{-Am}\|fw_\gamma\|_\infty]\end{aligned}$$

from which by applying (4.9) we deduce

$$\|[f-f_j]w_\gamma^2\|_\infty \le \mathcal{C}\left[\omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am}\|fw_\gamma\|_\infty\right].$$

The second term in (4.29) can be estimated by applying Lemma 4.2. Consequently we have only to bound the last term. To this end it is sufficient to note that $\mathcal{P}_{m+1}^{*(\alpha)}$ is a bounded operator and is a polynomial. Therefore by applying (4.32) we easily deduce

$$\|\mathcal{P}_{m+1}^{*(\alpha)}(f)w_\gamma^2\|_{L^\infty((x_j,+\infty))} \le \mathcal{C}e^{-Am}\|fw_\gamma\|_\infty$$

and the theorem follows. □

Proof of Corollary 3.1. By (1.5) we can write

$$\|[f-\mathcal{F}_{m+1}^*(f)]w_\gamma^2\|_\infty \le \|[f-\mathcal{P}_{m+1}^{*(\alpha)}(f)]w_\gamma^2\|_\infty + \sup_{x\ge 0}\left|w_\gamma^2(x)\sum_{k=1}^{j}\left(-1+\frac{2}{4m-x_k}\right)(x-x_k)f(x_k)\tilde{l}_k^2(x)\right|.$$

The first term can be estimated as in Theorem 3.1 and then we have only to bound the last one. By applying Proposition 4.2, by using (4.7), (4.8), (4.26) and taking into account that $|x-x_d| \sim \sqrt{\frac{x}{m}}$, $\sup_x \sqrt{x}w_\gamma(x) \le 1$, $(-1+\frac{2}{4m-x_k}) \le 1$ we have

$$\begin{aligned}
&\sup_{x\ge 0}\left|w_\gamma^2(x)\sum_{k=1}^{j}\left(-1+\frac{2}{4m-x_k}\right)(x-x_k)f(x_k)\tilde{l}_k^2(x)\right| \\
&\le \mathcal{C}\|fw_\gamma\|_\infty \sup_{\frac{a}{m}\le x\le 2m} w_\gamma(x)\left[|x-x_d|\tilde{l}_d^2(x)\frac{w_\gamma(x)}{w_\gamma(x_d)}\right. \\
&\qquad\qquad \left. + x^{\gamma-\alpha-\frac{1}{2}}\sum_{\substack{k=1\\k\ne d}}^{j}\frac{\Delta x_k}{|x-x_k|}\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}\right] \\
&\le \mathcal{C}\|fw_\gamma\|_\infty \sup_{\frac{a}{m}\le x\le 2m}\sqrt{x}w_\gamma(x)\left[\frac{1}{\sqrt{m}}+\frac{\log m}{\sqrt{m}}\right] \\
&\le \mathcal{C}\frac{\log m}{\sqrt{m}}\|fw_\gamma\|_\infty
\end{aligned}$$

and the corollary follows. □

Proof of Proposition 3.1. Let $w_\gamma(x) = x^\gamma e^{-x}$, $v(x) = w_\gamma(x)(1+x)^{-\lambda}$, $u(x) = w_\gamma(x)(1+x)^{\lambda}$. By applying Proposition 4.2 we have

$$\begin{aligned}\|\mathcal{P}_{m+1}^{*(\alpha)}(f)v\|_1 &\le \mathcal{C}\|\mathcal{P}_{m+1}^{*(\alpha)}(f)v\|_{L^1([\frac{a}{m},2m])}\\ &= \mathcal{C}\int_{\frac{a}{m}}^{2m}\left|v(x)\sum_{k=1}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)f(x_k)\tilde{l}_k^2(x)\right|dx\\ &\le \mathcal{C}\|fu\|_\infty\int_{\frac{a}{m}}^{2m}\sum_{k=1}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}_k^2(x)\frac{v(x)}{u(x_k)}dx.\end{aligned}$$

Then denoting by x_d a node closest to x with $x_{d-1} < x < x_d < x_{d+1}$ we get

$$\begin{aligned}\|\mathcal{P}_{m+1}^{*(\alpha)}(f)v\|_1 \le \mathcal{C}\|fu\|_\infty\Bigg[&\int_{\frac{a}{m}}^{2m}\left(\frac{(1+\alpha)x}{x_d}-\alpha\right)\tilde{l}_d^2(x)\frac{v(x)}{u(x_d)}dx\\ &+\int_{\frac{a}{m}}^{2m}\sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}_k^2(x)\frac{v(x)}{u(x_k)}dx\Bigg]\end{aligned}$$

from which by (4.8) we obtain

$$\begin{aligned}&\|\mathcal{P}_{m+1}^{*(\alpha)}(f)v\|_1\\ &\le \mathcal{C}\|fu\|_\infty\left[\int_0^\infty\frac{dx}{(1+x)^{2\lambda}}+\int_{\frac{a}{m}}^{2m}\sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}_k^2(x)\frac{v(x)}{u(x_k)}dx\right]\\ &\le \mathcal{C}\|fu\|_\infty\left[1+\int_{\frac{a}{m}}^{2m}\frac{1}{(1+x)^\lambda}\sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\frac{w_\gamma(x)}{w_\gamma(x_k)}\frac{\tilde{l}_k^2(x)dx}{(1+x_k)^\lambda}\right].\end{aligned}\tag{4.33}$$

Now let $\frac{a}{m} \le x \le 2m$ and consider the sum in the integral.

Being $\frac{1}{(1+x_k)^\lambda} < 1$ and by applying (4.7) we have

$$\sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}_k^2(x)\frac{w_\gamma(x)}{w_\gamma(x_k)}\frac{1}{(1+x_k)^\lambda}$$

$$\leq \mathcal{C}\left[x^{\gamma+\frac{1}{2}-\alpha}\sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{(x-x_k)^2}\frac{x_k^{\alpha-\gamma}}{\sqrt{4m-x}}\left(\frac{4m-x}{4m-x_k}\right)^2\right.$$

$$\left.+x^{\gamma-\alpha-\frac{1}{2}}\sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{(x-x_k)^2}\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}\left(\frac{4m-x}{4m-x_k}\right)^2\right]$$

$$:= \mathcal{C}[\Sigma_1(x)+\Sigma_2(x)].$$

We first consider $\Sigma_1(x)$. If $x < x_k$, since $\gamma > \alpha$, $(\frac{4m-x}{4m-x_k}) < \mathcal{C}$ and $x_d > \frac{1}{m}$ by (4.2) and (4.3) it results

$$\Sigma_1(x) \leq \sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{(x-x_k)^2}\sqrt{\frac{x_k}{4m-x_k}} \leq \sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{\Delta x_k}{x-x_k}\right)^2$$

$$\leq \sum_{\substack{k=1\\k\neq d}}^{j}\frac{1}{(|d-k|+1)^2} \leq \mathcal{C}.$$

If $x_k < x < 2m$ then $\frac{1}{\sqrt{4m-x_k}} \leq \frac{1}{\sqrt{2}\sqrt{m}}$ and being again $(\frac{4m-x}{4m-x_k}) < \mathcal{C}$, we can write

$$\Sigma_1(x) \leq \frac{\mathcal{C}}{\sqrt{m}}x^{\gamma+\frac{1}{2}-\alpha}\left\{\sum_{x_1\leq x_k\leq\frac{x}{2}}+\sum_{\frac{x}{2}\leq x_k\leq x_{d-1}}\right\}\frac{\Delta x_k}{(x-x_k)^2}x_k^{\alpha-\gamma}$$

$$:= \frac{\mathcal{C}}{\sqrt{m}}x^{\gamma+\frac{1}{2}-\alpha}[I_1(x)+I_2(x)]. \tag{4.34}$$

In the sum I_1 it results $\frac{1}{(x-x_k)^2} \leq \frac{4}{x^2}$ and being $\gamma < \alpha+1$ we obtain

$$I_1(x) \leq \frac{4}{x^2}\sum_{x_1\leq x_k\leq\frac{x}{2}}\left(\frac{1}{x_k}\right)^{\gamma-\alpha}\Delta x_k \leq \frac{\mathcal{C}}{x^2}\sum_{x_1\leq x_k\leq\frac{x}{2}}\int_{x_{k-1}}^{x_k}\left(\frac{1}{t}\right)^{\gamma-\alpha}dt$$

$$\leq \frac{\mathcal{C}}{x^2}\int_0^{\frac{x}{2}}\frac{dt}{t^{\gamma-\alpha}} \leq \mathcal{C}x^{\alpha-\gamma-1}.$$

For the sum $I_2(x)$ since $\gamma > \alpha$ and $\frac{x}{2} \le x_k$ we have

$$\begin{aligned} I_2(x) &\le \left(\frac{2}{x}\right)^{\gamma-\alpha} \sum_{\frac{x}{2}\le x_k\le x_{d-1}} \frac{\Delta x_k}{(x-x_k)^2} \\ &\le \left(\frac{2}{x}\right)^{\gamma-\alpha} \sum_{\frac{x}{2}\le x_k\le x_{d-1}} \int_{x_k}^{x_{k+1}} \frac{dt}{(x-t)^2} \\ &= \left(\frac{2}{x}\right)^{\gamma-\alpha} \int_{\frac{x}{2}}^{x_{d-1}} \frac{dt}{(x-t)^2} \\ &\le \mathcal{C}\ x^{\alpha-\gamma-1}. \end{aligned}$$

Then substituting the expression of I_1 and I_2 in (4.34) and taking into account that $x > \frac{a}{m}$ we obtain the boundedness of $\Sigma_1(x)$.
Now we consider $\Sigma_2(x)$. We write

$$\Sigma_2(x) = \mathcal{C} x^{\gamma-\alpha-\frac{1}{2}} \sum_{\substack{k=1\\k\neq d}}^{j} \frac{\Delta x_k}{(x-x_k)^2} \frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x_k}} \left(\frac{4m-x}{4m-x_k}\right)^{3/2}.$$

Being $(\frac{4m-x}{4m-x_k})^{3/2} < \mathcal{C}$ and taking into account that $\sqrt{\frac{x_k}{4m-x_k}} \sim \Delta x_k$ one deduces

$$\Sigma_2(x) \le \mathcal{C} x^{\gamma-\alpha-\frac{1}{2}} \sum_{\substack{k=1\\k\neq d}}^{j} \left(\frac{\Delta x_k}{x-x_k}\right)^2 x_k^{\frac{1}{2}+\alpha-\gamma}.$$

Hence by (4.3) and (4.1) we get

$$\begin{aligned} \Sigma_2(x) &\le \mathcal{C} x^{\gamma-\alpha-\frac{1}{2}} \sum_{\substack{k=1\\k\neq d}}^{j} \frac{1}{(|d-k|+1)^2} \left(\frac{1}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \\ &\le \mathcal{C} x^{\gamma-\alpha-\frac{1}{2}} m^{\gamma-\frac{1}{2}-\alpha} \sum_{\substack{k=1\\k\neq d}}^{j} \frac{1}{k^{2(\gamma-\alpha-\frac{1}{2})}(|d-k|+1)^2}. \end{aligned}$$

Therefore being $\max\{0, \alpha+\frac{1}{2}\} \le \gamma < \alpha+1$ and $x < x_d$ we obtain (see, for instance, [16, Lemma 9])

$$\Sigma_2(x) \le \mathcal{C} x^{\gamma-\alpha-\frac{1}{2}} x_d^{-\gamma+\alpha+\frac{1}{2}} \le \mathcal{C}.$$

Consequently the sum appearing in the integral (4.33) is bounded and we deduce

$$\|\mathcal{P}_{m+1}^{*(\alpha)}(f)v\|_1 \leq \mathcal{C}\|fu\|_\infty \left[1 + \int_0^\infty \frac{dx}{(1+x)^\lambda}\right] \leq \mathcal{C}\|fu\|_\infty$$

being $\lambda > 1$. □

In order to prove Theorem 3.2 we need the following lemma.

Lemma 4.3. *Let* $w_\gamma(x) = x^\gamma e^{-x}$, $v(x) = w_\gamma(x)(1+x)^{-\lambda}$, $u(x) = w_\gamma(x)(1+x)^\lambda$ *such that*

$$1 < \lambda < \frac{3}{2}, \quad \max\left\{0, \alpha + \frac{1}{2}\right\} \leq \gamma < \alpha + 1, \quad -1 < \alpha \leq 0.$$

Then for every function $f \in L_u^\infty$ *we have*

$$\|[Q_M - \mathcal{P}_{m+1}^{*(\alpha)}(Q_M)]v\|_1 \leq \mathcal{C}\left[\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u + \frac{\|fu\|_\infty}{\sqrt{m}}\right]$$

where $Q_M \in \mathbb{P}_M$ *with* $M \sim m$ *is the near best approximant polynomial of* f *defined in* (4.30) *and* $\mathcal{C} \neq \mathcal{C}(m, f)$.

Proof. First of all we recall that denoting by $H(f,x)$ the Hermite polynomial based on the zeros $x_1 < x_2 < ... < x_m < x_{m+1} = 4m$

$$H(f,x) = F_{m+1}(f,x) + \sum_{k=1}^{m+1}(x - x_k)\widetilde{l}_k^2(x)f'(x_k)$$

it results $f(x) = H(f,x)$ if $f \in \mathbb{P}_{2m-1}$.
Now let $f \in L_u^\infty$ and let us consider its near beat approximant polynomial Q_M with $M = [\frac{m}{3}]$. By Proposition 4.2 we have

$$\begin{aligned}\|[Q_M - \mathcal{P}_{m+1}^{*(\alpha)}(Q_M)]v\|_1 &\leq \mathcal{C}\ \|[Q_M - \mathcal{P}_{m+1}^{*(\alpha)}(Q_M)]v\|_{L^1([\frac{a}{m}, 2m])} \\ &= \mathcal{C}\int_{\frac{a}{m}}^{2m} |Q_M(x) - \mathcal{P}_{m+1}^{*(\alpha)}(Q_M, x)|v(x)dx \\ &= \mathcal{C}\int_{\frac{a}{m}}^{2m} |H(Q_M, x) - \mathcal{P}_{m+1}^{*(\alpha)}(Q_M, x)|v(x)dx\end{aligned}$$

from which

$$\|[Q_M - \mathcal{P}_{m+1}^{*(\alpha)}(Q_M)]v\|_1$$
$$= \mathcal{C}\int_{\frac{a}{m}}^{2m}\left|v(x)\sum_{k>j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)Q_M(x_k)\tilde{l}_k^2(x)\right|dx,$$
$$+\int_{\frac{a}{m}}^{2m}\left|v(x)\sum_{k=1}^{m+1}\left(Q'_M(x_k)+\left(-1+\frac{2}{4m-x_k}\right)Q_M(x_k)\right)(x-x_k)\tilde{l}_k^2(x)\right|dx$$
$$:= I_1 + I_2.$$

Now we immediately note that

$$I_1 \le \mathcal{C}\|Q_M u\|_{L^\infty([2m,+\infty)}\int_{\frac{a}{m}}^{2m}\frac{1}{(1+x)^{2\lambda}}\sum_{k>j}\left(\frac{(1+\alpha)x}{x_k}-\alpha\right)\tilde{l}_k^2(x)\frac{w_\gamma(x)}{w_\gamma(x_k)}dx$$

and this quantity tends to zero taking into account relation (4.32) and by estimating the sum by using (4.7).
About the integral I_2 we observe that since $-1+\frac{2}{4m-x_k}\le 1$ we can write

$$\begin{aligned}
I_2 \le & \int_{\frac{a}{m}}^{2m}\left|\sum_{k=1}^{j}Q'_M(x_k)(x-x_k)\tilde{l}_k^2(x)v(x)\right|dx \\
& + \int_{\frac{a}{m}}^{2m}\left|\sum_{k=1}^{j}Q_M(x_k)(x-x_k)\tilde{l}_k^2(x)v(x)\right|dx \\
& + \int_{\frac{a}{m}}^{2m}\left|\sum_{k>j}Q'_M(x_k)(x-x_k)\tilde{l}_k^2(x)v(x)\right|dx \\
& + \int_{\frac{a}{m}}^{2m}\left|\sum_{k>j}Q_M(x_k)(x-x_k)\tilde{l}_k^2(x)v(x)\right|dx \\
:= & B_1 + B_2 + B_3 + B_4. \qquad (4.35)
\end{aligned}$$

The integrals B_3 and B_4 can be estimated as already done for the integral I_1 deducing that they tend to zero. There remains to estimate the first two. We begin with B_2. By using (4.4) we have

$$B_2 \le \mathcal{C}\sum_{k=1}^{j}x_k w_\alpha(x_k)\Delta x_k Q_M(x_k)\int_{\frac{a}{m}}^{2m}\frac{p_m^2(w_\alpha,x)v(x)g(x)}{(x-x_k)}dx$$

where $g = sign(v(x)\sum_{k=1}^{j} Q_M(x_k)(x-x_k)\tilde{l}_k^2(x))$. Therefore we deduce

$$|B_2| \le \mathcal{C}\ \|Q_M u\|_\infty \sum_{k=1}^{j} x_k^{1+\alpha-\gamma} \frac{\Delta x_k}{(1+x_k)^\lambda} |\Pi(x_k)|$$

where

$$\Pi(t) = \int_{\frac{a}{m}}^{2m} \frac{p_m^2(w_\alpha, x)q(x) - p_m^2(w_\alpha, t)q(t)}{x-t} \frac{v(x)g(x)}{q(x)} dx \tag{4.36}$$

and q is a polynomial of degree ℓm (ℓ fixed).
Then by applying the Marcinkiewicz inequality (see, for instance, [7]) according to which

$$\sum_{k=1}^{j} \Delta x_k |P(x_k)x_k^\sigma| \le \mathcal{C} \int_{\frac{a}{m}}^{4m\theta_1} |P(x)x^\sigma| dx, \quad x^\sigma \in L^1, \quad P \in \mathbb{P}_{\ell m}, \quad \ell \text{ fixed}$$

where $x_j = \min\{x_k \ge 2m\}$ and $x_j \le 4m\theta_1$ with $\theta_1 \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned}
|B_2| &\le \mathcal{C}\ \|Q_M u\|_\infty \int_{\frac{a}{m}}^{4m\theta_1} \frac{t^{1+\alpha-\gamma}}{(1+t)^\lambda} |\Pi(t)| dt \\
&= \mathcal{C}\ \|Q_M u\|_\infty \left[\int_{\frac{a}{m}}^{4m\theta_1} \frac{t^{1+\alpha-\gamma}}{(1+t)^\lambda} \left| \int_{\frac{a}{m}}^{2m} \frac{p_m^2(w_\alpha, x)v(x)g(x)}{(x-t)} dx \right| dt \right. \\
&\qquad \left. + \int_{\frac{a}{m}}^{4m\theta_1} p_m^2(w_\alpha, t) \frac{t^{1+\alpha-\gamma}}{(1+t)^\lambda} q(t) \left| \int_{\frac{a}{m}}^{2m} \frac{v(x)g(x)}{(x-t)q(x)} dx \right| dt \right] \\
&:= \mathcal{C}\ \|Q_M u\|_\infty [A_1 + A_2]
\end{aligned} \tag{4.37}$$

where each integral is in the sense of the principal value.
We estimate A_1. To this end by (4.6) and being $\frac{1}{\sqrt{4m-x_k}} \le \frac{\mathcal{C}}{\sqrt{m}}$ we write

$$\begin{aligned}
&A_1 \\
&\le \frac{\mathcal{C}}{\sqrt{m}} \int_{\frac{a}{m}}^{4m\theta_1} \left(\frac{t}{1+t} \right)^{1+\alpha-\gamma} (1+t)^{-\lambda+1+\alpha-\gamma} \left| \int_{\frac{a}{m}}^{4m\theta_1} \frac{x^{\gamma-\alpha-\frac{1}{2}} dx}{(1+x)^\lambda (x-t)} \right| dt.
\end{aligned} \tag{4.38}$$

Now we recall the following inequality [14]

$$\int_0^\infty \left(\frac{t}{1+t}\right)^r (1+t)^s \int_0^\infty \left|\frac{F(x)}{x-t}dx\right| dt$$
$$\leq \mathcal{C}\left[1+\int_0^\infty \left(\frac{t}{1+t}\right)^R (1+t)^S |F(t)|(1+\log^+|F(t)|+\log^+(t))dt\right] \tag{4.39}$$

where

$$r>-1, \quad R\leq 0, \quad r\geq R, \quad s<0, \quad S\geq -1, \quad s\leq S \tag{4.40}$$

$\log^+(x)=\log\max(1,x)$, and $\mathcal{C}\neq\mathcal{C}(F)$.
In order to apply this inequality to estimate (4.38) we set $R=0$, $S=-\lambda+\frac{1}{2}$, $r=1+\alpha-\gamma$, $s=-\lambda+1+\alpha-\gamma$.
In fact with these definitions by the hypothesis on γ and λ, relations (4.40) are satisfied and we have

$$A_1\leq \frac{\mathcal{C}}{\sqrt{m}}\int_0^\infty \frac{t^{\gamma-\alpha-\frac{1}{2}}}{(1+t)^{2\lambda-\frac{1}{2}}}\left(1+\log^+\left(\frac{t^{\frac{1}{2}-\alpha+\gamma}}{(1+t)^\lambda}\right)\right)dt\leq \frac{\mathcal{C}}{\sqrt{m}}.$$

Now we estimate A_2. To this end we choose the polynomial $q(x)\sim x^{\alpha+\frac{1}{2}}e^{-x}$ (see, for instance, [8, pp. 485-493] and [17, p. 335]) and we use (4.6). Hence we get

$$A_2\leq \frac{\mathcal{C}}{\sqrt{m}}\int_{\frac{a}{m}}^{4m\theta_1}\frac{t^{1+\alpha-\gamma}}{(1+t)^\lambda}\left|\int_{\frac{a}{m}}^{4m\theta_1}\frac{x^{\gamma-\alpha-\frac{1}{2}}}{(x-t)(1+x)^\lambda}dx\right|dt.$$

This integral is equal to that of (4.38) and then is less than $\frac{\mathcal{C}}{\sqrt{m}}$. Therefore we deduce by (4.37)

$$|B_2|\leq \frac{\mathcal{C}}{\sqrt{m}}\|fu\|_\infty.$$

Now we consider the integral B_1. By proceeding as already done for B_2 we have

$$|B_1|\leq \mathcal{C}\|Q'_M\varphi u\|_\infty \sum_{k=1}^{j} x_k^{\frac{1}{2}+\alpha-\gamma}\frac{\Delta x_k}{(1+x_k)^\lambda}|\Pi(x_k)|$$

where $\varphi(t)=\sqrt{t}$ and $\Pi(t)$ is the polynomial defined in (4.36). Moreover following the same procedure adopted to estimate the integral B_2 we find

$$|B_1|\leq \frac{\mathcal{C}}{\sqrt{m}}\|Q'_m\varphi u\|_\infty$$

from which (see, for instance, [1])

$$|B_1| \leq \mathcal{C}\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u.$$

Hence replacing this expression in (4.35) we have

$$|I_2| \leq \mathcal{C}\left[\frac{\|fu\|_\infty}{\sqrt{m}} + \omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u\right]$$

and the lemma follows. □

Proof of Theorem 3.2. Let Q_M with $M = [\frac{m}{3}]$,the near best approximant polynomial of f and write

$$\begin{aligned}\|[f - \mathcal{P}^{*(\alpha)}_{m+1}(f)]v\|_1 \leq \|[f - Q_M]v\|_1 + \|\mathcal{P}^{*(\alpha)}_{m+1}(f - Q_M)v\|_1 \\ + \|[Q_M - \mathcal{P}^{*(\alpha)}_{m+1}(Q_M)]v\|_1.\end{aligned}$$

By applying Proposition 3.1 to the second term and Lemma 4.3 to the last one we have

$$\begin{aligned}\|[f - \mathcal{P}^{*(\alpha)}_{m+1}(f)]v\|_1 &\leq \mathcal{C}\left[\|[f - Q_M]u\|_\infty \int_0^\infty \frac{dx}{(1+x)^{2\lambda}} + \|[f - Q_M]u\|_\infty\right. \\ &\quad \left. + \omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u + \frac{\|fu\|_\infty}{\sqrt{m}}\right] \\ &\leq \mathcal{C}\left[E_M(f)_u + \omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_u + \frac{\|fu\|_\infty}{\sqrt{m}}\right].\end{aligned}$$

Then by applying (4.9) we have the statement. □

Proof of Corollary 3.2. By (1.5) and by applying Proposition 4.2 we have

$$\|[f - \mathcal{F}^*_{m+1}(f)]v\|_1 \leq \mathcal{C}\,\|[f - \mathcal{P}^{*(\alpha)}_{m+1}]v\|_1 + \int_{\frac{a}{m}}^{2m}\left|v(x)\sum_{k=1}^{j}(x - x_k)f(x_k)\tilde{l}_k^2(x)\right|dx.$$

Hence the corollary follows by applying Theorem 3.2 to the first term and by observing that the second is bounded by $\frac{\mathcal{C}}{\sqrt{m}}\|fu\|_\infty$. Indeed it is equal to the integral B_2 defined in (4.35) with $f = Q_M$. □

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Maximal Operators of Logarithmic Means of One-Dimensional Walsh-Fourier Series

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Abstract. The main aim of this paper is to investigate (H_p, L_p)-type inequality for maximal operators of logarithmic means of one-dimensional Walsh-Fourier series.

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Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G, \quad I_n(x) := I_n(x_0, ..., x_{n-1}) \\ &:= \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...)\}, \end{aligned}$$

$$(x \in G, n \in \mathbf{N}).$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Set $e_n := (0, ..., 0, 1, 0, ...) \in G$ the n th coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$. Let $\overline{I}_n := G \setminus I_n$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N})$$

the k-th Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ $(i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x \in I_n, \\ 0, \text{ if } x \in \overline{I}_n. \end{cases} \tag{1}$$

The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M f(x) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x),$$

where the number

$$\widehat{f}(i) = \int_G f(x) w_i(x) d\mu(x)$$

is said to be the i th Walsh-Fourier coefficient of the function f.

The norm (or quasinorm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \left(\int_G |f(x)|^p d\mu(x) \right)^{1/p} \quad (0 < p < +\infty).$$

The space weak-$L_p(G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G)} := \sup_{\lambda>0} \lambda\mu\left(|f|>\lambda\right)^{1/p} < +\infty.$$

The σ-algebra generated by the I_k dyadic intervale of measure 2^{-k} will be denoted by $F_k\,(k\in\mathbf{N})$.

Denote by $f=\left(f^{(n)},n\in\mathbf{N}\right)$ martingale with respect to $(F_n,n\in\mathbf{N})$ (for details see, e. g. [16]). The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbf{N}}\left|f^{(n)}\right|.$$

In case $f\in L_1(G)$, the maximal function can also be given by

$$f^*(x)=\sup_{n\in\mathbf{N}}\frac{1}{\mu\left(I_n(x)\right)}\left|\int_{I_n(x)} f(u)\,d\mu(u)\right|,\ x\in G.$$

For $0<p<\infty$ the Hardy martingale space $H_p(G)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f\in L_1(G)$ then it is easy to show that the sequence $(S_{2^n}(f):n\in\mathbf{N})$ is a martingale. If f is a martingale, that is $f=(f^{(0)},f^{(1)},...)$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i)=\lim_{k\to\infty}\int_G f^{(k)}(x)\,w_i(x)\,d\mu(x).$$

The Walsh-Fourier coefficients of $f\in L_1(G)$ are the same as the ones of the martingale $(S_{2^n}(f):n\in\mathbf{N})$ obtained from f.

For $n=1,2,...$ and a martingale f the Fejér means of order n of the Walsh-Fourier series of the function f is given by

$$\sigma_n f(x)=\frac{1}{n}\sum_{j=1}^{n} S_j f(x).$$

For the martingale f we consider maximal operators

$$\sigma^* f=\sup_{n\in\mathbf{P}}|\sigma_n f(x)|.$$

The Fejér kernel of order n of the Walsh-Fourier series defined by

$$K_n(x) := \frac{1}{n}\sum_{k=1}^{n} D_k(x).$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a d\mu = 0$;

b) $\|a\|_\infty \leq \mu(I)^{-1/p}$;

c) supp $a \subset I$.

The first result with respect to the a.e. convergence of the Walsh-Fejér means σ_n is due to Fine [1]. Later, Schipp [9] showed that the maximal operator σ^* is of weak type (1, 1), from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$. However, Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [10]). Fujii's theorem was extended by Weisz [15]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [11] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In [7] the author proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$. In the endpoint case $p = 1/2$ Weisz [17] proved that σ^* is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$.

In the literature, it is known the notion of Riesz's logarithmic means of a Fourier series. The n-th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n f(x) := \frac{1}{l_n}\sum_{k=1}^{n} \frac{S_k f(x)}{k},$$

where $l_n := \sum_{k=1}^{n} (1/k)$.

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mentioned, for instance, the paper by Szasz [13] and Yabuta [14]. This means with respect to the Walsh and Vilenkin systems was discussed by Simon [11] and Gát [4].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n-th

Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n}\sum_{k=0}^{n-1} q_{n-k}S_k f,$$

where

$$Q_n := \sum_{k=1}^{n} q_k.$$

If $q_k = k$, then we get the Nörlund logarithmic means

$$L_n f\,(x) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f\,(x)}{n-k}.$$

In this paper we call it logarithmic mean altough, it is a kind of "reverse" Riesz's logarithmic mean.

For the martingale f we consider the following maximal operators

$$R^* f(x) = \sup_{n\in\mathbf{P}} |R_n f\,(x)\,|, \qquad L^* f(x) = \sup_{n\in\mathbf{P}} |L_n f\,(x)\,|.$$

Moricz and Siddiqi [8] investigates the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p functions in norm. The case when $q_k = 1/k$ is excluded, since the methods of Móricz and Siddiqi are not applicable to logarithmic means. In [5] Gát and Goginava proved some convergence and divergence properties of the logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1. Amoung others, we gave a negative answer to the question of Móricz and Siddiqi [8]. Gát and Goginava [6] proved that for each measurable function $\Phi\,(u) = o\left(u\sqrt{\log u}\right)$ there exists an integrable function f such that

$$\int_G \Phi\left(|f\,(x)\,|\right) d\mu\,(x) < \infty$$

and there exists a set with positive measure, such that the Walsh-logarithmic means of the function diverge on this set.

The main aim of this paper is to inverstigate (H_p, L_p)-type inequality for maximal operators of logarithmic means of one-dimensional Walsh-Fourier series. In paricular, we prove that the following are true

Theorem 1 *a) Let $p > 1/2$. Then the maximal operator R^* is bounded from the Hardy space $H_p\,(G)$ to the space $L_p\,(G)$.*

b) Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p(G)$ for which

$$\|R^* f\|_p = +\infty.$$

Corollary 2 *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p(G)$ for which*

$$\|\sigma^* f\|_p = +\infty.$$

Theorem 3 *Let $0 < p \leq 1$. Then there exists a martingale $f \in H_p(G)$ for which*

$$\|L^* f\|_p = +\infty.$$

In order to prove theorems we need the following

Theorem A (Weisz [16]). A martingale $f = \left(f^{(n)} : n \in \mathbf{N}\right)$ is in $H_p\,(0 < p \leq 1)$ if and only if there exists a sequence $(a_k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that for every $n \in \mathbf{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \tag{2}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (2).

Proof of Theorem 1. a) Using Abel transformation we obtain

$$L_n f\,(x) = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\sigma_j f\,(x)}{j+1} + \frac{\sigma_n f\,(x)}{l_n}.$$

Consequently,

$$L^* f \leq c\sigma^* f. \tag{3}$$

On the other hand, Weisz [15] proved that σ^* is bounded from the martingale Hardy space $H_p\,(G)$ to the space $L_p\,(G)$, when $p > 1/2$. Hence, from (3) we conclude that L^* is bounded from the martingale Hardy space $H_p\,(G)$ to the space $L_p\,(G)$, when $p > 1/2$.

b) Let $\{m_k : k \in \mathbf{P}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=1}^{\infty} \frac{1}{m_k^{p/2}} < \infty, \tag{4}$$

$$\sum_{l=0}^{k-1} \frac{2^{2m_l/p}}{\sqrt{m_l}} < \frac{2^{2m_k/p}}{\sqrt{m_k}}, \tag{5}$$

$$\frac{2^{2m_{k-1}}}{\sqrt{m_{k-1}}} < \frac{2^{m_k}}{m_k}. \tag{6}$$

Let

$$f^{(A)}(x) := \sum_{\{k:2m_k<A\}} \lambda_k a_k(x),$$

where

$$\lambda_k := \frac{2}{\sqrt{m_k}}$$

and

$$a_k(x) := 2^{2m_k(1/p-1)-1}\left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right).$$

It is easy to show that the martingale $f = \left(f^{(0)}, f^{(1)}, ..., f^{(A)}, ...\right) \in H_p(G)$. Indeed, since

$$\|a_k\|_{\infty} = 2^{2m_k(1/p-1)-1}2^{2m_k+1} = 2^{2m_k/p} = (\operatorname{supp}(a_k))^{1/p},$$

$$S_{2^A}a_k(x) = \begin{cases} 0, A \le 2m_k \\ a_k, A > 2m_k \end{cases},$$

$$f^{(A)}(x) := \sum_{\{k:2m_k<A\}} \lambda_k a_k(x) = \sum_{k=0}^{\infty} \lambda_k S_{2^A}a_k(x)$$

from (4) and Theorem A we conclude that $f \in H_p(G)$.

Let

$$q_{A,s} := 2^{2A} + 2^{2s}, A > s.$$

Then we can write

$$\begin{aligned} R_{q_{m_k,s}}f(x) &= \frac{1}{l_{q_{m_k,s}}} \sum_{j=1}^{q_{m_k,s}} \frac{S_j f(x)}{j} \qquad (7) \\ &= \frac{1}{l_{q_{m_k,s}}} \sum_{j=1}^{2^{2m_k-1}} \frac{S_j f(x)}{j} + \frac{1}{l_{q_{m_k,s}}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} \frac{S_j f(x)}{j} = I + II. \end{aligned}$$

Let $j \in \left\{2^{2m_k}, ..., 2^{2m_k+1}-1\right\}$ for some $k \in \mathbf{P}$. Then

$$\widehat{f}(j) = \lim_{A\to\infty} \widehat{f^{(A)}}(j) = \frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}} \tag{8}$$

and

$$\widehat{f}(j) = 0, \;\; j \notin \bigcup_{l=1}^{\infty} \left\{2^{2m_l}, ..., 2^{2m_l+1}-1\right\}. \tag{9}$$

Let $j < 2^{2m_k}$. Then from (5), (8) and (9) we have

$$\begin{aligned} |S_j f(x)| &\leq \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \left|\widehat{f}(v)\right| \leq \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2m_l(1/p-1)}}{\sqrt{m_l}} \qquad (10)\\ &= \sum_{l=0}^{k-1} \frac{2^{2m_l/p}}{\sqrt{m_l}} < \frac{2\cdot 2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}}. \end{aligned}$$

Consequently,

$$I \leq \frac{1}{l_{q_{m_k}}} \sum_{j=1}^{2^{2m_k-1}} \frac{|S_j f(x)|}{j} < \frac{c}{m_k} \frac{2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}} \sum_{j=1}^{2^{2m_k-1}} \frac{1}{j} < \frac{c2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}}. \tag{11}$$

Let $2^{2m_k} \leq j < q_{m_k,s}$. Then we have the following

$$\begin{aligned} S_j f(x) &= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}(v) w_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}(v) w_v(x) \qquad (12)\\ &= \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{\sqrt{m_l}} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right)\\ &\quad + \frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}} \left(D_j(x) - D_{2^{2m_k}}(x)\right). \end{aligned}$$

This gives that

$$\begin{aligned} II &= \frac{1}{l_{q_{m_k,s}}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} \frac{1}{j} \left(\sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{\sqrt{m_l}} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right)\right) \qquad (13)\\ &\quad + \frac{1}{l_{q_{m_k,s}}} \frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} \frac{D_j(x) - D_{2^{2m_k}}(x)}{j}\\ &= II_1 + II_2. \end{aligned}$$

To discuss II_1, we use (1) and (5). Thus we can write

$$II_1 \leq c\sum_{l=0}^{k-1}\frac{2^{2m_l/p}}{\sqrt{m_l}} < \frac{c2^{2m_{k-1}}}{\sqrt{m_{k-1}}}. \tag{14}$$

Since

$$D_{j+2^{2m_k}} - D_{2^{2m_k}} = w_{2^{2m_k}}D_j, \;\; 0 \leq j < 2^{2m_k}$$

for II_2 we have

$$II_2 = \frac{1}{l_{q_{m_k},s}}\frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}}w_{2^{2m_k}}(x)\sum_{j=0}^{2^{2s}}\frac{D_j(x)}{j+2^{2m_k}}. \tag{15}$$

From

$$R_{q_{m_k},s}f(x) = I + II_1 + II_2$$

and (6), (7), (11) and (13)-(15) we have

$$\begin{aligned}\left|R_{q_{m_k},s}f(x)\right| &\geq |II_2| - |I| - |II_1| \geq |II_2| - \frac{c2^{2m_{k-1}}}{\sqrt{m_{k-1}}} \\ &\geq \frac{c}{m_k}\frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}}\left|\sum_{j=0}^{2^{2s}}\frac{D_j(x)}{j+2^{2m_k}}\right| - \frac{c2^{m_k}}{m_k}.\end{aligned}$$

Let $x \in I_{2s}\backslash I_{2s+1}$. Then it is evident that

$$\left|\sum_{j=0}^{2^{2s}}\frac{D_j(x)}{j+2^{2m_k}}\right| \geq \frac{c2^{4s}}{2^{2m_k}},$$

$$\begin{aligned}\left|R_{q_{m_k},s}f(x)\right| &\geq \frac{c}{m_k}\frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}}\frac{2^{4s}}{2^{2m_k}} - \frac{c2^{m_k}}{m_k} \\ &\geq \frac{c2^{2m_k(1/p-2)}}{m_k^{3/2}}2^{4s} - \frac{c2^{m_k}}{m_k} \\ &\geq \frac{c2^{2m_k(1/p-2)}}{m_k^{3/2}}2^{4s}, \;\; s = [m_k/2], ..., m_k.\end{aligned}$$

Consequently $(0 < p \leq 1/2)$,

$$\begin{aligned}
&\int_G (R^* f(x))^p \, d\mu(x) \\
&\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} (R^* f(x))^p \, d\mu(x) \\
&\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left| R_{q_{m_k,s}} f(x) \right|^p d\mu(x) \\
&\geq c \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left(\frac{2^{2m_k(1/p-2)}}{m_k^{3/2}} 2^{4s} \right)^p d\mu(x) \\
&\geq c \sum_{s=[m_k/2]}^{m_k} 2^{2s(2p-1)} \frac{2^{2m_k(1-2p)}}{m_k^{(3/2)p}} \\
&\geq \begin{cases} c m_k^{1/4}, \ p = 1/2 \\ \frac{c 2^{m_k(1-2p)}}{m_k^{(3/2)p}}, 0 < p < 1/2 \end{cases} \to \infty \ \text{as} \ k \to \infty.
\end{aligned}$$

Theorem 1 is proved.

Proof of Theorem 3. We write

$$\begin{aligned}
L_{q_{m_k,s}} f(x) &= \frac{1}{l_{q_{m_k,s}}} \sum_{j=0}^{q_{m_k,s}-1} \frac{S_j f(x)}{q_{m_k,s} - j} \qquad (16) \\
&= \frac{1}{l_{q_{m_k,s}}} \sum_{j=1}^{2^{2m_k}-1} \frac{S_j f(x)}{q_{m_k,s} - j} + \frac{1}{l_{q_{m_k,s}}} \sum_{j=2^{2m_k}}^{q_{m_k,s}-1} \frac{S_j f(x)}{q_{m_k,s} - j} \\
&= III + IV.
\end{aligned}$$

Since (see (10))

$$|S_j f(x)| \leq \frac{c 2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}}, \quad j < 2^{2m_k}$$

for III we can write

$$|III| \leq \frac{c}{m_k} \sum_{j=1}^{2^{2m_k}-1} \frac{1}{q_{m_k,s} - j} \frac{2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}} < \frac{c 2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}}. \qquad (17)$$

For IV we have (see (12))

$$\begin{aligned} & IV \qquad (18)\\ = & \frac{1}{l_{q_{m_k,s}}}\sum_{j=2^{2m_k}}^{q_{m_k,s}-1}\frac{1}{q_{m_k,s}-j}\left(\sum_{l=0}^{k-1}\frac{2^{2m_l(1/p-1)}}{\sqrt{m_l}}\left(D_{2^{2m_l+1}}(x)-D_{2^{2m_l}}(x)\right)\right)\\ & +\frac{1}{l_{q_{m_k,s}}}\frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}}\sum_{j=2^{2m_k}}^{q_{m_k,s}-1}\frac{D_j(x)-D_{2^{2m_k}}(x)}{q_{m_k,s}-j}\\ = & IV_1+IV_2. \end{aligned}$$

It is easy to show that

$$|IV_1|<\frac{c2^{2m_{k-1}/p}}{\sqrt{m_{k-1}}}, \qquad (19)$$

$$IV_2=\frac{1}{l_{q_{m_k,s}}}\frac{2^{2m_k(1/p-1)}}{\sqrt{m_k}}w_{2^{2m_k}}(x)\sum_{j=0}^{2^{2s}-1}\frac{D_j(x)}{2^{2s}-j}. \qquad (20)$$

Let $x\in I_{2s}\backslash I_{2s+1}$. Then $D_j(x)=j$, $j<2^{2s}$. Consequently,

$$\sum_{j=0}^{2^{2s}-1}\frac{D_j(x)}{2^{2s}-j}=\sum_{j=0}^{2^{2s}-1}\frac{j}{2^{2s}-j}=\sum_{j=0}^{2^{2s}-1}\left(\frac{2^{2s}}{2^{2s}-j}-1\right)\geq cs2^{2s},$$

$$IV_2\geq\frac{c2^{2m_k(1/p-1)}}{m_k^{3/2}}s2^{2s},\ \ x\in I_{2s}\backslash I_{2s+1}. \qquad (21)$$

Combining (6), (16)-(21) we have

$$\begin{aligned} \left|L_{q_{m_k,s}}f(x)\right| &\geq \frac{c2^{2m_k(1/p-1)}}{m_k^{3/2}}s2^{2s}-\frac{c2^{m_k}}{m_k}\\ &\geq \frac{c2^{2m_k(1/p-1)}}{m_k^{3/2}}s2^{2s},\quad x\in I_{2s}\backslash I_{2s+1},\ s=[m_k/2],...,m_k. \end{aligned}$$

Then we have

$$\begin{aligned}
&\int_G (L^* f(x))^p \, d\mu(x) \\
&\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s}\setminus I_{2s+1}} (L^* f(x))^p \, d\mu(x) \\
&\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s}\setminus I_{2s+1}} \left| L_{q_{m_k},s} f(x) \right|^p d\mu(x) \\
&\geq c \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s}\setminus I_{2s+1}} \left(\frac{2^{2m_k(1/p-1)}}{m_k^{3/2}} s 2^{2s} \right)^p d\mu(x) \\
&\geq c \sum_{s=[m_k/2]}^{m_k} 2^{s(2p-2)} \frac{2^{2m_k(1-p)} s^p}{m_k^{(3/2)p}} \\
&\geq \begin{cases} c m_k^{1/2}, & p = 1 \\ \frac{c 2^{m_k(1-p)}}{m_k^{(1/2)p}}, & 0 < p < 1 \end{cases} \to \infty \text{ as } k \to \infty.
\end{aligned}$$

Theorem 3 is proved.

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Coconvex pointwise approximation

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Dedicated to Professor Giuseppe Mastroianni on occasion of his 70th birthday

Abstract. There are two kinds of estimates of the degree of approximation of continuous functions on $[-1,1]$ by algebraic polynomials, Nikolskii-type pointwise estimates and Jackson-type uniform estimates, involving either ordinary moduli of smoothness, or the Ditzian-Totik (DT) ones, or the recent estimates involving the weighted DT-moduli of smoothness. The purpose of this paper is to complete the table of the validity or invalidity of the pointwise estimates in the case of coconvex polynomial approximation. This will enable us to compare this table with a corresponding one for coconvex Jackson-type (uniform) estimates.

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1 Introduction

There are two kinds of estimates of the degree of approximation of continuous functions on $[-1,1]$ by algebraic polynomials, Nikolskii-type pointwise estimates and Jackson-type uniform estimates, involving either ordinary moduli of smoothness, or the Ditzian-Totik (DT) ones, or the recent estimates involving the weighted DT-moduli of smoothness. Specifically, if

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$\omega_k(f, 1/n)$ denotes the ordinary modulus of smoothness (of order k), then the uniform estimates took the form

$$(1)\quad E_n(f) := \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_{C[-1,1]} \le c(k,r) n^{-r} \omega_k(f^{(r)}, 1/n), \quad n \ge k + r,$$

where $f \in C^r[-1,1] =: C^r$, the space of r times continuously differentiable functions on $[-1,1]$ (when $r = 0$ we suppress the superscript r, that is, we write $C := C[-1,1]$ instead of $C^0[-1,1]$), $E_n(f)$ is the degree of approximation of f by polynomials of degree $< n$, the class of which we denote by $\mathcal{P}_n$, and $c(k,r)$ is a constant which may depend on k and r but is independent of f and n. Similar estimates hold for the other moduli of smoothness mentioned above.

In the sequel, we denote by c constants, which may depend only on the parameters indicated in the parentheses.

While we are dealing with the notation of function spaces, we denote as usual by $L_\infty := L_\infty[-1,1]$ the space of functions essentially bounded in $[-1,1]$, equipped with the norm $\|\cdot\| := \|\cdot\|_{L_\infty[-1,1]} := \operatorname{esssup} f$ and, in particular, if $f \in C$, then $\|f\| = \|f\|_{C[-1,1]}$. For $r \in \mathbb{N}$ we let W^r be the subspace of all functions $f \in C$, possessing an absolutely continuous $(r-1)$-st derivative in $(-1,1)$ and such that $f^{(r)} \in L_\infty$.

Since the early papers by Lorentz, Zeller, DeVore, and Newman, on shape preserving approximation most papers (many of them by some or all the authors of this paper), were dedicated to the question of the validity of analogs of these estimates for comonotone and coconvex approximations. Namely, for each triple (k, r, s), where k and r are as above and s is the number of changes of monotonicity or convexity, respectively, of the function f, it is now known whether or not (1) is valid where the approximation is restricted to polynomials which are comonotone or coconvex, respectively, with f (see, e.g., [15] and [17], respectively). Estimates for the analogue of (1) for the weighted DT-moduli of smoothness are also known (see e.g. [15, 12, 13]). Finally, in [15] are given the complete tables of validity for comonotone pointwise estimates (see [3] for details).

Recently, Dzyubenko and collaborators have dealt with coconvex Nikolskii-type (pointwise) estimates (see references) and have covered most but not all cases. The purpose of this paper is to complete the table of the validity or invalidity of the estimates in the case of coconvex polynomial approximation. This will enable us to compare this table with a corresponding one for coconvex Jackson-type (uniform) estimates.

2 Pointwise and uniform coconvex approximation: tables of validity

Let $f \in C[a,b]$, where $a < b$. For $k \in \mathbb{N}$, denote by

$$\omega_k(f,t,[a,b]) \tag{2}$$

$$:= \sup_{h\in[0,t]} \sup_{a\le x-kh/2<x+kh/2\le b} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x-kh/2+jh) \right|,$$

the ordinary k-th modulus of smoothness of f, in $[a,b]$. For the sake of uniformity in notation we also denote $\omega_0(f,t,[a,b]) := \|f\|_{[a,b]}$. When $f \in C$ we suppress mentioning the interval, namely, we write $\omega_k(f,t) := \omega_k(f,t,[-1,1])$.

For $n \in \mathbb{N}$, let

$$\rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}, \qquad x \in [-1,1].$$

If $f \in C^r$, then by the classical Nikolskii-type direct pointwise estimates a sequence $\{P_n\}_{n=k+r}^{\infty}$ of polynomials $P_n \in \mathcal{P}_n$, exists, such that

$$|f(x) - P_n(x)| \le c(k,r)\rho_n^r(x)\omega_k(f^{(r)},\rho_n(x)), \qquad x \in [-1,1]. \tag{3}$$

This direct theorem was proved by Timan for $k = 1$, by Dzyadyk and, independently, by Freud, for $k = 2$, and by Brudnyi for $k \ge 3$ (for details, see [1], p. 381). Note that since $\rho_n(x) \le \frac{2}{n}$, (3) implies (1).

Everywhere below we assume that $f \notin \mathcal{P}_{k+r}$, and for such an f, we find it convenient to rewrite (3) in the form

$$E_{n,k,r}(f) := \inf_{P_n\in\mathcal{P}_n} \left\| \frac{f-P_n}{\rho_n^r \omega_k(f^{(r)},\rho_n)} \right\| \le c(k,r), \qquad n \ge k+r.$$

It follows immediately by (3) that if $f \in W^r$, then a sequence $\{P_n\}_{n=r}^{\infty}$ of polynomials $P_n \in \mathcal{P}_n$, exists, such that

$$|f(x) - P_n(x)| \le c(r)\rho_n^r(x)\|f^{(r)}\|, \qquad x \in [-1,1].$$

In this paper we compare the validity or invalidity of the uniform and pointwise estimates for coconvex approximation, for a given triplet (k,r,s), where $k \ge 0$ is the order of the modulus of smoothness of $f^{(r)}$, $r \ge 0$, and s

is the number of changes of convexity. More precisely, denote by $\mathbb{Y}_s$, $s \in \mathbb{N}$, the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points y_i, such that

$$-1 < y_s < \cdots < y_1 < 1.$$

For $Y_s \in \mathbb{Y}_s$ denote by $\Delta^{(2)}(Y_s)$ the collection of functions $f \in C$, that change convexity at the points y_i, and are convex in $[y_1, 1]$. That is $f \in \Delta^{(2)}(Y_s)$ if and only if f is convex in $[y_1, 1]$, concave in $[y_2, y_1]$, convex in $[y_3, y_2]$, and so on. Note that if f is twice differentiable in $[-1, 1]$, then $f \in \Delta^{(2)}(Y_s)$ if and only if

$$f''(x)\Pi(x) \geq 0, \quad x \in [-1, 1],$$

where

$$\Pi(x) := \prod_{i=1}^{s}(x - y_i).$$

To unify the notation we set $Y_0 := \emptyset$, $\mathbb{Y}_0 = \{\emptyset\}$, and denote by $\Delta^{(2)}(Y_0)$ the set of convex functions $f \in C$.

For $f \in \Delta^{(2)}(Y_s)$ let

$$E_n^{(2)}(f, Y_s) := \inf_{P_n \in \mathcal{P}_n \cap \Delta^{(2)}(Y_s)} \|f - P_n\|,$$

denote the degree of best uniform coconvex polynomial approximation, and denote for $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$),

$$E_{n,k,r}^{(2)}(f, Y_s) := \inf_{P_n \in \mathcal{P}_n \cap \Delta^{(2)}(Y_s)} \left\| \frac{f - P_n}{\rho_n^r \omega_k(f^{(r)}, \rho_n)} \right\|,$$

so that we always have

$$E_n^{(2)}(f, Y_s) \leq 2^{k+r} n^{-r} \omega_k(f^{(r)}, 1/n) E_{n,k,r}^{(2)}(f, Y_s).$$

Thus, with this notation the purpose of the paper is for a given triplet (k, r, s), to establish whether or not the inequality

$$E_{n,k,r}^{(2)}(f, Y_s) \leq c, \qquad n \geq N, \tag{4}$$

is valid for a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$), and $f \notin \mathcal{P}_{k+r}$.

With $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $(k, r, s) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ and $k + r \neq 0$. We say that

a) (4) is not valid with both c and N independent on Y_s, if for each $A > 0$ there is a positive integer N, such that for every $n \geq N$ there are, a collection $Y_s \in \mathbb{Y}_s$ and a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$), satisfying

$$E^{(2)}_{n,k,r}(f, Y_s) \geq A; \tag{5}$$

b) (4) is not valid with $c = c(k, r, s)$ and N independent of f, if for each $A > 0$ there are, a positive integer N and a collection $Y_s \in \mathbb{Y}_s$, such that for every $n \geq N$, a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$) exists, satisfying (5);
c) (4) is not valid for any Y_s with both c and N independent of f, if for each $A > 0$ and $Y_s \in \mathbb{Y}_s$, there is a positive integer N, such that for every $n \geq N$, a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$) exists, satisfying (5);
d) (4) cannot be had even if we allow both constants c and N to depend on all parameters k, r, Y_s and f, if for each $Y_s \in \mathbb{Y}_s$ there is a function $f \in C^r \cap \Delta^{(2)}(Y_s)$, such that

$$\limsup_{n\to\infty} E^{(2)}_{n,k,r}(f, Y_s) = \infty.$$

It turns out we have to distinguish between five different cases for a triplet (k, r, s). Namely,

Definition 1. *Let $(k, r, s) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ and $k + r \neq 0$. We write*
1. *$(k, r, s) \in$ "$+$" if (4) holds with $c = c(k, r, s)$ and $N = k + r$;*
2. *$(k, r, s) \in$ "$\oplus$" if (4) holds with $c = c(k, r, Y_s)$ and $N = k + r$, as well as, with $c = c(k, r, s)$ and $N = (k, r, Y_s)$, but (4) is not valid with both c and N independent on Y_s;*
3. *$(k, r, s) \in$ "$\oslash$" if (4) holds with $c = c(k, r, Y_s)$ and $N = k + r$, as well as, with $c = c(k, r, s)$ and $N = (k, r, Y_s, f)$, but (4) is not valid with $c = c(k, r, s)$ and N independent of f;*
4. *$(k, r, s) \in$ "$\ominus$" if (4) holds with $c = c(k, r, s)$ and $N = (k, r, Y_s, f)$ but (4) is not valid for any Y_s, with both c and N independent of f;*
5. *$(k, r, s) \in$ "$-$": (4) cannot be had even if we allow both constants c and N to depend on all parameters k, r, Y_s and f.*

We summarize the results in the following theorem.

Theorem 1.
1. *$(k, r, s) \in$ "$+$", if $s = 0$ and either $k \leq 3$ and $r \leq 3 - k$, or $k \geq 0$ and $r \geq 2$; or $s = 1$ and $k \leq 2$ and $r \leq 2 - k$;*

2. $(k,r,s) \in$ "$\oplus$", *if* $s \geq 2$ *and* $k \leq 2$ *and* $r \leq 2-k$;
3. $(k,r,s) \in$ "$\oslash$", *if* $s \geq 2$ *and either* $k=3$ *and* $r=0$, *or* $k=2$ *and* $r=1$, *or* $1 \leq k \leq 3$ *and* $r=2$, *or* $k \geq 0$ *and* $r \geq 3$;
4. $(k,r,s) \in$ "$\ominus$", *if* $s=1$ *and either* $k=3$ *and* $r=0$, *or* $k=2$ *and* $r=1$, *or* $1 \leq k \leq 3$ *and* $r=2$, *or* $k \geq 0$ *and* $r \geq 3$.
5. $(k,r,s) \in$ "$-$", *otherwise.*

We find it easier and more convenient for the reader to comprehend Theorem 1 by describing the results in the following truth tables for the pairs (k,r), for "pointwise estimates, $s=0$", "pointwise estimates, $s=1$" and "pointwise estimates, $s \geq 2$". We also take the opportunity to compare these tables with tables for "uniform" coconvex approximation (see [17], p. 110 and 114), namely, when the estimates

$$E_n^{(2)}(f, Y_s) \leq cn^{-r}\omega_k(f^{(r)}, 1/n), \quad n \geq N, \tag{6}$$

are valid.

We do this before proceeding to give references to the already known results and to prove the new ones.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	+	$\cdots$
1	+	+	+	−	−	−	$\cdots$
0		+	+	+	−	−	$\cdots$
	0	1	2	3	4	5	k

Pointwise, $s=0$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	+	+	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	+	+	$\cdots$
1	+	+	+	$\ominus$	−	−	−	$\cdots$
0		+	+	+	$\ominus$	−	−	$\cdots$
	0	1	2	3	4	5	6	k

Uniform, $s=0$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
3	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
2	+	$\ominus$	$\ominus$	$\ominus$	−	−	$\cdots$
1	+	+	$\ominus$	−	−	−	$\cdots$
0		+	+	$\ominus$	−	−	$\cdots$
	0	1	2	3	4	5	k

Pointwise, $s=1$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	+	+	+	+	+	+	$\cdots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	$\oplus$	−	−	$\cdots$
1	+	+	$\oplus$	−	−	−	$\cdots$
0		+	+	$\oplus$	−	−	$\cdots$
	0	1	2	3	4	5	k

Uniform, $s=1$

r	⋮	⋮	⋮	⋮	⋮	⋮	
4	⊘	⊘	⊘	⊘	⊘	⊘	$\cdots$
3	⊘	⊘	⊘	⊘	⊘	⊘	$\cdots$
2	⊕	⊘	⊘	⊘	−	−	$\cdots$
1	⊕	⊕	⊘	−	−	−	$\cdots$
0		⊕	⊕	⊘	−	−	$\cdots$
	0	1	2	3	4	5	k

Pointwise, $s \geq 2$

r	⋮	⋮	⋮	⋮	⋮	⋮	
4	⊕	⊕	⊕	⊕	⊕	⊕	$\cdots$
3	⊕	⊕	⊕	⊕	⊕	⊕	$\cdots$
2	⊕	⊕	⊕	⊕	−	−	$\cdots$
1	⊕	⊕	⊕	−	−	−	$\cdots$
0		⊕	⊕	⊕	−	−	$\cdots$
	0	1	2	3	4	5	k

Uniform, $s \geq 2$

Note that the "Uniform" tables do not contain the case " ⊘ ".

Credits. Theorem 1.1 (that is, the " + " case) was proved for $s = 0$, first by Leviatan [14] for $k \leq 2$ and $r \leq 2 - k$, and later by Kopotun [10] for $k \leq 3$ and $r \leq 3 - k$, and by Mania and Shevchuk (see [1], Theorem 7.6.5) for $k = 0$ and $r \geq 2$. For $s = 1$, $k \leq 2$ and $r \leq 2 - k$, it was proved by Dzyubenko, Gilewicz and Shevchuk [4].

The positive result in Theorem 1.2 (that is, the " ⊕ " cases) was proved in [4], and the negative result in [16].

The first variant of the positive result in Theorem 1.3 (that is, the " ⊘ " cases) was proved in [7, 8] and the second one for $r \geq 2$ follows from Remark 1.3 in [6] ("... the arguments of this paper can be easily extended to the case $s > 1$, so that the exact analogs of Theorems 1.1 through 2.2 hold also for $s > 1$."). We prove here the second variant for $r < 2$, that is, for $s > 1$ and either $k = 3$ and $r = 0$, or $k = 2$ and $r = 1$. The negative result was proved in [5].

The positive result in Theorem 1.4 (that is, the " ⊖ " cases) for $r \geq 2$ was proved in [6]. We are going to prove it here for $r < 2$, that is, for $s = 1$ and either $k = 3$ and $r = 0$, or $k = 2$ and $r = 1$. The negative result was proved in [5].

Finally, Theorem 1.5 (that is, the " − " case) was proved by Wu and Zhou [18, 20], for $s = 0$ and either $k \geq 5$ and $r = 0$, or $k \geq 4$ and $r = 1$; and for $s \geq 1$ and either $k \geq 4$ and $r = 0$, or $k \geq 3$ and $r = 1$. It was proved by Gilewicz and Yushchenko [9] for $s \geq 1$ and $k \geq 4$ and $r = 2$, and by Yushchenko [19] for $s = 0$ and either $k = 4$ and $r = 0$, or $k = 3$ and $r = 1$.

3 The case $(k = 3, r = 0, s \geq 1)$

Our aim is to prove,

Theorem 2. *For each $Y_s \in \mathbb{Y}_s$ and every function $f \in \Delta^{(2)}(Y_s)$ there are a number $N = N(Y_s, f)$ and a sequence $\{P_n\}_{n=N}^{\infty}$ of polynomials $P_n \in \mathcal{P}_n$, such that*

$$|f(x) - P_n(x)| \le c\omega_3(f, \rho_n(x)), \quad x \in [-1, 1],$$

where c is an absolute constant.

Theorem 2 readily follows from Theorems 3 and 4 below, but in order to formulate them we need some notations.

Let $x_{j,n} := \cos(j\pi/n)$, $j = 0, \dots, n$, be the Chebyshev partition of $[-1, 1]$. For $n \in \mathbb{N}$ and $Y_s = \{y_i\}_{i=1}^{s} \in \mathbb{Y}_s$, denote by

$$O_{i,n} := (x_{j+1,n}, x_{j-2,n}), \quad \text{if} \quad y_i \in [x_{j,n}, x_{j-1,n}),$$

where $x_{n+1,n} := -1$ and $x_{-1,n} := 1$. Let

$$O = O(n, Y_s) := \bigcup_{i=1}^{s} O_{i,n}.$$

Set $I_{j,n} := [x_{j,n}, x_{j-1,n}]$, $j = 1, \dots, n$, and let $|I_{j,n}| = x_{j-1,n} - x_{j,n}$, be its length. It is well known that

$$|I_{j\pm1,n}| < 3|I_{j,n}|, \tag{7}$$

and

$$\rho_n(x) < |I_{j,n}| < 5\rho_n(x), \quad x \in I_{j,n}. \tag{8}$$

In particular

$$|O_{i,n}| < c_1\rho_n(x), \quad x \in O_{i,n}, \tag{9}$$

where $|O_{i,n}|$ denotes the length of $O_{i,n}$ and $c_1 < 65$ is an absolute constant.

Denote by $\Sigma_{k,n}$ the collection of all continuous piecewise polynomials of degree $< k$, on the Chebyshev partition $\{x_{j,n}\}_{j=0}^{n}$, and by $\Sigma_{k,n}(Y_s)$, the subset of $\Sigma_{k,n}$ consisting of those continuous piecewise polynomials $S \in \Sigma_{k,n}$ such that for every $i = 1, \dots, s$, $S|_{O_{i,n}}$ is a polynomial.

The following Theorem allows us to reduce the proofs of the direct estimates for (co)convex polynomial approximation to those for (co)convex piecewise polynomial approximation, proofs which are much easier.

Theorem 3[2]. *Let $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$. For every $Y_s \in \mathbb{Y}_s$, an $f \in \Delta^{(2)}(Y_s)$, an $n \ge N(k, Y_s)$, and a piecewise polynomial $S \in \Sigma_{k,n}(Y_s) \cap \Delta^{(2)}(Y_s)$, there is a polynomial $P_n \in \Delta^{(2)}(Y_s)$ of degree $< c^*(k, s)n$, such that*

$$|f(x) - P_n(x)| \le c(k, s)\left(1 + \left\|\frac{f - S}{\omega_k(f, \rho_n)}\right\|\right)\omega_k(f, \rho_n(x)), \quad x \in [-1, 1].$$

Thus, our main result is,

Theorem 4. *If $f \in \Delta^{(2)}(Y_s)$, then there exists an $N = N(f, Y_s)$, such that for each $n \geq N$, a piecewise polynomial $S \in \Sigma_{3,n}(Y_s) \cap \Delta^{(2)}(Y_s)$ exists, such that*

$$|f(x) - S(x)| \leq c\omega_3(f, \rho_n(x)), \quad x \in [-1, 1],$$

where c is an absolute constant.

4 Proof of Theorem 4

We begin with three lemmas.

Everywhere below c_i are absolute constants.

Lemma 1. *For each convex (concave) function $f \in C[a, b]$ there exists a number $d = d([a, b], f) > 0$, such that for each h,*

$$|h| \leq E_2(f)_{[a,b]} := \inf_{l \in \mathcal{P}_2} \|f - l\|_{[a,b]},$$

there is convex (concave) function $g \in C[a, b]$, $g(x) = g(x, h, f, [a, b])$, satisfying

$$g(x) = f(x) + h, \quad x \in [a, a + d]; \qquad g(x) = f(x), \quad x \in [b - d, b],$$

and

$$\|f - g\|_{[a,b]} = |h|.$$

Proof. We prove Lemma 1 for a convex function f. Let l be a linear function of best approximation of the function f, that is $\|f - l\|_{[a,b]} = E_2(f)_{[a,b]} =: E$. Set $f^* := f - l$, and note that $f^*(a) = f^*(b) = E$. If $h = 0$, then there is nothing to prove. Hence, we may assume that $0 < |h| \leq E \neq 0$. Denote by x_1 and x_5 the (exactly two) roots of the equation $f^*(x) = 0$, $x_1 < x_5$, and let x_2 and x_4 be the (exactly two) roots of the equation $f^*(x) = |h| - E$, $x_2 < x_4$. Finally, let x_3 be a root of the equation $f^*(x) = -E$. Clearly we have

$$a < x_1 < x_2 < x_3 < x_4 < x_5 < b.$$

Then Lemma 1 is valid with $d = \min\{x_1 - a, b - x_5\}$, and the evidently convex function,

$$g(x) = l(x) + \begin{cases} f^*(x) + h, & \text{if} \quad x \in [a, x_3], \\ h - E & \text{if} \quad x \in [x_3, x_4], \\ f^*(x) & \text{if} \quad x \in [x_4, b], \end{cases}$$

if $h > 0$, and

$$g(x) = l(x) + \begin{cases} f^*(x) + h, & \text{if} \quad x \in [a, x_2], \\ -E & \text{if} \quad x \in [x_2, x_3], \\ f^*(x) & \text{if} \quad x \in [x_3, b], \end{cases}$$

if $h < 0$. □

Denote by

$$[t_1, t_2; f] := \frac{f(t_2) - f(t_1)}{t_2 - t_1},$$

the first divided difference of the function f at the points t_1 and t_2.

Lemma 2. *Given the interval $[a, b]$, let $y \in (a, b)$ be such that $(b - y)/3 < y - a < 3(b - y)$. If a function $f \in C[a, b]$ is concave on $[a, y]$ and convex on $[y, b]$, then there is a linear function $l(x) = l(x, f, [a, b], y)$, such that*

$$l' \le [a, y; f] \quad \textit{and} \quad l' \le [y, b; f], \tag{10}$$

we may choose $l(a) = f(a)$ or $l(b) = f(b)$, and

$$\|f - l\|_{[a.b]} \le c_2 \omega_3(f, b - a, [a, b]), \tag{11}$$

where the modulus of smoothness for the interval $[a, b]$ was defined in (2).

Proof. Since f is concave on $[a, y]$ and convex on $[y, b]$, then (see [11]) $E_2(f)_{[a,b]} \le c_3 E_3(f)_{[a,b]} := c_3 \inf_{p \in \mathcal{P}_3} \|f - p\|_{[a,b]}$, whence $E_2(f)_{[a,b]} \le c_4 \omega_3(f, b - a, [a, b])$. Therefore, it readily follows that the choice of either $l_1(x) = f(a) + (x - a)[a, y; f]$, or $l_2(x) = f(b) + (x - b)[y, b; f]$, according to which of the two divided differences is smaller, clearly fulfils (10), while (11) follows immediately by Whitney's Theorem and the fact that l_j $j = 1$ or $j = 2$, interpolates f at two points, the distance between which is proportional to $b - a$. In order to guarantee that $l(a) = f(a)$, if the choice is l_1, then we take $l = l_1$. Otherwise, if $h := l_2(a) - f(a)$, then by (11) we may take $l = l_2 - h$, which satisfies (10) and (11), and we obtain $l(a) = f(a)$. Guaranteeing, instead, that $l(b) = f(b)$ is done in a similar way. □

Lemma 3. *For $f \in \Delta^{(2)}(Y_s)$, there exists an $N = N(f, Y_s)$, such that for each $n \geq N$, a function $f_n \in \Delta^{(2)}(Y_s)$ exists, such that $f_n|_{0_{i,n}}$ is linear for all $i = 1, \ldots, s$, and*

$$|f(x) - f_n(x)| \leq c_5\, \omega_3\left(f, \rho_n(x)\right), \quad x \in [-1, 1].$$

Proof. Set

$$y_i^+ := \min\{y_i + \frac{1}{2}(1 - |y_i|), \frac{1}{2}(y_{i-1} + y_i)\},$$

and

$$y_i^- := \max\{y_i - \frac{1}{2}(1 - |y_i|), \frac{1}{2}(y_i + y_{i+1})\},$$

where $y_0 := 1$ and $y_{s+1} := -1$. Now write

$$J_i^- := [y_i^-, y_i] \quad \text{and} \quad J_i^+ := [y_i, y_i^+].$$

We divide $1 \leq i \leq s$ into two sets. We write $i \in A$, if there are two linear functions l_{i-} and l_{i+}, such that,

$$f|_{J_i^-} = l_{i-} \qquad \text{and} \qquad f|_{J_i^+} = l_{i+}.$$

Otherwise $i \notin A$.

Let N_0 be so big that for $n \geq N_0$,

$$O_{i,n} \subset J_i^- \cup J_i^+ = [y_i^-, y_i^+], \quad 1 \leq i \leq s,$$

and note that in particular $O_{i,n} \cap O_{i+1,n} = \emptyset$, $1 \leq i < s$. Denote by $y_{i,n}^+$ and $y_{i,n}^-$ the right and left ends of $O_{i,n}$, respectively. Let $n \geq N_0$ and $i \in A$. We denote by $L_{i,n}^+$, the polygonal line consisting of three segments, such that

$$L_{i,n}^+(-1) = L_{i,n}^+(y_i) = L_{i,n}^+(1) = 0 \quad \text{and} \quad L_{i,n}^+(y_{i,n}^+) = 1,$$

and similarly, we denote by $L_{i,n}^-$, the polygonal line consisting of three linear pieces, such that,

$$L_{i,n}^-(-1) = L_{i,n}^-(y_i) = L_{i,n}^-(1) = 0 \quad \text{and} \quad L_{i,n}^-(y_{i,n}^-) = 1,$$

and define

$$L_{i,n} := \begin{cases} (l_{i-}(y_{i,n}^+) - l_{i+}(y_{i,n}^+))L_{i,n}^+, & \text{if} \quad (-1)^i(l'_{i-} - l'_{i+}) \geq 0; \\ (l_{i+}(y_{i,n}^-) - l_{i-}(y_{i,n}^-))L_{i,n}^-, & \text{otherwise.} \end{cases}$$

Clearly,

$$L_{i,n} \in \Delta^{(2)}(Y_s), \tag{12}$$

and either $(f + L_{i,n})|_{O_{i,n}} = l_{i-}$, or $(f + L_{i,n})|_{O_{i,n}} = l_{i+}$, so that $f + L_{i,n}$ is linear on $O_{i,n}$. Hence, with

$$L_n := \sum_{i \in A} L_{i,n},$$

we conclude that $(f + L_n)|_{O_{i,n}}$ is linear for each $i \in A$. Also, clearly, $L_n|_{O_{i,n}}$ is linear for each $i \notin A$.

If $M := \max_{i \in A} |l'_{i+} - l'_{i-}|$, then by (9),

$$|l'_{i+} - l'_{i-}|(y^+_{i,n} - y_i) \le M(y^+_{i,n} - y_i) \le M|O_{i,n}| \le c_1 M \rho_n(y^+_{i,n}),$$

and

$$|l'_{i+} - l'_{i-}|(y_i - y^-_{i,n}) \le c_1 M \rho_n(y^-_{i,n}).$$

Hence, it follows that

$$|L_{i,n}(x)| \le c_1 M \rho_n(x), \quad x \in [-1, 1],$$

which in turn implies,

$$|L_n(x)| \le s c_1 M \rho_n(x), \quad x \in [-1, 1]. \tag{13}$$

On the other hand, for each $i \in A$ there exists $t^0_i > 0$, such that

$$\omega_3(f, t, [y^-_i, y^+_i]) = |l'_{i+} - l'_{i-}|t, \quad t \le t^0_i.$$

Therefore, if $A \neq \emptyset$, then we have

$$\omega_3(f, t) \ge Mt, \quad t \le t^0 := \min_{i \in A} t^0_i.$$

Combining with (13) this in turn yields, for all $n \ge N^0 := \max\{N_0, [1/t^0]\}$,

$$|L_n(x)| \le s c_1 \omega_3(f, \rho_n(x)), \quad x \in [-1, 1], \tag{14}$$

Next, for $i \notin A$, denote $J_i := J^-_i$, if $f|_{J^+_i}$ is linear. Otherwise, we put $J_i := J^+_i$.

Now we need an auxiliary function g_n, which coincides with f on a major part of the interval $[-1, 1]$. In fact, it differs from f only on the $O_{i,n}$'s and on the J_i's, where $i \notin A$.

We fix $i \notin A$ and assume that $J_i := J_i^+$ and that i is odd, that is, f is convex on J_i. The other cases are similar. By Lemma 2, there is a linear function $l_{i,n}$, such that

$$l'_{i,n} \le [y_{i,n}^-, y_i; f], \quad l'_{i,n} \le [y_i, y_{i,n}^+; f], \tag{15}$$

$$f(y_{i,n}^-) = l_{i,n}(y_{i,n}^-), \tag{16}$$

and

$$\|f - l_{i,n}\|_{\overline{O}_i} \le c_2 \omega_3(f, |O_{i,n}|), \tag{17}$$

in particular,

$$h_{i,n} := l_{i,n}(y_{i,n}^+) - f(y_{i,n}^+) \le c_2 \omega_3(f, |O_{i,n}|).$$

Since $i \notin A$, it follows that $E_2(f)_{J_i} > 0$, and we may apply Lemma 1 with $d_i := d(J_i, f)$ and $g(x, h_{i,n}, f, J_i)$, guaranteed by that lemma.

We take $N_i \ge N^0$ so big that for all $n \ge N_i$ we have

$$y_{i,n}^+ - y_i < d_i \quad \text{and} \quad c_2 \omega_3(f, |O_{i,n}|) < E_2(f)_{J_i},$$

and we define

$$g_{i,n}(x) := \begin{cases} f(x), & \text{if} \quad x \in J_i^- \setminus O_{i,n}; \\ l_{i,n}, & \text{if} \quad x \in O_{i,n}; \\ g(x, h_{i,n}, f, J_i), & \text{if} \quad x \in J_i \setminus O_{i,n}. \end{cases}$$

Then by (17),

$$|f(x) - g_{i,n}(x)| \le c_2 \omega_3(f, |O_{i,n}|), \quad x \in J_i^- \cup J_i.$$

Hence, the inequality

$$\max_{x \in J_i^- \cup J_i^+} \rho_n(x) \le 2 \min_{x \in J_i^- \cup J_i^+} \rho_n(x),$$

combined with (7) through (9), yields

$$|f(x) - g_{i,n}(x)| \le c_4 \omega_3(f, \rho_n(x)), \quad x \in J_i^- \cup J_i. \tag{18}$$

Also, $g_{i,n}$, is continuous on $J_i^- \cup J_i$, it is concave on J_i^- and convex on J_i, and $g_{i,n}(x) = f(x)$ near both ends of $J_i^- \cup J_i$.

Thus, for each $n \geq N := \max_{i \notin A} N_i$, the function

$$g_n(x) := \begin{cases} g_{i,n}(x), & \text{if} \quad x \in O_{i,n} \cup J_i \quad \text{with} \quad i \notin A; \\ f(x), & \text{otherwise} \end{cases}$$

satisfies

$$g_n \in \Delta^{(2)}(Y_s), \tag{19}$$

$$|f(x) - g_n(x)| \leq c_9 \omega_3(f, \rho_n(x)), \quad x \in [-1, 1], \tag{20}$$

and

$$g_n \quad \text{is a linear function on each} \quad O_i \quad \text{with} \quad i \notin A. \tag{21}$$

Combining (19) through (21) with (12) and (14), we conclude that

$$f_n := g_n + L_n,$$

is the required function. This completes the proof of Lemma 3. □

Denote by

$$[t_1, t_2, t_3; f] := \frac{[t_1, t_2; f] - [t_2, t_3; f]}{t_1 - t_3},$$

the second divided difference of the function f at the points t_1, t_2, t_3.

Proof of Theorem 4. Let N and f_n be defined by Lemma 3. For each $n \geq N$ and $j = 2, \ldots, n-1$ let

$$\begin{aligned} p_{j,n} \quad &:= f_n(x_{j,n}) + (x - x_{j,n})[x_{j,n}, x_{j-1,n}; f_n] \\ &\quad + (\mathrm{sgn}\Pi(x_{j,n}))(x - x_{j,n})(x - x_{j-1,n}) \\ &\quad \cdot \min\{|[x_{j,n}, x_{j-1,n}, x_{j-2,n}; f_n]|, |[x_{j,n}, x_{j-1,n}, x_{j+1,n}; f_n]|\}, \end{aligned}$$

be the quadratic polynomial that interpolates f_n at the points $x_{j,n}$, $x_{j-1,n}$ and either $x_{j-2,n}$ or $x_{j+1,n}$. It is readily seen (see e.g.,[10]), that the required piecewise polynomial S may be taken in the form

$$S|_{I_{j,n}} = p_{j,n}, \quad j = 2, \ldots, n-1, \quad S|_{I_{1,n}} = p_{2,n} \quad \text{and} \quad S|_{I_{n,n}} = p_{n-1,n}.$$

This concludes the proof.

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Strong and weak estimates for fractional integral operators on some Herz-type function spaces

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Dedicated to Professor Kôzô Yabuta on his seventieth birthday

Abstract. We consider the boundedness of the fractional integral operators on $B^p_q(\mathbb{R}^n)$, which is some generalization of the $B^p(\mathbb{R}^n)$.

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1 Introduction

A. Beurling [2] introduced the Beurling algebras and their dual spaces $B^p(\mathbb{R}^n)$. Since then, these spaces has been studied by many mathematicians (see, for example, [1], [3], [4], [5], [6], [7], [10], [11], [12], [13] and [14]). H. Feichtinger [4] showed that the B^p space $B^p(\mathbb{R}^n)$ is a special case of the non-homogeneous Herz spaces $K^{\alpha,r}_p(\mathbb{R}^n)$, introduced by C. Herz [10], precisely speaking $B^p(\mathbb{R}^n) = K_p^{-n/p,\infty}(\mathbb{R}^n)$ (see [7]). However, in order to observe the boundedness of several operators on $B^p(\mathbb{R}^n)$, we can't use the general theory about the Herz spaces $K^{\alpha,r}_p(\mathbb{R}^n)$ with $\alpha \neq -n/p$: For example, let $1 < p < \infty$, $0 < r \leq \infty$ and $-n/p < \alpha < n(1 - 1/p)$, and let T

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be a sublinear operator satisfying that for any integrable function f with a compact support,

$$|Tf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \operatorname{supp} f,$$

where $c > 0$ is independent of f and x. If T is bounded on $L^p(\mathbb{R}^n)$, then T is bounded on $K_p^{\alpha,r}(\mathbb{R}^n)$ (see [14]).

Y. Chen and K. Lau [3] and J. García-Cuerva [6] introduced the CMO spaces $CMO^p(\mathbb{R}^n)$, which are the dual spaces of the Beurling-type Hardy spaces, and showed that the Calderón-Zygmund operators are bounded from $B^p(\mathbb{R}^n)$ to $CMO^p(\mathbb{R}^n)$, where $1 < p < \infty$ (see also [1]). Furthermore, J. García-Cuerva and M.J. L. Herrero [7] generalized these spaces and introduced the B_q^p space $B_q^p(\mathbb{R}^n)$, which is another special case of the non-homogeneous Herz spaces $K_p^{\alpha,r}(\mathbb{R}^n)$, and $CMO_q^p(\mathbb{R}^n)$ for a close study about the Hardy spaces (see also [13]). And we [13] proved that the standard singular integral operators are bounded from $B_q^p(\mathbb{R}^n)$ to $CMO_q^p(\mathbb{R}^n)$, where $1 < p < \infty$ and $n/(n+1) < q \leq 1$. Moreover, we introduced a new function space "weak CMO", i.e. $WCMO_q^1(\mathbb{R}^n)$, and proved the boundedness from $B_q^1(\mathbb{R}^n)$ to $WCMO_q^1(\mathbb{R}^n)$, where $n/(n+1) < q \leq 1$.

In this paper, we will investigate the strong and weak estimates for the fractional integral operator $\widetilde{I}_s$, i.e.

$$\begin{aligned}\widetilde{I}_s f(x) &= I_s f(x) - \int_{|y|\geq 1} \frac{f(y)}{|y|^{n-s}} dy \\ &= \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy - \int_{|y|\geq 1} \frac{f(y)}{|y|^{n-s}} dy,\end{aligned}$$

on $B_q^p(\mathbb{R}^n)$.

Note that for the homogeneous spaces, we have the corresponding results to those obtained in this paper. But, since their proofs are similar to those of the results in this paper, we omit the details.

2 Definitions and results

In the following, we denote for a measurable set $E \subset \mathbb{R}^n$, the characteristic function of E by χ_E and the Lebesgue measure of E by $|E|$, and the open ball in $\mathbb{R}^n$, having center 0 and radius $R > 0$, by B_R. Also the letter C is a positive constant which is independent of essential parameters and not necessarily the same one.

First, we define a fractional integral operator I_s and its modified fractional integral operator $\widetilde{I}_s$.

Definition 2.1 For $0 < s < n$, the fractional integral operator I_s is defined by

$$I_s f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy.$$

Definition 2.2 For a fractional integral operator I_s, we define the modified (shifted) fractional integral operator $\widetilde{I}_s$ as follows:

$$\begin{aligned}\widetilde{I}_s f(x) &= I_s f(x) - \int_{|y|\geq 1} \frac{f(y)}{|y|^{n-s}} dy \\ &= \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-s}} - \frac{1}{|y|^{n-s}} \chi_{\{y\in\mathbb{R}^n : |y|\geq 1\}}(y) \right) f(y) dy,\end{aligned}$$

where $0 < s < n$.

Note that the modified operator $\widetilde{I}_s$ of a fractional integral operator I_s is well-defined for every $f \in \bigcup_{1\leq p\leq\infty} L^p(\mathbb{R}^n)$, where $0 < s < 1$.

Next, we recall a weak L^p space $WL^p(\mathbb{R}^n)$ and a Lipschitz space $Lip_\epsilon(\mathbb{R}^n)$.

Definition 2.3 For $1 \leq p < \infty$,

$$WL^p = \left\{ f : \|f|WL^p\| = \sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right|^{1/p} < \infty \right\}.$$

In order to define a Lipschitz space $Lip_\epsilon(\mathbb{R}^n)$, we introduce the local Lipschitz norm and the Lipschitz norm.

Definition 2.4 For a ball $B \subset \mathbb{R}^n$ and $0 \leq \epsilon < 1$,

$$\|f|Lip_\epsilon(B)\| = \sup_Q \frac{1}{|Q|^{1+\epsilon/n}} \int_Q |f(x) - f_Q| dx.$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$ and the supremum is taken over all balls $Q \subset B$.

Similarly, when $B = \mathbb{R}^n$, we define

$$\|f|Lip_\epsilon\| = \|f|Lip_\epsilon(\mathbb{R}^n)\|.$$

Definition 2.5 For $0 \leq \epsilon < 1$,

$$Lip_\epsilon(\mathbb{R}^n) = \{f : \|f|Lip_\epsilon\| < \infty\}.$$

In particular, when $\epsilon = 0$, we denote

$$BMO(\mathbb{R}^n) = Lip_0(\mathbb{R}^n).$$

Then, the following strong and weak estimates for a fractional integral operator I_s on $L^p(\mathbb{R}^n)$ are well-known (see [9], [8], [15] and [16]).

Proposition 2.6 *If $0 < s < n$, $1 < p_1 < n/s$ and $1/p_2 = 1/p_1 - s/n$, then the fractional integral operator I_s is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$, that is for $f \in L^{p_1}(\mathbb{R}^n)$,*

$$\|I_s f|L^{p_2}\| \le C\|f|L^{p_1}\|.$$

Proposition 2.7 *If $0 < s < n$ and $1/p = 1 - s/n$, then the fractional integral operator I_s is bounded from $L^1(\mathbb{R}^n)$ to $WL^p(\mathbb{R}^n)$, that is for $f \in L^1(\mathbb{R}^n)$,*

$$\|I_s f|WL^p\| \le C\|f|L^1\|.$$

Proposition 2.8 *If $0 < s < 1$, $n/s \le p < \infty$ and $\epsilon = s - n/p$, then the fractional integral operator I_s is bounded from $L^p(\mathbb{R}^n)$ to $Lip_\epsilon(\mathbb{R}^n)$, that is for $f \in L^1(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|Lip_\epsilon\| \le C\|f|L^p\|.$$

Second, we define a B_q^p space $B_q^p(\mathbb{R}^n)$ and a CMO space $CMO_q^p(\mathbb{R}^n)$, which were introduced by J. García-Cuerva and M.J. L. Herrero [7].

Definition 2.9 For $1 \le p < \infty$ and $0 < q \le 1$,

$$B_q^p(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f|B_q^p\| < \infty \right\},$$

where

$$\|f|B_q^p\| = \sup_{R \ge 1} |B_R|^{1-1/q} \left(\frac{1}{|B_R|} \int_{B_R} |f(x)|^p dx \right)^{1/p}.$$

In particular, when $q = 1$, we denote

$$B^p(\mathbb{R}^n) = B_1^p(\mathbb{R}^n).$$

Definition 2.10 For $1 \le p < \infty$ and $n/(n+1) < q \le 1$,

$$CMO_q^p(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f|CMO_q^p\| < \infty \right\},$$

where

$$\|f|CMO_q^p\| = \sup_{R \ge 1} \inf_{c \in \mathbb{C}} |B_R|^{1-1/q} \left(\frac{1}{|B_R|} \int_{B_R} |f(x) - c|^p dx \right)^{1/p}.$$

In particular, when $q = 1$, we denote

$$CMO^p(\mathbb{R}^n) = CMO_1^p(\mathbb{R}^n).$$

Remark 2.11 About the definition of CMO_q^p of the case $0 < q \leq n/(n+1)$, see [7]. In this paper, we consider the case $n/(n+1) < q \leq 1$ for simplicity.

Note that for $1 \leq p < \infty$ and $0 < q \leq 1$,

$$L^\infty(\mathbb{R}^n) \subset B_q^p(\mathbb{R}^n) \subset CMO_q^p(\mathbb{R}^n).$$

Next, we introduce the weak CMO space $WCMO_q^p(\mathbb{R}^n)$.

Definition 2.12 For $1 \leq p < \infty$ and $0 < q \leq 1$,

$$WCMO_q^p(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n) : \|f|WCMO_q^p\| < \infty \right\},$$

where

$$\|f|WCMO_q^p\| = \sup_{R \geq 1} |B_R|^{1-1/p-1/q} \inf_c \sup_{\lambda > 0} \lambda |\{x \in B_R : |f(x) - c| > \lambda\}|^{1/p}.$$

Furthermore, we define a "new" function space $CMO_{q,\epsilon}^\infty$.

Definition 2.13 For $n/(n+1) < q \leq 1$ and $0 \leq \epsilon < 1$,

$$CMO_{q,\epsilon}^\infty(\mathbb{R}^n) = \left\{ f : \|f|CMO_{q,\epsilon}^\infty\| < \infty \right\},$$

where

$$\|f|CMO_{q,\epsilon}^\infty\| = \sup_{R \geq 1} |B_R|^{1-1/q} \|f|Lip_\epsilon(B_R)\|.$$

Then, our results for a fractional integral operator I_s are the following strong and weak estimates for $\widetilde{I}_s$ on $B_q^p(\mathbb{R}^n)$.

Theorem 2.14 *If $0 < s < 1$, $1 < p_1 < n/s$, $n/(n+1-s) < q_1 \leq 1$, $1/p_2 = 1/p_1 - s/n$, $1/q_2 = 1/q_1 + s/n$ and I_s is a fractional integral operator, then the modified fractional integral operator $\widetilde{I}_s$ is bounded from $B_{q_1}^{p_1}(\mathbb{R}^n)$ to $CMO_{q_2}^{p_2}(\mathbb{R}^n)$, that is for $f \in B_{q_1}^{p_1}(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|CMO_{q_2}^{p_2}\| \leq C \|f|B_{q_1}^{p_1}\|.$$

Remark 2.15 For a fractional integral operator I_s, in general, $\widetilde{I}_s$ is not bounded from $B_{q_1}^1(\mathbb{R}^n)$ to $CMO_{q_2}^p(\mathbb{R}^n)$ (see Counterexample 4.1 in Section 4).

Theorem 2.16 *If $0 < s < 1$, $n/(n+1-s) < q_1 \leq 1$, $1/p = 1 - s/n$, $1/q_2 = 1/q_1 + s/n$ and I_s is a fractional integral operator, then the modified fractional integral operator $\widetilde{I}_s$ is bounded from $B^1_{q_1}(\mathbb{R}^n)$ to $WCMO^p_{q_2}(\mathbb{R}^n)$, that is for $f \in B^1_{q_1}(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|WCMO^p_{q_2}\| \leq C\|f|B^1_{q_1}\|.$$

Theorem 2.17 *If $0 < s < 1$, $n/s \leq p < \infty$, $n/(n+1-s) < q_1 \leq 1$, $1/q_2 = 1/p + 1/q_1$, $\epsilon = s - n/p$ and I_s is a fractional integral operator, then the modified fractional integral operator $\widetilde{I}_s$ is bounded from $B^p_{q_1}(\mathbb{R}^n)$ to $CMO^\infty_{q_2,\epsilon}(\mathbb{R}^n)$, that is for $f \in B^p_{q_1}(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|CMO^\infty_{q_2,\epsilon}\| \leq C\|f|B^p_{q_1}\|.$$

Remark 2.18 In Theorems 2.14, 2.16 and 2.17, $n/(n+1) < q_2 < 1$.

In particular, when $q = 1$, for a fractional integral operator I_s, we get the following strong and weak estimates for $\widetilde{I}_s$ on $B^p(\mathbb{R}^n)$.

Corollary 2.19 *If $0 < s < 1$, $1 < p_1 < n/s$, $1/p_2 = 1/p_1 - s/n$, $1/q = 1 + s/n$ and I_s is a fractional integral operator, then the modified fractional integral operator $\widetilde{I}_s$ is bounded from $B^{p_1}(\mathbb{R}^n)$ to $CMO^{p_2}_q(\mathbb{R}^n)$, that is for $f \in B^{p_1}(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|CMO^{p_2}_q\| \leq C\|f|B^{p_1}\|.$$

Corollary 2.20 *If $0 < s < 1$, $1/p = 1 - s/n$, $1/q = 1 + s/n$ and I_s is a fractional integral operator, then the modified fractional integral operator $\widetilde{I}_s$ is bounded from $B^1(\mathbb{R}^n)$ to $WCMO^p_q(\mathbb{R}^n$, that is for $f \in B^1(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|WCMO^p_q\| \leq C\|f|B^1\|.$$

Corollary 2.21 *If $0 < s < 1$, $n/s \leq p < \infty$, $1/q = 1/p + 1$, $\epsilon = s - n/p$ and I_s is a fractional integral operator, then the modified fractional integral operator $\widetilde{I}_s$ is bounded from $B^p(\mathbb{R}^n)$ to $CMO^\infty_{q,\epsilon}(\mathbb{R}^n)$, that is for $f \in B^p(\mathbb{R}^n)$,*

$$\|\widetilde{I}_s f|CMO^\infty_{q,\epsilon}\| \leq C\|f|B^p\|.$$

3 Proofs of theorems

Throughout this section, let B_R be a fixed ball for some $R \geq 1$, and decompose

$$f = f\chi_{B_{2R}} + f\chi_{\mathbb{R}^n \setminus B_{2R}} = f_1 + f_2, \quad \text{say.}$$

The following lemma is important for our proofs.

Lemma 3.1 *Let $0 < s < 1$, $1 \le p < \infty$ and $n/(n+1-s) < q \le 1$. If $f \in B_q^p(\mathbb{R}^n)$, then for any $R \ge 1$,*

$$\int_{|y|\ge 2R} \frac{|f(y)|}{|y|^{n+1-s}} dy \le C R^{n/q-n-1+s} \|f|B_q^p\|.$$

Proof. By Hölder's inequality, we can get

$$\begin{aligned}
\int_{|y|\ge 2R} \frac{|f(y)|}{|y|^{n+1-s}} dy &= \sum_{j=1}^{\infty} \int_{B_{2^{j+1}R}\setminus B_{2^jR}} \frac{|f(y)|}{|y|^{n+1-s}} dy \\
&\le \sum_{j=1}^{\infty} \frac{1}{(2^jR)^{n+1-s}} \int_{B_{2^{j+1}R}} |f(y)| dy \\
&\le C \sum_{j=1}^{\infty} \frac{1}{(2^jR)^{n+1-s}} |B_{2^{j+1}R}|^{1/q} \|f|B_q^p\| \\
&\le C R^{n/q-n-1+s} \|f|B_q^p\|,
\end{aligned}$$

because $n/(n+1-s) < q$. □

The proof of Theorem 2.14 is similar to that of Theorem 2.16. In the following proof of Theorem 2.16, use Proposition 2.6 to estimate I. Therefore we omit the proof.

Now, we begin to prove Theorems 2.16 and 2.17.

Proof. [Proof of Theorem 2.16] Let $0 < s < 1$, $n/(n+1-s) < q_1 \le 1$, $1/p = 1 - s/n$, $1/q_2 = 1/q_1 + s/n$.

Now, we have for $x \in B_R$,

$$\begin{aligned}
\widetilde{I}_s f(x) &= \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-s}} - \frac{1}{|y|^{n-s}} \chi_{|y|\ge 1}(y) \right) f(y) dy \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-s}} - \frac{1}{|y|^{n-s}} \chi_{|y|\ge 1}(y) \right) f_1(y) dy \\
&\quad + \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-s}} - \frac{1}{|y|^{n-s}} \chi_{|y|\ge 1}(y) \right) f_2(y) dy \\
&= I_s(f\chi_{B_{2R}})(x) - \int_{1\le|y|\le 2R} \frac{f(y)}{|y|^{n-s}} dy \\
&\quad + \int_{|y|\ge 2R} \left(\frac{1}{|x-y|^{n-s}} - \frac{1}{|y|^{n-s}} \right) f(y) dy.
\end{aligned}$$

Hence, putting

$$C_R = -\int_{1\le|y|\le 2R} \frac{f(y)}{|y|^{n-s}} dy,$$

it follows that

$$\begin{aligned}
&\sup_{\lambda>0}(2\lambda)^p\left|\left\{x\in B_R : |\widetilde{I}_s f(x)-C_R|>2\lambda\right\}\right| \\
&\quad\le 2^p\Bigg\{\sup_{\lambda>0}\lambda^p\left|\{x\in B_R : |I_s(f\chi_{B_{2R}})(x)|>\lambda\}\right| \\
&\qquad+\sup_{\lambda>0}\lambda^p\left|\left\{x\in B_R : \left|\int_{|y|\ge 2R}\left(\frac{1}{|x-y|^{n-s}}-\frac{1}{|y|^{n-s}}\right)f(y)dy\right|>\lambda\right\}\right|\Bigg\} \\
&\quad= 2^p\left(I+II\right), \quad \text{say.}
\end{aligned}$$

First, to estimate I, we apply Proposition 2.7. Then,

$$I\le C\|f\chi_{B_{2R}}|L^1\|^p\le C|B_R|^{p/q_1}\|f|B^1_{q_1}\|^p.$$

Next, we estimate II. Using the assumption $n/(n+1-s)<q_1$, it follows from Lemma 3.1 that for $x\in B_R$,

$$\begin{aligned}
\left|\int_{|y|\ge 2R}\left(\frac{1}{|x-y|^{n-s}}-\frac{1}{|y|^{n-s}}\right)f(y)dy\right| &\le CR\int_{|y|\ge 2R}\frac{|f(y)|}{|y|^{n+1-s}}dy \\
&\le CR\cdot R^{n/q_1-n-1+s}\|f|B^1_{q_1}\| = CR^{n(1/q_1-1/p)}\|f|B^1_{q_1}\|.
\end{aligned}$$

Therefore,

$$II\le CR^{np(1/q_1-1/p)}\|f|B^1_{q_1}\|^p\cdot|B_R|\le C|B_R|^{p/q_1}\|f|B^1_{q_1}\|^p.$$

Thus,

$$|B_R|^{1-1/p-1/q_2}\sup_{\lambda>0}\lambda\left|\left\{x\in B_R : |\widetilde{I}_s f(x)-C_R|>\lambda\right\}\right|^{1/p}\le C\|f|B^1_{q_1}\|.$$

This proves the theorem. □

Proof. [Proof of Theorem 2.17] Let $0<s<1$, $n/s\le p<\infty$, $n/(n+1-s)<q_1\le 1$, $1/q_2=1/p+1/q_1$ and $\epsilon=s-n/p$.

Now, we have

$$\begin{aligned}
&|B_R|^{1-1/q_2}\|\widetilde{I}_s f|Lip_\epsilon(B_R)\| \\
&\quad= |B_R|^{1-1/q_2}\sup_{B'}\frac{|B'|^{-\epsilon/n}}{|B'|}\int_{B'}|\widetilde{I}_s f(x)-(\widetilde{I}_s f)_{B'}|dx \\
&\quad\le |B_R|^{1-1/q_2}\left\{\|\widetilde{I}_s f_1|Lip_\epsilon(B_R)\|+\|\widetilde{I}_s f_2|Lip_\epsilon(B_R)\|\right\} \\
&\quad= I+II, \quad \text{say.}
\end{aligned}$$

First, to estimate I, we apply Proposition 2.8. Then,

$$I \le C|B_R|^{1-1/q_2}\|f\chi_{B_{2R}}|L^p\| \le C\|f|B^p_{q_1}\|.$$

Next, we estimate II. Since for $x_1, x_2 \in B_R$,

$$|\widetilde{I}_s f_2(x_1) - \widetilde{I}_s f_2(x_2)| \le C|x_1 - x_2| \int_{|y|\ge 2R} \frac{|f(y)|}{|y|^{n+1-s}}dy.$$

we obtain

$$\begin{aligned}\|\widetilde{I}_s f_2|Lip_\epsilon(B_R)\| &= \sup_{B'} \frac{|B'|^{-\epsilon/n}}{|B'|}\int_{B'} |\tilde{I}_s f_2(x) - (\tilde{I}_s f_2)_{B'}|dx \\ &\le C \sup_{B'} |B'|^{-\epsilon/n+1/n} \int_{|y|\ge 2R} \frac{|f(y)|}{|y|^{n+1-s}}dy.\end{aligned}$$

Therefore, it follows from Lemma 3.1 that

$$II \le C|B_R|^{1-1/q_2} \cdot |B_R|^{-\epsilon/n+1/n} \cdot R^{n/q_1-n-1+s}\|f|B^p_{q_1}\| = C\|f|B^p_{q_1}\|.$$

Thus, by the estimates of I and II, we get the theorem. □

4 Counterexample

Counterexample 4.1 *Let $0 < s < 1$, $1/(2-s) < q_1 \le 1$, $1/p = 1 - s$ and $1/q_2 = 1/q_1 + s$. Then, the modified fractional integral operator $\widetilde{I}_s$ is not bounded from $B^1_{q_1}(\mathbb{R}^1)$ to $CMO^p_{q_2}(\mathbb{R}^1)$.*

Proof. Let

$$f(x) = \begin{cases} \dfrac{1}{(1-x)|\log(1-x)|^{3/2}} & \text{for } \dfrac{1}{2} < x < 1, \\ 0 & \text{otherwise.}\end{cases}$$

Then, $f \in B^1_{q_1}(\mathbb{R}^1)$.

Now, for $1 < x < 3/2$, putting $x = 1 + h$ (i.e. $0 < h < 1/2$), then

$$|\widetilde{I}_s f(x)| \ge \frac{1}{(2h)^{1-s}}\int_{1-h}^{1} f(y)dy = \frac{1}{(2h)^{1-s}|\log h|^{1-s}}.$$

Thus, letting $B = (-R, R)$ be a fixed open interval for some $R > 3/2$ and $0 < s < 1/2$, we obtain

$$\left(\int_B |\widetilde{I}_s f(x)|^p dx\right)^{1/p} = \left(\int_1^{3/2} |\widetilde{I}_s f(x)|^{1/(1-s)}dx\right)^{1-s} = \infty,$$

i.e. $\widetilde{I}_s f \notin CMO^p_{q_2}(\mathbb{R}^1)$. □

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Approximation by weighted means of Walsh-Kaczmarz-Fourier series

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Abstract. In this article we investigate the rate of the approximation by weighted means of one and two-dimensional Walsh-Kaczmarz-Fourier series of a function in L^p and, in particular, in $\mathrm{Lip}(\alpha, p)$ $(1 \leq p \leq \infty, \alpha > 0)$. In case $p = \infty$, by L^p we mean C_W, the collection of uniformly W-continuous functions. As special case, we obtain the earlier result by Skvortsov [20].

The approximation properties of weighted means of the Walsh-Fourier series was discussed earlier by Móricz and Rhoades [11].

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1 Introduction

Let $\{p_k : k \geq 0\}$ be a sequence of nonnegative numbers. The weighted means and kernels for the Walsh-(Kaczmarz)-Fourier series are defined by

$$t_n^{\psi}(f,x) := \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k^{\psi}(f,x), \quad L_n^{\psi}(x) := \frac{1}{P_n}\sum_{k=0}^{n} p_k D_k^{\psi}(x),$$

where $P_n := \sum_{k=0}^{n} p_k$ $(n \geq 0,\ \psi = w \text{ or } \kappa)$. In particular case t_n^{ψ} are the Fejér means (set $p_k = 1$ for all k).

This paper is in final form and no version of it will be submitted for publication elsewhere.

We always assume that $p_0 > 0$ and

$$\lim_{n\to\infty} P_n = \infty, \tag{1.1}$$

which is the condition for regularity [12]. The weighted mean method was studied by a lot of authors [9, 11, 12, 13].

In the paper [11] the rate of the approximation by weighted means of Walsh-Fourier series of a function in L^p (in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \le p \le \infty$) was studied. In case $p = \infty$, by L^p we mean C_W, the collection of the uniform W-continuous functions. As special cases Móricz and Rhoades obtained the earlier results by Yano [23] for $0 < \alpha < 1$, and by Jastrebova [8] for $\alpha = 1$.

In this article we investigate the rate of the approximation by weighted means of Walsh-Kaczmarz-Fourier series of a function in L^p $(1 \le p \le \infty)$. As a special case, we have the result of Skvortsov given in [20].

In 1948 Šneider [21] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \ge C > 0$$

holds a.e. In 1974 Schipp [18] and Young [24] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [20] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f.

In 1998 Gát [2] proved, for any integrable function, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. In 2004, Gát's result was generalized by Simon [19].

Now, we give a brief introduction to the Walsh-Fourier analysis [1, 17].

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is $1/2$. Let $G := \overset{\infty}{\underset{k=0}{\times}} \mathbf{Z}_2$, G is called the Walsh group. The elements of G are sequences $x = (x_0, x_1, ..., x_k, ...)$ with coordinates $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure is the product measure (denoted by μ) and the topology is the pruduct topology. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1}...)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G.

Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$. Let L^p denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C_W, where C_W is the collection of the uniformly W-continuous functions over G and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$. Next, we define the modulus of continuity in $L^p, 1 \leq p \leq \infty$, of a function $f \in L^p$ by

$$\omega_p(\delta, f) := \sup_{|t|<\delta} \|f(. + t) - f(.)\|_p, \quad \delta > 0.$$

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\mathrm{Lip}(\alpha, p) := \{f \in L^p : \omega_p(\delta, f) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \ (i \in \mathbf{N}),$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{P}$ and $\kappa_0 = w_0$.

A relation (given by V. A. Skvortsov, see [20]) between the Walsh-Kaczmarz functions and the Walsh-Paley functions can be given by the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{P}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k,$$

where $\psi_n = w_n$ or κ_n $(n \in \mathbf{P})$, $D_0^{\psi} := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [17])

$$D_{2^n}^{w} = D_{2^n}^{\kappa} = D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbf{N}). \end{cases} \tag{1.2}$$

The nth Fejér mean and Fejér kernel of the Fourier series of a function f is defined by

$$\sigma_n^{\psi}(f;x) := \frac{1}{n} \sum_{i=0}^{n} S_i^{\psi}(f;x), \quad K_n^{\psi}(x) := \frac{1}{n} \sum_{k=0}^{n} D_k^{\psi}(x) \quad (x \in G)$$

where $\psi_n := w_n$ or κ_n and $K_0^{\psi} := 0$.

2 The rate of the approximation of the one-dimensional weighted means

Now, we introduce the notation

$$R_n^{w} := \frac{1}{P_n} \sum_{i=1}^{n-2^{|n|}} p_{2^{|n|}+i} D_i^{w}$$

and decompose the nth weighted kernel of the Walsh-Kaczmarz-Fourier series.

Lemma 2.1 *Let $|n| = A \geq 1$, then*

$$\begin{aligned} P_nL_n^{\kappa}(x) &= (P_n - P_{2^A-1})D_{2^A}(x) + r_A(x)P_nR_n^w(\tau_A(x)) \\ &+ \sum_{l=0}^{A-1}(P_{2^{l+1}-1} - P_{2^l-1})D_{2^l}(x) \\ &+ \sum_{l=0}^{A-1} r_l(x)\sum_{j=1}^{2^l-2}(p_{2^l+j} - p_{2^l+j+1})jK_j^w(\tau_l(x)) \\ &+ \sum_{l=0}^{A-1} r_l(x)p_{2^{l+1}-1}(2^l-1)K_{2^l-1}^w(\tau_l(x)). \end{aligned}$$

Proof. During the proof of Lemma 2.1 we use the following equation:

$$D_{2^A+j}^{\kappa}(x) = D_{2^A}(x) + r_A(x)D_j^w(\tau_A(x)), \quad j = 0, 1, ..., 2^A - 1. \tag{2.1}$$

Let $|n| = A$, then we write

$$P_nL_n^{\kappa}(x) = \sum_{k=1}^{2^A-1} p_kD_k^{\kappa}(x) + \sum_{k=2^A}^{n} p_kD_k^{\kappa}(x) =: I + II.$$

By the help of (2.1), we decompose II.

$$\begin{aligned} II &= \sum_{l=0}^{n-2^A} p_{2^A+l}D_{2^A+l}^{\kappa}(x) \\ &= (P_n - P_{2^A-1})D_{2^A}(x) + r_A(x)\sum_{l=1}^{n-2^A} p_{2^A+l}D_l^w(\tau_A(x)). \end{aligned}$$

We use (2.1) and Abel's transformation for the discussion of I.

$$\begin{aligned} I &= \sum_{l=0}^{A-1}\sum_{k=2^l}^{2^{l+1}-1} p_kD_k^{\kappa}(x) = \sum_{l=0}^{A-1}\sum_{j=0}^{2^l-1} p_{2^l+j}D_{2^l+j}^{\kappa}(x) \\ &= \sum_{l=0}^{A-1}(P_{2^{l+1}-1} - P_{2^l-1})D_{2^l}(x) \\ &+ \sum_{l=0}^{A-1} r_l(x)\left(\sum_{j=1}^{2^l-2}(p_{2^l+j} - p_{2^l+j+1})jK_j^w(\tau_l(x))\right. \\ &\left. + p_{2^{l+1}-1}(2^l-1)K_{2^l-1}^w(\tau_l(x))\right). \end{aligned}$$

This completes the proof of Lemma 2.1. □

By the help of this lemma we state our main theorem in the following form:

Theorem 2.2 *Let $f \in L^p$, $1 \leq p \leq \infty$, and let $|n| = A \geq 1$.*

If $\{p_k\}$ is nondecreasing sequence of nonnegative numbers, in sign $p_k \uparrow$, such that the condition

$$\frac{np_n}{P_n} = O(1) \tag{2.2}$$

holds, then

$$\|t_n^{\kappa}(f) - f\|_p \leq \frac{c}{P_n} \sum_{l=0}^{A-1} p_{2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

If $\{p_k\}$ is nonincreasing, in sign $p_k \downarrow$, then

$$\|t_n^{\kappa}(f) - f\|_p \leq \frac{c}{P_n} \sum_{l=0}^{A-1} p_{2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

This Theorem states that the behavior of the weighted means of Walsh-Kaczmarz-Fourier series is so good as the behavior of the weighted means of Walsh-Paley-Fourier series [11] in this special view.

To prove our theorem we need the following lemmas given by Yano [22], Simon [19] and Móricz and Rhoades [11].

Lemma 2.3 (Yano) *Let $n \geq 1$, then*

$$\|K_n^w\|_1 \leq 2.$$

Lemma 2.4 (Simon) *There exists a constant C such that*

$$\|K_n^{\kappa}\|_1 \leq C, \quad n \geq 1.$$

Lemma 2.5 (Móricz, Rhoades) *Let $\{p_k\}$ be a sequence of nonnegative numbers either nondecreasing which satisfies the condition (2.2) or merely nonincreasing, and let R_n^w be defined by $R_n^w := \frac{1}{P_n} \sum_{i=1}^{n-2^{|n|}} p_{2^{|n|}+i} D_i^w$. Then there exists a constant C such that*

$$\|R_n^w\|_1 \leq C, \quad n \geq 1.$$

Proof of Theorem 1: We present the proof for $1 \leq p < \infty$, for $p = \infty$ (with notation $L^{\infty} = C_W$) the proof goes in a similar way.

Let $n \in \mathbf{N}$ be fixed and set $|n| = A$. By Lemma 2.1 and Minkowski inequality we may write that

$$\begin{aligned}
P_n\|t_n^{\kappa}(f) - f\|_p &\leq (P_n - P_{2^A-1}) \left\| \int_G (f(.+x) - f(.)) D_{2^A}(x) d\mu(x) \right\|_p \\
&+ P_n \left\| \int_G (f(.+x) - f(.)) r_A(x) R_n^w(\tau_A(x)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} (P_{2^{l+1}-1} - P_{2^l-1}) \left\| \int_G (f(.+x) - f(.)) D_{2^l}(x) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} \sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j \left\| \int_G (f(.+x) - f(.)) r_l(x) K_j^w(\tau_l(x)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} p_{2^{l+1}-1}(2^l - 1) \left\| \int_G (f(.+x) - f(.)) r_l(x) K_{2^l-1}^w(\tau_l(x)) d\mu(x) \right\|_p \\
&=: A_{n,1} + A_{n,2} + A_{n,3} + A_{n,4} + A_{n,5}.
\end{aligned}$$

Now, we discuss $A_{n,1}$. By (1.2) we find that

$$\begin{aligned}
&\left\| \int_G (f(.+x) - f(.)) D_{2^A}(x) d\mu(x) \right\|_p \leq \qquad (2.3) \\
&\leq \int_{I_A} D_{2^A}(x) \left(\int_G |f(y+x) - f(y)|^p d\mu(y) \right)^{1/p} d\mu(x) \leq \omega_p(2^{-A}, f).
\end{aligned}$$

Thus, we immediately have

$$A_{n,1} \leq (P_n - P_{2^A-1}) \omega_p(2^{-A}, f)$$

and

$$A_{n,3} \leq \sum_{l=0}^{A-1} (P_{2^{l+1}-1} - P_{2^l-1}) \omega_p(2^{-l}, f).$$

In the case $p_k \uparrow$, we have

$$(P_{2^{l+1}-1} - P_{2^l-1}) \leq 2^l p_{2^{l+1}-1},$$

and

$$\begin{aligned} A_{n,3} &\leq \sum_{l=0}^{A-2} 2^l p_{2^{l+1}} \omega_p(2^{-l}, f) + (P_{2^A-1} - P_{2^{A-1}-1})\omega_p(2^{-A+1}, f) \\ &\leq c\sum_{j=1}^{A-1} 2^j p_{2^j} \omega_p(2^{-j}, f) + c(P_{2^A-1} - P_{2^{A-1}-1})\omega_p(2^{-A}, f). \quad (2.4) \end{aligned}$$

While, in the case $p_k \downarrow$, we have

$$(P_{2^{l+1}-1} - P_{2^l-1}) \leq 2^l p_{2^l} \quad \text{and} \quad A_{n,3} \leq \sum_{l=0}^{A-1} 2^l p_{2^l} \omega_p(2^l, f).$$

Now, we discuss $A_{n,4}, A_{n,5}$.

In the paper [14] the following is proved (see Lemma 2.3, too)

$$\begin{aligned} B_j^l &:= \left\| \int_G (f(.+x) - f(.)) r_l(x) K_j^w(\tau_l(x)) d\mu(x) \right\|_p \\ &\leq c\|K_j^w \circ \tau_l\|_1 \omega_p(2^{-l}, f) \leq c\|K_j^w\|_1 \omega_p(2^{-l}, f) \\ &\leq c\omega_p(2^{-l}, f). \quad (2.5) \end{aligned}$$

for $|j| \leq l$.

Thus,

$$\begin{aligned} A_{n,4} &\leq c\sum_{l=0}^{A-1}\sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j B_j^l \\ &\leq c\sum_{l=0}^{A-1}\sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j \omega_p(2^{-l}, f). \end{aligned}$$

In the case $p_k \uparrow$

$$\sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j = (2^l - 2)p_{2^{l+1}-1} - \sum_{j=1}^{2^l-2} p_{2^l+j} \leq 2^l p_{2^{l+1}-1}$$

and

$$\begin{aligned} A_{n,4} &\leq c\sum_{l=0}^{A-1} 2^l p_{2^{l+1}} \omega_p(2^{-l}, f) \\ &= c\sum_{l=0}^{A-2} 2^l p_{2^{l+1}} \omega_p(2^{-l}, f) + 2^{A-1} p_{2^A} \omega_p(2^{-A+1}, f). \quad (2.6) \end{aligned}$$

The condition (2.2) completes this case.

In the case $p_k \downarrow$

$$\sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j = \sum_{j=1}^{2^l-2} p_{2^l+j} - (2^l-2)p_{2^{l+1}-1} \leq \sum_{j=1}^{2^l-2} p_{2^l+j} \leq 2^l p_{2^l}$$

and

$$A_{n,4} \leq c \sum_{l=0}^{A-1} 2^l p_{2^l} \omega_p(2^{-l}, f).$$

Moreover,

$$A_{n,5} \leq c \sum_{l=0}^{A-1} p_{2^{l+1}-1} 2^l B^l_{2^l-1} \leq c \sum_{l=0}^{A-1} p_{2^{l+1}-1} 2^l \omega_p(2^{-l}, f).$$

The method of (2.6) completes the case when $p_k \uparrow$.

In the case $p_k \downarrow$, we immediately get that

$$A_{n,5} \leq c \sum_{l=0}^{A-1} p_{2^l} 2^l \omega_p(2^{-l}, f).$$

At last, we investigate $A_{n,2}$. To do this we discuss the integral

$$\| \int_G (f(.+x) - f(.)) r_A(x) R^w_n(\tau_A(x)) d\mu(x) \|_p.$$

Recall the definition of the function R^w_n. The definition really say that the function R^w_n is $\mathcal{A}_{|n-2^{|n|}|}$-measurable. Thus, $R^w_n \circ \tau_A$ is $\mathcal{A}_A$-measurable (with $|n| = A$). The method of the investigation of B^l_j for $l = A$ (see (2.5) and [14]) implies that

$$\begin{aligned} \left\| \int_G (f(.+x) - f(.)) r_A(x) R^w_n(\tau_A(x)) d\mu(x) \right\|_p &\leq c\omega_p(2^{-A}, f) \|R_n \circ \tau_A\|_1 \\ &\leq c\omega_p(2^{-A}, f) \|R_n\|_1. \end{aligned}$$

From Lemma 2.5, we get that

$$A_{n,2} \leq c P_n \omega_p(2^{-A}, f).$$

Summarising our results on $A_{n,i}$ $(i = 1, ..., 5)$ we complete the proof of this theorem. □

We note that if $\{p_k\}$ is nondecreasing ($p_k \uparrow$), the condition (2.2) is satiesfied if

$$p_k = k^\beta \text{ or } (\log k)^\beta \text{ for some } \beta > 0.$$

Examples for nonincreasing $\{p_k\}$ ($p_k \downarrow$):

bi.) $p_k \asymp k^{-\beta}$ for some $0 < \beta \leq 1$, ($P_n \asymp n^{1-\beta}$ for $0 < \beta < 1$, and $P_n \asymp \log n$ for $\beta = 1$).

bii.) $p_k \asymp (\log k)^{-\beta}$ for some $0 < \beta$ ($P_n \asymp n(\log n)^{-\beta}$).

For more details see [11].

The following theorem is a generalization of the theorem of Skvortsov in [20] (set $p_k := 1$ for all k), and its analogue was proved for Walsh-Paley system by Móricz and Rhoades in [11].

Theorem 2.6 *Let $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$.*

Let $\{p_k : k \geq 0\}$ be a sequence of nonnegative numbers such that in case $p_k \uparrow$ the condition (2.2) is satisfied, then

$$\|t_n^\kappa(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(n^{-1}\log n), & \text{if } \alpha = 1, \\ O(n^{-1}), & \text{if } \alpha > 1. \end{cases}$$

If $p_k \downarrow$, then

$$\|t_n^\kappa(f) - f\|_p = O\left(\frac{1}{P_n}\sum_{j=0}^{|n|-1} 2^{(1-\alpha)j} p_{2^j} + 2^{-\alpha|n|}\right).$$

The proof goes analogous as Móricz and Rhoades did in [11]. So, it is left to the reader.

Let $\{p_k : k \geq 0\}$ be a sequence of nonnegative numbers such that in case $p_k \downarrow$ the condition bi) or bii.) is satisfied, then in these cases we could get so good results as Móricz and Rhoades did for the Walsh system (for more details see [11], page 3).

3 On the two-dimensional weighted means

On G^2 we consider the two-dimensional system as $\{\psi_{n^1}(x^1) \times \psi_{n^2}(x^2) : n := (n^1, n^2) \in \mathbf{N}^2\}$ (where $\psi = w$ or κ). The two-dimenional Fourier coefficients, the rectangular partial sums of the Fourier series and Dirichlet

kernels are defined in the usual way. Define the n-th Marcinkiewicz mean $\mathcal{M}_n^{\psi}$ and kernel $\mathcal{K}_n^{\psi}$ by

$$\mathcal{M}_n^{\psi}(f, x^1, x^2) := \frac{1}{n}\sum_{k=0}^{n} S_{k,k}^{\psi}(f, x^1, x^2) \quad (x = (x^1, x^2) \in G^2),$$

$$\mathcal{K}_n^{\psi}(x^1, x^2) := \frac{1}{n}\sum_{k=0}^{n} D_k^{\psi}(x^1) D_k^{\psi}(x^2) \quad (x = (x^1, x^2) \in G^2).$$

The a.e. convergence of Marcinkiewicz means was discussed by Weisz and Goginava for Walsh system [25, 7], by Gát for Vilenkin system [3], by the author for Walsh-Kaczmarz system [16].

For $x = (x^1, x^2) \in G^2$ we define $|x|$ by $|x|^2 := |x^1|^2 + |x^2|^2$. Thus, for $f \in L^p(G^2)$ $(1 \le p \le \infty)$ the modulus of continuity $\omega_p(\delta, f)$ is well defined for $\delta > 0$. We define the mixed modulus of continuity as follows:

$$\omega_{1,2}^p(\delta_1, \delta_2, f) :=$$

$$\sup\{\|f(.+x^1, .+x^2) - f(.+x^1, .) - f(., .+x^2) + f(., .)\|_p : |x^1| \le \delta_1, |x^2| \le \delta_2\},$$

where $\delta_1, \delta_2 > 0$.

Recently, Gát and Goginava [4] discussed the uniform and L-convergence of the Nörlund logarithmic means of cubical partial sums of double Walsh-Fourier series. Moreover, Goginava discussed the approximation properties of Cesàro (C, α)-means of cubical partial sums of Walsh-Fourier series [6].

Motivated by the work of Gát and Goginava, we investigate the approximation properties of the weighted means of cubical partial sums of double Walsh-Kaczmarz-Fourier series. These weighted means and kernels are defined by

$$\mathbf{t}_n^{\psi}(f, x^1, x^2) := \frac{1}{P_n}\sum_{k=1}^{n} p_k S_{k,k}^{\psi}(f, x^1, x^2),$$

$$\mathcal{L}_n^{\psi}(x^1, x^2) := \frac{1}{P_n}\sum_{k=1}^{n} p_k D_k^{\psi}(x^1) D_k^{\psi}(x^2)$$

for $\psi = w$ or κ. The means $\mathbf{t}_n^{\psi}$ could be called weighted means of Marcinkiewicz type. As special case (set $q_k := 1$ for all k) we obtain the Marcinkiewicz means and kernels.

To prove our Theorem on two-dimensional weighted means we need the following decomposition lemma:

Lemma 3.1 *Let $|n| = A \geq 1$, then*

$$
\begin{aligned}
& P_n\mathcal{L}_n^\kappa(x^1,x^2) = \\
= \;& (P_n - P_{2^A-1})D_{2^A}(x^1)D_{2^A}(x^2) + D_{2^A}(x^1)r_A(x^2)P_nR_n^w(\tau_A(x^2)) \\
+ \;& D_{2^A}(x^2)r_A(x^1)P_nR_n^w(\tau_A(x^1)) + r_A(x^1)r_A(x^2)P_n\mathcal{R}_n^w(\tau_A(x^1),\tau_A(x^2)) \\
+ \;& \sum_{l=0}^{A-1}(P_{2^{l+1}-1} - P_{2^l-1})D_{2^l}(x^1)D_{2^l}(x^2) \\
+ \;& \sum_{l=0}^{A-1} D_{2^l}(x^1)r_l(x^2)\sum_{j=1}^{2^l-2}(p_{2^l+j} - p_{2^l+j+1})jK_j^w(\tau_l(x^2)) \\
+ \;& \sum_{l=0}^{A-1} D_{2^l}(x^2)r_l(x^1)\sum_{j=1}^{2^l-2}(p_{2^l+j} - p_{2^l+j+1})jK_j^w(\tau_l(x^1)) \\
+ \;& \sum_{l=0}^{A-1} D_{2^l}(x^1)r_l(x^2)p_{2^{l+1}-1}(2^l-1)K_{2^l-1}^w(\tau_l(x^2)) \\
+ \;& \sum_{l=0}^{A-1} D_{2^l}(x^2)r_l(x^1)p_{2^{l+1}-1}(2^l-1)K_{2^l-1}^w(\tau_l(x^1)) \\
+ \;& \sum_{l=0}^{A-1} r_l(x^1)r_l(x^2)\sum_{j=1}^{2^l-2}(p_{2^l+j} - p_{2^l+j+1})j\mathcal{K}_j^w(\tau_l(x^1),\tau_l(x^2)) \\
+ \;& \sum_{l=0}^{A-1} r_l(x^1)r_l(x^2)p_{2^{l+1}-1}(2^l-1)\mathcal{K}_{2^l-1}^w(\tau_l(x^1),\tau_l(x^2)),
\end{aligned}
$$

with the notation R_n^w of previous section and

$$\mathcal{R}_n^w(x^1,x^2) := \frac{1}{P_n}\sum_{l=1}^{n-2^{|n|}} p_{2^{|n|}+l}D_l^w(x^1)D_l^w(x^2).$$

Proof.

$$P_n\mathcal{L}_n^\kappa(x^1,x^2) = \sum_{k=1}^{2^A-1} p_kD_k^\kappa(x^1)D_k^\kappa(x^2) + \sum_{k=2^A}^{n} p_kD_k^\kappa(x^1)D_k^\kappa(x^2) =: I + II.$$

We use (2.1) and Abel's transformation to study I.

$$
\begin{aligned}
I &= \sum_{l=0}^{A-1}\sum_{k=2^l}^{2^{l+1}-1} p_k D_k^{\kappa}(x^1)D_k^{\kappa}(x^2) = \sum_{l=0}^{A-1}\sum_{j=0}^{2^l-1} p_{2^l+j} D_{2^l+j}^{\kappa}(x^1)D_{2^l+j}^{\kappa}(x^2)\\
&= \sum_{l=0}^{A-1} D_{2^l}(x^1)D_{2^l}(x^2)\sum_{j=0}^{2^l-1} p_{2^l+j}\\
&+ \sum_{l=0}^{A-1} D_{2^l}(x^1)r_l(x^2)\sum_{j=0}^{2^l-1} p_{2^l+j}D_j^w(\tau_l(x^2))\\
&+ \sum_{l=0}^{A-1} D_{2^l}(x^2)r_l(x^1)\sum_{j=0}^{2^l-1} p_{2^l+j}D_j^w(\tau_l(x^1))\\
&+ \sum_{l=0}^{A-1} r_l(x^1)r_l(x^2)\sum_{j=0}^{2^l-1} p_{2^l+j}D_j^w(\tau_l(x^1))D_j^w(\tau_l(x^2))\\
&= \sum_{l=0}^{A-1}(P_{2^{l+1}-1}-P_{2^l-1})D_{2^l}(x^1)D_{2^l}(x^2)\\
&+ \sum_{l=0}^{A-1} D_{2^l}(x^1)r_l(x^2)\left[\sum_{j=1}^{2^l-2}(p_{2^l+j}-p_{2^l+j+1})jK_j^w(\tau_l(x^2))\right.\\
&\qquad\left. + p_{2^{l+1}-1}(2^l-1)K_{2^l-1}^w(\tau_l(x^2))\right]\\
&+ \sum_{l=0}^{A-1} D_{2^l}(x^2)r_l(x^1)\left[\sum_{j=1}^{2^l-2}(p_{2^l+j}-p_{2^l+j+1})jK_j^w(\tau_l(x^1))\right.\\
&\qquad\left. + p_{2^{l+1}-1}(2^l-1)K_{2^l-1}^w(\tau_l(x^1))\right]\\
&+ \sum_{l=0}^{A-1} r_l(x^1)r_l(x^2)\sum_{j=1}^{2^l-2}(p_{2^l+j}-p_{2^l+j+1})j\mathcal{K}_j^w(\tau_l(x^1),\tau_l(x^2))\\
&+ \sum_{l=0}^{A-1} r_l(x^1)r_l(x^2)p_{2^{l+1}-1}(2^l-1)\mathcal{K}_{2^l-1}^w(\tau_l(x^1),\tau_l(x^2)).
\end{aligned}
$$

By the help of (2.1), we discuss II.

$$\begin{aligned}
II &= \sum_{l=0}^{n-2^A} p_{2^A+l} D^{\kappa}_{2^A+l}(x^1) D^{\kappa}_{2^A+l}(x^2) \\
&= (P_n - P_{2^A-1}) D_{2^A}(x^1) D_{2^A}(x^2) \\
&+ D_{2^A}(x^1) r_A(x^2) \sum_{l=0}^{n-2^A} p_{2^A+l} D^{w}_{l}(\tau_A(x^2)) \\
&+ D_{2^A}(x^2) r_A(x^1) \sum_{l=0}^{n-2^A} p_{2^A+l} D^{w}_{l}(\tau_A(x^1)) \\
&+ r_A(x^1) r_A(x^2) \sum_{l=0}^{n-2^A} p_{2^A+l} D^{w}_{l}(\tau_A(x^1)) D^{w}_{l}(\tau_A(x^2)) \\
&= (P_n - P_{2^A-1}) D_{2^A}(x^1) D_{2^A}(x^2) + D_{2^A}(x^1) r_A(x^2) P_n R^{w}_{n}(\tau_A(x^2)) \\
&+ D_{2^A}(x^2) r_A(x^1) P_n R^{w}_{n}(\tau_A(x^1)) + r_A(x^1) r_A(x^2) P_n \mathcal{R}^{w}_{n}(\tau_A(x^1), \tau_A(x^2)).
\end{aligned}$$

This completes the proof of this Lemma. □

By the help of this decomposition Lemma we state our main Theorem in the following form:

Theorem 3.2 *Let $f \in L^p$, $1 \le p \le \infty$, and let $|n| = A \ge 1$.*

If $\{p_k\}$ is nondecreasing sequence of nonnegative numbers, in sign $p_k \uparrow$, such that the condition (2.2) *holds, then*

$$\|\mathbf{t}^{\kappa}_{n}(f) - f\|_p \le \frac{c}{P_n} \sum_{l=0}^{A-1} p_{2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

If $\{p_k\}$ is nonincreasing, in sign $p_k \downarrow$, then

$$\|\mathbf{t}^{\kappa}_{n}(f) - f\|_p \le \frac{c}{P_n} \sum_{l=0}^{A-1} p_{2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

This theorem states that the behavior of the weighted means of double Walsh-Kaczmarz-Fourier series is so good as the behavior of the weighted means of one-dimensional Walsh-Kaczmarz-Fourier series in this special view.

To prove our theorem we need the following lemma given by Glukhov [5].

Lemma 3.3 (Glukhov) *There exists a constant C such that*

$$\|\mathcal{K}_n^w\|_1 \leq C \quad (n \geq 1).$$

Proof of Theorem 3: We make the proof for $1 \leq p < \infty$, for $p = \infty$ (with notation $L^\infty = C_W$) the proof goes analogously.

Let $n \in \mathbf{P}$ be fixed and set $|n| := A$.

First, we prove that if the sequence $\{p_k\}$ is either nondecreasing such that the condition (2.2) holds or nonincreasing, then there exists a C such that

$$\|\mathcal{R}_n^w\|_1 \leq C \quad (n \geq 1). \tag{3.1}$$

The definition of $\mathcal{R}_n^w$ and Abel's transformation give that

$$\mathcal{R}_n^w := \frac{1}{P_n} \sum_{l=1}^{n-2^A-1} (p_{2^A+l} - p_{2^A+l+1}) l \mathcal{K}_l^w + \frac{p_n(n-2^A)}{P_n} \mathcal{K}_{n-2^A}^w.$$

The Lemma 3.3 of Glukhov yields that

$$\|\mathcal{R}_n^w\|_1 \leq \frac{c}{P_n} \sum_{l=1}^{n-2^A-1} |p_{2^A+l} - p_{2^A+l+1}| l + \frac{c p_n (n-2^A)}{P_n}.$$

If $p_k \downarrow$, then

$$\frac{c}{P_n} \sum_{l=1}^{n-2^A-1} |p_{2^A+l} - p_{2^A+l+1}| l + \frac{c p_n (n-2^A)}{P_n} \leq \frac{c}{P_n} \sum_{j=2^A+1}^{n} p_j \leq C.$$

While, in case $p_k \uparrow$ the condition (2.2) gives that

$$\frac{c}{P_n} \sum_{l=1}^{n-2^A-1} |p_{2^A+l} - p_{2^A+l+1}| l + \frac{c p_n (n-2^A)}{P_n} \leq$$

$$\leq \frac{c}{P_n} (2p_n(n-2^A) - \sum_{j=2^A+1}^{n-1} p_j) \leq \frac{c p_n n}{P_n} = O(1).$$

By the help of Lemma 3.1, the Minkowski inequality and with the notation $x = (x^1, x^2) \in G^2$ we write that

$$P_n \|\mathbf{t}_n^\kappa(f) - f\|_p \leq$$

$$\leq (P_n - P_{2^A-1}) \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^A}(x^1) D_{2^A}(x^2) d\mu(x) \right\|_p$$

$$+ \sum_{l=0}^{A-1} (P_{2^{l+1}-1} - P_{2^l-1}) \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^l}(x^1) D_{2^l}(x^2) d\mu(x) \right\|_p$$

$$
\begin{aligned}
&+ P_n \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^A}(x^1) r_A(x^2) R_n^w(\tau_A(x^2)) d\mu(x) \right\|_p \\
&+ P_n \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^A}(x^2) r_A(x^1) R_n^w(\tau_A(x^1)) d\mu(x) \right\|_p \\
&+ P_n \left\| \int_{G^2} (f(.+x) - f(.)) r_A(x^1) r_A(x^2) \mathcal{R}_n^w(\tau_A(x^1), \tau_A(x^2)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} \sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j \times \\
&\qquad \times \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^l}(x^1) r_l(x^2) K_j^w(\tau_l(x^2)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} \sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j \times \\
&\qquad \times \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^l}(x^2) r_l(x^1) K_j^w(\tau_l(x^1)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} p_{2^{l+1}-1}(2^l - 1) \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^l}(x^1) r_l(x^2) K_{2^l-1}^w(\tau_l(x^2)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} p_{2^{l+1}-1}(2^l - 1) \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^l}(x^2) r_l(x^1) K_{2^l-1}^w(\tau_l(x^1)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} \sum_{j=1}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j \times \\
&\qquad \times \left\| \int_{G^2} (f(.+x) - f(.)) r_l(x^1) r_l(x^2) \mathcal{K}_j^w(\tau_l(x^1), \tau_l(x^2)) d\mu(x) \right\|_p \\
&+ \sum_{l=0}^{A-1} p_{2^{l+1}-1}(2^l - 1) \times \\
&\qquad \times \left\| \int_{G^2} (f(.+x) - f(.)) r_l(x^1) r_l(x^2) \mathcal{K}_{2^l-1}^w(\tau_l(x^1), \tau_l(x^2)) d\mu(x) \right\|_p \\
&=: \sum_{i=1}^{11} A_{n,i}.
\end{aligned}
$$

First, we discuss $A_{n,1}, A_{n,2}$. By (1.2) we obtain that

$$\left\| \int_{G^2} (f(.+x)-f(.))D_{2^A}(x^1)D_{2^A}(x^2)d\mu(x) \right\|_p \leq$$

$$\leq \int_{I_A^2} D_{2^A}(x^1)D_{2^A}(x^2) \left(\int_{G^2} |f(y+x)-f(y)|^p d\mu(y) \right)^{1/p} d\mu(x)$$

$$\leq c\omega_p(2^{-A},f). \tag{3.2}$$

This immediately yields

$$A_{n,1} \leq c(P_n - P_{2^A-1})\omega_p(2^{-A},f)$$

and

$$A_{n,2} \leq c\sum_{l=0}^{A-1}(P_{2^{l+1}-1}-P_{2^l-1})\omega_p(2^{-l},f).$$

For sequence $p_k \uparrow$

$$P_{2^{l+1}-1}-P_{2^l-1} \leq 2^l p_{2^{l+1}-1} \tag{3.3}$$

and for sequence $p_k \downarrow$

$$P_{2^{l+1}-1}-P_{2^l-1} \leq 2^l p_{2^l}. \tag{3.4}$$

The following was proved in the paper [15]:

$$B_j^k := \left\| \int_{G^2} (f(.+x)-f(.))D_{2^k}(x^2)r_k(x^1)K_j^w(\tau_k(x^1))d\mu(x) \right\|_p$$

$$\leq c\|K_j^w \circ \tau_k\|_1 \omega_p(2^{-k},f) \leq c\omega_p(2^{-k},f), \tag{3.5}$$

where $|j| \leq k$ (we used Lemma 2.3, too). This yields

$$A_{n,6}, A_{n,7} \leq \sum_{l=0}^{A-1}\sum_{j=0}^{2^l-2} |p_{2^l+j}-p_{2^l+j+1}| j B_j^l$$

$$\leq c\sum_{l=0}^{A-1}\omega_p(2^{-l},f)\sum_{j=0}^{2^l-2}|p_{2^l+j}-p_{2^l+j+1}|j.$$

If $\{p_k\}$ is nondecreasing ($p_k \uparrow$), then we have

$$\sum_{j=0}^{2^l-2}|p_{2^l+j}-p_{2^l+j+1}|j \leq 2^l p_{2^{l+1}-1}. \tag{3.6}$$

If $\{p_k\}$ is nonincreasing ($p_k \downarrow$), then we have

$$\sum_{j=0}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j \leq \sum_{j=1}^{2^l-2} p_{2^l+j} \leq 2^l p_{2^l}. \tag{3.7}$$

By the inequality (3.5) we get again

$$A_{n,8}, A_{n,9} \leq \sum_{l=0}^{A-1} p_{2^{l+1}-1}(2^l-1)B^l_{2^l-1} \leq c \sum_{l=0}^{A-1} p_{2^{l+1}-1} 2^l \omega_p(2^{-l}, f).$$

The inequality (2.6) completes these cases on $A_{n,6} - A_{n,9}$ for nondecreasing sequence $\{p_k\}$.

The method of the discussion (3.5), the measurability property of $R^w_n \circ \tau_A$ (with $|n| = A$) and Lemma 2.5 imply that

$$A_{n,3}, A_{n,4} \leq cP_n \|R^w_n \circ \tau_A\|_1 \omega_p(2^{-A}, f) \leq cP_n \omega_p(2^{-A}, f).$$

In the paper [15] it was showed that

$$\begin{aligned} F^k_j &:= \left\| \int_{G^2} (f(.+x) - f(.)) r_k(x^1) r_k(x^2) \mathcal{K}^w_j(\tau_k(x^1), \tau_k(x^2)) d\mu(x) \right\|_p \\ &\leq c \|\mathcal{K}^w_j \circ (\tau_k \times \tau_k)\|_1 \omega^p_{1,2}(2^{-k}, 2^{-k}, f) \leq c\omega_p(2^{-k}, f) \end{aligned} \tag{3.8}$$

for $|j| \leq k$ (we used Lemma 3.3, too). By this inequality

$$A_{n,11} \leq \sum_{l=0}^{A-1} p_{2^{l+1}-1}(2^l-1)F^l_{2^l-1} \leq c \sum_{l=0}^{A-1} p_{2^{l+1}-1} 2^l \omega_p(2^{-l}, f)$$

and

$$A_{n,10} \leq \sum_{l=0}^{A-1} \sum_{j=0}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j F^l_j \leq c \sum_{l=0}^{A-1} \omega_p(2^{-l}, f) \sum_{j=0}^{2^l-2} |p_{2^l+j} - p_{2^l+j+1}| j.$$

The inequalities (3.6) and (3.7) complete the discussion of this case (see the inequality (2.6) for nondecreasing sequence $\{p_k\}$, too).

At last, we have to discuss $A_{n,5}$. The method of the discussion (3.8), the measurability property of $\mathcal{R}^w_n \circ (\tau_A \times \tau_A)$ ($|n| = A$) and the inequality (3.1) gives that

$$A_{n,5} \leq cP_n \|\mathcal{R}^w_n \circ (\tau_A \times \tau_A)\|_1 \omega^p_{1,2}(2^{-A}, 2^{-A}, f) \leq cP_n \omega_p(2^{-A}, f).$$

These facts complete the proof of Theorem 1. □

At last, we note that the analogue of Theorem 2 and of corollaries of Theorem 2 could be stated for Walsh-Kaczmarz weighted means of Marcinkiewicz type, too.

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New kernels of Sard type and applications

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Abstract. New useful kernels of Sard type are derived by applying a linear functional to the two-point Taylor polynomial expansion, for some real functions possessing a sufficient number of derivatives. These kernels are used to calculate error representations for some cubature and linear approximation formulae. In all considered examples kernels are of one sign, thus accurate error estimates are obtained.

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1 Introduction

Peano's theorem is a well-known result that concerns the representation of a linear functional in terms of a differential operator. Sard ([9]) generalized the theorem to the multivariate case by applying a bounded linear functional to an appropriate Taylor expansion. By using Sard's kernel theorem it is possible to express and calculate sharp bounds for approximation errors. The errors are expressed in terms of the partial derivatives of the underlying function. Particularly, Sard kernels are used ([2], [7], [11]) to bound cubature errors for certain cubature rules.

In [4] the authors applied a linear functional to the expansion in Bernoulli polynomials of a real function and obtained kernels which are different from Sard's for a wide class of linear functionals.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In this paper we consider the expansion in bivariate two-point Taylor polynomials for real functions possessing a sufficient number of derivatives and we obtain new kernels of Sard type. We restrict our attention to functions of only two variables; the extension to dimensions greater then 2 is straightforward.

We use these kernels in order to derive error representations for some cubature and approximation formulae. In all considered examples kernels are of one sign, thus accurate error estimates are obtained.

The paper is organized as follows: in section 2 we quote some remarks on the two-point Taylor formula in one and two dimensions. In section 3 we introduce the new kernels by applying a particular functional to the expansion of some real functions. In the last section we present some applications.

2 Remarks on the two-point Taylor formula

In this section we shall give some known results about the two-point Taylor formula for univariate functions, which is sometimes referred to as the piecewise Hermite interpolation.

For n a positive integer, let $P^{2n-1}([a,b])$ be the set of all real polynomials of degree at most $2n-1$, defined on $[a,b]$.

Theorem 2.1 *Given $f \in C^{2n}([a,b])$, the following relation holds ([8])*

$$f(x) = f_n(x) + R_n(f;x) \tag{2.1}$$

where

$$f_n(x) = \sum_{j=0}^{n-1} \left[C_{n,j}(x) f^{(j)}(a) + D_{n,j}(x) f^{(j)}(b) \right], \qquad f_n \in P^n([a,b]), \tag{2.2}$$

$$\begin{aligned} C_{n,j}(x) &= \frac{(b-x)^n}{j!} \sum_{k=0}^{n-j-1} \binom{n+k-1}{k} \frac{(x-a)^{k+j}}{(b-a)^{n+k}}, \\ D_{n,j}(x) &= (-1)^j C_{n,j}(a+b-x) \end{aligned} \tag{2.3}$$

and

$$R_n(f;x) = (x-a)^n (b-x)^n \frac{f^{(2n)}(\xi_x)}{(2n)!} \qquad \xi_x \in (a,b). \tag{2.4}$$

Remark 2.2 *f_n is the unique polynomial which satisfies the interpolation conditions*

$$f_n^{(j)}(a) = f^{(j)}(a), \qquad f_n^{(j)}(b) = f^{(j)}(b), \qquad j = 0, \dots, n-1.$$

¿From (2.4) an upper bound for the remainder $R_n(f;x)$ and its derivatives can be obtained, in terms of $\|f^{(2n)}\|_\infty = \max\limits_{a\le x\le b} |f^{(2n)}(x)|$ ([12]):

Theorem 2.3 *Let $f \in C^{2n}([a,b])$. Then*

$$|R_n^{(k)}(f;x)| \le \frac{[(x-a)(b-x)]^{n-k}}{k!(2n-2k)!}\|f^{(2n)}\|_\infty \qquad k = 0, \dots, 2n-1.$$

This bound is best possible for $k = 0$ only.

In [3] the authors determined sharp bounds for derivatives of interpolation errors for general piecewise Hermite interpolation in intervals and in rectangular polygons.

For the remainder (2.4) we can also have an integral representation. First we give some preliminary definitions.

For any positive integer n let $C_{2n}^*([a,b])$ be the space of linear functionals of $C^{2n}([a,b])$ which are finite linear combinations of functionals of the form

$$L(f) = \sum_{i=0}^{n-1} \int_a^b a_i(x) f^{(i)}(x)dx + \sum_{i=0}^{n-1} b_i f^{(i)}(x_i)$$

where $a_i(x)$ are piecewise continuous functions on $[a,b]$, $b_i \in \mathbb{R}$ and $x_i \in [a,b]$ are fixed points, such that $L(p) = 0$ for all $p \in P^{2n-1}([a,b])$.

Now we observe that for each $x \in (a,b)$ the error

$$L(f) = f(x) - f_n(x),$$

is a functional in $C_{2n}^*([a,b])$. Thus from Peano's Lemma ([5]) we have

Proposition 2.4 . *For all f in $C^{2n}([a,b])$*

$$L(f) = R_n(f;x) = \int_a^b f^{(2n)}(s) K_n(x,s)ds \tag{2.5}$$

with

$$K_n(\cdot, s) = \frac{1}{(2n-1)!} L\left[(\cdot - s)_+^{2n-1}\right].$$

By simple calculations we can derive an explicit expression for the kernels $K_n(x,s)$

$$K_n(x,s) = \frac{(x-s)_+^{2n-1}}{(2n-1)!} - \sum_{j=0}^{n-1} \frac{D_{n,j}(x)}{(2n-j-1)!}(b-s)^{2n-j-1}. \tag{2.6}$$

The generalization to the two dimensional case is straightforward.

2.1 The two dimensional case

Let T be the rectangle $[a,b]\times[c,d]$. For n, m positive integers, let $P^{(2n-1,2m-1)}(T$ be the set of all real polynomials in the variables x and y of degree at most $2n-1$ in x and $2m-1$ in y, defined on T.

We define $C^{(n-1,m-1)}(T)$ as the space of functions $f : T \to \mathbb{R}$ with continuous partial derivatives

$$f^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y), \qquad (x,y) \in T$$

for all (i,j) such that $0 \le i \le n-1$ and $0 \le j \le m-1$.

Let

$$\overline{C}_{m,j}(y) = \frac{(d-y)^m}{j!} \sum_{k=0}^{m-j-1} \binom{m+k-1}{k} \frac{(y-c)^{k+j}}{(d-c)^{m+k}}$$

and

$$\overline{D}_{m,j}(y) = (-1)^j \overline{C}_{m,j}(c+d-y).$$

Observe that $\overline{C}_{m,j}(y)$ and $\overline{D}_{m,j}(y)$ are polynomials of degree $2m-1$ in y with $c \le y \le d$, while $C_{n,j}(x)$ and $D_{n,j}(x)$ defined in (2.3) are polynomials of degree $2n-1$ in x with $a \le x \le b$.

Then the following theorem holds.

Theorem 2.5 *Let $f \in C^{(2n,2m)}(T)$, $n,m \ge 1$. Then, for each $(x,y) \in T$,*

$$f(x,y) = f_{n,m}(x,y) + R_{n,m}(f;(x,y)) \tag{2.7}$$

where

$$f_{n,m}(x,y) =$$

$$= \sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\Big[C_{n,j}(x)\overline{C}_{m,k}(y)f^{(j,k)}(a,c) + C_{n,j}(x)\overline{D}_{m,k}(y)f^{(j,k)}(a,d) +$$

$$+D_{n,j}(x)\overline{C}_{m,k}(y)f^{(j,k)}(b,c) + D_{n,j}(x)\overline{D}_{m,k}(y)f^{(j,k)}(b,d)\Big] \tag{2.8}$$

and

$$R_{n,m}\left(f;(x,y)\right) =$$

$$= \sum_{k=0}^{m-1}\int_a^b \left[\overline{C}_{m,k}(y)f^{(2n,k)}(s,c) + \overline{D}_{m,k}(y)f^{(2n,k)}(s,d)\right] K_n(x,s)ds +$$

$$+\sum_{j=0}^{n-1}\int_c^d \left[C_{n,j}(x)f^{(j,2m)}(a,t) + D_{n,j}(x)f^{(j,2m)}(b,t)\right] \overline{K}_m(y,t)dt +$$

$$+\int_a^b\int_c^d f^{(2n,2m)}(s,t)K_n(x,s)\overline{K}_m(y,t)dsdt\ . \tag{2.9}$$

$K_n(x,s)$ is the Peano kernel defined in (2.6) and

$$\overline{K}_m(y,t) = \frac{(y-t)_+^{2m-1}}{(2m-1)!} - \sum_{j=0}^{m-1}\frac{\overline{D}_{m,j}(y)}{(2m-j-1)!}(d-t)^{2m-j-1}.$$

Proof. In order to obtain (2.8) and (2.9), we apply (2.2)-(2.3) to the function $x \mapsto f(x,y)$ in the variable x

$$f(x,y) = \sum_{j=0}^{n-1}\Big[C_{n,j}(x)f^{(j,0)}(a,y) + D_{n,j}(x)f^{(j,0)}(b,y)\Big] + \int_a^b f^{(2n)}(s,y)K_n(x,s)ds. \tag{2.10}$$

Then we apply (2.2)-(2.3) to $f^{(j,0)}(a,y)$, $f^{(j,0)}(b,y)$ and to $f^{(2n)}(s,y)$ in the variable y and replace in (2.10). □

Remark 2.6 *The approximation formula (2.7) has degree of exactness $(2n-1, 2m-1)$, i.e.*

$$R_{n,m}\left(p;(x,y)\right) = 0 \qquad \text{for each } p \in P^{(2n-1,2m-1)}(T).$$

Remark 2.7 *$f_{n,m}$ satisfies the interpolation conditions*

$$f_{n,m}^{(i,j)}(\overline{x},\overline{y}) = f^{(i,j)}(\overline{x},\overline{y}) \qquad i=0,\dots,n-1,\ j=0,\dots,m-1\,, \tag{2.11}$$

where $(\overline{x},\overline{y})$ is one of the vertices of T.

The interpolate $f_{n,m}$ is determined by the $4nm$ linear conditions (2.11) and it is known ([1]) that this interpolate is unique.

Proposition 2.8 *The remainder (2.9) can be expressed in the compact form*

$$\begin{aligned} R_{nm}(f;(x,y)) \;&= \int_a^b f^{(2n,0)}(s,y)K_n(x,s)ds + \int_c^d f^{(0,2m)}(x,t)H_m(y,t)dt - \\ &- \int_a^b \int_c^d f^{(2n,2m)}(s,t)K_n(x,s)H_m(y,t)dsdt. \end{aligned} \tag{2.12}$$

Proof. From (2.2) it follows that

$$\begin{aligned} \sum_{k=0}^{m-1} \left[\overline{C}_{m,k}(y)f^{(2n,k)}(s,c) + \overline{D}_{m,k}(y)f^{(2n,k)}(s,d)\right] \;&= \\ &= f^{(2n,0)}(s,y) - R_m\left(f^{(2n,0)}(s,\cdot);y\right), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \sum_{j=0}^{n-1} \left[C_{n,j}(x)f^{(j,2m)}(a,t) + D_{n,j}(x)f^{(j,2m)}(b,t)\right] \;&= \\ &= f^{(0,2m)}(x,t) - R_n\left(f^{(0,2m)}(\cdot,t);x\right). \end{aligned} \tag{2.14}$$

Thus form (2.12) of the remainder is obtained by replacing in (2.9) the (2.13)-(2.14) and by observing that

$$R_m\left(f^{(2n,0)}(s,\cdot);y\right) = \int_c^d f^{(2n,2m)}(s,t)H_m(y,t)dt$$

and

$$R_n\left(f^{(0,2m)}(\cdot,t);x\right) = \int_a^b f^{(2n,2m)}(s,t)K_n(x,s)ds. \qquad \square$$

3 New Sard type kernels

Let us define by $C^*_{2n,2m}(T)$ the space of functionals on $C^{(2n,2m)}(T)$ which are finite linear combinations of functionals of the form

$$\begin{aligned} L(f) = & \sum_{i=0}^{2n}\sum_{j=0}^{2m}\int_a^b\int_c^d a_{ij}(x,y)f^{(i,j)}(x,y)dxdy+ \\ & +\sum_{i=0}^{2n}\sum_{j=0}^{2m}\int_a^b b_i(x)f^{(i,j)}(x,y_i)\,dx+ \\ & +\sum_{i=0}^{2n}\sum_{j=0}^{2m}\int_c^d c_j(y)f^{(i,j)}(x_j,y)\,dy+\sum_{i=0}^{2n}\sum_{j=0}^{2m}d_{ij}f^{(i,j)}(x_i,y_j) \end{aligned} \tag{3.1}$$

where $a_{ij}(x,y), b_i(x), c_j(y)$ are piecewise continuous functions respectively on $T, [a,b], [c,d]$, $x_i \in [a,b], y_j \in [c,d]$. $(x_i, y_j) \in T$ are fixed points, such that $L(p) = 0$ for all $p \in P^{(n,m)}(T)$. Hence a Sard's extension of Peano's kernel theorem is applicable.

Theorem 3.1 (Sard's type kernel Theorem). *If $L \in C^*_{2n,2m}(T)$, then, for all $f \in C^{(2n,2m)}(T)$*

$$\begin{aligned} L(f) &= L\left(R_{n,m}(f;(x,y))\right) = \\ &= \sum_{k=0}^{m-1}\left\{\int_a^b f^{(2n,k)}(s,c)K^C_{nk}(s)ds+\int_a^b f^{(2n,k)}(s,d)K^D_{nk}(s)ds\right\}+ \\ &\quad +\sum_{j=0}^{n-1}\left\{\int_c^d f^{(j,2m)}(a,t)K^C_{jm}(t)dt+\int_c^d f^{(j,2m)}(b,t)K^D_{jm}(t)dt\right\}+ \\ &\quad +\int_a^b\int_c^d f^{(2n,2m)}(s,t)K_{nm}(s,t)dsdt \end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
K^C_{nk}(s) &= L_{xy}\left[\overline{C}_{m,k}(y)K_n(x,s)\right] = L_{xy}\left[\overline{C}_{m,k}(y)\tfrac{(x-s)_+^{2n-1}}{(2n-1)!}\right]\\
K^D_{nk}(s) &= L_{xy}\left[\overline{D}_{m,k}(y)K_n(x,s)\right] = L_{xy}\left[\overline{D}_{m,k}(y)\tfrac{(x-s)_+^{2n-1}}{(2n-1)!}\right]\\
K^C_{jm}(t) &= L_{xy}\left[C_{n,j}(x)H_m(y,t)\right] = L_{xy}\left[C_{n,j}(x)\tfrac{(y-t)_+^{2m-1}}{(2m-1)!}\right]\\
K^D_{jm}(t) &= L_{xy}\left[D_{n,j}(x)H_m(y,t)\right] = L_{xy}\left[D_{n,j}(x)\tfrac{(y-t)_+^{2m-1}}{(2m-1)!}\right]\\
K_{nm}(s,t) &= L_{xy}\left[K_n(x,s)H_m(y,t)\right].
\end{aligned}
\tag{3.3}
$$

The notation $L_{xy}[\cdot]$ means that the functional L is applied to its argument considered as a function of (x,y).

Proof. The result follows by applying the functional L to both sides of expansion (2.7) with remainder (2.9). Note that we are working under the hypothesis which allow an interchange of the functional with the integral. □

In analogy with the well known Sard's kernel theorem ([6], [9]) we call the functions (3.3) two-point Taylor kernels of the functional L.

Remark 3.2 *In order to obtain an error bound in terms of the sup-norm we put*

$$M_{i,j} = \sup_{(x,y)\in T} |f^{(i,j)}(x,y)|$$

and we have

$$
\begin{aligned}
|L(f)| \quad \leq & \sum_{k=0}^{m-1} M_{2n,k}\int_a^b \left(|K^C_{nk}(s)| + |K^D_{nk}(s)|\right) ds\\
& + \sum_{j=0}^{n-1} M_{j,2m}\int_c^d \left(|K^C_{jm}(t)| + |K^D_{jm}(t)|\right) dt\\
& + M_{2n,2m}\int_a^b\int_c^d |K_{nm}(s,t)|dsdt.
\end{aligned}
$$

Remark 3.3 *If all kernels of the remainder $L(f)$ are of one sign, by the general mean-value theorem, there exist points (μ_k,c), (ν_k,d), (a,ξ_j), (b,η_j),*

$(s_1, t_1) \in T$ *such that*

$$\begin{aligned} L(f) &= \sum_{k=0}^{m-1} \left\{ f^{(2n,k)}(\mu_k, c) \int_a^b K_{nk}^C(s)ds + f^{(2n,k)}(\nu_k, d) \int_a^b K_{nk}^D(s)ds \right\} + \\ &+ \sum_{j=0}^{n-1} \left\{ f^{(j,2m)}(a, \xi_j) \int_c^d K_{jm}^C(t)dt + f^{(j,2m)}(b, \eta_j) \int_c^d K_{jm}^D(t)dt \right\} + \\ &+ f^{(2n,2m)}(s_1, t_1) \int_a^b \int_c^d K_{nm}(s,t)dsdt \end{aligned}$$

where μ_k, ν_k, s_1 *lie between* a *and* b *and* ξ_j, η_j, t_1 *lie between* c *and* d.

4 Application

Now we calculate the error representation for some cubature and approximation formulae. In each of the examined cases the remainder is a functional of the form (3.1) and we can apply Theorem 5.

Example 4.1 (Cubature). *Let's consider the functional*

$$E(f) = \int_0^1 \int_0^1 f(x,y)dxdy - f\left(\frac{1}{2}, \frac{1}{2}\right) \tag{4.1}$$

for $f \in C^{(2,2)}([0,1] \times [0,1])$.

Since the formula has degree of exactness $(1,1)$*, its error representation (3.2) takes the following form*

$$\begin{aligned} E(f) &= \int_0^1 \left[f^{(2,0)}(s,0)K_{10}^C(s) + f^{(2,0)}(s,1)K_{10}^D(s) \right] ds + \\ &+ \int_0^1 \left[f^{(0,2)}(0,t)K_{01}^C(t) + f^{(0,2)}(1,t)K_{01}^D(t) \right] dt + \\ &+ \int_0^1 \int_0^1 f^{(2,2)}(s,t)K_{11}(s,t)dsdt. \end{aligned} \tag{4.2}$$

¿From (3.3), after some calculations, we have

$$K_{10}^C(s) = K_{10}^D(s) = \begin{cases} \frac{1}{4}s^2 & 0 \le s \le \frac{1}{2} \\ \frac{1}{4}(1-s)^2 & otherwise \end{cases}$$

$$K_{01}^C(t) = K_{01}^D(t) = \begin{cases} \frac{1}{4}t^2 & 0 \le t \le \frac{1}{2} \\ \frac{1}{4}(1-t)^2 & otherwise \end{cases}$$

$$K_{11}(s,t) = \begin{cases} \frac{1}{4}st(st-s-t) & 0 \le s,t \le \frac{1}{2} \\ \frac{1}{4}s(1-t)(t-st-1) & 0 \le s \le \frac{1}{2}, \frac{1}{2} \le t \le 1 \\ \frac{1}{4}t(1-s)(s-st-1) & 0 \le t \le \frac{1}{2}, \frac{1}{2} \le s \le 1 \\ \frac{1}{4}(1-s)(1-t)(st-1) & \frac{1}{2} \le s,t \le 1 \end{cases} \tag{4.3}$$

We can easily see that $K_{11}(s,t) \le 0$*, while the other kernels are* ≥ 0*. As they have constant sign, there exist points* $(\xi_1, 0)$*,* $(\xi_2, 1)$*,* $(0, \eta_1)$*,* $(1, \eta_2)$*,* (s_1, t_1) *such that the error representation is*

$$E(f) = \frac{1}{48}\left[f^{(2,0)}(\xi_1,0) + f^{(2,0)}(\xi_2,1) + f^{(0,2)}(0,\eta_1) + f^{(0,2)}(1,\eta_2)\right] dt - \frac{5}{576} f^{(2,2)}(s_1,t_1).$$

Figure (1) shows the graphic representation of the kernel (4.3).

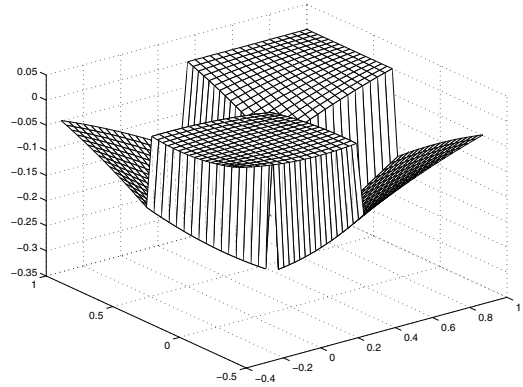


Figure 1: Kernel K_{11}.

In [6] an integral representation of the remainder (4.1) has been computed by means of Sard kernels. But in that case the $K_{11}(s,t)$ *Sard kernel*

$$K_{11}(s,t) = \begin{cases} st-s-t & 0 \le s,t \le \frac{1}{2} \\ (1-s)(1-t) & otherwise \end{cases}$$

is not of one sign, and the error is

$$E(f) = \frac{1}{24}\left[f^{(2,0)}(\xi_1,0) + f^{(0,2)}(0,\eta_1)\right] + \int_0^1\int_0^1 f^{(1,1)}(s,t)K_{11}(s,t)dsdt.$$

In [4] the error is given only in terms of partial derivatives of order one, but also in this case kernels change sign.

Example 4.2 (Cubature of degree (3,3)). *We calculate the error representation of the functional*

$$E(f) = \int_0^1\int_0^1 f(x,y)dxdy - \left[\frac{1}{6}f\left(1,\frac{1}{2}\right) + \frac{1}{6}f\left(0,\frac{1}{2}\right) + \frac{1}{3}f\left(\frac{1}{2},u_1\right) + \frac{1}{3}f\left(\frac{1}{2},u_2\right)\right]$$

where

$$u_1 = \frac{1}{2}\left(\sqrt{\frac{1}{2}}+1\right), \qquad u_2 = \frac{1}{2}\left(1-\sqrt{\frac{1}{2}}\right)$$

and $f \in C^{(4,4)}([0,1]\times[0,1])$.

Since the formula has degree of exactness $(3,3)$*, its error representation (3.2) is*

$$\begin{aligned} E(f) \;&= \int_0^1 \left[f^{(4,0)}(s,0)K_{20}^C(s) + f^{(4,0)}(s,1)K_{20}^D(s)\right]ds + \\ &+ \int_0^1 \left[f^{(4,1)}(s,0)K_{21}^C(s) + f^{(4,1)}(s,1)K_{21}^D(s)\right]ds + \\ &+ \int_0^1 \left[f^{(0,4)}(0,t)K_{02}^C(t) + f^{(0,4)}(1,t)K_{02}^D(t)\right]dt + \\ &+ \int_0^1 \left[f^{(1,4)}(0,t)K_{12}^C(t) + f^{(1,4)}(1,t)K_{12}^D(t)\right]dt + \\ &+ \int_0^1\int_0^1 f^{(4,4)}(s,t)K_{22}(s,t)dsdt \end{aligned} \tag{4.4}$$

with

$$K_{20}^C(s) = K_{20}^D(s) = \begin{cases} \dfrac{1}{144}s^3(3s-2) & 0 \le s \le \frac{1}{2} \\ \dfrac{1}{144}(s-1)^3(3s-1) & \textit{otherwise} \end{cases}$$

$$K_{21}^C(s) = K_{21}^D(s) = \begin{cases} \dfrac{s^2(1-2s)^2}{1152} & 0 \le s \le \frac{1}{2} \\ \dfrac{\left(1-3s+2s^2\right)^2}{1152} & \textit{otherwise}\,. \end{cases}$$

We can easily see that $K_{20}^C(s) = K_{20}^D(s) \leq 0$, $K_{21}^C(s) = K_{21}^D(s) \geq 0$. *If we compute the other kernels we have that all of them are non-negative:*

$$K_{02}^C(t) = K_{02}^D(t) \geq 0 \qquad K_{12}^C(t) = K_{12}^D(t) \geq 0, \qquad K_{22}(s,t) \geq 0.$$

Not all classical Sard kernels are of one sign. For example

$$K_{11}(u) = \frac{1}{6}(1-u)^3 - \begin{cases} \frac{1}{12}(1-u)^2 + \frac{1}{3}\left(\frac{1}{2}-u\right)^2 & 0 \leq u < \frac{1}{2} \\ \frac{1}{12}(1-u)^2 & \frac{1}{2} \leq u \leq 1\,. \end{cases}$$

Example 4.3 (Lagrange interpolation). *We calculate the error representation of the functional*

$$L(f) = f(x,y) - \sum_{i=0}^{n}\sum_{j=0}^{m} l_i^n(x) l_j^m(y) f\left(x_i, y_j\right)$$

where $l_i^n(x)$ *and* $l_j^m(y)$ *are the fundamental Lagrange polynomials,* $0 \leq x, y \leq 1$.

For $n = m = 1$ *we have the representation (4.2) with*

$$K_{10}^C(s) = (1-y)\left[(x-s)_+ - x(1-s)\right], \quad K_{10}^D(s) = y\left[(x-s)_+ - x(1-s)\right],$$

$$K_{01}^C(t) = (1-x)\left[(y-t)_+ - y(1-t)\right], \quad K_{10}^D(t) = x\left[(y-t)_+ - y(1-t)\right].$$

All these kernels are non-positive, while

$$K_{11}(s,t) = \left[(x-s)_+ - x(1-s)\right]\left[(y-t)_+ - y(1-t)\right]$$

is non-negative.

Also in this case not all Sard kernels are of constant sign. For example the Sard kernel $K_{10}(s)$ *is*

$$K_{10}(s) = \begin{cases} x - s - xy(1-s) & 0 \leq s \leq x \\ xy(s-1) & x \leq s \leq 1. \end{cases}$$

Example 4.4 (Bernstein approximation). *Let's consider the functional*

$$L(f) = f(x,y) - \sum_{i=0}^{n}\sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f\left(\frac{i}{n}, \frac{j}{m}\right)$$

where

$$p_{v,r}(t) = \binom{v}{r} t^r (1-t)^{v-r}.$$

Since the formula has degree of exactness $(1,1)$*, its error representation takes the form (4.2), where*

$$K_{10}^{C}(s) = L_{xy}\left[\overline{C}_{1,0}(y)(x-s)_+\right] = (1-y)\left[(x-s)_+ - \sum_{i=0}^{n} p_{n,i}(x)\left(\frac{i}{n}-s\right)_+\right].$$

Similarly we can obtain explicit expressions for $K_{10}^{D}(s)$*,* $K_{01}^{C}(t)$ *and* $K_{01}^{D}(t)$*.*

$$K_{11}(s,t) = (x-s)_+(y-t)_+ - \sum_{i=0}^{n} p_{n,i}(x)\left(\frac{i}{n}-s\right)_+ \sum_{j=0}^{m} p_{m,j}(y)\left(\frac{j}{m}-t\right)_+ .$$

They are all non-positive.

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On Q-like properties in Complete and Advertibly Complete Algebras

Lourdes Palacios

Abstract.

In the frame of commutative, unital, m-convex algebras, W. Zelazko gave some equivalent properties with Q-algebras, assuming completeness. Previously, S. Warner had related similar properties with the Q-algebras that were advertibly complete. In this survey paper, we examine the properties concerning complete Q-algebras and advertibly complete algebras.

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1 Introduction

A Q-algebra is a topological algebra such that the group of its invertible elements is an open set. Q-algebras mantain many of the fundamental properties of Banach algebras and in several cases these properties characterize what Kaplanski [5] called in 1947 the Q-property. This was done in the frame of topological rings. Q-algebras are not complete, but in some cases they behave as if they were.

Every Banach algebra is a Q-algebra but this is not the case, in general, for normed algebras. Generally speaking, not every normed algebra is a

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Q-algebra. We can consider, for example, the algebra A of all polynomials with complex coeficients on the complex unit disc, with the pointwise algebraic operations and the sup norm $\|\cdot\|_\infty$. Then $(A, \|\cdot\|_\infty)$ is a normed algebra which is not a Q-algebra.

Q-algebras are involved, for example, in differential geometry [8] and [10].

Advertibly complete algebras were introduced in [12] by S. Warner in 1956. He considered this notion in the context of the theory of locally m-convex algebras. He realized that "locally m-convex algebras possessing advertibly completeness, posses also most of the fundamental properties of Banach algebras". A Hausdorff locally m-convex algebra which is either complete or a Q-algebra, is advertibly complete. There do exist, however, locally m-convex algebras which are advertibly complete but are neither complete, nor Q-algebras.

Recently, there has been attention given to locally m-convex Q-algebras, particularly to Fréchet algebras. In this respect, W. Zelazko [13], [14], also got some remarkable relations between them, ideals and Q -algebras.

In this survey article, we will start from a well known theorem given by Zelazko [13] (1976) and another one by Warner [12] (1956) and examine some properties concerning complete Q-algebras and advertibly complete algebras.

2 Preliminaries

A **topological algebra** over a field F (where $F = \mathbb{R}$ or $F = \mathbb{C}$) is a topological linear space with an associative jointly continuous multiplication that makes A an algebra over F.

A **semitopological algebra** over a field F (where F=$\mathbb{R}$ or F=$\mathbb{C}$) is a topological linear space with an associative separately continuous multiplication that makes A an algebra over F.

These two concepts coincide when A is an m-convex algebra (see below.)

A **locally convex algebra** is a topological algebra A which is a locally convex space. In this case its topology can be given by a family $\{\|\cdot\|_\alpha : \alpha \in \Lambda\}$ of seminorms satisfying that for each $\alpha \in \Lambda$ there exists β such that

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta \tag{1}$$

for all x, $y \in A$.

A locally convex algebra is said to be **multiplicatively convex** (shortly

m-convex) if every seminorm is submultiplicative i.e. (1) can be replaced by

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha \tag{2}$$

for each $\alpha \in \Lambda$ and all $x, y \in A$.

If $(A, \{\|\cdot\|_\alpha : \alpha \in \Lambda\})$ is an m-convex algebra, then for each $\alpha \in \Lambda$ we denote by A_α the completion of the normed algebra $A/\ker\|\cdot\|_\alpha$ and by π_α the canonical homomorphism from A to A_α given by the composition $A \twoheadrightarrow A/\ker\|\cdot\|_\alpha \hookrightarrow A_\alpha$.

An **F-algebra** is a topological algebra which is complete and metrizable.

If A is a topological algebra with unit e, the set of all the invertible elements in A is denoted by $G(A)$. If $G(A)$ is an open set, then A is **a Q-algebra**.

Let A be a topological algebra with unit e. A net $(a_\lambda)_{\lambda\in\Lambda}$ is **right (resp. left) advertibly convergent** with respect to $a \in A$ if $aa_\lambda \to e$ (resp. $a_\lambda a \to e$). In this case, the net is a **right (resp. left) topological inverse** of a.

An element of A is **topologically invertible** if it has right and left topological inverses. The set of all topologically invertible elements in A is denoted by $G_t(A)$. It is clear that $G(A) \subseteq G_t(A)$ always holds.

If a right (resp. left) topological inverse of $a \in A$ converges, then its limit is the right (resp. left) inverse of a. Thus, an element is invertible in A if it has convergent right and left topological inverses in A.

A net $(a_\lambda)_{\lambda\in\Lambda}$ is **advertibly convergent** (shortly **advertible**) if it is right and left advertibly convergent with respect to some $a \in A$. The topological algebra A is **advertibly complete** if every advertible and Cauchy net in A converges in A.

For $x \in A$, the set $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin G(A)\}$ is **the spectrum of** x in A.

The **spectral radius** is defined as $\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$ if $\sigma(x) \neq \emptyset$ and $\rho(x) = 0$ otherwise.

We denote by $\mathcal{M}(A)$ the space of all non-zero continuous multiplicative linear functionals on A, endowed with the weak*-topology.

For any $x \in A$, $\widehat{x}$ stands for its **Gelfand transform**: $\widehat{x}(f) = f(x)$ for all $f \in \mathcal{M}(A)$. Recall that $\widehat{x}(\mathcal{M}(A)) \subseteq \sigma(x)$ for all $x \in A$.

For general definitions and results about Topological Algebras, the reader is referred to [16], [3], [6], [9] and [7].

3 Related Properties

Many theorems on Banach algebras have generalizations to locally convex algebras which are not necessarilly complete, but have some weaker completeness condition. In this survey we will state a Theorem by W. Zelazko, 1976 [13], and see what happens when we weaken or drop some of the hypothesis, mainly completeness.

W. Zelazko gives a characterization of commutative, complete, unital, m -convex algebras in which all maximal ideals are of codimension one.

Theorem 3.1 (Zelazko) *Let A be a complete, complex, unital, m-convex algebra. Then the following conditions are equivalent:*

1. *Every maximal ideal of A is of codimension 1.*
2. *For each element $x \in A$ the spectrum $\sigma(x)$ is bounded.*
3. *For each element $x \in A$ the spectrum $\sigma(x)$ is compact.*
4. *A is a complete m-convex Q-algebra under a topology stronger than the original one.*
5. *A is a complete m-convex Q-algebra under some topology.*
6. *The space $\mathcal{M}^{\#}(A)$ of all non-zero multiplicative-linear functionals on A is compact in the weak star topology.*

Corollary 3.2 *The algebra A possesses a dense maximal ideal of infinite codimension if and only if it possesses an element with unbounded spectrum.*

For barreled algebras we have a stronger version of the theorem, also due to Zelazko in the same paper.

Theorem 3.3 *Let A be a commutative, complex, complete, m-convex algebra with unit e, which is a barreled space. Then the following conditions are equivalent:*

1. *Every maximal ideal of A is of codimension one.*
2. *Every maximal ideal of A is closed.*
3. *The spectrum $\sigma(x)$ of each element $x \in A$ is bounded.*
4. *The spectrum $\sigma(x)$ of each element $x \in A$ is compact.*
5. *A is a Q-algebra.*
6. *The space $\mathcal{M}(A)$ is compact.*

The notion for an algebra to be advertibly complete weakens the situation in Zelazko's Theorem and yields similar results. This theorem appeared in 1956 in a paper by Warner [12].

Theorem 3.4 (Warner) *Let A be a commutative, locally m-convex algebra with unit e. Then the following conditions are equivalent:*

1. *For any $x \in A$, if $f(x) \neq 1$ for all $f \in \mathcal{M}(A)$, then x is invertible, and $\mathcal{M}(A)$ is equicontinuous.*
2. *A is a Q-algebra.*
3. *A is advertibly complete and $\mathcal{M}(A)$ is equicontinuous.*
4. *Every maximal ideal of A is closed and $\mathcal{M}(A)$ is equicontinuous.*

We present an example of a complex, commutative, metrizable m-convex algebra with unit e, in which all maximal ideals are closed but it is not a Q-algebra. This example is due to Choukri [4].

Example 3.5 *Let A be the algebra of all complex rational functions $r(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are complex polynomials and $q(n) \neq 0$ for each natural number n.*

We consider the family of submultiplicative seminorms $\{p_n\}_{n\in\mathbb{N}}$ defined by:

$$p_n(r(z)) = |r(n)|,\ r(z) \in A \text{ and } n \in \mathbb{N}.$$

A is an m-convex, metrizable algebra where all its ideals are principal; in fact, its maximal ideals are precisely $(z-n)A$ with $n \in \mathbb{N}$. These maximal ideals are kernels of seminorms, so they are closed.

But $\sigma(z) = \mathbb{N}$, which is not a bounded set; therefore A is not a Q-algebra.

If the algebra is not necessarily m-convex, the most general situation concerning maximal ideals is the following:

Theorem 3.6 (Mallios) *Let A be a commutative semi-topological Q-algebra with unit. Then each maximal ideal of A is closed.*

It can be easily proved that

Theorem 3.7 *Let A be a commutative topological algebra with unit. If each maximal ideal of A is closed, then $G_t(A) = G(A)$.*

We note earlier in this survey that $\widehat{x}(\mathcal{M}(A)) \subseteq \sigma(x)$ for any algebra A and $x \in A$. For some algebras, e.g. commutative, complex, Banach algebras, we have that $\widehat{x}(\mathcal{M}(A)) = \sigma(x)$. Arizmendi and Valov [2] considered the following general condition:

(m_2) $\qquad \rho(x) = \sup\{|\lambda| \mid \lambda \in \widehat{x}(\mathcal{M}(A))\}$ for every $x \in A$.

So we have the next:

Theorem 3.8 *Let A be a commutative algebra with unit and suppose that A satisfies one of the following conditions:*

1. $\sigma(x) = \widehat{x}(\mathcal{M}(A))$
2. $\rho(x) = \sup\limits_{f \in \mathcal{M}(A)} (|f(x)|)$ $\qquad (m_2)$

If $\sigma(x)$ is bounded for each $x \in A$, then A is a Q-algebra for some topology τ.

There has been much attention given to *lmc*-Q-algebras, particularly to F-algebras; in this respect we have the following:

Theorem 3.9 *Let A be a commutative unital F-algebra (complete metrizable). If each maximal ideal is closed, then A is a Q-algebra (and the inversion $x \longmapsto x^{-1}$ is continuous).*

Theorem 3.10 *Let A be a commutative unital F-algebra. If the inversion $x \longmapsto x^{-1}$ is not continuous, then $G(A) \subsetneq G_t(A)$.*

Theorem 3.11 (Zelazko) *If A is a commutative F-algebra with a discontinuous inversion, then A has a proper dense principal ideal.*

Ideal Theory is one of the main chapters of the Theory of Topological Algebras; it is natural to ask: under which conditions all maximal ideals (left, right or two-sided) are closed? W. Zelazko deals with this question for maximal left (or right) ideals in unital F-algebras (generally speaking, such an algebra may have dense ideals).

Theorem 3.12 (Zelazko) *Let A be a unital F-algebra. Then the following conditions are equivalent:*

1. *All maximal left ideals in A are closed.*
2. *The set $G_l(A)$ of all its left invertible elements is open.*

3. *The set $G(A)$ of all its invertible elements is open, that is, A is a Q-algebra.*
4. *The set $G_r(A)$ of all its right invertible elements is open.*
5. *All maximal right ideals in A are closed.*

In particular, as a corollary of the last theorem we have:

Corollary 3.13 *Let A be commutative unital F-algebra; then A is a Q-algebra if and only if each maximal ideal of A is closed.*

The continuity of the inversion plays an important role: we present an example of an algebra that is a Q-algebra with a closed maximal ideal of infinite codimension:

Example 3.14 *Let us consider the algebra $\mathbb{C}(z)$ of all complex rational functions $\frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are complex polynomials and $q(z) \neq 0$.*

$\mathbb{C}(z)$ is a division ring, in fact, it is a field. We endow it with the maximal locally convex topology $\tau^{lc}_{\max}$; i.e, the topology given by means of all seminorms.

Then $(\mathbb{C}(z), \tau^{lc}_{\max})$ is a complete locally convex Q -algebra. This is a semitopological algebra where the inversion $x \longmapsto x^{-1}$ is not continuous. However, in this algebra, $\{0\}$, the zero ideal, is a closed maximal ideal of infinite codimension.

By definition, Q-algebras are not complete, but they are advertibly complete and the two concepts are equivalent on normed algebras. One of the classical properties of m-convex algebras is the following:

Theorem 3.15 *Let A be an m-convex algebra with unit e; then the following conditions are equivalent:*

a) A is advertibly complete.

b) $a \in A$ is invertible if and only if $\pi_\alpha(a)$ is invertible in A_α for each $\alpha \in \Lambda$.

Theorem 3.16 *Let A be a commutative, m-convex algebra with unit e. Then the following conditions are equivalent:*

a) A is advertibly complete

b) x is an invertible element in A iff $f(x) \neq 0$ for each $f \in \mathcal{M}(A)$.

Sumarizing the previous discussion, we have the following:

Properties. 1. If A is complete, then it is advertibly complete
2. Every Q-algebra is advertibly complete
3. If all its maximal ideals are closed, then A is advertibly complete

Nevertheless, these classes of algebras are far to cover the class of advertibly complete algebras. In fact, we have the following:

Theorem 3.17 (Arizmendi et al) *Let A be a topological algebra with unit e. Then properties a) to d) below satisfy a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d).*

a) A is a Q-algebra.

b) Every maximal bilateral ideal of A is closed.

c) $G_t(A) = G(A)$.

d) A is advertibly complete.

Moreover, in [12] it is proved the following:

Theorem 3.18 *Let A be a normed, unital, algebra. Then properties a) to d) are equivalent.*

Remark 3.19 *Concerning Theorem 3.17, we have:*

1. *b) $\not\Rightarrow$ a) in general; Example 5 given above shows that there exists topological algebras that are not Q-algebras in which each maximal ideal is closed.*
2. *c) $\not\Rightarrow$ b) in general. If A is a complete, m-convex algebra with unit e, condition c) always holds. On the other hand, let $\mathcal{E}$ be the algebra of all entire complex functions endowed with the compact-open topology. $\mathcal{E}$ satisfies c) but it has a dense maximal ideal of infinite codimension (see Corollary 3.2).*
3. *d) $\not\Rightarrow$ a) in general, as it is shown in the next:*

Example 3.20 *Let E be the algebra of all continuous function from $\mathbb{C}$ into $\mathbb{C}$ vanishing at infinity, equipped with the compact-open topology. E is an m-convex algebra that is advertibly complete but it is neither a complete, nor a Q-algebra.*

But we have the following:

Theorem 3.21 *Let A be a commutative, m-convex algebra with unit e. Then the following conditions are equivalent:*

a) A is advertibly complete.

b) $G_t(A) = G(A)$.

c) $\sigma(x) = \widehat{x}(\mathcal{M}(A))$ for each $x \in A$.

Moreover, each one of these equivalent conditions implies condition (m_2) defined above.

Remark If we drop the hypothesis of commutativity in the above theorem, conditions a) and b) remain equivalent; moreover, each one of the conditions a) to c), although not necessarily equivalent, still implies condition (m_2).

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On the maximal value of Dirichlet kernels with respect to representative product systems

Rodolfo Toledo

Abstract. Dirichlet kernels play a prominent role in the study of Fourier series. For Walsh and Vilenkin systems the maximal value of the n-th Dirichlet kernel is exactly n. This situation is very different if we generalize these systems on the complete product of arbitrary groups, not necessarily commutative groups.

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Representative product systems are systems given by the finite products of normalized coordinate functions considering the complete product of arbitrary groups, not necessarily commutative groups. These systems are ordered as the order introduced by Paley for the Walsh system.

In Section 1 we introduce basic concepts in the study of representative product systems and Fourier analysis. We also give some examples of these systems showing the properties of them. Representative product systems were discussed in [1], [2], [6] and [7]. Section 2 contains the results for the maximal value of the n-th Dirichlet kernel, denoted by D_n. We prove that

$$n \leq D_n \leq M_{A+1},$$

if $M_A \leq n \leq M_{A+1}$. Then we study the condition by which $D_n = n$ holds, like in the commutative cases, and also we find the first number n such that $D_n = M_{A+1}$ holds.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Throughout this work denote by $\mathbf{N}$, $\mathbf{P}$, $\mathbf{C}$ the set of nonnegative, positive integers and complex numbers, respectively. In order to simplicity we always use the multiplication to denote the group operation and use the symbol e to denote the identity of the groups. We use the notation of Hewitt and Ross [3].

1 Representative product systems

Let $m := (m_k,\ k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k, $(k \in \mathbf{N})$. Suppose that each group has discrete topology and normalized Haar measure μ_k. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ are sequences $x := (x_0, x_1, ...)$, where $x_k \in G_k$, $(k \in \mathbf{N})$. We call this sequence the *expansion* of x. The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the expansion of n with respect to m. We often use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ and

$$n_{(k)} := \sum_{j=0}^{k-1} n_k M_k, \qquad n^{(k)} := \sum_{j=k}^{\infty} n_k M_k.$$

Denote by Σ_k the dual object of the finite group G_k $(k \in \mathbf{N})$. Thus each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$. Let d_σ be the dimension of its representation space and let $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \zeta_i, \zeta_j \rangle \qquad (i, j \in \{1, \dots, d_\sigma\},\ x \in G_k)$$

are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$. In this manner for each $\sigma \in \Sigma_k$ we obtain d_σ^2 number of coordinate functions, in total m_k number of functions for the whole dual object of G_k. The L^2-norm of these functions is $1/\sqrt{d_\sigma}$.

Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be a system of all *normalized coordinate functions* of the group G_k. Now, we do not decide the order of the system φ, only suppose that φ_k^0 is always the character 1. Thus for every $0 \leq s < m_k$

there exists a $\sigma \in \Sigma_k$, $i,j \in \{1,...,d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. Thus we say that ψ is the *representative product system* of φ. The Weyl-Peter's theorem (see [3]) gives that the system ψ is orthonormal and complete on $L^2(G)$.

The functions ψ_n $(n \in \mathbf{N})$ are not necessary uniformly bounded, so define

$$\Psi_k := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbf{N}).$$

Ψ_k is the multiplication of the greatest product of L^∞-norm and L^1-norm of the functions φ_j appeared in all group G_{m_j} for $0 \le j < k$. Namely

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

It seems that the boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

For an integrable complex function f defined in G we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f\overline{\psi}_k \, d\mu \quad (k \in \mathbf{N}), \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbf{P}).$$

According to the theorem of Banach-Steinhauss, $S_n f \to f$ as $n \to \infty$ in L^p norm for all $f \in L^p(G)$ if and only if there exists a $C_p > 0$ such that

$$\|S_n f\|_p \le C_p \|f\|_p, \qquad (f \in L^p(G)).$$

Thus, we say that the operator S_n is of *type* (p,p). Since system ψ forms an orthonormal base in the Hilbert space $L^2(G)$, it is obvious that S_n is of type $(2,2)$.

The *Dirichlet kernels* are defined as follows:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{P}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x,y) d\mu(y). \tag{1.1}$$

This shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series. The following lemmas were proved in [2].

Lemma 1.1 *If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_n(x,y) = \sum_{k=0}^{\infty} D_{M_k}(x,y) \left(\sum_{s=0}^{n_k - 1} \varphi_k^s(x) \overline{\varphi}_k^s(y) \right) \psi_{n^{(k+1)}}(x) \overline{\psi}_{n^{(k+1)}}(y),$$

where $(n_0, n_1, ...)$ is the expansion of n and $x = (x_0, x_1, ...)$, $y = (y_0, y_1, ...)$.

Lemma 1.2 (Paley lemma) *If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

The Paley lemma is used to prove that the $S_{M_n} f$ partial sequence of Fourier sums converge to f in L^p-norm and a.e., if $f \in L^p(G)$, $p \geq 1$. Indeed, the

$$S_{M_n} f(x) = \int_G f(y) D_{M_n}(x,y) d\mu(y) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu$$

operator is the conditional expectation operator with respect to the σ-algebra generated by the sets $I_n(x)$, $x \in G$. Thus, according to the martingale convergence theorem, for each $f \in L^p(G)$, $p \geq 1$, $S_{M_n} f$ converge to f in L^p-norm, and a.e.

The representative product systems are the generalization of the well-known Walsh-Paley and Vilenkin systems. Indeed, we obtain the Walsh-Paley system if $m_k = 2$ and $G_k := \mathcal{Z}_2$, the cyclic group of order 2 for all $k \in \mathbf{N}$. The characters of $\mathcal{Z}_2$ are the Rademacher functions:

$$\varphi^s(x) = (-1)^{sx} \qquad (s \in \{0,1\},\ x \in \mathcal{Z}_2).$$

Moreover, we obtain the Vilenkin systems if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k := \mathcal{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbf{N}$. The characters of $\mathcal{Z}_{m_k}$ are the generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi \imath s x / m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1).$$

Now, we consider the complete product of $\mathcal{S}_3$ where $\mathcal{S}_3$ is the symmetric group on 3 elements. Thus $m_k = 6$ for all $k \in \mathbf{N}$. $\mathcal{S}_3$ has two characters and a 2-dimensional representation. The values of the system φ obtained from the 2-dimensional representation depend on the chosen basis. Table 1 contains the values of a possible system φ.

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Table 1: The system φ for $\mathcal{S}_3$

Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. This facts encumber the study of this systems. On the other hand, $\max_{0\le s<6} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$, thus $\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$ if $k \to \infty$.

Let $m_k = 8$ for all $k \in \mathbf{N}$ and $\mathcal{Q}_2$ be the the *quaternion group* of order 8, i.e.

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}.$$

Let $G_k = \mathcal{Q}_2$ for all $k \in \mathbf{N}$. $\mathcal{Q}_2$ has four characters and a 2-dimensional representation ($8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$). Table 2 contains the values of a possible system φ.

Notice that $|\varphi^s|$ can only take the values 0 or the square root of the corresponding dimension because the representation is monomial. Hence, $\Psi_k = 1$ for $k \in \mathbf{N}$, but the group G is not abelian.

Finally, we write the values of a possible system φ for the alternating group $\mathcal{U}_4$. These values appear in Table 3, where $\alpha = \exp(2\pi\imath/3)$. We use this system in the results of Section 2.

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

Table 2: The system φ for $\mathcal{Q}_2$

2 The maximal value of Dirichlet kernels

Define by

$$D_n := \sup_{x,\, y\in G} |D_n(x,y)| \qquad (n \in \mathbf{P})$$

the maximal value of the Dirichlet kernel D_n. Using the inequality of Cauchy-Bunyakovszkij we have

$$\left|\sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y)\right|^2 \le \sum_{k=0}^{n-1} |\psi_k(x)|^2 \sum_{k=0}^{n-1} |\psi_k(y)|^2$$
$$\le \max{}^2 \left\{\sum_{k=0}^{n-1} |\psi_k(x)|^2, \sum_{k=0}^{n-1} |\psi_k(y)|^2\right\} \qquad (x,\, y \in G).$$

Moreover,

$$D_n \ge \sup_{x\in G} |D_n(x,x)| = \sup_{x\in G} \sum_{k=0}^{n-1} |\psi_k(y)|^2.$$

Consequently,

$$D_n = \sup_{x\in G} D_n(x,x) = \sup_{x\in G} \sum_{k=0}^{n-1} |\psi_k(x)|^2 \qquad (n \in \mathbf{P}) \tag{2.1}$$

	e	(12)(34)	(13)(24)	(14)(23)	(123)	(142)	(134)	(234)	(124)	(132)	(234)	(143)
φ^0	1	1	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	α	α	α	α	α^2	α^2	α^2	α^2
φ^2	1	1	1	1	α^2	α^2	α^2	α^2	α	α	α	α
φ^3	$\sqrt{3}$	$-\sqrt{3}$	0	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^4	$\sqrt{3}$	$-\sqrt{3}$	0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
φ^5	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	0	0	0	0	0	0	0	0
φ^6	0	0	$\sqrt{3}$	$-\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^7	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^8	0	0	0	0	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^9	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^{10}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^{11}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$

Table 3: The system φ for $\mathcal{U}_4$

from which we have that D_n is monotone increasing sequence.

For the commutative case $D_n = n$ for all $n \in \mathbf{P}$, but the general case is a bit more different.

Theorem 2.1 *If $n \in \mathbf{P}$ and $A := \max\{k \in \mathbf{N} : n_k \neq 0\}$, then*

$$n \leq D_n \leq M_{A+1}.$$

Proof. Let G_k, $k \in \mathbf{N}$, be an arbitrary finite group appeared in the product to obtain G. The unitary property of the representations implies (see [6])

$$\sum_{s=0}^{j} |\varphi_k^s(x_k)|^2 \leq m_k \qquad (j < m_k,\, x_k \in G_k). \tag{2.2}$$

Moreover, let $n \in \mathbf{N}$, x, $y \in G$. Thus, we have

$$D_n(x,y) = D_{M_A}(x,y)\left(\sum_{s=0}^{n_A-1} \varphi_A^s(x_A)\overline{\varphi}_A^s(y_A)\right) + \varphi_A^{n_A}(x_A)\overline{\varphi}_A^{n_A}(y_A)D_{n_{(A)}}(x,y).$$

By Paley lemma and (2.1) we obtain

$$D_n(x,x) = M_A\left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2\right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x,x). \tag{2.3}$$

Observe, $n = \sum_{k=0}^{A} n_k M_k$ and $n_{(A)} = \sum_{k=0}^{A-1} n_k M_k$. By induction on A we prove $D_n(x,x) \le M_{A+1}$ for all $x \in G$. Indeed, by (2.2)

$$D_{n_{(1)}}(x,x) = \sum_{s=0}^{n_0-1} |\varphi_0^s(x_0)|^2 \le m_0 = M_1$$

and supposing $D_{n_{(A)}}(x,x) \le M_A$, by (2.2) and (2.3) we have

$$\begin{aligned} D_{n_{(A+1)}}(x,x) = D_n(x,x) &\le M_A\left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2\right) + |\varphi_A^{n_A}(x_A)|^2 M_A \\ &= M_A\left(\sum_{s=0}^{n_A} |\varphi_A^s(x_A)|^2\right) \le M_A m_A = M_{A+1}, \end{aligned}$$

from which the inequality $D_n \le M_{|n|+1}$ follows.

On the other hand, by the orthonormality of the system ψ we have

$$\int_G \sum_{k=0}^{n-1} |\psi_k|^2 \, d\mu = \sum_{k=0}^{n-1} \int_G |\psi_k|^2 \, d\mu = n$$

and by the $\mu(G) = 1$ property

$$\int_G \sum_{k=0}^{n-1} |\psi_k|^2 \, d\mu \le \sup_{x \in G} \sum_{k=0}^{n-1} |\psi_k|^2 = D_n \tag{2.4}$$

from which we obtain the property $D_n \ge n$. This completes the proof of the lemma. □

Figure 1 illustrates the statements of Theorem 2.1 with respect to the system φ on $\mathcal{S}_3$ appeared in Table 1.

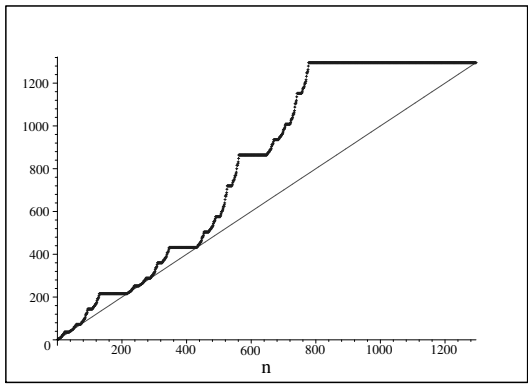


Figure 1: D_n $(n \leq 6^4)$ on the complete product of $\mathcal{S}_3$

Notice that $D_n = n$ for some values of n. Let $n \in \mathbf{P}$ with expansion $(n_0, n_1, \dots)$. Denote the indexes A and B by

$$A := \max\{k \in \mathbf{N} : n_k \neq 0\}, \quad \text{and} \quad B := \min\{k \in \mathbf{N} : n_k \neq 0, k \leq A\}.$$

Thus $B = A$ if the expansion of n has only one nonzero coordinate. By Lemma 1.1 it is not hard to see that $D_n = n$ when n satisfies the following two properties

(i) $\sum_{s=0}^{n_B-1} |\varphi_B^s(x_B)|^2 = n_B$ for all $x_B \in G_B$,

(ii) $B = A$ or all of $\varphi_i^{n_i}$ are characters such that $\sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 = n_i$ for all $B < i \leq A$ and $x_i \in G_i$.

Indeed, if the above properties are true we have

$$D_n(x,x) = \sum_{i=B}^{A} M_i \left(\sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 \right) |\psi_{n^{(i+1)}}(x)|^2 = \sum_{i=B}^{A} M_i n_i = n$$

for all $x \in G$. However, property (ii) is too strong. To show this fact we consider the complete product of finite groups where G_0 is the dyadic group and G_1 is the alternating group $\mathcal{U}_4$ with the system appeared in Table 3 but in different order, i.e. $\{\varphi^0, \varphi^8, \varphi^5, \varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^6, \varphi^7, \varphi^9, \varphi^{10}, \varphi^{11}\}$. The rest of finite groups in the product will be arbitrary. Thus $D_5 = 5$, but property (ii) is not true because $5 = (1, 2, 0, 0, \dots)$ and $\varphi_1^2 = \varphi^5$ is not a character.

In order to obtain necessary and sufficient conditions we generalize the property (ii). Denote by $d_i^{(s)}$ the dimension of the representation corresponded to the normalized coordinate function φ_i^s. We say that the positive integer n satisfies the property (ii*) if $B = A$ or there exist $r_i < d_i^{(n_i)}$ nonnegative integers such that

$$(d_i^{(n_i)} - 1)n_{(i)} = r_i M_i \tag{2.5}$$

and

$$n_i - \sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 = \begin{cases} 0 & \text{if } d_i^{(n_i)} = 1 \\ \frac{r_i}{d_i^{(n_i)}-1}\left(|\varphi_i^{n_i}(x_i)|^2 - 1\right) & \text{if } d_i^{(n_i)} > 1 \end{cases} \tag{2.6}$$

for all $B < i \leq A$ and $x_i \in G_i$. It is clear, if φ_i^s are characters for all $k < i \leq A$, then property (ii) follows from property (ii*).

Lemma 2.2 *Let $n \in \mathbf{P}$. $D_n = n$ if and only if n satisfies properties (i) and (ii*).*

Proof. By (2.4) $D_n = n$ is possible if and only if

$$\sum_{k=0}^{n-1} |\psi_k(x)|^2 = n$$

for all $x \in G$. Thus by (2.3) the sum

$$D_n(x,x) = M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x,x)$$

does not depend on the value of x_A and x. Since there is an $x_A \in G_A$ such that $\varphi_A^{n_A}(x_A) \neq 0$, the value of $D_{n_{(A)}}(x,x)$ does not depend on the value of x, that is

$$D_{n_{(A)}}(x,x) = n_{(A)} \tag{2.7}$$

for all $x \in G$. Since $n = M_A n_A + n_{(A)}$ we also obtain

$$M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 n_{(A)} = M_A n_A + n_{(A)} \tag{2.8}$$

for all $x_A \in G_A$. If $B = A$ (in this case $n_{(A)} = 0$) or $\varphi_A^{n_A}$ is a character, then (2.8) is reduced to

$$\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 = n_A \tag{2.9}$$

for all $x_A \in G_A$. Otherwise, denoting by $e = (e_1, e_2, \dots)$ the identity of G we have

$$\sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2 \le n_A \quad \text{and} \quad |\varphi_A^{n_A}(e_A)|^2 = d_A^{(n_A)}. \tag{2.10}$$

Indeed, the values of $|\varphi_A^s(e_A)|^2$ are integers, so if $\sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2 \ge n_A+1$ then $D_n(e,e) \ge M_A(n_A+1) > n$, but $D_n(e,e) = n$. On the other hand, if $|\varphi_A^{n_A}(e_A)|^2 \ne d_A^{(n_A)}$ then $|\varphi_A^{n_A}(e_A)|^2 = 0$. Thus we substitute x_A by e_A in (2.8) to obtain $\sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2 > n_A$ which is not possible.
Denote

$$r_A := n_A - \sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2.$$

Thus r_A is a nonnegative integer. After a simple computation in (2.8) for $x_A = e_A$, we have

$$(d_A^{(n_A)} - 1)n_{(A)} = r_A M_A \tag{2.11}$$

from which we have that $r_A = 0$ if and only if $d_A^{(n_A)} = 1$, and also for $r_A > 0$ we have that $r_A < d_A^{(n_A)} - 1$ and $d_A^{(n_A)} > 2$. In the last case we calculate the value of $n_{(A)}$ in (2.11) and we replace it in (2.8) to obtain

$$\frac{r_A}{d_A^{(n_A)} - 1} \left(|\varphi_A^{n_A}(x_A)|^2 - 1\right) = n_A - \sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \tag{2.12}$$

for all $x_A \in G_A$.
Summarizing the above results we obtain that $D_n = n$ is possible if and only if $B = A$ and (2.9) is true or $B < A$, (2.7) is true and

$$(d_i^{(n_A)} - 1)n_{(A)} = r_A M_A$$

and

$$n_A - \sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 = \begin{cases} 0 & \text{if } d_A^{(n_A)} = 1 \\ \frac{r_A}{d_A^{(n_A)}-1} \left(|\varphi_A^{n_A}(x_A)|^2 - 1\right) & \text{if } d_A^{(n_A)} > 1 \end{cases}$$

for all $x_A \in G_A$. By (2.7) we use a similar method to the applied in the proof of Theorem 2.1, so repeating the above method for k from A down to B we prove Lemma 2.2. □

Under some conditions of the group G the properties (i) and (ii) will be enough. Indeed, by 2.11 we obtain that $d_A^{(n_A)} \neq 2$, because if $B < A$ then $0 < n_{(A)} < r_A M_A$. For this reason we can state the following theorem.

Theorem 2.3 *Let G the complete product of finite groups where the dimensions of the representations appeared in the finite groups do not exceed the value 2. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).*

In some cases the values of the systems φ only permit (2.8) if the properties (ii) is valid. Suppose that the square of the modulus of systems φ can only take integer values, for instance when the finite groups are all monomials (see Table 2). In this case we obtain a similar statement.

Theorem 2.4 *Let G the complete product of finite groups where the square of the modulus of all of systems φ can only take integer values. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).*

Proof. If the square of the modulus of systems φ can only take integer values, then we obtain

$$\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \leq n_A$$

for all $x_A \in G_A$ in a similar way to prove (2.10) in the proof of Lemma 2.2. Thus the orthonormality of the systems φ only allow the equality (2.9) for all $x_A \in G_A$ and by (2.8) we obtain

$$|\varphi_A^{n_A}(x_A)|^2 n_{(A)} = n_{(A)}.$$

for all $x_A \in G_A$. Then $B = A$, so $n_{(A)} = 0$ or $B < A$ and $|\varphi_A^{n_A}(x_A)| = 1$ for all $x_A \in G_A$, so $\varphi_A^{n_A}$ is a character. This prove that the property (ii) is necessary and this complete the proof of the theorem. □

In another cases the order of the systems φ is determinant. Suppose we order the systems φ such that the positive values of the identity are at the beginning of the systems. Thus it is easy to see that $D_n = D_n(e, e)$, so we have our statement from the fact that the values of $|\varphi_A^s(e_A)|^2$ are integers. Furthermore, if the characters are at the beginning of all of the systems φ,

like the examples in Table 1, 2 and 3, the equation (2.9) is only possible if and only if all of the functions $\varphi_i^{n_i}$ are characters for all $i \in \mathbf{N}$. Thus we obtain the following statement.

Theorem 2.5 *Suppose we order all of the systems φ such that the positive values of the identity are at the beginning of the systems. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).*
Furthermore, suppose also that the characters are at the beginning of all of the systems φ. Then $D_n = n$ if and only if all of the functions $\varphi_i^{n_i}$ are characters for all $i \in \mathbf{N}$.

On the other hand, it is interesting to study the upper estimation in the inequality of Theorem 2.1, so we found all of positive integers n for which $D_n = M_{A+1}$ holds. In this regard suppose there is a finite group G_k such that

$$\varphi_k^{m_k-1}(x_k) = 0, \text{ for some } x_k \in G_k. \tag{2.13}$$

It is meaning that the value 0 appears at least one time at the last row of the table containing the values of the system φ_k. In fact property (2.13) only depend on the last member of the system φ_k^s, $0 \le s < m_k$, but G_k can not be a commutative group (characters can not take the value 0). Several finite groups G_k can have property (2.13), so denote by N the smallest index of them, that is G_N is the first group having property (2.13).

Notice that we can not always found the index N. For instance we can not obtain it in the commutative cases when we only have characters or the order all systems φ does not allow property (2.13) for any group G_k, $k \in \mathbf{N}$. Although for all $k \in \mathbf{N}$ we can find the first number $r \in \{1, 2, \dots, m_k - 1\}$ such that there exists an $x_k \in G_k$ for which

$$\sum_{s=0}^{r} |\varphi_k^s(x_k)|^2 = m_k \quad \text{and} \quad \varphi_k^r(x_k) \neq 0$$

and denote this number by k^*.

We only obtain an equality in the upper estimation of Theorem 2.1 in case of the existence of the index N.

Theorem 2.6 *Let $A \in \mathbf{N}$ and $M_A \le n < M_{A+1}$. If the index N exists then $D_n = M_{A+1}$ if and only if*

$$\sum_{k=N}^{A} k^* M_k < n < M_{A+1}$$

If the index N does not exist then $D_n < M_{A+1}$.

Proof. Let $A \in \mathbf{N}$ and $M_A \leq n < M_{A+1}$. Then by (2.3) $D_n = M_{A+1}$ if and only if there exists an $x \in G$ such that

$$M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x,x) = M_{A+1} = m_A M_A.$$

Let $k = A$ and write the above equation in the following form.

$$|\varphi_k^{n_k}(x_k)|^2 D_{n_{(k)}}(x,x) = \left(m_k - \sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2 \right) M_k. \tag{2.14}$$

The both sides in (2.14) only can be 0 if $\sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2 = m_k$, so $\varphi_k^s(x_k) = 0$ for all $s \geq n_k$. Otherwise from the

$$|\varphi_k^{n_k}(x_k)|^2 \leq m_k - \sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2 \quad \text{and} \quad D_{n_{(k)}}(x,x) \leq M_k$$

inequalities we obtain that (2.14) is only possible if there exists an $x \in G$ such that

$$\sum_{s=0}^{n_k} |\varphi_k^s(x_k)|^2 = m_k \quad \text{and} \quad D_{n_{(k)}}(x,x) = M_k. \tag{2.15}$$

Thus by the definition of k^* if $n_k > k^*$ then (2.14) holds, if $n_k < k^*$ then (2.14) does not hold and if $n_k = k^*$ then (2.14) holds if and only if $D_{n_{(k)}}(x,x) = M_k$. In the last case we have to repeat the above idea for $k = A - 1$ obtaining the same conclusion for n_{k-1} and so on.
If $A < N$ or the index N does not exist then $k^* = m_k - 1$ for all $0 \leq k \leq A$, so we can not find an index k such that $n_k > k^*$ and then (2.14) holds. Thus $D_n < M_{A+1}$.
If $N \leq A$ we have to stop the process for $k = N$, because $D_{n_{(k)}}(x,x) = M_k$ does not hold for $k < N$. Thus $n_k = k^*$ for all $N < k \leq A$. If we choose $n_N = N^* + 1$ we obtain that

$$1 + \sum_{k=N}^{A} k^* M_k$$

is the smaller number for which (2.14) holds. This completes the proof of the theorem. □

The study of the quotients $\frac{D_n}{n}$ is important in order to estimate the Dirichlet kernels. This quotients are 1 in the commutative cases, but they

can be unbounded if the dimensions of the representations appeared in the finite groups are also unbounded. To show this property we order the normalized coordinate functions φ of the finite groups such that let $\varphi_k^0 \equiv 1$ and the representation corresponding to φ_k^1 has the greatest dimension of all representations of the group G_k and denoting this dimension by d_k we also suppose that $\varphi_k^1(e_k) = \sqrt{d_k}$. Thus, it is easy to calculate that

$$\frac{D_n}{n} = \frac{1+d_k}{2} \qquad \text{if } n = 2M_k$$

from which we have that under this order the quotients $\dfrac{D_n}{n}$ are not bounded for all $n \in \mathbf{P}$ if the dimensions d_k are not bounded. Figure 2 illustrates that the order of the systems φ play an important role. The new order here is $\{\varphi^0, \varphi^1, \varphi^4, \varphi^2, \varphi^5, \varphi^3\}$. of Table 1. Compare it with Figure 1.

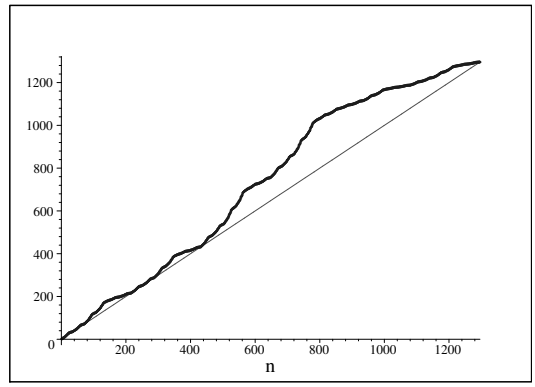


Figure 2: D_n $(n \leq 6^4)$ on the complete product of $\mathcal{S}_3$ using another order of the system φ.

The following lemma states that the boundedness of the quotients $\frac{D_n}{n}$ is given by the boundedness of this quotients on the finite groups.

Lemma 2.7 *Let G the complete product of finite groups G_k. The quotients $\dfrac{D_n}{n}$ are bounded for all $n \in \mathbf{P}$ if and only if the quotients*

$$\frac{\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2}{r} \tag{2.16}$$

are bounded for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$.

Proof. Let $n = rM_k$. By (2.3) we have

$$\frac{D_n}{n} = \sup_{x\in G_k} \frac{M_k\left(\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2\right)}{rM_k} = \sup_{x\in G_k} \frac{\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2}{r}.$$

Thus if the quotients $\dfrac{D_n}{n}$ are bounded then the quotients (2.16) are bounded for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$.
To prove the inverse statement let $n = \sum_{k=0}^{A} n_k M_k$. By Lemma 1.1 and (2.3) there exist an $x \in G$ such that

$$D_n = M_A\left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2\right) + |\varphi_A^{n_A}(x_A)|^2 M_{A-1}\left(\sum_{s=0}^{n_{A-1}-1} |\varphi_{A-1}^s(x_{A-1})|^2\right)$$
$$+ |\varphi_A^{n_A}(x_A)|^2 |\varphi_{A-1}^{n_{A-1}}(x_{A-1})|^2 M_{A-1}\left(\sum_{s=0}^{n_{A-2}-1} |\varphi_{A-2}^s(x_{A-2})|^2\right) + \dots$$

Let c a positive number for which (2.16) is less than c for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$. Thus

$$\frac{D_n}{n} \leq \frac{D_n}{n_A M_A} = \frac{\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2}{n_A} + \frac{|\varphi_A^{n_A}(x_A)|^2\left(\sum_{s=0}^{n_{A-1}-1} |\varphi_{A-1}^s(x_{A-1})|^2\right)}{n_A m_{A-1}}$$
$$+ \frac{|\varphi_A^{n_A}(x_A)|^2 |\varphi_{A-1}^{n_{A-1}}(x_{A-1})|^2\left(\sum_{s=0}^{n_{A-2}-1} |\varphi_{A-2}^s(x_{A-2})|^2\right)}{n_A m_{A-1} m_{A-2}} + \dots$$

From (2.16) we obtain that $\frac{|\varphi_A^{n_A}(x_A)|^2}{n_A} \leq 2c$. Indeed, $\frac{\sum_{s=0}^{n_A} |\varphi_A^s(x_A)|^2}{n_A+1} \leq c$ follows $\frac{|\varphi_A^{n_A}(x_A)|^2}{n_A+1} \leq c$. Thus $\frac{|\varphi_A^{n_A}(x_A)|^2}{n_A} \leq \frac{n_A+1}{n_A} c \leq 2c$. On the other hand

$$\frac{\sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2}{m_k} \leq 1 \quad \text{and} \quad \frac{|\varphi_k^{n_k}(x_k)|^2}{m_k} \leq \frac{1}{2}.$$

Thus

$$\frac{D_n}{n} \leq c + 2c + 2c\frac{1}{2} + 2c\frac{1}{4} + \dots < 5c$$

from which we have that the quotients $\dfrac{D_n}{n}$ are bounded for all $n \in \mathbf{P}$. This completes the proof of the lemma. □

Finally, notice that if the group G is bounded or the dimensions of the representations appeared in the finite groups are uniformly bounded

then the conditions of Lemma 2.7 hold. Otherwise we can try to order the systems φ putting the representations with big dimensions at the end of the systems φ.

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Locally convex quasi C*-algebras: basic aspects and commutative case

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Abstract. A locally convex quasi C*-algebra is obtained as the completion of a C*-algebra $\mathfrak{A}_0[\|\cdot\|_0]$ with respect to a locally convex topology τ on $\mathfrak{A}_0$, coarser than the norm topology but coupled in suitable way with the initial norm topology. In this paper, after a short review of the main features of this structure, some new results on the possibility of representing a commutative locally convex quasi C*-algebra as a locally convex quasi C*-algebra of functions is presented.

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1 Introduction and Preliminaries

The completion of a C*-algebra $\mathfrak{A}_0[\|\cdot\|_0]$ under a locally convex topology τ coarser than the topology induced by the C*-norm, is not in general, an algebra, if the multiplication is only separately continuous with respect to τ. This fact leads to the notions of *locally convex quasi *-algebra* or locally convex *quasi C*-algebra*, if some additional conditions, pairing τ more strictly with the initial norm, are satisfied. This structure, as shown in [3, 4], behaves quite reasonably, in the sense that several results that hold for C*-algebras have natural extensions to them. In particular, it has been shown that if $\mathfrak{A}_0[\|\cdot\|_0]$ possesses sufficiently many τ-continuous

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positive invariant sesquilinear forms (or equivalently, sufficiently many *-representations) then it can be continuously embedded in a locally convex quasi C*-algebra of operators. One would expect that, under similar conditions, a commutative locally convex quasi C*-algebra could be embedded in a locally convex quasi C*-algebra of functions. This problem was also considered in [3, 4] but the result we got is not complete: the Gel'fand transform of $\mathfrak{A}_0$, in fact, cannot be extended to the whole quasi *-algebra but only to its quasi-positive cone. In this paper, after a short overview on locally convex quasi C*-algebras, we examine once more the commutative case and we make some steps further toward a representation into a locally convex quasi C*-algebra of functions, obtaining some new partial result.

Let $\mathfrak{A}_0[\|\cdot\|_0]$ be a C*-algebra, with unit *1*. We shall use the symbol $\|\cdot\|_0$ of the C*-norm to denote the corresponding topology. Suppose that τ is a topology on $\mathfrak{A}_0$ such that $\mathfrak{A}_0[\tau]$ is a locally convex $*$-algebra. Then, the topologies τ, $\|\cdot\|_0$ on $\mathfrak{A}_0$ are *compatible* whenever each Cauchy net in both topologies that converges with respect to one of them, also converges with respect to the other one. The symbol $\widetilde{\mathfrak{A}_0[\tau]}$ denotes the completion of $\mathfrak{A}_0[\tau]$.

A *partial $*$-algebra* is a vector space $\mathfrak{A}$ equipped with a vector space involution $*:\mathfrak{A}\to\mathfrak{A}:x\mapsto x^*$ and a partial multiplication defined on a set $\Gamma\subset\mathfrak{A}\times\mathfrak{A}$ in such a way that:

(i) $(x,y)\in\Gamma$ implies $(y^*,x^*)\in\Gamma$;

(ii) $(x,y_1),(x,y_2)\in\Gamma$ and $\lambda,\mu\in\mathbb{C}$ imply $(x,\lambda y_1+\mu y_2)\in\Gamma$;

(iii) for every $(x,y)\in\Gamma$, a product $xy\in\mathfrak{A}$ is defined, such that xy depends linearly on y and satisfies the equality $(xy)^*=y^*x^*$.

Whenever $(x,y)\in\Gamma$, we say that x is a *left multiplier* of y and y a *right multiplier* of x and we write $x\in L(y)$, respectively $y\in R(x)$.

Quasi $*$-algebras are important examples of partial $*$-algebras.

If $\mathfrak{A}$ is a vector space and $\mathfrak{A}_0$ is a subspace of $\mathfrak{A}$ such that is also a $*$-algebra, then $\mathfrak{A}$ is said to be a *quasi $*$-algebra over* $\mathfrak{A}_0$ whenever the following properties hold:

(i)$'$ The multiplication of $\mathfrak{A}_0$ is extended on $\mathfrak{A}$ as follows: The assignments

$\mathfrak{A}\times\mathfrak{A}_0\to\mathfrak{A}:(a,x)\mapsto ax$ (left multiplication of x by a) and

$\mathfrak{A}_0\times\mathfrak{A}\to\mathfrak{A}:(x,a)\mapsto xa$ (right multiplication of x by a)

are always defined and are bilinear;

(ii)$'$ $x_1(x_2a)=(x_1x_2)a,(ax_1)x_2=a(x_1x_2)$ and $x_1(ax_2)=(x_1a)x_2$, for all $x_1,x_2\in\mathfrak{A}_0$ and $a\in\mathfrak{A}$;

(iii)$'$ the involution $*$ of $\mathfrak{A}_0$ is extended on $\mathfrak{A}$, denoted also by $*$, such that $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$, for all $x \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$.

For further details see [1].

If $\mathfrak{A}_0[\tau]$ is a locally convex $*$-algebra, with separately continuous multiplication, its completion $\widetilde{\mathfrak{A}_0}[\tau]$ is a quasi $*$-algebra over $\mathfrak{A}_0$ under the following operations: Given $x \in \mathfrak{A}_0$ and $a \in \widetilde{\mathfrak{A}_0}[\tau]$

- $ax := \lim\limits_{\alpha} x_\alpha x$ (right multiplication)
- $xa := \lim\limits_{\alpha} x x_\alpha$ (left multiplication)

with $\{x_\alpha\}_{\alpha\in\Delta}$ a net in $\mathfrak{A}_0$ such that $a = \tau\text{-}\lim\limits_{\alpha} x_\alpha$.

- The involution of $\widetilde{\mathfrak{A}_0}[\tau]$, like in (iii)$'$, is the continuous extension of the involution given on $\mathfrak{A}_0$.

A $*$-invariant subspace $\mathfrak{A}$ of $\widetilde{\mathfrak{A}_0}[\tau]$ containing $\mathfrak{A}_0$ is called a *quasi $*$-subalgebra* of $\widetilde{\mathfrak{A}_0}[\tau]$ if ax, xa belong to $\mathfrak{A}$, for any $x \in \mathfrak{A}_0$, $a \in \mathfrak{A}$. Then, one easily shows that $\mathfrak{A}$ is a quasi $*$-algebra over $\mathfrak{A}_0$. Moreover, $\mathfrak{A}[\tau]$ is a locally convex space that contains $\mathfrak{A}_0$ as a dense subspace and for every fixed $x \in \mathfrak{A}_0$, the maps $L_x : \mathfrak{A}[\tau] \to \mathfrak{A}[\tau]$, with $L_x(a) = xa$, and $R_x : \mathfrak{A}[\tau] \to \mathfrak{A}[\tau]$, with $R_x(a) = ax$, are continuous on $\mathfrak{A}$. In this case, $\mathfrak{A}$ is called *locally convex quasi $*$-algebra over* $\mathfrak{A}_0$. In particular, if the topology τ of $\mathfrak{A}$ comes from a norm $\|\cdot\|$, we say that the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is a normed quasi *-algebra if the involution is isometric w. r. to $\|\cdot\|$ i.e., if $\|a^*\| = \|a\|$, $\forall a \in \mathfrak{A}$. We say that $(\mathfrak{A}, \mathfrak{A}_0)$ is a *Banach quasi $*$-algebra* over $\mathfrak{A}_0$ (see [7, 8]) if, in addition, $\mathfrak{A}[\|\cdot\|]$ is a Banach space.

Concerning $*$-representations of partial $*$-algebras and quasi $*$-algebras, start with a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ and denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all linear operators X from $\mathcal{D}$ to $\mathcal{H}$ such that $\mathcal{D}(X^*) \supset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial $*$-algebra with respect to the usual sum, scalar multiplication and involution $X^\dagger = X^*\restriction_{\mathcal{D}}$ and the (weak) partial multiplication $X \Box Y = X^\dagger{}^* Y$, defined whenever X is a left multiplier of Y ($X \in L(Y)$), that is, iff $Y\mathcal{D} \subset \mathcal{D}(X^\dagger{}^*)$ and $X^\dagger\mathcal{D} \subset \mathcal{D}(Y^*)$. A (partial) $*$-subalgebra of the partial $*$-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is said to be a *partial O^*-algebra* on $\mathcal{D}$. A *$*$-representation* of a partial $*$-algebra $\mathfrak{A}$ is a $*$-homomorphism π of $\mathfrak{A}$ into a partial O*-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, in the sense of ([1], Definition 2.1.6), satisfying $\pi(\mathit{1}) = I$, whenever $\mathit{1} \in \mathfrak{A}$.

In this case, the spaces $\mathcal{D}$ and $\mathcal{H}$ will be denoted by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$ respectively.

2 Locally convex quasi C*-algebras: A short survey

In this section we introduce the notion of locally convex quasi C*-algebra and give a short overview of its structure properties, that have been studied at length in [3, 4, 5]. It is worth pointing out that this object is important not only from the mathematical point of view, but also for applications: completions of locally convex *-algebras under a topology weaker than the norm topology are very frequently introduced in the so called algebraic approach to quantum theories [1, Ch.11], [2, 6] .

Let $\mathfrak{A}_0$ be a C*-algebra with norm $\|\cdot\|_0$. Let τ be a locally convex topology on $\mathfrak{A}_0$ defined by a directed family of seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$. We assume that the following hold:

(T_1) $\mathfrak{A}_0[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication;

(T_2) $\tau \preceq \|\cdot\|_0$;

(T_3) For each $\lambda\in\Lambda$, there exists $\lambda'\in\Lambda$ such that

$$p_\lambda(xy)\leq \|x\|_0 p_{\lambda'}(y), \forall\ x,y\in\mathfrak{A}_0 \text{ with } xy=yx;$$

(T_4) The set $\mathcal{U}(\mathfrak{A}_0)_+ := \{x\in(\mathfrak{A}_0)_+ : \|x\|_0\leq 1\}$ is τ-complete.

Definition 2.1. A quasi $*$-subalgebra $\mathfrak{A}$ of the locally convex quasi $*$-algebra $\widetilde{\mathfrak{A}_0}[\tau]$ over $\mathfrak{A}_0$, where $\mathfrak{A}_0[\|\cdot\|_0]$ is a unital C*-algebra and τ a locally convex topology on $\mathfrak{A}_0$ satisfying the conditions (T_1)-(T_4), is said to be a *locally convex quasi C*-algebra over* $\mathfrak{A}_0$.

Example 2.2. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$ Let $\mathcal{M}_0$ be a unital C*-algebra of operators over $\mathcal{H}$ such that $\mathcal{M}_0\mathcal{D}\subset\mathcal{D}$. Then, $\mathcal{M}_0\restriction\mathcal{D}$ is an O^*-algebra on $\mathcal{D}$. To make notations lighter we identify an operator X of $\mathcal{M}_0$ with its restriction $X\restriction\mathcal{D}$ to $\mathcal{D}$.

Let us endow $\mathcal{M}_0$ with the *strong*-topology* τ_{s^*} defined by the seminorms

$$p_\dagger^{\mathcal{M}}(X) := \sup_{\xi\in\mathcal{M}}\{\|X\xi\|+\|X^\dagger\xi\|\},\quad X\in\mathcal{M}_0.$$

where $\mathcal{M}$ is a finite subset of $\mathcal{D}$.

Then the completion $\mathcal{M} := \widetilde{\mathcal{M}_0}^{\tau_{s^*}}$ of $\mathcal{M}_0$, w. r. to the strong*-topology τ_{s^*}, is a locally convex quasi C*-algebra of operators over $\mathcal{M}_0$. The τ_{s^*}-completeness of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ implies that $\widetilde{\mathcal{M}_0}^{\tau_{s^*}}\subset\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

Locally convex quasi C*-algebras enjoy a series of interesting structure properties that make clear their strong closedness with C*-algebras. In order to give the reader a more intense flavor of this structure, we mention the following ones concerning (quasi) positivity.

Definition 2.3. An element a of $\widetilde{\mathfrak{A}_0}[\tau]$ is called *quasi-positive* (resp. *commutatively quasi-positive*) if there is a net (resp. commuting net) $(x_\alpha)_{\alpha\in\Delta}$ of the positive cone $(\mathfrak{A}_0)_+$ of the C*-algebra $\mathfrak{A}_0[\|\cdot\|_0]$, which converges to a with respect to the topology τ.

Proposition 2.4. *Let $\mathfrak{A}_0[\|\cdot\|_0]$ be a unital C*-algebra and τ a locally convex topology on $\mathfrak{A}_0$. Suppose that τ fulfils the conditions* (T_1)-(T_4). *Then, the locally convex quasi C*-algebra $\widetilde{\mathfrak{A}_0}[\tau]$ over $\mathfrak{A}_0$ satisfies the properties:*

(1) *For every $a \in \widetilde{\mathfrak{A}_0}[\tau]_{cq+}$, the element $1+a$ is invertible and its inverse $(1+a)^{-1}$ belongs to $\mathcal{U}(\mathfrak{A}_0)_+$.*

(2) *For a given $a \in \widetilde{\mathfrak{A}_0}[\tau]_{cq+}$ and any $\varepsilon > 0$, let*

$$a_\varepsilon = a(1+\varepsilon a)^{-1}.$$

Then, $\{a_\varepsilon\}_{\varepsilon>0}$ is a commuting net in $(\mathfrak{A}_0)_+$ such that $a - a_\varepsilon \in \widetilde{\mathfrak{A}_0}[\tau]_{cq+}$ and $a = \tau\text{-}\lim_{\varepsilon\to 0} a_\varepsilon$.

(3) $\widetilde{\mathfrak{A}_0}[\tau]_{cq+} \cap (-\widetilde{\mathfrak{A}_0}[\tau]_{cq+}) = \{0\}$.

(4) *If $a \in \widetilde{\mathfrak{A}_0}[\tau]_{cq+}$ and $b \in (\mathfrak{A}_0)_+$ such that $b - a \in \widetilde{\mathfrak{A}_0}[\tau]_{q+}$, then $a \in (\mathfrak{A}_0)_+$.*

As shown in [4, Section 6], the continuous functional calculus, familiar for C*-algebras, extends to the wedge of quasi-positive elements of a locally convex quasi C*-algebra.

Let now φ be a sesquilinear form on $\mathfrak{A}\times\mathfrak{A}$. Then, φ is called

- *positive*, if $\varphi(a,a) \geq 0$, for each $a \in \mathfrak{A}$;
- *invariant*, if $\varphi(ax,y) = \varphi(x,a^*y)$, for all $a \in \mathfrak{A}$ and $x,y \in \mathfrak{A}_0$;
- *τ-continuous*, if $|\varphi(a,b)| \leq p(a)p(b)$ for some τ-continuous seminorm p on $\mathfrak{A}$ and all $a,b \in \mathfrak{A}$.

Further, if φ is a τ-continuous positive invariant sesquilinear form on $\mathfrak{A}_0\times\mathfrak{A}_0$, then, φ extends by continuity to a τ-continuous positive invariant sesquilinear form $\tilde{\varphi}$ on $\mathfrak{A}\times\mathfrak{A}$. Moreover, let $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ be the GNS-construction for φ (see, for instance, [1, Section 9.1]). Then, π_φ is extended on $\mathfrak{A}$, as follows:

$$\pi_\varphi(a)\lambda_\varphi(x) := \lim_\alpha \pi_\varphi(x_\alpha)\lambda_\varphi(x), \quad \forall\ x \in \mathfrak{A}_0, \tag{2.1}$$

where $\{x_\alpha\}$ is a net in $\mathfrak{A}[\tau]$ with $a = \tau\text{-}\lim_\alpha x_\alpha$. By the very definitions and the τ-continuity of φ, it follows that π_φ is a (τ, τ_{s^*})-continuous $*$-representation of $\mathfrak{A}$. Now, put

$$\mathcal{S}(\mathfrak{A}_0) := \{\tau\text{-continuous positive invariant sesquilinear forms } \varphi \text{ on } \mathfrak{A}_0 \times \mathfrak{A}_0\}.$$

We shall say that the set $\mathcal{S}(\mathfrak{A}_0)$ is *sufficient*, whenever

$$a \in \mathfrak{A} \text{ with } \tilde{\varphi}(a,a) = 0, \forall\ \varphi \in \mathcal{S}(\mathfrak{A}_0), \text{ implies } a = 0.$$

Example 2.5. In the locally convex quasi C*-algebra $\mathcal{M}$ over $\mathcal{M}_0$, constructed in Example 2.2, the family of forms $\{\varphi_\xi; \xi \in \mathcal{D}\}$, where $\varphi_\xi(X,Y) = \langle X\xi | Y\xi\rangle$, $X, Y \in \mathcal{M}_0$, is a sufficient family of τ_{s^*}-continuous, positive, invariant sesquilinear forms.

The next theorem states that every locally convex quasi C*-algebra, with sufficiently many continuous, positive, invariant sesquilinear forms, can be realized as a locally convex quasi C*-algebra of operators as described in Example 2.2.

Theorem 2.6. *Let $\mathfrak{A}[\tau]$ be a locally convex quasi C*-algebra over a unital C*-algebra $\mathfrak{A}_0$. The following statements are equivalent:*

(1) *There exists a faithful (τ, τ_{s^*})-continuous $*$-representation π of $\mathfrak{A}$.*
(2) *The set $\mathcal{S}(\mathfrak{A}_0)$ is sufficient.*

3 Commutative case: a representation theorem

As it is well known, every commutative C*-algebra with unit is isometrically *-isomorphic to the C*-algebra $C(X)$ of continuous functions on a compact Hausdorff space X via the so-called Gel'fand transform. In [3] we tried to extend this classical result to the case of commutative locally convex quasi C*-algebras. The result we got is the following.

Theorem 3.1. *Let $\mathfrak{A}[\tau]$ be a locally convex commutative quasi C*-algebra over a unital C*-algebra $\mathfrak{A}_0$. Then the wedge $\mathfrak{A}[\tau]_{q+}$ of all quasi positive elements of $\mathfrak{A}$ is isomorphic to a wedge of $\mathbb{C}^*$-valued positive continuous functions on a compact space X, which take the value ∞ on at most a nowhere dense subset of X.*

Here $\mathbb{C}^*$ stands for the one-point compactification of $\mathbb{C}$.

The starting point of the proof of Theorem 3.1 consists in extending the Gel'fand transform of $\mathfrak{A}_0$, as much as it is possible, to other elements of $\mathfrak{A}$. More precisely, let

$$\mathfrak{M}(\mathfrak{A}_0, \mathfrak{A}[\tau]_{q+}) := \{ax + y : a \in \mathfrak{A}[\tau]_{q+},\ x, y \in \mathfrak{A}_0\},$$

and denote by $\mathcal{M}(\mathfrak{A}_0)$ the Gel'fand space of $\mathfrak{A}_0$, i.e. the set of all nonzero multiplicative linear functionals on $\mathfrak{A}_0$, endowed with the weak*-topology $\sigma(\mathcal{M}(\mathfrak{A}_0), \mathfrak{A}_0)$. Now, let $a \in \mathfrak{A}[\tau]_{q+}$ and $x, y \in \mathfrak{A}_0$. Suppose x is hermitian. Then, by continuous functional calculus, x is uniquely decomposed in the following way:

$$\begin{aligned} &x = x_+ - x_-, \quad x_+, \quad x_- \in (\mathfrak{A}_0)_+, \quad x_+x_- = 0 \\ &|x| \equiv (x^*x)^{1/2} = x_+ + x_- \in (\mathfrak{A}_0)_+. \end{aligned}$$

These two facts, together with Proposition 2.4, imply that

$$(ax + y)(1 + a|x|)^{-1} \in \mathfrak{A}_0,\ \forall\ a \in \mathfrak{A}[\tau]_{q+} \text{ and } \ x, y \in \mathfrak{A}_0.$$

Hence, the elements $\varphi((1 + a|x|)^{-1})$ and $\varphi((ax + y)(1 + a|x|)^{-1})$ are complex numbers for each $\varphi \in \mathcal{M}(\mathfrak{A}_0)$, so that we can consider the correspondence

$$\varphi' : \mathfrak{M}(\mathfrak{A}_0, \mathfrak{A}[\tau]_{q+}) \longrightarrow \mathbb{C}^* \equiv \mathbb{C} \cup \{\infty\}, \text{ with}$$
$$ax + y \mapsto \varphi'(ax + y) = \begin{cases} \frac{\varphi((ax+y)(1+a|x|)^{-1})}{\varphi((1+a|x|)^{-1})} & \text{if } \varphi((1 + a|x|)^{-1}) \neq 0 \\ \infty & \text{if } \varphi((1 + a|x|)^{-1}) = 0. \end{cases}$$

Then, as shown in [4], for every $\varphi \in \mathcal{M}(\mathfrak{A}_0)$, φ' is well defined and has the following properties:

(1) $\varphi' \supset \varphi$ (i.e., φ' is an extension of φ);

(2) $\varphi'(ax + y) = \varphi'(a)\varphi(x) + \varphi(y)$ and $\varphi'(ax) = \varphi'(a)\varphi(x)$, whenever $a \in \mathfrak{A}[\tau]_{q+}$ and $x, y \in \mathfrak{A}_0$ such that $\varphi'(a)\varphi(x) \neq \infty \cdot 0$;

(3) $\varphi'(a + b) = \varphi'(a) + \varphi'(b)$, for all $a, b \in \mathfrak{A}[\tau]_+$;

(4) $\varphi'(\lambda a) = \lambda\varphi'(a)$, for all $\lambda \in \mathbb{C}$ and $a \in \mathfrak{A}[\tau]_+$, where $0 \cdot \infty = 0$.

Now, for any $a \in \mathfrak{A}[\tau]_{q+}$ and $x, y \in A_0$, we fix the notation:

$$\widehat{ax + y}(\varphi) \equiv \varphi'(ax + y), \quad \varphi \in \mathcal{M}(\mathfrak{A}_0).$$

Then, $\widehat{ax + y}$ is a $\mathbb{C}^*$-valued continuous function on the compact Hausdorff space $\mathcal{M}(\mathfrak{A}_0)$, which takes the value ∞ on at most a nowhere dense subset

of $\mathcal{M}(\mathfrak{A}_0)$. The definitions and facts summarized here are the building blocks of the proof of Theorem 3.1: we refer the reader to [4] for more details.

As is clear, Theorem 3.1 provides an extension to positive elements of $\mathfrak{A}[\tau]$ the Gel'fand transform of $\mathfrak{A}_0$. This extension has, however, some drawbacks: first, it cannot be extended to the whole algebra; second, it is not continuous, in general.

In this section we will give a new representation result which shows that every commutative locally convex quasi C*-algebra can be realized as a locally convex quasi *-algebra of functions. But before going forth, we want to make the reader aware of the fact that while given a compact Hausdorff space, the *-algebra $C(X)$ of continuous functions on X is always a C*-algebra (and by the Gel'fand - Naimark theorem, it is the prototype of any commutative C*-algebra with identity), it is, in general false that by completing $C(X)$ with respect to locally convex topology weaker than the supremum norm $\|\cdot\|_\infty$, we get a locally convex quasi C*-algebra. The main reason is the failure of condition (T_4) (for instance, in the case where $X = [0,1]$, it is very easy to exhibit examples of sequences in the set $\{f \in C(X)_+ : \|f\|_\infty \leq 1\}$ which converge with the L^1-norm to a discontinuous function). On the other hand, if we consider the C*-algebra $L^\infty([0,1])$ and complete it with respect to the L^p-norm ($p \geq 1$) then we get a Banach quasi C*-algebra. This is due to the fact that $L^\infty([0,1])$ is not only a C*-algebra but also a W*-algebra.

Lemma 3.2. *Let $\mathfrak{A}_0[\|\cdot\|_0]$ be a W*-algebra and $\mathfrak{X}$ its predual. Assume that a second norm $\|\cdot\|$ is defined on $\mathfrak{A}_0$, such that*

$$\mathfrak{A}_0[\|\cdot\|_0] \hookrightarrow \mathfrak{A}_0[\|\cdot\|]^\sharp \hookrightarrow \mathfrak{X},$$

where $\mathfrak{A}_0[\|\cdot\|]^\sharp$ is the Banach space dual of $\mathfrak{A}_0[\|\cdot\|]$ and $\hookrightarrow$ denotes a continuous embedding with dense range. Then, the unit ball $\mathcal{U}(\mathfrak{A}_0)$ of $\mathfrak{A}_0$ is $\|\cdot\|$-complete.

Proof. In order to keep the notation lighter, we will consider the embeddings as inclusions. Denote by $<\cdot,\cdot>$ the bilinear form defining the duality between $\mathfrak{X}$ and $\mathfrak{A}_0$ and let $\{x_n\}$ be a sequence in $\mathcal{U}(\mathfrak{A}_0)$ which is Cauchy w. r. to $\|\cdot\|$. Then, the sequence $\{L_n\}$ of bounded linear functionals on $\mathfrak{X}$ defined by

$$L_n(y) =< y, x_n >, \quad y \in \mathfrak{A}_0$$

is also Cauchy in $\mathfrak{A}_0[\|\cdot\|]^\sharp$ and so $L_n \to L$, where L is a $\|\cdot\|$-bounded functional on $\mathfrak{A}_0$. On the other hand,

$$|L(y)| = \lim_{n\to\infty} |L_n(y)| = \lim_{n\to\infty} |< y, x_n >| \leq \lim_{n\to\infty} \|y\|_{\mathfrak{X}}\| \, \|x_n\|_0 = \|y\|_{\mathfrak{X}}.$$

Hence, there exists $x \in \mathfrak{A}_0$ such that

$$L(y) =< y, x >, \quad y \in \mathfrak{X}.$$

The density of $\mathfrak{A}_0$ in $\mathfrak{X}$ implies, finally, the statement. □

From the previous discussion and from Lemma 3.2 we have

Proposition 3.3. *Let $\mathfrak{A}_0[\|\cdot\|_0]$ be a W*-algebra. Assume that a second norm $\|\cdot\|$ is defined on $\mathfrak{A}_0$ such that*

(i) $\|x\| \leq C\|x\|_0$, for every $x \in \mathfrak{A}_0$ and some constant $C > 0$;

(ii) the involution $x \to x^*$ is $\|\cdot\|$-continuous;

(iii) if $\{x_n\}$ $\|\cdot\|$-converges to x and $x_n \geq 0$, for every $n \in \mathbb{N}$, then $x \geq 0$;

(iv) $\mathfrak{A}_0[\|\cdot\|_0] \hookrightarrow \mathfrak{A}_0[\|\cdot\|]^\sharp \hookrightarrow \mathfrak{X}$.

Then $\widetilde{\mathfrak{A}_0}[\|\cdot\|]$ is a Banach quasi C^*-algebra.

Recalling that the algebra $C(X)$ of continuous functions on a compact Hausdorff X is a W*-algebra if, and only if, X is extremely disconnected, we get

Corollary 3.4. *Let X be an extremely disconnected compact Hausdorff space and μ a Borel measure on X. Then $L^p(X,\mu)$ is a Banach quasi C^*-algebra over $C(X)$.*

Let X be a compact Hausdorff space and $\mathcal{M} = \{\mu_\alpha;\ \alpha \in \mathcal{I}\}$ be a family of Borel measures on X. We will assume that the family $\mathcal{M}$ is directed by the order relation $\preceq$ defined as follows:

$$\mu_\alpha \preceq \mu_\beta \Leftrightarrow \int_X f(t)d\mu_\alpha(t) \leq \int_X f(t)d\mu_\beta(t),\ \forall f \in C(X)_+.$$

Let us denote by $\|\cdot\|_{p,\alpha}$ the norm in $L^p(X,\mu_\alpha)$. We endow $C(X)$ with the locally convex topology $\mathsf{t}_{\mathcal{M}}$ defined by the family of norms $\{\|\cdot\|_{p,\alpha};\ \alpha \in \mathcal{I}\}$ and denote by $L^p_\sharp(X,\mathcal{M})$ the completion of $C(X)[\mathsf{t}_{\mathcal{M}}]$. Then, we have

(i) $\|f^*\|_{p,\alpha} = \|f\|_{p,\alpha}, \quad \forall f \in C(X)$

(ii) $\|fg\|_{p,\alpha} \leq \|f\|_{p,\alpha}\|g\|_\infty, \quad \forall f, g \in C(X).$

Therefore $L^p_\sharp(X,\mathcal{M})$ is a commutative locally convex quasi *-algebra over $C(X)$ (but not necessarily a locally convex quasi C*-algebra, as we have seen). It is clear that $L^p_\sharp(X,\mathcal{M})$ can be identified with a subspace of $L^p(X,\mu_\alpha)$, for every $\alpha\in\mathcal{I}$. Furthermore, it is obvious that $L^p_\sharp(X,\mathcal{M})$ contains also noncontinuous functions.

A second interesting space can be constructed if the measures of $\mathcal{M}$ are uniformly bounded; i.e. $\mu_\alpha(X)\leq C$, for every $\alpha\in\mathcal{I}$ and for some $C>0$. In this case, if $p\geq 1$, we can define

$$\|f\|_{p,\mathcal{I}}=\sup_{\alpha\in\mathcal{I}}\|f\|_{p,\alpha},$$

which is a norm on $C(X)$. Then the completion of $C(X)[\|\cdot\|_{p,\mathcal{I}}]$ is a Banach quasi *-algebra and, if X is extremely disconnected, a Banach quasi *-algebra over $C(X)$. We denote this space by $\mathsf{L}^p_{\mathcal{I}}(X,\mathcal{M})$.

Theorem 3.5. *Let $\mathfrak{A}$ be a commutative locally convex quasi C*-algebra over $\mathfrak{A}_0$, with unit 1. Then there exists a directed family $\mathcal{M}$ of Borel measures on the compact space $X:=\mathcal{M}(\mathfrak{A}_0)$ of all nonzero multiplicative linear functionals on $\mathfrak{A}_0$ and a map $\Phi: a\in\mathfrak{A}\to\Phi(a):=\hat{a}\in L^2_\sharp(X,\mathcal{M})$ with the properties*

(i) Φ *extends the Gel'fand transform of elements of* $\mathfrak{A}_0$ *and* $\Phi(\mathfrak{A})\supseteq C(X)$*;*

(ii) Φ *is linear, one-to-one and continuous;*

(iii) $\Phi(ax)=\Phi(a)\Phi(x),\quad \forall a\in\mathfrak{A},\ x\in\mathfrak{A}_0$*;*

(iv) $\Phi(a^*)=\Phi(a)^*\quad\forall a\in\mathfrak{A}$.

Thus $\mathfrak{A}$ can be identified with a subspace of $L^2_\sharp(X,\mathcal{M})$

Proof. Define first Φ on $\mathfrak{A}_0$ as the usual Gel'fand transform

$$\Phi:\ x\in\mathfrak{A}_0\to\hat{x}\in C(X)$$

where X is the space of characters of $\mathfrak{A}_0$.

As is known, the Gel'fand transform is an isometric *-isomorphism of $\mathfrak{A}_0$ onto $C(X)$.

Let $\varphi\in\mathcal{S}(\mathfrak{A}_0)$. Define the linear functional ω on $C(X)$ by

$$\omega(\hat{x})=\varphi(x,1).$$

It is easy to check that ω is bounded on $C(X)$; then by the Riesz representation theorem, there exists a unique positive Borel measure μ_φ on X such that

$$\omega_\varphi(\hat{x})=\varphi(x,1)=\int_X\hat{x}(t)d\mu_\varphi(t),\quad\forall x\in\mathfrak{A}_0.$$

Let $\mathcal{M} \equiv \{\mu_\varphi : \varphi \in \mathcal{S}(\mathfrak{A}_0)\}$. This family is directed since $\mathcal{S}(\mathfrak{A}_0)$ is directed by $\leq$, with $\varphi \leq \psi \Leftrightarrow \varphi(x,x) \leq \psi(x,x)$, for every $x \in \mathfrak{A}_0$.

Let $L^2_\sharp(X, \mathcal{M})$ be a locally convex quasi *-algebra constructed as above. Now, if $a \in \mathfrak{A}$ there exists a net $\{x_j\}_{j\in\Delta} \subset \mathfrak{A}_0$ converging to a w. r. to $\mathfrak{t}_\mathcal{M}$. We have then

$$\|\hat{x}_j - \hat{x}_{j'}\|^2_{2,\varphi} = \varphi(x_j - x_{j'}, x_j - x_{j'}) \leq p(x_j - x_{j'})^2 \to 0$$

for some continuous seminorm p.

Let $\hat{a}$ be the $\mathfrak{t}_\mathcal{M}$-limit in $L^2_\sharp(X, \mathcal{M})$ of $\{\hat{x}_j\}$ and define

$$\Phi(a) = \hat{a}.$$

Evidently, $\|\hat{a}\|_{2,\varphi} = \tilde{\varphi}(a,a)$. This implies that if $\hat{a} = 0$, then $\tilde{\varphi}(a,a) = 0$, for every $\varphi \in \mathcal{S}(\mathfrak{A}_0)$ and thus $a = 0$ for $\mathcal{S}(\mathfrak{A}_0)$ is *-sufficient. Thus, Φ is injective. The proofs of (ii), (iii) and (iv) are straightforward. □

In very similar way, one can prove

Theorem 3.6. *Let $\mathfrak{A}$ be a commutative Banach quasi C^*-algebra over the W^*-algebra $\mathfrak{A}_0$, with unit 1. Then there exists a uniformly bounded directed family $\mathcal{M}$ of Borel measures on the extremely disconnected compact space $X := \mathcal{M}(\mathfrak{A}_0)$ of all multiplicative linear functionals on $\mathfrak{A}_0$ and a linear map $\Phi : \mathfrak{A} \to \mathsf{L}^p_\mathcal{I}(X, \mathcal{M})$, which is injective, continuous and preserves the involution and the algebraic operations. The restriction of Φ to $\mathfrak{A}_0$ coincides with the usual Gel'fand transform.*

Of course these results are very far from the beautiful characterizazion of Gel'fand - Naimark, but we couldn't reasonably expect much more: the essential uniqueness of the structure of a C*-algebra is lost in our situation: several nonequivalent locally convex topologies can be given on $\mathfrak{A}_0$ to construct, in principle, very different locally convex quasi C*-algebras.

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